Wanted:

A Test for FSD Optimality of a Given Portfolio

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ABSTRACT AND KEYWORDS	
Abstract	FIRST-ORDER STOCHASTIC DOMINANCE (FSD) is one of the fundamental concepts of decision making under uncertainty, relying only on the assumption of nonsatiation, or decision makers preferring more to less. There exist well-known, simple algorithms for establishing FSD relationships between a pair of choice alternatives. Unfortunately, these algorithms have limited use in applications with more than two choice alternatives. The analysis of investment portfolios is one such application; investors generally can form a large number of portfolios by diversifying across individual assets. For such applications, there is a need to develop an algorithm for establishing if a given portfolio represents the optimal solution for at least some nonsatiable investor, i.e., is in the FSD optimal set.
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A test for FSD optimality of a given portfolio

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FIRST-ORDER STOCHASTIC DOMINANCE (FSD) is one of the fundamental concepts of decision making under uncertainty, relying only on the assumption of nonsatiation, or decision makers preferring more to less. There exist well-known, simple algorithms for establishing FSD relationships between a pair of choice alternatives. Unfortunately, these algorithms have limited use in applications with more than two choice alternatives. The analysis of investment portfolios is one such application; investors generally can form a large number of portfolios by diversifying across individual assets. For such applications, there is a need to develop an algorithm for establishing if a given portfolio represents the optimal solution for at least some nonsatiable investor, i.e., is in the FSD optimal set.

Bawa *et al*. (1985) and Kuosmanen (2004) propose FSD tests that apply under more general conditions than a pairwise test does. The two tests differ in subtle way. While Bawa *et al*. consider all convex combinations of the distribution functions of a given set of assets, Kuosmanen considers the distribution function of all convex combinations of a given set of assets. Each of these two tests captures an important aspect of portfolio choice that is not captured in a pairwise FSD test. Still, as we will show in this note, both aspects are needed for a true test of FSD optimality and hence both tests generally give a necessary but not sufficient condition for establishing if a given portfolio is in the FSD optimal set. We therefore call for extensions of the two tests that do give necessary and sufficient conditions for FSD optimality. Meanwhile, the researcher is probably well-advised to use both tests in combination; the one test may correct erroneous optimality classifications by the other test.

I. Preliminaries

The investment universe consists of N assets, associated with return vector $\boldsymbol{x} \in D$ from closed and convex domain $D \subset \mathbb{R}^N$ and associated with cumulative distribution function (CDF) $\Phi(\boldsymbol{x}) : D \to [0,1]$. Investors may diversify between the assets, and we will use $\boldsymbol{\lambda} \in \Lambda_N$ for a vector of portfolio weights from a portfolio possibilities set Λ_N that takes the form of a basic simplex in $\mathbb{R}^N$.[1] Apart from Λ_N, we will also use the set of individual assets $\mathrm{X} \equiv \Lambda_N \cap \{0,1\}^N$. For a given portfolio $\boldsymbol{\lambda} \in \Lambda_N$, we may describe the return distribution by the marginal CDF $\Phi_{\boldsymbol{\lambda}}(y) \equiv \int_{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\lambda} \le y} \partial\Phi(\boldsymbol{x})$. Investors are nonsatiable and are characterized by utility functions $u \in U_1$, where U_1 represents the set of increasing von Neumann-Morgenstern utility functions. Apart from U_1, we will also use the set of increasing and concave utility functions U_2 and the set of increasing and convex utility functions U_R.

[1] The same analysis applies for an arbitrary polytope of general form, provided we replace the individual assets with the vertices of the polytope.

The classical definition of the FSD rule compares a pair of choice alternatives. A given portfolio $\boldsymbol{\tau} \in \Lambda_N$ is FSD dominated by another portfolio $\boldsymbol{\lambda} \in \Lambda_N$ if and only if all nonsatiable investors prefer $\boldsymbol{\lambda}$ to $\boldsymbol{\tau}$:

$$E[u(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\lambda})] \geq E[u(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\tau})] \quad \forall u \in U_1 \tag{1}$$

with a strict inequality for at least some $u \in U_1$.[2]

Bawa *et al.* and Kuosmanen generalize this definition to the case where more than two portfolios can be formed. To allow for a compact presentation, it is useful to introduce the following two statistics for any $U \subseteq U_1$, $\Lambda \subseteq \Lambda_N$:

$$\xi(\boldsymbol{\tau}, U, \Lambda) \equiv \min_{u \in U} \max_{\boldsymbol{\lambda} \in \Lambda} E[u(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\lambda})] - E[u(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\tau})] \tag{2}$$

$$\psi(\boldsymbol{\tau}, U, \Lambda) \equiv \max_{\boldsymbol{\lambda} \in \Lambda} \min_{u \in U} E[u(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\lambda})] - E[u(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\tau})] \tag{3}$$

DEFINITION 1 Portfolio $\boldsymbol{\tau} \in \Lambda$ is FSD optimal relative to $\Lambda \subseteq \Lambda_N$ iff $\xi(\boldsymbol{\tau}, U_1, \Lambda) = 0$; the portfolio is FSD dominated iff $\xi(\boldsymbol{\tau}, U_1, \Lambda) > 0$.

DEFINITION 2 Portfolio $\boldsymbol{\tau} \in \Lambda_N$ is FSD admissible iff $\xi(\boldsymbol{\tau}, U_1, \Lambda_N) = 0$; the portfolio is FSD inadmissible iff $\xi(\boldsymbol{\tau}, U_1, \Lambda_N) > 0$.[3]

II. Necessary and sufficient conditions

Bawa *et al.* (1985) basically provide a test for FSD optimality relative to all assets X ($\xi(\boldsymbol{\tau}, U_1, \mathrm{X}) = 0$), or "convex FSD" (CFSD; Fishburn, 1974). Kuosmanen basically provides a test for FSD admissibility ($\psi(\boldsymbol{\tau}, U_1, \Lambda_N) = 0$), which he coins "FSD efficiency".[4]

Both tests are important additions to the stochastic dominance methodology. Still, they generally do not provide necessary and sufficient conditions for FSD optimality relative to all portfolios Λ_N. The Bawa et al. test obviously gives a necessary condition for FSD optimality. Specifically, $\mathrm{X} \subseteq \Lambda_N$ directly implies

PROPOSITION 1 $\xi(\boldsymbol{\tau}, U_1, \Lambda_N) = 0 \Rightarrow \xi(\boldsymbol{\tau}, U_1, \mathrm{X}) = 0$

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[2] For the sake of compactness, we will abstract from the well-established equivalent formulations of the FSD rule in terms of the CDFs of the two portfolios.

[3] According to the precise definition of admissibility, a portfolio $\boldsymbol{\tau} \in \Lambda_N$ is FSD admissible iff no other portfolio $\boldsymbol{\lambda} \in \Lambda_N$ FSD dominates the evaluated portfolio $E[u(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\lambda})] \geq E[u(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\tau})] \quad \forall u \in U_1$ with a strong inequality for at least some $u \in U_1$. By contrast, $\xi(\boldsymbol{\tau}, U_1, \Lambda_N) = 0$ requires strong inequality for all $u \in U_1$. In practice, the two definitions are indistinguishable for a continuous return distribution, because arbitrarily small data perturbations suffice to change the classification based on a weak inequality.

[4] We don't use the term "efficient" in this note, because "efficient" is sometimes used to mean "admissible" and sometimes to mean "optimal".

Similarly, the Kuosmanen test gives a necessary condition. By construction, if the evaluated portfolio is optimal for some utility function than there exists no portfolio that yields a higher expected utility for all utility functions. Hence,

PROPOSITION 2 $\xi(\tau, U_1, \Lambda_N) = 0 \Rightarrow \psi(\tau, U_1, \Lambda_N) = 0$.

However, the tests generally do not give sufficient conditions. To understand the problem, it is insightful to first consider subsets of U_1 for which the test do give necessary and sufficient optimality conditions. For risk seekers, or $u \in U_R$, expected utility $E[u(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\lambda})]$ is a quasiconvex function of the portfolio weights $\boldsymbol{\lambda}$. Hence, only the vertices of the portfolio set are relevant and we can harmlessly replace Λ_N with X. Hence,

PROPOSITION 3 $\xi(\tau, U_R, \Lambda_N) = \xi(\tau, U_R, \mathrm{X})$.

Similarly, for risk averters, or $u \in U_2$, expected utility $E[u(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\lambda})]$ is a quasiconcave function of the portfolio weights $\boldsymbol{\lambda}$. Also, expected utility is a quasiconvex function of the utility functions and U_2 and Λ_N are convex. Hence, for risk averters, the conditions for Sion's (1958) Minimax Theorem are satisfied and we can harmlessly change the order of the two optimization operators in (2), to find:[5]

PROPOSITION 4 $\xi(\tau, U_2, \Lambda_N) = \psi(\tau, U_2, \Lambda_N)$.

From Proposition 1—4, we can see the source of the problem: the Bawa *et al.* test ignores diversified portfolios (it uses X instead of Λ_N), which in general is allowed only for risk seekers, and the Kuosmanen test uses admissibility rather than optimality, which in general is allowed only for risk averters.

III. Numerical example

A numerical example helps to illustrate our point. Table 1 shows the returns to three assets (X, Y and Z) in four states (1, 2, 3 and 4). We will denote the returns to the three assets by $(x_1 \cdots x_4)$, $(y_1 \cdots y_4)$, and $(z_1 \cdots z_4)$, respectively. Also shown is the equal-weighted portfolio $\mathrm{E} = 0.5(\mathrm{X} + \mathrm{Y})$.

Table 1: Example showing that the Bawa et al. test and Kuosmanen test do not give a sufficient condition for FSD optimality. The table shows the return profile of the three individual assets (X, Y and Z) and the equal-weighted portfolio $\mathrm{E} = 0.5(\mathrm{X} + \mathrm{Y})$. No convex combination of X and Y FSD dominates Z and hence Z is in the FSD admissible set of all portfolios. Still, Z is in the FSD dominated set of all portfolios; see Table 2.

t	X	Y	Z	E
1	2	4	2	3
2	2	4	3	3
3	6	0	3	3
4	6	2	5	4

[5] The same result is used by Post (2003; Thm 1).

At first sight, Z seems an unattractive investment for every nonsatiable investor. Z is a mean-preserving spread of the portfolio E, making it unattractive for risk averters. Also, risk seekers are better off by holding X, which offers a higher mean and a higher spread. Still, it is not immediately clear whether Z will be selected by investors with local risk aversion and local risk seeking. Therefore, it is interesting to employ the Bawa *et al.* and Kuosmanen tests.

To implement the Kuosmanen test, we need to solve the following LP problem for each of the $4! = 12$ permutations of $(z_1 \cdots z_4)$, $j = 1, \cdots, 12$:[6]

$$\psi_j \equiv \max_{\lambda_1, \lambda_2} \frac{1}{4} \sum_{t=1}^{4} (\lambda_1 x_t + \lambda_2 y_t - z_{jt}) \tag{4}$$

$$\text{s.t. } (\lambda_1 x_t + \lambda_2 y_t) \geq z_{jt} \quad t = 1, \cdots, 4 \tag{5}$$

$$\lambda_1 + \lambda_2 = 1 \tag{6}$$

$$\lambda_1, \lambda_2 \geq 0 \tag{7}$$

We find $\psi_j = 0$, $j = 1, \cdots, 12$, and hence Z is in the FSD admissible set (not FSD dominated by any convex combination of X and Y).

To implement the Bawa *et al*. test, we need to establish if some convex combination of the CDFs of X and Y dominates the CDF of Z (see Bawa *et al*. (1985, p. 421, Eq. 5)). Table 2 shows the CDFs of the three assets (Φ_X, Φ_Y and Φ_Z) . Note that these CDFs need to be evaluated only at the observed return levels $\{y_j\}_{j=1}^{7} = \{0,1,2,3,4,5,6\}$.

Table 2: Example showing that the Bawa et al. test and Kuosmanen test do not give a sufficient condition for FSD optimality—continued. The table shows the CDFs of the three individual assets (X, Y and Z) and the equal-weighted portfolio $E = 0.5(X + Y)$., as well as the equal-weighted average of the CDFs of X and the CDF of the equal-weighted portfolio. No convex combination of Φ_X and Φ_Y dominates Φ_Z and hence Z is in the FSD optimal set of the individual assets. Still, $0.5(\Phi_X + \Phi_E)$ dominates Φ_Z and hence Z is not in the FSD optimal set of all portfolios.

y	Φ_X	Φ_Y	Φ_Z	Φ_E	$0.5(\Phi_X + \Phi_E)$
0	0	0.25	0	0	0
1	0	0.25	0	0	0
2	0.5	0.5	0.25	0	0.25
3	0.5	0.5	0.75	0.75	0.625
4	0.5	1	0.75	1	0.75
5	0.5	1	1	1	0.75
6	1	1	1	1	1

To test CFSD, we need to solve the following LP problem:[7]

[6] See Kuosmanen (2004) Section 4.1, Eq. 1.
[7] See *Bawa et al*. (1985), Section IC, LP problem at the bottom of p. 423.

$$\xi \equiv \max_{\lambda_1,\lambda_2} \sum_{j=1}^{7} (\Phi_Y(y_j) - \lambda_1 \Phi_X(y_j) - \lambda_2 \Phi_Y(y_j)) \tag{8}$$

$$\text{s.t. } \lambda_1 \Phi_X(y_j) + \lambda_2 \Phi_Y(y_j) \le \Phi_Z(y_j) \quad j = 1,\cdots,7 \tag{9}$$

$$\lambda_1 + \lambda_2 = 1 \tag{10}$$

$$\lambda_1, \lambda_2 \ge 0 \tag{11}$$

Solving this problem, we find $\xi = 0$, and hence Z is in the optimal set of assets (not every nonsatiable investor will prefer X or Y to Z).

Based on the positive outcomes of the two tests, we may be tempted to conclude that Z is the optimal portfolio for some nonsatiable investor. Perhaps surprisingly, this conclusion is wrong. Although there currently exists no test to implement FSD optimality (this is the reason for this note), we may show that Z belongs to the FSD dominated set by adding the equal-weighted portfolio $E = 0.5(X + Y)$ to the Bawa *et al*. test. As is shown in the last column of Table 2 and in Figure 1, $0.5(\Phi_X + \Phi_E)$ dominates Φ_Z and hence every nonsatiable investor would prefer either X or E to Z. It follows directly that Z belongs to the FSD dominated set.

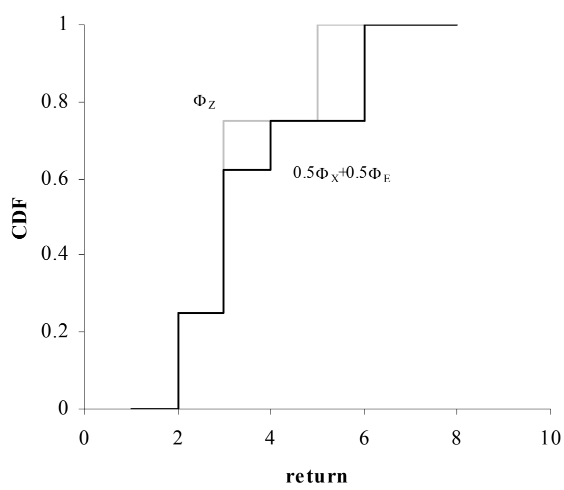

Figure 1: Example showing that the Bawa *et al.* test and Kuosmanen test do not give a sufficient condition for FSD optimality—continued. The figure displays the CDF of Z and the equal-weighted average of the CDF of the equal-weighted portfolio $E = 0.5(X + Y)$. Clearly, the latter CDF dominates the former and hence Z is not in the FSD optimal set of all portfolios. Still, the Bawa *et al.* test and Kuosmanen test do not classify Z as dominated (see Table 1 and 2).

IV. Concluding remarks

To conclude, we find that the Bawa *et al.* test and Kuosmanen test do not give necessary conditions for FSD optimality. Both tests miss some key aspect of a proper optimality test: the Bawa *et al.* test does not account for diversification, while the Kuosmanen test relies on admissibility rather than optimality. Thus both tests may fail to classify a portfolio as FSD dominated. We therefore call for developing a test that does give necessary and sufficient conditions for FSD optimality. This seems a nontrivial task. Kuosmanen's FSD admissibility test already is complex, because we need to account for changes to the ranking of observations as the portfolio weights change, a task that requires integer programming. A true FSD optimality test introduces an additional layer of complexity: we need to consider convex combinations of CDFs in addition to convex combinations of asset returns. In the absence of a FSD optimality test, the researcher is probably well-advised to use the two existing tests in combination; the one test may correct erroneous optimality classifications by the other test. It follows from Proposition 3 and 4 that the combined test are sufficient for risk averters and risk seekers. For investors with local risk aversion and local risk seeking, the combination may give a good approximation.

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