A note on "Khouja and Park, Optimal Lot Sizing under continuous price decrease, Omega 31 (2003)"

Ruud Teunter*

February 11, 2004

Econometric Institute Report EI2004-04

Abstract

Khouja and Park [1] analyze the problem of optimizing the lot size under continuous price decrease. They show that the classic EOQ formula can lead to far from optimal solutions and develop an alternative lot size formula using the software package Mathematica. This formula is more exact, but also more complicated. In this note, we study the net present value formulation of the model, and thereby gain an insight that leads to the proposal of a modified EOQ formula. In an extensive numerical experiment, we show that it leads to nearly optimal solutions. It is therefore a good alternative to the formula developed by Khouja and Park, especially if mathematical complexity may hamper implementation.

Keywords: Lot sizing, Economic Order Quantity, price decrease

1 Introduction

Khouja and Park [1] analyze the problem of optimizing the lot size under continuous price decrease. This problem is relevant for the high-tech industry and especially the PC assembly industry, where the prices of components decrease at significant rates. They study

*Erasmus University Rotterdam, Econometric Institute, Burg. Oudlaan 50, PO Box 800, 3000 DR Rotterdam, The Netherlands. Tel. +31 (0)10 4801523. Fax +31 (0)10 4089162. E-mail: teunter@few.eur.nl.

a single-item model with a constant lead time, constant demand, no quantity discounts, and no shortages allowed. However, their model deviates from the standard *Economic Order Quantity (EOQ)* model in two ways: (i) there is a finite planning horizon, and (ii) the purchase price decreases at a constant rate.

Khouja and Park develop an expression for the total cost over the planning horizon using a mixture of the *average cost (AC)* approach and the *Net Present Value (NPV)* approach (see Section 2.1 for details). By setting the derivative of that expression to zero and using a Taylor series approximation for one of the exponential terms, they derive a complex optimality condition for the number of orders during the planning horizon. Using the software package Mathematica, they then find an expression for the number of orders during the planning horizon, which leads to nearly optimal solutions for realistic values of the model parameters. They also develop the corresponding expression for a nearly optimal order quantity.

For a specific example, Khouja and Park illustrate that their order quantity formula indeed leads to a nearly optimal solution. They further show for this example, that the classic EOQ formula, with holding cost per unit of inventory value per time unit equal to the interest rate, results in a far from optimal solution.

As mentioned above, Khouja and Park use a mixture of the AC approach and the NPV approach in deriving their total cost expression. In this note, we instead develop a 'pure' NPV expression. Although the numerical difference between the expressions is small for examples with realistic parameter settings, the pure NPV expression leads to the important insight that the holding cost per unit of inventory value per time unit in the 'corresponding' AC approximation is equal to the interest rate plus the rate of price decrease. We therefore propose a modified version of the classic EOQ formula with holding cost per unit of inventory value per time unit equal to the interest rate plus the rate of price decrease.

We illustrate for the example of Khouja and Park, that the modified EOQ formula leads to a nearly optimal solution. An extensive numerical experiment shows that this result also holds in general. Combining this near-optimality with the simple structure of the EOQ formula that many practitioners are familiar with, we conclude that the modified

EOQ formula has great practical value.

The remainder of this paper is organized as follows. In Section 2 we review the model and the results of Khouja and Park [1]. In Section 4 we apply the pure NPV approach and present our results. We end with conclusions in Section 5.

2 Model and results of Khouja and Park

Section 2.1 describes the model analyzed by Khouja and Park. Their analysis and results are summarized in Section 3.

2.1 Model of Khouja and Park

The model is based on the following standard EOQ assumptions.

- No quantity discount are given.
- The lead time is constant.
- No shortages are allowed.
- The demand rate is constant.

But the model differs from the classical EOQ model in the following ways.

- The price decreases at a constant percentage over time.
- The planning horizon is finite.

The objective is to minimize the total cost over the planning horizon. Relevant costs are: the ordering cost (per order), the purchase cost (per unit of product), and the holding cost (per unit of inventory value per time unit). Note that the entire holding cost is expressed per unit of inventory value. From now on, we will refer to this cost as the *interest cost*. This will avoid confusion in Section 4, where we argue that the holding cost should also include the rate of price decrease.

The notations that Khouja and Park use are listed in Table 1. That table also includes additional notations that will be used in Section 4.

INSERT TABLE 1

3 Results of Khouja and Park

Since the price of the product is C_0 at time 0 and decreases by u per cent per time unit, it holds that

$$C(t) = C_0 e^{-bt},$$

where

$$b = -\ln(1 - u/100) \tag{1}$$

is the *continuous price decrease rate*.

The total cost over the planning horizon is therefore

$$\begin{aligned} TC &= nS + \sum_{i=0}^{n-1} \left[\frac{DTC_0 e^{-biT/n}}{n} + e^{-biT/n} rC_0 \int_{s=0}^{T/n} (T/n - s) D ds \right] && (2) \\ &= nS + \sum_{i=0}^{n-1} \left[\frac{DTC_0 e^{-biT/n}}{n} + \frac{DTC_0 e^{-biT/n}}{2n} \frac{rT}{n} \right] \\ &= nS + \frac{C_0 DT e^{-bT} e^{bT/n} (e^{bT} - 1)(2n + rT)}{2n^2 (e^{bT/n} - 1)} && (3) \end{aligned}$$

By setting the differential of (3) to zero and using a two-term Taylor series approximation of $e^{bT/n}$, a complicated (5th degree polynomial with exponential terms) optimality condition for n is obtained. Using the software package Mathematica, the following approximation for n (ignoring its discreteness) is then derived.

$$\tilde{n} = \sqrt{\frac{C_0 DT(b + r)(e^{bT} - 1)}{2e^{bT} bS}} - \frac{bT}{2}. \tag{4}$$

The corresponding approximation for the optimal order quantity is

$$\tilde{Q} = \frac{DT}{\tilde{n}} = \frac{2e^{bT/2} D\sqrt{bST}}{\sqrt{2C_0 D(b + r)(e^{bT} - 1)} - be^{bT/2}\sqrt{bST}}. \tag{5}$$

However, since n should be discrete, Khouja and Park propose to round $\tilde{n}$ to the closest integer and adjust the order quantity accordingly. This is illustrated in Example 1. In that example, the optimal order quantity (5) is also compared to the classic Economic Order Quantity (EOQ) defined by Khouja and Park as

$$EOQ_c = \sqrt{\frac{2SD}{rC_0}}, \tag{6}$$

We remark that to avoid dependency of the classic EOQ on the unit of time, an alternative definition is

$$EOQ'_c = \sqrt{\frac{2SD}{aC_0}}, \tag{7}$$

where, similar to (1), the continuous interest rate a is defined as (note that r is not defined as a percentage)

$$a = -\ln(1-r). \tag{8}$$

This alternative definition will turn out to be insightful in Section 4. For realistic values of r (smaller than 0.25), we have $a = -\ln(1-r) \approx r$ and hence $EOQ'_c \approx EOQ_c$.

Example 1 *Consider the following example: demand* $D = 100,000$ *units/year, ordering cost* $S = 300$ *\$/order, interest cost* $r = 8\%/\$/\text{year}$*, price decrease* $u = 1\%/\text{week}$*, initial price* $C_0 = 8\$$*, planning horizon* $T = 1$ *year, and time unit 1 year.*

Applying (1) gives a continuous price decrease per week of 0.01005034*. However, since the time unit is 1 year, we get* $b = 52 \times 0.01005034 = 0.52262$*. We remark that Khouja and Park falsely state that* b *is equal to* 0.01005034*, but that the other reported results are based on the correct value* $b = 0.52262$*.*

Applying (4), (5), and (3) gives $\tilde{n} = 24.8$*,* $\tilde{Q} = 4,040$*, and* $TC = 638,112$*. The adjusted solution with* n *discrete is:* $n = 25$*,* $Q = 4,000$*, and* $TC = 638,113$*. This is, in fact, the global optimal solution.*

The classic EOQ is $EOQ_c = 9,682$ *with corresponding number of orders* $n = DT/9,682 = 10.3$ *and total cost* $TC = 644,540$*. The adjusted classic EOQ with* n *discrete is:* $n = 10$*,* $Q = 10,000$*, and* $TC = 645,050$*. So the total cost of the adjusted classic EOQ is* $645,050 - 638,112 = 6,938$ *(or 1.1%) higher than that of the global optimal solution.*

Note that the cost increase of 1.1% is very small, considering that the adjusted classic EOQ is $10,000$ *and hence 150% higher than the optimal adjusted optimal lot size of* $4,000$*. This is explained by the fact that a large part of the total cost in (3) can not be influenced by the lot size. Indeed, since no shortages are allowed, there is a non-influential purchase cost of*

$$\int_0^T DC_0 e^{-bt} dt = \frac{DC_0}{b}\left(1 - e^{-bT}\right),$$

which is equal to 623,069 for this example. So, using the classic EOQ increases the influential cost from 638,113-623,069=15,043 to 645,050-623,069=21,981, i.e., increases the influential cost by a considerable 46.1%.

4 A pure NPV formulation

The cost expression (2) from Khouja and Park contains both Average Cost (AC) and Net Present Value (NPV) elements. As in any AC inventory model, there is no proper discounting of purchase and ordering costs. Instead, an interest cost is r is charged per unit of inventory value per unit of time. However, the price decrease is modelled NPV-like using a continuous discounting rate b.

In this section, we will apply the 'pure' NPV approach. The NPV can easily be written as

$$\begin{aligned} NPV &= \sum_{i=0}^{n-1}\left[Se^{-aiT/n}\right] + \sum_{i=0}^{n-1}\left[\frac{DTC_0e^{-biT/n}}{n}e^{-aiT/n}\right] && (9)\\ &= \sum_{i=0}^{n-1}\left[Se^{-aiT/n}\right] + \sum_{i=0}^{n-1}\left[\frac{DTC_0e^{-(a+b)iT/n}}{n}\right] && (10)\\ &= \frac{S(1-e^{-aT})}{1-e^{aT/n}} + \frac{C_0DT(1-e^{-(a+b)T})}{n(1-e^{(a+b)T/n})}. && (11)\end{aligned}$$

For realistic values of the model parameters, NPV and TC are approximately the same up to a constant. This implies that they have approximately the same optimal value for n. The main advantage of using the NPV, however, is that the above expressions are more insightful than (2). Indeed, note from (10) that the order cost are discounted by factor a, whereas purchase costs are discounted by factor $a+b$. We will use this insight to propose a modification of the classic EOQ formula.

The logic underlying the usual AC approximation of an NPV model is to include a holding cost per unit of inventory value per time unit that is equal to the discount factor for purchase costs (since those determine the inventory value) in the NPV model. Since the purchase costs are discounted by factor $a+b$ in (10), that factor should be used as the holding cost in the AC approximation. For the interested reader, a more formal argument is given in the appendix. This reasoning leads us to propose the following modified EOQ formula:

$$EOQ_m = \sqrt{\frac{2SD}{(a+b)C_0}}. \tag{12}$$

A comparison with (7) shows that EOQ_m differs considerably from EOQ_c if b is large compared to a, i.e. if the rate of price decrease is large compared to the interest rate. In Example 2, we compare the modified EOQ to the classic EOQ and to the optimal order quantity in a continuation of Example 1.

Example 2 *The settings of the model parameters are the same as for Example 1. Applying (12) gives $EOQ_m = 3512$. The corresponding value for n is 28.4. Rounding n gives 28 orders of size 3571. In Table 2 this solution corresponding to the modified EOQ is compared to the solutions (with n discrete) corresponding to EOQ_c and $\widetilde{Q}$. We remark that for this example, the solution corresponding to $\widetilde{Q}$ minimizes both TC and NPV.*

INSERT TABLE 2

The modified EOQ performs well for this example. The increase in TC or NPV compared to the optimal solution is less than 0.02%.

To check whether the modified EOQ performs well in general, we test it in an extensive numerical experiment. The initial price is normalized at 10. All other parameters can have three different values: $D = \{1000, 10000, 100000\}$, $S = \{100, 300, 1000\}$, $r = \{0.05, 0.1, 0.2\}$, $T = \{0.5, 1, 2\}$, and $52u = \{0.2, 0.4, 0.6\}$. This gives $3^5 = 243$ examples in total. Table 3 summarizes the results.

INSERT TABLE 3

The table shows that the performance of the modified EOQ is well in general. The average increase in the NPV compared to the optimal solution is only 0.03%. Moreover, the increase was less than a half per cent for each of the 243 examples. In comparison, the average increase in the NPV for the classic EOQ is 0.98% and the maximum increase is more than 10%.

5 Conclusion

A Net Present Value (NPV) formulation of Khouja and Park's model with a continuously decreasing price, lead to the insight that the holding cost per unit of inventory value per time unit in the corresponding Average Cost (AC) model should be equal to the sum of the interest rate and the rate of price decrease. This lead to the proposal of a modified EOQ formula.

In an extensive numerical experiment, it was shown that the modified EOQ is a good approximation of the optimal order quantity. The average increase in the NPV compared to the optimal solution was only 0.03%, which is much lower than the 0.98% for the classic EOQ.

The order quantity formula proposed by Khouja and Park is even more accurate. In the same experiment, it gives an average increase in the NPV of 0.001%. However, that formula is rather complex, which may hamper its implementation in situations where users have limited mathematical skills. In such situations, the modified EOQ formula is a good alternative, especially if users are already familiar with the classic EOQ formula.

Acknowledgement: The research of Dr. Ruud H. Teunter has been made possible by a fellowship of the Royal Netherlands Academy of Arts and Sciences.

A AC approximation of the NPV model with an infinite planning horizon

The NPV is given by

$$NPV = \frac{S(1-e^{-aT})}{1-e^{aT/n}} + \frac{C_0 DT(1-e^{-(a+b)T})}{n(1-e^{(a+b)T/n})}. \tag{13}$$

For the infinite horizon case ($n \to \infty$) this can be simplified using $Q = DT/n$ to

$$NPV^{\infty} \quad = \quad \frac{S}{1-e^{aQ/D}} + \frac{C_0 Q}{1-e^{(a+b)Q/D}}. \tag{14}$$

A McLaurin expansion of the ordering cost term gives

$$\frac{S}{1-e^{aQ/D}} = S\left(\frac{D}{aQ} + \frac{1}{2} + O(a)\right) = \frac{SD}{aQ} + \frac{S}{2} + O(a)$$

and the corresponding *annuity stream* (discount factor a) is

$$a\left(\frac{SD}{aQ}+\frac{S}{2}+O(a)\right)=\frac{SD}{Q}+\frac{aS}{2}+O(a^2).$$

A McLaurin expansion of the purchase cost term gives

$$\frac{C_0Q}{1-e^{(a+b)Q/D}}=C_0Q\left(\frac{D}{(a+b)Q}+\frac{1}{2}+O(a+b)\right)=\frac{C_0D}{(a+b)}+\frac{C_0Q}{2}+O(a+b)$$

and the corresponding annuity stream (discount factor $a+b$) is

$$(a+b)\left(\frac{C_0D}{(a+b)}+\frac{C_0Q}{2}+O(a+b)\right)=C_0D+\frac{(a+b)C_0Q}{2}+O((a+b)^2).$$

The *linearization* of the total annuity stream is therefore

$$\frac{SD}{Q}+\frac{aS}{2}+C_0D+\frac{(a+b)C_0Q}{2}.$$

Note that, except for the term $aS/2$ which is independent of the order quantity Q, the same expression results from an AC approach if the holding cost per unit of inventory value per time unit is set to $a+b$. Therefore, minimizing the linearized total annuity stream leads to the modified EOQ formula in (12).

References

[1] Khouja and Park. Optimal lot sizing under continuous price decrease. *Omega: The International Journal of Management Science*, 31:539–545, 2003.

D	Demand per unit of time
S	Ordering cost
r	Interest cost per unit of inventory value per unit of time
a	Continuous interest rate corresponding to r
u	Per cent price decrease per unit of time
b	Continuous price decrease rate corresponding to u
T	Length of the planning horizon
n	Number of orders during the planning horizon
C_0	Price per unit of product at time 0

Table 1: Notation.

Method	n	Q	TC	NPV
EOQ_c	10	9682	645,050	621,325
EOQ_m	28	3571	638,200	614,878
$\widetilde{Q}$	25	4000	638,112	614,763

Table 2: Comparison of the solutions (with n discrete) corresponding to EOQ_c, EOQ_m, and $\widetilde{Q}$ for Example 2 (continuation of Example 1).

	Average error percentage	
Method	n	NPV
EOQ_c	48.27%	0.978%
EOQ_m	11.09%	0.026%
$\widetilde{Q}$	1.86%	0.001%

Table 3: Summary of the results for the numerical experiment with 243 examples. The solutions (with n discrete) corresponding to EOQ_c, EOQ_m, and $\widetilde{Q}$ are compared. For each solution, the average error percentages compared to the optimal solution (that minimizes the NPV) are given.