An easy derivation of the order level optimality condition for inventory systems with backordering

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Abstract

We analyze the classical inventory model with backordering, where the inventory position is controlled by an order level, order quantity policy. The cost for a backorder contains a fixed and a time-proportional component. Demand can be driven by any discrete process. Order lead times may be stochastic and orders are allowed to cross. The optimality condition for the order-level, given some predetermined order quantity, is derived using an easy and insightful marginal cost analysis. It is further shown how this condition can easily be (approximately) rewritten in well-known forms for special cases.

1 Introduction

We analyze the classical continuous review inventory system with backordering, where the inventory position is controlled by an order level, order quantity policy, in a rather general setting. We derive the optimality condition for the order level r, given some predetermined order quantity Q. Other authors have done so in the past, but our *marginal cost approach* is easier and more insightful. The main advantage of this approach is that there is no need to find an expression for the average cost (per time unit) associated with an (r, Q) policy. The analysis in Section 4.7 of Hadley and Whitin [1] shows how complicated that is, even for the relatively simple case where demand is driven by a unit Poisson process and the replenishment order lead time is deterministic. Furthermore, as we will show, the optimality condition that is obtained using the marginal approach can easily be (approximately) rewritten in well-known forms for special cases. Together with a derivation of the EOQ formula, our derivation and rewriting of the optimality condition for r could be explained to managers and students with a limited mathematical background.

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2 System description

We study an inventory system with backordering, where the inventory position is controlled by an order level, order quantity (r, Q) policy. Demand can be driven by any discrete process (e.g. unit Poisson or compound Poisson). Demand splitting is applied. That is, all available stock on hand is used to satisfy (backordered) demands, either fully or partially. Demands are satisfied on a first come first served (FCFS) basis. There are no assumptions about the process that generates replenishment order lead times. So, order lead times may be stochastic and orders are allowed to cross.

The order level, order quantity (r, Q) policy places a replenishment order of size Q whenever the inventory position (on hand + on order - backorders) drops to or below the order level r. The order quantity Q is assumed to be predetermined, for instance using the well-known EOQ formula. See, for instance, Section 5.2 of Silver, Pyke and Peterson [2] for a derivation of the EOQ-formula.

The objective is to find an optimality condition for the order level r. That condition should determine the order level that minimizes the average cost (per time unit) over an infinite planning horizon, i.e., in period $[0, \infty)$. The relevant costs are the cost of holding inventory and the backorder cost. The cost for holding one item during t time units is ht $(h > 0)$. The cost for backordering one demanded item during t time units is $b + \hat{b}t$ $(b \geq 0, \hat{b} \geq 0, b + \hat{b} > 0)$.

We remark that because of this (per item) backorder cost structure and because demand splitting is applied, a demand for multiple items can be treated as multiple demands for a single item that occur simultaneously. So a compound demand process can be treated as a unit demand process. For ease of presentation and without loss of generality, we will therefore assume that there is a unit demand process in what remains.
The following terminology is used:

D	average demand per time unit
$\pi(r, Q)$	policy with order level r and fixed order quantity Q
r^*	optimal order level
h	holding cost per item per time unit
b	cost per demand not met directly (fixed backorder cost)
$\hat{b}$	backorder cost per item per time unit (time-proportional backorder cost)
Δ	difference in average cost between policies $\pi(r, Q)$ and $\pi(r-1, Q)$
$\Pr_1$	probability associated with policy $\pi(r, Q)$ that the arrival of a replenishment order increases the stock on hand from zero to a positive value
$\Pr_2$	probability associated with policy $\pi(r, Q)$ of zero stock on hand at a random time

3 Derivation of the order level optimality condition

In this section, we use a marginal cost approach to derive an optimality condition for r^*. We find an expression for the marginal cost Δ, i.e. for the difference in average cost between policies $\pi(r, Q)$ and $\pi(r-1, Q)$, by comparing those two policies in period $[0, \infty)$. In doing so, we assume that the inventory position of policy $\pi(r, Q)$ is 1 higher at time 0. It is further assumed that both policies have the same number of outstanding replenishment orders, and that the remaining lead times of those orders are also equivalent. As a result, the difference in the net inventory at time 0 is also exactly one item. We remark that it is not unfair to compare these two policies with different starting inventory positions, since those positions do not effect the average cost over the infinite planning horizon.

The key observation is the following. *For any possible sequence of demand realizations after time t, the ordering times of policies $\pi(r, Q)$ and $\pi(r-1, Q)$ coincide, and the difference in both the inventory position and the net inventory is always 1 item.* That is, increasing the order level by one item results in shifting both the net inventory and the inventory position up by one item. This is illustrated in Figure 1.

INSERT FIGURE 1 ABOUT HERE

Hence, the only cost-effecting differences between policies $\pi(r, Q)$ and $\pi(r-1, Q)$ are the following.

- Policy $\pi(r, Q)$ has one more item on hand if its net inventory is positive (i.e. if there are items on hand) and one less item backordered otherwise.
- Those demands that reduce the stock on hand to 0 under policy $\pi(r, Q)$, are satisfied immediately under policy $\pi(r, Q)$ but not under policy $\pi(r-1, Q)$.

Obviously, there is exactly one replenishment order arrival that increases the stock on hand from 0 to a positive value between two successive demands that reduce it to 0, and vice versa. So we get

$$\begin{aligned}\Delta &= h(1 - \mathrm{Pr}_2) - \hat{b}\mathrm{Pr}_2 - b\mathrm{Pr}_1 \frac{D}{Q} \\ &= h - (h + \hat{b})\mathrm{Pr}_2 - b\mathrm{Pr}_1 \frac{D}{Q}. \end{aligned} \tag{1}$$

If this marginal cost is increasing in r (from a negative value for $r = -\infty$ to a positive value for $r = \infty$), then r^* is the largest value of r for which $\Delta < 0$, i.e.,

$$\begin{array}{c} r^* \text{ is the largest value of } r \text{ for which} \\ (h + \hat{b})\mathrm{Pr}_2 + b\mathrm{Pr}_1 \frac{D}{Q} > h. \end{array} \tag{2}$$

For cases with $b = 0$, i.e., if there is no fixed backorder cost, it is easy to see that the marginal cost is indeed increasing in r. For cases with $b > 0$, however, this is not necessarily true and hence (2) does not necessarily hold. Of course, even when the marginal cost is not increasing in r, (1) can still be used to determine the optimal order level.

In general, the probabilities Pr_1 and Pr_2 are not easy to calculate. If replenishment orders do not cross, then Pr_1 is equal to the probability that lead time demand is at least

r and at most $r+Q-1$. If the lead time is deterministic (say L) and $Q=1$, then $\Pr_2$ is equal to the probability that demand during a random period of length L is at least $r+1$. The latter follows from the well-known result for systems with a deterministic lead time L, that the net inventory at a random time t is equal to the inventory position at time $t-L$, which is always $r+1$ if $Q=1$, minus the demand in period $[t-L,t]$.

We will not further discuss the calculation of $\Pr_1$ and $\Pr_2$. Instead, we will (approximately) rewrite the order level optimality condition (2) in an easier form for the two special cases with respectively $b=0$ and $\hat{b}=0$. We additionally discuss how that easier form relates to a (nearly) optimal service level.

For the special case that $b=0$, (2) can be simplified to

$$\Pr_2 > \frac{h}{h+\hat{b}}. \tag{3}$$

Noting that the *ready rate* is $1-\Pr_2$, it follows that the ready rate associated with the optimal order level is approximately $h/(h+\hat{b})$ if $b=0$. We remark that the ready rate is equal to the *fill rate* if demand is driven by a Poisson process. This is due to the Poisson Arrivals See Time Averages (PASTA) property. See e.g. Section 1.8 of Tijms [3].

For the special case that $\hat{b}=0$ and using the approximation $\Pr_2 \approx 0$, (2) can be approximately rewritten as

$$\Pr_1 = \frac{hQ(1-\Pr_2)}{bD} \approx \frac{hQ}{bD}. \tag{4}$$

If the order quantity is large, then the arrival of an order will almost surely increase the stock level to a positive value, and hence the *cycle service level* associated with the optimal order level is approximately $1-\Pr_1 \approx 1-hQ/bD$.

References

[1] G. Hadley and T.M. Whitin. *Analysis of Inventory Systems.* Prentice-Hall, Englewood Cliffs, New York, 1963.

[2] E.A. Silver, D.F. Pyke, and R. Peterson. *Inventory Management and Production Planning and Scheduling.* John Wiley and Sons, New York, 1998.

[3] H.C. Tijms. *Stochastic modelling and analysis : a computational approach.* Wiley series in probability and mathematical statistics, 1986.

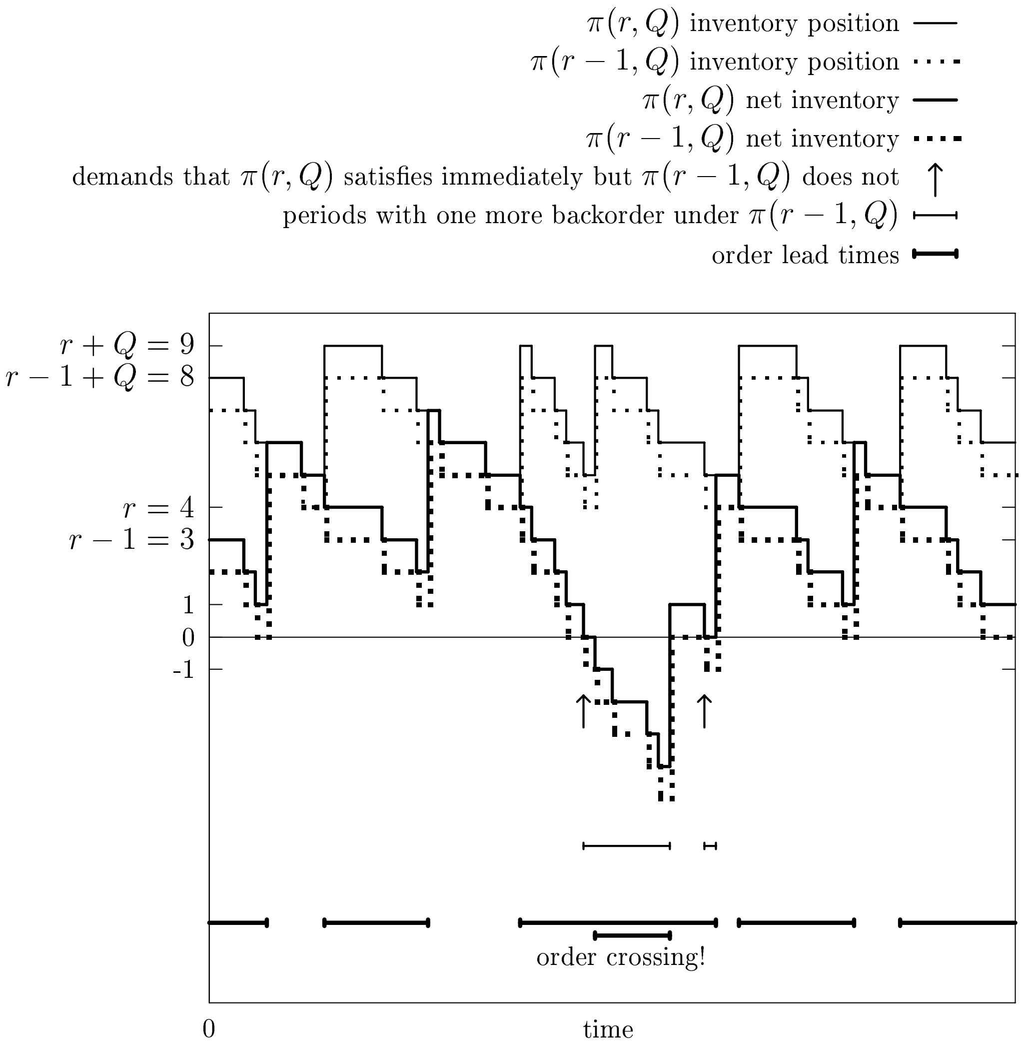

Figure 1: A comparison of policies $\pi(r,Q)$ and $\pi(r-1,Q)$ for $r=4$ and $Q=5$. The (remaining) lead time realizations and demand realizations are chosen arbitrarily.