UPDATING MULTI-FACTOR RISK MODELS AND PORTFOLIO UPDATES

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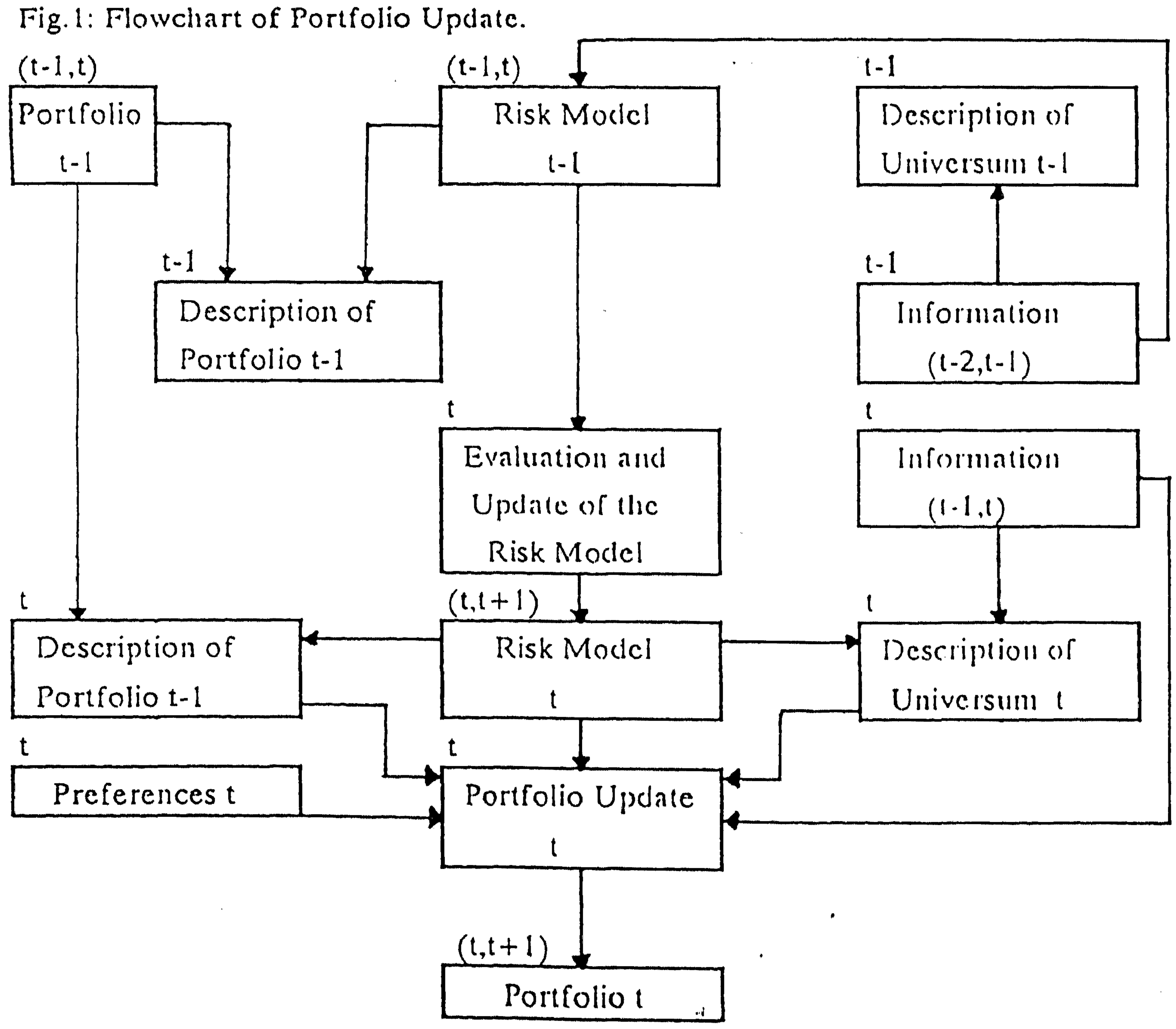
1.Introduction.

In this paper we address the problem of updating portfolios, which are described either partially or in whole by means of a multi factor risk model. Let us, at time t, consider a portfolio, which was constructed at time t-1, for the period (t-1,t). Clearly, during the interval (t-1,t), new information has arrived which can be used (1) to update the portfolio and (2) to re-evaluate the risk model which was originally used in constructing the portfolio at time (t-1). Of course, the portfolio update and the reconsideration of the risk model are interrelated activities, as is shown in figure 1.

Fig.1: Flowchart of Portfolio Update.



New information is the main input for the update of the existing risk model and the evaluation of portfolio attributes. A risk model built using information from the preceding period is supposed to be reliable for the next period, until the changes of the environment require another revision. The same considerations hold for the portfolio composition that is periodically revised.

The revision of the risk model precedes that of the portfolio. The risk model allows evaluations about the actual qualities of the portfolio, about the investment opportunities offered in the market and an estimation of the qualities of the risk model forecasts.

Apart from its risk/return characteristics, other portfolio attributes may enter the decision process (e.g. dividend yield). In addition, the investor's preferences may play their role.

The paper is organized as follows. In sections 2 and 3 we define the components of the risk model and describe their interrelationships. In section 4 and 5, some of the elements of a general framework for the updating process by means of multi-factorial risk modelling are discussed. Some examples of the possible influence of an intermediate update of expectations are given in a technical appendix.

2. Risk models: the multi-factor model.

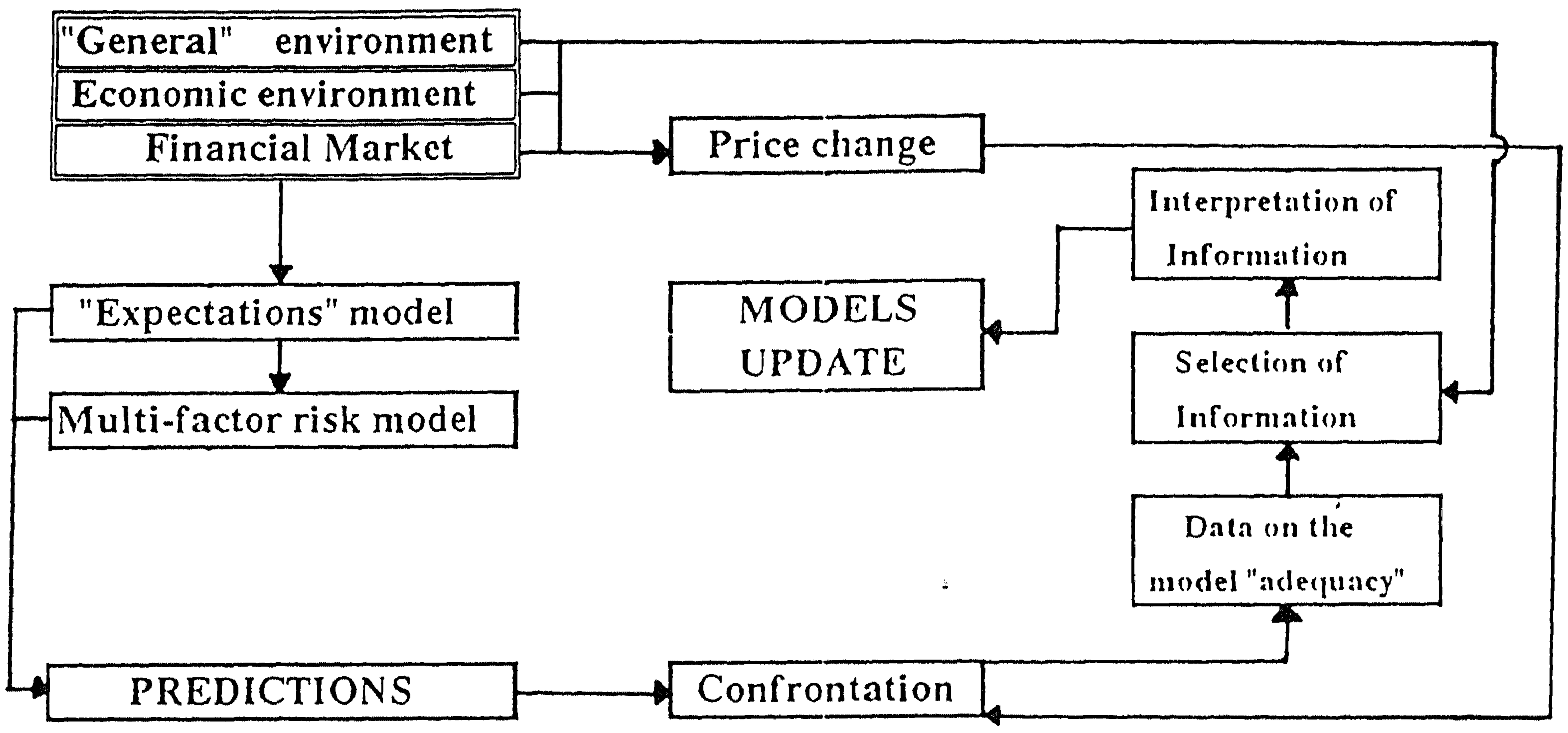
The relation between stock market returns and some economic forces has been analysed by means of multi-factor models. These models show the reaction of returns to unexpected movements of relevant factors. Multi-factor models can be used in risk management. To use these models in practical risk management, the relevant model structure has to be defined, estimated and validated.

The theory behind the model gives a guideline for its implementation. The well known Arbitrage Pricing Theory (see e.g. Ross, 1976) exploits the connection between unexpected components of factors and returns, but from an equilibrium point of view. The APT looks for all the structurable components within the covariance matrix of returns. It assumes the existence of a number of factors, without identifying them. By now, many authors have made empirical studies of the APT. An example is the article of Chen, Roll and

Ross, 1986. The multi-factor models we refer to in this paper are different from the APT models in the sense that they are used to measure the sensitivities of returns for unexpected changes in factors, even if these sensitivities are not priced by the market.

The logic process of model contruction and update is described in figure 2.

Fig.2: Flowchart of Risk Model Update



Price formation is the output of the processes in the financial market that can be thought of as a subset of the economic environment or of the general environment.

Information from all of those entities is used to model the process of **expectations** about returns and factors. **Sensitivities** are derived from expectations as a residual component and expressed through the estimated parameters of the multi-factor model. Sensitivities of returns measured by means of the risk model can be combined into a multidimensional risk profile of an investment possibility. Thus, we switch from a generic probability distribution of returns to a combination of probability distributions of unexpected movements of factors.

The two connected models, the expectations and the sensitivities model, identify some behavioral characteristics of the financial markets that have to be checked over time. Because of new information from the environment, predictions should be compared with realized reactions. As a consequence, the models may need to be re-estimated and sometimes maybe even re-structured.

The structure of the model is the following:

$$r_{(a,b)} - \bar{r}_{(a,b)} = \beta \Delta f_{(a,b)} + \varepsilon_{(a,b)}; \qquad (1)$$

where $r_{(a,b)}$ stands for the return over the period (a,b), $\bar{r}_{(a,b)}$ is the expected value of it, β stands for the sensitivity coefficient, $\Delta f_{(a,b)}$ stands for the value of the unexpected changes of a factor over the period (a,b) and $\varepsilon_{(a,b)}$ are the residuals over the same period.
The model links the unexpected component of returns through a sensitivity coefficient to unexpected changes of a factor.

The model may be specified in different ways. For instance the sensitivity factor may be constant or time dependent, the same holds for the distribution term and the model may relate to only one or to a multiplicity of factors.

A Multi-Factor model strucutere is defined at the beginning of the period, taking account of all the available information at that time, and is supposed to be valid until the end of the period.

We describe the main choices to be made using a multi-factorial model in risk management.

Some choices have to be made at the very beginning, such as the definition of returns and the choice between an additive and a multiplicative formulation. Furthermore, it has to be decided how to handle the data on factor movements. That is because the information can come from different sources and may refer to different time intervals. Another choice concerns the procedure to select the expected components of returns and factors through the expectations model.

Other choices have to be made as the "reference" environment changes over time. It has to be decided which new information is relevant and when it is necessary to use it for an updating.

3. Definitions.

Return: $r_{(a,b)}$; A return expressed as a percentage is a ratio between the price at the end of the period, p_t, and that one of the beginning, p_{t-1}, making the assumption that no dividends or other payments are distributed in the mean time. As it is shown in R. A. Haugen, 1990, one can easily adapt this definition to take account of dividends and other payments.
In our notation $r^t_{(a,b)}$ stands for the return, as calculated/estimated at time t, over the period (a,b).

Let p_0, the price at time 0, be the fixed basis. Then

$$r^{a}_{(a,\tau)} = (p_\tau - p_a)/p_0; \qquad r^{a}_{(\tau,b)} = (p_b - p_\tau)/p_0;$$
$$r^{a}_{(a,b)} = (p_b - p_a)/p_0 = (p_b - p_\tau + p_\tau - p_a)/p_0 = r^{a}_{(a,\tau)} + r^{a}_{(\tau,b)}; \tag{2}$$

Thus assuming returns as price indices, these can be added in order to get the return over the combined period.

Unexpected movements of factors: $f_{(a,b)}$; The expected and unexpected movements of factors may also refer to different time periods. We define $\Delta f^{*}_{(a,b)}$ as the realized change, $\Delta f_{(a,b)}$ as the unexpected change and $\bar{\Delta} f^{t}_{(a,b)}$ as the expected change, all over the period (a,b). The superscript refers to the time at which the expectation is formulated. For the unexpected changes the superscript is dropped in case of coincidence with the beginning of the period.

It is easily seen that the following relations hold:

$$\Delta f_{(a,b)} = \Delta f^{*}_{(a,b)} - \bar{\Delta} f^{a}_{(a,b)};$$
$$\Delta f_{(a,b)} = \Delta f^{*}_{(a,\tau)} + \Delta f^{*}_{(\tau,b)} - \bar{\Delta} f^{a}_{(a,b)} \qquad a<\tau<b; \tag{3}$$

Expected returns: $\bar{r}_{(a,b)}$; In general, if our expectations with respect to factor values change over time, then also our expectations with respect to the return will change. If we assume that the sensitivities and the properties of the distributions of the residuals do not change over time and

the model for formulating the expectation $\bar{r}$ does not change, after rewriting (1)

$$r_{(a,\tau)} = \bar{r}^{a}_{(a,\tau)} + \beta \Delta f_{(a,\tau)} + \varepsilon_{(a,\tau)}; \tag{4}$$

and assuming that our expectation with respect to the factor movements change at time τ, we get:

$$\bar{r}^{\tau}_{(\tau,b)} = \phi_\tau \,[\, I^{a},\ \Delta f_{(a,\tau)}, (b-\tau)] \quad \text{if} \quad I^{\tau} = I^{a} \tag{5}$$

In which, I^t denotes the information set at time t. In equation (8) we assume that the only new information from time a to time τ is the unexpected change of factors recorded by the model over the same period. In the general case we assume that $I^a \subseteq I^b$ if a<t<b and $\bar{r}^{\,t}_{(a,b)} = \phi_t[I^t,(b-a)]$;

Expected changes of factors: $\Delta f_{(a,b)}$; We can build the relation between $\bar{\Delta} f^{\,a}_{(\tau,b)}$ and $\bar{\Delta} f^{\,\tau}_{(\tau,b)}$ in the same way we did with expected returns, saying that:

$$\bar{\Delta} f^{\,\tau}_{(\tau,b)} = \bar{\Delta} f^{\,a}_{(\tau,b)} + \psi_\tau(\Delta f_{(a,\tau)}); \tag{6}$$

With this structure we use the unexpected change as an input for the update of expectations through a function that can vary over time, depending on the perceived importance of past experience for future results.

The general case can be expressed as a function of the available new information with $\bar{\Delta} f^{\,t}_{(a,b)} = \bar{\Delta} f^{\,a}_{(a,b)} + \psi_t(I^t - I^a)$.

4. Information process and risk models.

We present a procedure for the evaluation and use of new information in risk management.

The multi-factor model allows for returns which differ from the return predicted by the sum of the expectations component and the multi-factor component. However, one has to take account of the possibility that the multi-factor model has not been specified correctly. For instance, one may have overlooked some of the systematic risk factors. Also, the assumed linear form of the model might be incorrect. For the moment, it is assumed that the use of a linear specification is justified.

Apart from possible misspecifications, it may well be that the sensitivities in the model are changing over time.

The portfolio manager who wants to verify (ex-post) the adequacy of the multi-factor model used, can observe different entities:

(1) the actual return

(2) the actual change of factors included in the model

By deducting the original expectation from the return, the actual unexpected return is found. Part of this can be explained by the unexpected change of the factors in the model (found by deducting the expected change of those factors from their actual change) multiplying these by their corresponding sensitivities. The remainder is explained by the model through the disturbance term.

On basis of this information some questions can be raised.
The first question is whether the actual unexpected price change is acceptable or not.
The same questions apply to the actual unexpected changes of factors and to the rest term. Of course, the answers to these three questions are mutually interdependent.

The actual unexpected return depends on the expectations model that is part of the multi factor model, the actual unexpected changes in factors are the inputs used by the multi factor model to explain as much as possible of the unexpected price change, the rest term corresponds to the unexplained part.

Let us assume, first, that the structural form of the model is correct and we have identified all the factors. The analysis starts by considering the disturbance term. The actual disturbance should be compared with the estimated disturbance distribution.The actual disturbance is determined by the following expression:

$$\varepsilon_{(a,b)} = \bar{r}_{(a,b)} - r_{(a,b)} - \beta \Delta f_{(a,b)}; \tag{7}$$

The actual disturbance can be expressed in terms of the estimated standard deviation of the disturbance distribution. Although it is hard to judge from one observation only, one would feel more confortable with small actual disturbances than with bigger disturbances. And, of course, if the actual disturbance would be larger than, say, three times the estimated standard deviation there is a strong incentive to critically analyse the other actual values.

The next elements to consider are the actual unexpected changes of the individual factors. Again, one has to decide whether these actual unexpected changes are acceptable or not, given the historical records of unexpected changes of these factors. In case an actual unexpected change is considered to be unacceptable, the question arises whether the estimated sensitivity for this factor is still valid for such an unusual factor change.

Next, the acceptability of combinations of factor changes should be considered. It is difficult, if not impossible, to give a clear-cut answer to this question. Both statistical and economic arguments might be relevant, but one easily runs into a Gordian knot of reasoning. Of course, if an answer could be given, the same question with respect to the validity of the estimated sensitivities arises.

The last variable to be analysed, still assuming the correctness of the model structure, is the return expectation. It may be that the expectation model used was not the correct one.
If the models for factors expectations need some update, it may be that also the model for return expectation has to be reconsidered.

Thusfar, we assumed that all factors have been identified. However, it may well be that some "sleeping" factors came into action during the period under consideration, causing a further unexpected change in price. In this case another entity is added to the list of observable variables:

(3) the actual change of some factors which were not included, but which might have been relevant.

Past experience and economic theory provide an indication about the possible entering factors. As before, one should first decide whether the unexpected change of these factors were of any significance. The factors resulting from this analysis may then be used to restructure the original model ex-post, leading to a new set of sensitivities which possibly gives a better explanation of the unexpected return.
The restructured model may be adopted for next period, but the main question to be answered is whether the "new" factors represented an exceptional contingency or they represent a systematic change in market behavior.

Undeniably, the different steps in the analysis described above, are to a certain extent, qualitative by nature. It is therefore important to make records of the outcomes of the analysis, thus creating a "knowledge base" which may lead to new insights in the future. Of course, this not only holds for the model at hand but for any model describing a constantly changing reality.

The model is assumed to be able to explain the acceptable changes and disturbances. Information about "model adequacy" should be used to improve the efficacy of the model in the following periods. This can be done through an update of the estimation of sensitivities, without changes in definitions and number of factors, or modifying also the structure of the model. In case we restructure the model, the use of the past experience for adequacy evaluation is quite difficult or even impossible.

The reaction of returns to economic forces movements follows the perception of those movements. Data refers to a certain instant in time, but

it is possible that the market was aware of them at a different moment. For a correct use of the model, we need an estimation of market perceptions.
We can define the characteristics of all data we need, but not all are available. Using a risk model, we should be aware of the limited availability of information as a constraint.
Information is not obtained for free. Some useful data are obtained as a product of the risk management process, with no additional costs. Other relevant information can be captured only outside the system, carring significant financial costs.
The evaluation of the accuracy of new information costs time and money. The use of new information carries additional imprecision risk.

We can distinguish some successive steps in using new information in risk management and each of them presents some critical aspects.
In collecting information, its availability plays a major role for all the subsequent results.
The collected information has to be interpreted. The interpretation process can be partially structured. In this case the connected costs are not high. Otherwise, the interpretation of information carries additional costs. Interpreting new data, we always face additional imprecision risk.
Then, we update the model. The update can be a cheap re-estimation of its parameters or a costly restructure of its formulation. The two possibilities carry different appreciations of the additional mispredicition risk.
By means of predictions provided by the updated model, we can adapt our portfolio. All market transactions are expensive because of transaction costs. Imprecisions of the model will have their impact on the realized returns of the investment. The decision of adaptation can be a "calculated" choice if we compare the appreciation of additional costs and the expected qualities of our predictions, with the expected improvement of returns, due to adaptation.

The time horizon of the investor can also give a guideline for the frequency of adaptation. As it become shorter, the attention of the risk manager is devoted to more detailed predictions of investment opportunities, especially if their risk is not priced.

5. A procedure for updating portfolio.

Given the updated risk model and the other information described in the preceding section, we can structure the decision process of updating portfolios.

The problem of portfolio update can arise at different moments. It can be at the end of the "standard" period of analysis. It can also be that an important event occurs, which may have an influence on expectations, posing a threat to the current strategy or opening new opportunities, thus, requiring a reconsideration of past moves.

The first decision is to update the portfolio or not. Clearly, the portfolio composition should only be changed if it may assumed worthwhile. This is decided comparing the expected advantages of an update with expected costs and additional risks.

If one decide to update the portfolio, information is required with respect to:

a) Description of the characteristics of the present portfolio: expected return, the risk profile according to the most recent risk model and other attributes, e.g. yield, time distribution of dividends etc.

b) Description of available investment opportunities, as above.

c) Description of the set of constraints. This step can be limited to "objective" constraints, such as the maximum amount to be invested, liquidity needed for maturing debts etc. In some cases, more "subjective" constraints can also be considered, e.g. a maximum risk level that can be accepted, a minimum yield etc.

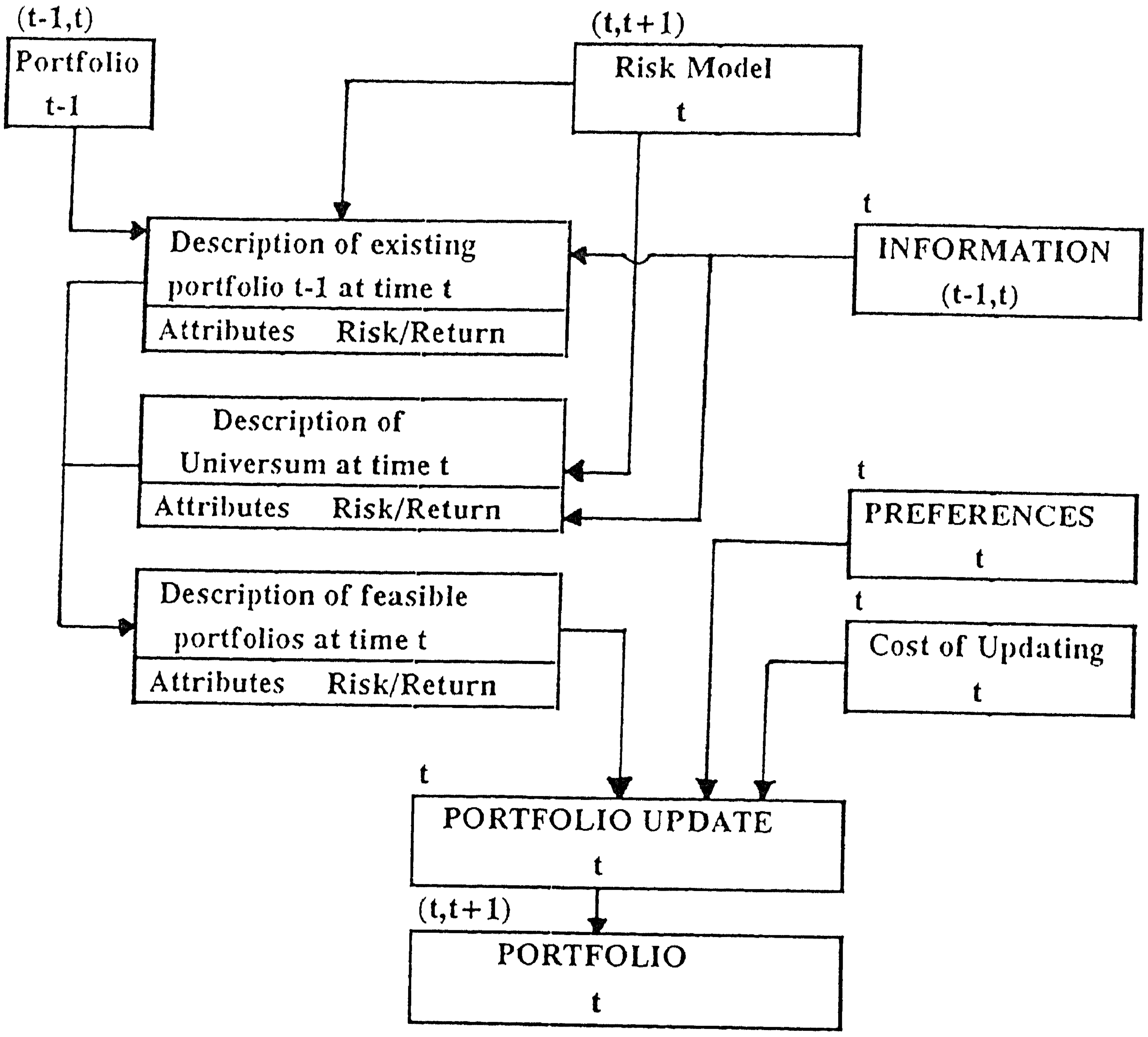
d) Costs of updating: transaction costs.

e) Preference structure of the investor. The subjective constraints already reflect a part of the investor's preferences. Other information about the preferences can be expressed in terms of decision criteria, such as risk, market value, etc.

The decision process of updating portfolios is illustrated in Fig. 3:

Obviusly, there is a large subjective element in the decisions concerning the portfolio composition. Therefore, the proposed procedure is only intended to give a framework for organization and interpretation of the available information.

Fig. 3: Procedure for portfolio update.



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Appendix: contribution of intermediate update of expectations.

The assumptions on the process of expectations' updating are crucial for the results of an updating process:

If we define:

$$\Delta f_{(a,\tau)} = \Delta f^*_{(a,\tau)} - \bar{\Delta} f^{a}_{(a,\tau)};$$

$$\Delta f_{(\tau,b)} = \Delta f^*_{(\tau,b)} - \bar{\Delta} f^{\tau}_{(\tau,b)}; \qquad (8)$$

$$\Delta f_{(a,b)} = \Delta f^*_{(a,\tau)} + \Delta f^*_{(\tau,b)} - \bar{\Delta} f^{a}_{(a,b)} \qquad a<\tau<b;$$

Than the relation:

$$\Delta f_{(a,\tau)} + \Delta f_{(\tau,b)} = \Delta f_{(a,b)} \quad \text{iff} \quad \bar{\Delta} f^{\tau}_{(\tau,b)} = \bar{\Delta} f^{a}_{(\tau,b)}; \qquad (9)$$

should hold

$$\bar{\Delta} f^{a}_{(a,\tau)} + \bar{\Delta} f^{\tau}_{(\tau,b)} = \bar{\Delta} f^{a}_{(a,b)} \qquad \Leftrightarrow$$

$$\Leftrightarrow \quad \bar{\Delta} f^{a}_{(a,\tau)} + \bar{\Delta} f^{\tau}_{(\tau,b)} = \bar{\Delta} f^{a}_{(a,\tau)} + \bar{\Delta} f^{a}_{(\tau,b)} \qquad \Leftrightarrow$$

$$\Leftrightarrow \quad \bar{\Delta} f^{\tau}_{(\tau,b)} = \bar{\Delta} f^{a}_{(\tau,b)}$$

If you follow an updating procedure in which you use all the information available at the beginning of each period, the sum of the unexpected changes in each period is generally different from the unexpected change over the whole period which you would get without updating.

Next, using the definitions of the preceding sections and assuming a time constant value of the sensitivity β, we will identify the contribution that n intermediate updating can give to the final result.

Combining (8) and (6), we get:

$$\Delta f_{(a,b)} = \Delta f^*_{(a,b)} - \bar{\Delta} f^{a}_{(a,b)} =$$

$$= \Delta f^*_{(a,\tau)} + \Delta f^*_{(\tau,b)} - \bar{\Delta} f^{a}_{(a,\tau)} - \bar{\Delta} f^{a}_{(\tau,b)}$$

$$= \Delta f_{(a,\tau)} + \Delta f_{(\tau,b)} + \psi_{\tau}(\Delta f_{(a,\tau)}); \qquad (10)$$

For the return expressions, we get:

$$r^{a}_{(a,\tau)} = \phi_{a}(I^{a}, \tau - a) + \beta\Delta f_{(a,\tau)} + \varepsilon_{(a,\tau)};$$

$$r^{\tau}_{(\tau,b)} = \phi_{\tau}(I^{a}, \Delta f_{(a,\tau)}, b-\tau) + \beta\Delta f_{(\tau,b)} + \varepsilon_{(\tau,b)}; \qquad (11)$$

$$r^{a}_{(a,b)} = \phi_{a}(I^{a}, b-a) + \beta[\Delta f_{(a,\tau)} + \Delta f_{(\tau,b)} + \psi_{\tau}(\Delta f_{(a,\tau)})] + \varepsilon_{(a,b)}; (12)$$

$$r^{a}_{(\tau,b)} = \phi_{a}(I^{a}, b-\tau) + \beta[\Delta f_{(\tau,b)} + \psi_{\tau}(\Delta f_{(a,\tau)})] + \varepsilon^{a}_{(\tau,b)}; \qquad (13)$$

then:

$$r^{a}_{(\tau,b)} - r^{\tau}_{(\tau,b)} = \phi_{a}(I^{a}, b-\tau) - \phi_{\tau}(I^{a}, \Delta f_{(a,\tau)}, b-\tau) + \beta[\psi_{\tau}(\Delta f_{(a,\tau)})] + \varepsilon^{a}_{(\tau,b)} - \varepsilon_{(\tau,b)}; \qquad (14)$$

Changes over the period (a,τ), recorded at time τ are relevant for all variables in the model.

The update of the expected return is expressed through the interpretation of the unexpected change at time τ. If the model to review expectations is time constant, it is possible to make a calculation of the derivative with respect to the recorded changes, or do some simulation.
The second term, given the assumed constancy of the sensitivity, records the adjustment of expectations about factor changes following the past experience and the adaptation of the expectation model.
For the error term, the $\sigma^{2}_{\varepsilon}(t)$ can be supposed to be increasing with respect to time, but this is not sure if the interpretation of new information gives a less predictable situation.

The main result is that changing expectations, which cannot be recorded directly, drive the entire process. It may be that some improvement can be obtained looking to the characteristics of their probability distribution.

The difference of estimations for the total period can be easily formalized as follows:

$$r^{a}_{(a,b)} - r^{\tau}_{(a,b)} = r^{a}_{(\tau,b)} - r^{\tau}_{(\tau,b)}; \qquad (15)$$