WILCO VAN DEN HEUVEL

The Economic Lot-Sizing Problem

New Results and Extensions

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Het economische lot-sizing probleem:

Nieuwe resultaten en uitbreidingen

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Wilco van den Heuvel
Rotterdam, October 2006

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Chapter 1

Introduction

1.1 General introduction and motivation

The economic order quantity (EOQ) model is probably the most well-known model in inventory theory. Although the economic lot-sizing model is not as well-known as the EOQ model, it has been of great influence in (deterministic) production and inventory planning literature. At the celebration of "50 years of *Management Science*" (one of the leading journals in Operations Research and Management Science) in 2004, the seminal paper on the economic lot-sizing model by Wagner and Whitin (1958) was voted among the ten most influential papers (by INFORMS members). Moreover, it was number two on the list of most cited papers (592 citations).

The economic lot-sizing model will be the main theme in this thesis. We will study issues related to this classical model as well as extensions of the model. In the literature the model is also known as the dynamic lot-sizing model, where 'dynamic' refers to the dynamic nature of the demand in contrast to the constant demand rate as assumed in the EOQ model. In this thesis we will refer to the model as the economic lot-sizing (ELS) model.

Whereas in the EOQ model (Harris, 1913) it is assumed that there is infinite time horizon with a constant demand rate over time, in the ELS model there is discrete and finite time horizon with in each period a (possibly different) quantity of demand. This demand has to be satisfied by ordering in such a way that total costs are minimized. Costs include a fixed order cost for each period an order is made, a unit cost for each item purchased and a unit cost for each time period an item is held in stock. Instead of considering the model in an 'ordering environment', the model can also be considered in a 'production environment'. In this case demand has to be satisfied by producing items. Besides holding cost, there is a fixed setup cost associated with each production period and a unit cost for each item produced.

Because the ELS model is relatively simple and because of new developments in production and inventory management during the last decades, several extensions of the ELS model have been proposed in the literature. Examples of extensions are:

- more general cost functions (e.g., Zangwill (1968), Chan et al. (2002)),
- capacity constraints on production (e.g., Florian et al. (1980), Bitran and Yanasse (1982), Van den Heuvel and Wagelmans (2006a)) and inventory (e.g., Love (1973), Atamtürk and Küçükyavuz (2005)),
- multiple items (e.g., Eppen and Martin (1987), Federgruen et al. (2006), Jans and Degraeve (2006)),
- integration of decisions at different levels of the supply chain (e.g., Zangwill (1969), Van Hoesel et al. (2005)),
- incorporation in game theoretical models (e.g., Federgruen and Meissner (2005), Van den Heuvel et al. (2007)).

For surveys on lot-sizing models we refer to Kuik et al. (1994), Drexl and Kimms (1997), Brahimi et al. (2006) and Jans and Degraeve (2007). In this thesis we will study two other extensions of the ELS model.

In the first extension we include pricing decisions in the ELS model. The demand that a manufacturer has to satisfy is usually created by activities of its marketing department (assuming that the manufacturer has some market power). Instead of taking the marketing and production decisions more or less independently, it may be beneficial to integrate these decisions. This leads to a model that is more complex than when we are only concerned with optimal production decisions. As an example, suppose that the selling price of the item still has to be set and that the demand functions are given for the periods under consideration. Then the planning problem consists of deciding simultaneously how high to set the price and how much to produce in each period such that the total profit is maximized. In this thesis we will consider a joint pricing and lot-sizing model, where demand is assumed to be a deterministic function of price.

In the second extension we incorporate a remanufacturing option in the ELS model. Because of environmental legislation and for economic reasons, companies take back used products from the customer more often. These returned products can be reused in the production process, which is known as a type of 'reversed logistics'. Instead of making new products from scratch, it may be cheaper to remanufacture returned items. Therefore, we develop a model where demand can be satisfied by manufacturing new items or by remanufacturing returned items.

1.2 Outline

This thesis consists of three parts, where in turn each part consists of two chapters. The chapters are written such that they can be read more or less independently. To this end, each chapter starts with an abstract.

In Part I of this thesis we will focus on the classical ELS model with time-invariant cost parameters. In Chapter 2 we are interested in the composition of holding cost and order cost in an optimal solution. It is a well-known property for the EOQ model that holding cost and order cost are equal in an optimal order cycle. The question is whether this property also holds to some extent for the ELS model. In particular, we are interested whether there exists a bound on the total holding cost in an optimal order interval, where an order interval is defined as a consecutive number of periods for which demand is satisfied by a single order.

The analysis of the above problem resulted in the development of a new heuristic for the ELS problem. In the last part of Chapter 2 we analyze the worst case performance of this heuristic and derive some theoretical properties. Unfortunately, it turns out that the heuristic does not generate optimal solutions in the constant demand case (in contrast to several other heuristics). Therefore, we perform a worst case analysis on the algorithm for this case.

In Chapter 3 we analyze the performance of a general class of heuristics for the ELS problem with time-invariant cost parameters. Although the ELS problem can be solved to optimality in polynomial time, many heuristics have been proposed in the literature. One of the reasons for this is that the ELS problem is often solved in a so-called rolling horizon environment. That is, only demands (or demand estimates) for a limited number of periods (the planning horizon) are known. In this case the ELS problem is solved for the planning horizon and the first production decision is implemented in the production schedule. Then the horizon is 'rolled forward' to the period where the next decision has to be made. In this way a complete production schedule for the 'real' horizon can be constructed. It turns out that solving the ELS problem for the short planning horizon by a heuristic may result in better schedules for the real horizon (Stadtler, 2000) than when the problem in the short planning horizon is solved to optimality.

We analyze a class of heuristics which is suitable to be applied in a rolling horizon environment, because decisions are made on a period-by-period basis. By this property the heuristics can be considered as on-line heuristics, since decisions are made while not all future information is known (or used). We develop procedures to systematically construct worst case instances for a fixed time horizon and use them to derive worst case problem instances for an infinite time horizon.

In Part II of this thesis we extend the ELS model by incorporating pricing decisions. We start this part with a general introduction and provide an overview on the literature most related to the integration of pricing and lot-sizing decisions.

In Chapter 4 we consider an ELS model in which it is assumed that a manufacturer can affect his demand by pricing, where in each period demand is a (deterministic) function of price. The problem is now to decide simultaneously how much to produce and what prices to set in each period such that total profit is maximized. In this chapter we focus on a special case of the problem with time-invariant demand and cost parameters considered by Bhattacharjee and Ramesh (2000). They proposed two heuristics for the problem. However, we show that the problem can be solved to optimality by applying existing results in the literature. Application of (a slight modification of) the approach by Thomas (1970) solves the problem in a (practically) efficient way. Moreover, a faster algorithm can be developed by applying the results on a special partition problem derived by Orlin (1985).

In Chapter 5 we impose an additional restriction on the pricing model of Chapter 4. In practice it may not be desirable to set different prices in each time period, because customers may speculate on price changes or price changes may be expensive to communicate to the customers. Therefore, we consider a model with the restriction that prices must be constant over time. Kunreuther and Schrage (1973) proposed a heuristic algorithm for this problem, while Gilbert (1999) proposed a polynomial time algorithm for a special case of the problem. In this chapter, we generalize the work of Kunreuther and Schrage (1973) and Gilbert (1999) by developing a polynomial time algorithm for the general problem.

In Part III we allow for a remanufacturing option in the ELS model. We assume that products return from the customers and these returned products can be remanufactured to satisfy demand (in addition to 'normal' manufacturing). We start this part of the thesis with a general introduction and an overview of the literature on lot-sizing models with a remanufacturing option.

In the model of Chapter 6 we assume that there is a joint setup cost in each period that manufacturing or remanufacturing occurs. This can be the case when manufacturing and remanufacturing are performed on the same production line. We will develop an efficient algorithm to solve this problem under the assumption of time-invariant cost parameters. Furthermore, we show the relation between the ELS problem with a remanufacturing option and the ELS problem with capacities on inventory.

In Chapter 7 we consider a similar model as in Chapter 6 except that we assume a separation in setup cost. In this model there is a setup cost for both manufacturing and remanufacturing. This may be the case when manufacturing and remanufacturing are performed on different production lines. It turns out that under this assumption the problem becomes more complicated. We will show that the problem is already $\mathcal{NP}$-hard in the case of time-invariant cost parameters. Furthermore, we will derive complexity results for some related problems. Finally, we will develop a genetic algorithm for the problem and test it by performing a numerical experiment.

The thesis is ended in Chapter 8. In this chapter we give an overview of the main results.

Part I

New results on the classical problem

Chapter 2

An upper bound on the holding cost for the economic lot-sizing problem with time-invariant cost parameters

Abstract

In this chapter we derive a new property for an optimal solution of the economic lot-sizing problem with time-invariant cost parameters. We show that the total holding cost in an order interval of an optimal solution is bounded from above by a quantity proportional to the setup cost and the logarithm of the number of periods in the interval. Furthermore, we show how this property may be used for the improvement of existing heuristics and for the development of new heuristics. We propose a new heuristic with worst case ratio 2. Furthermore, we show the relation between the number of setups generated by the heuristic and an optimal procedure.

2.1 Introduction

The Economic Order Quantity (EOQ) model is probably the most well-known model in inventory management. There is a constant demand rate D over a continuous infinite horizon, which has to be satisfied by placing orders. For every order there is an order cost K and there is a holding cost h for each item held in inventory for a time unit. If an order occurs every T periods, then the average cost per time unit equals

$$\frac{K}{T} + \frac{1}{2}ThD,$$

where the fist part represents the order cost and the second part represents the holding cost. The optimal length between two orders that minimizes the average cost equals

$T^* = \sqrt{\frac{2K}{Dh}}$. It is a well-known property that setup cost and holding cost are equal for this solution.

In this chapter we consider the discrete version of the EOQ model introduced by Wagner and Whitin (1958), which is often referred to as the economic lot-sizing (ELS) problem in the literature. The model has a finite and discrete time horizon of T periods and in each period t there is a demand d_t ($t = 1, \ldots, T$). As in the EOQ model we have a fixed setup cost K, unit holding cost h and no unit production cost. (Note that the terms setup cost and order cost are both used in the literature dependent on the context of the problem.) The problem is to determine the order periods and quantities such that total costs are minimized. Although the ELS problem can be solved efficiently, many heuristics have been proposed in the literature. A number of those heuristics utilizes some optimality property of the EOQ model. For example, the Silver-Meal (SM) heuristic minimizes the cost per period, the Least Unit Cost (LUC) heuristic minimizes the cost per item, and the Part Period Balancing (PPB) heuristic balances setup cost and holding cost.

In this chapter we are interested in the question whether the properties of the EOQ model also hold for the ELS problem (to some extent) and hence whether it is justified to apply heuristics based on such properties. In particular, we are interested in the relation between holding cost and setup cost in an optimal solution. Clearly, we can have zero holding cost in case setup cost is sufficiently small (resulting in an optimal solution with an order in each period). This raises the question whether there also exist problem instances for which the total holding cost is relatively large compared to the total setup cost.

In this chapter we will show that the total holding cost in an optimal order interval is bounded from above by a quantity proportional to the setup cost and the logarithm of the number of periods in the interval. An order interval is defined as the number of consecutive periods for which demand is satisfied by a single order. In Section 2.2 we will derive this bound. In Section 2.3 we show how this property can be used for the design of heuristics. We propose two new heuristics, analyze the worst case performances, and derive a property on the number of setups. In Section 2.4 we analyze the performance of the best of the two heuristics for the constant demand case. The chapter is completed in Section 2.5 with the conclusion.

2.2 The main result

It is well known that there exists an optimal solution for the ELS problem that satisfies the zero-inventory property, i.e., a new order is placed only if the inventory drops down to zero. This means that in any order period demand is ordered for an integral number of consecutive periods. Consider some order interval in an optimal solution and w.l.o.g. assume it consists of periods $1, \ldots, t$. In this section we will derive an upper bound on the total holding costs in these periods. The idea of the proof is that in an optimal solution it is never profitable to add a setup in any period $p+1$ with $p \in \{1, \ldots, t-1\}$. The total holding cost for the order interval of length t equals

$$H(t) = h \sum_{i=2}^{t} (i-1) d_i.$$

Furthermore, adding a setup in period $p+1$ leads to a saving in holding cost of

$$ph \sum_{i=p+1}^{t} d_i.$$

The following lemmas are used to derive our main theorem.

Lemma 2.1 *Assume there exists a constant $c \geq 0$ and a period $p \in \{1, \ldots, t-1\}$ such that*

$$cph \sum_{i=p+1}^{t} d_i \geq h \sum_{i=2}^{t} (i-1) d_i. \tag{2.1}$$

Then in an optimal solution $H(t) \leq cK$.

Proof Assume that $H(t) > cK$. Then it follows that

$$ph \sum_{i=p+1}^{t} d_i \geq \frac{H(t)}{c} > K.$$

Now adding a setup in period $p+1$ leads to a solution with cost

$$K + H(t) + K - ph \sum_{i=p+1}^{t} d_i < K + H(t).$$

This means that an additional setup in period $p+1$ leads to a cost reduction, which contradicts the fact that the order interval is part of an optimal solution. □

It follows from Lemma 2.1 that if we can find a c that satisfies (2.1) for all demand sequences $d = d_1, \ldots, d_t$, then we have found a bound on the total holding cost in an order interval. Note that a trivial upper bound on c is $t-1$. It is easy to see that (2.1) holds for $c = t-1$ and $p = 1$, which implies that $H(t) \leq (t-1)K$ in an optimal solution. However, this upper bound on the holding cost is immediately obtained from the solution with setups in periods $2, \ldots, t$. Because the order interval is part of an optimal solution, holding cost must be smaller than the cost of the solution with a setup in each period. So we are looking for a bound c that is lower than this trivial bound. In the following lemma we will give a lower bound on the value of c and Lemma 2.3 shows that this bound is tight.

Lemma 2.2 *Given the demand sequence d^0 defined by*

$$\begin{array}{l} d_1^0 > 0, \\ d_i^0 = \frac{1}{i(i-1)} = \frac{1}{i-1} - \frac{1}{i}, \quad i = 2, \ldots, t-1 \\ d_t^0 = \frac{1}{t-1}. \end{array}$$

Then for the sequence d^0 it holds

$$\left(\sum_{i=1}^{t-1} \frac{1}{i}\right) p \sum_{i=p+1}^{t} d_i^0 = \sum_{i=2}^{t} (i-1) d_i^0 \text{ for } p = 1, \ldots, t-1.$$

Proof First, it holds that

$$\sum_{i=2}^{t} (i-1) d_i^0 = \sum_{i=2}^{t-1} \frac{1}{i} + 1 = \sum_{i=1}^{t-1} \frac{1}{i}.$$

Second, for any period $p \in \{1, \ldots, t-1\}$ it holds

$$p \sum_{i=p+1}^{t} d_i^0 = p \left(\sum_{i=p+1}^{t-1} \left(\frac{1}{i-1} - \frac{1}{i} \right) + \frac{1}{t-1} \right) = p \left(\left(\frac{1}{p} - \frac{1}{t-1} \right) + \frac{1}{t-1} \right) = 1,$$

which completes the proof. □

Note that for the problem instance of Lemma 2.2, adding a setup to any period p leads to the same reduction in holding cost. Because the problem instance of Lemma 2.2 is a specific problem instance, for an arbitrary instance we must have $c \geq \sum_{i=1}^{t-1} \frac{1}{i}$. However, the following lemma shows that the bound $\sum_{i=1}^{t-1} \frac{1}{i}$ is tight.

Lemma 2.3 *For any demand sequence $d = d_1, \ldots, d_t$ there exists a period $p \in \{1, \ldots, t-1\}$ such that*

$$\left(\sum_{i=1}^{t-1} \frac{1}{i}\right) p \sum_{i=p+1}^{t} d_i \geq \sum_{i=2}^{t} (i-1) d_i. \tag{2.2}$$

Proof In case $d_i = 0$ for $i = 2, \ldots, t$ the lemma trivially holds. Let $\alpha > 0$ be such that

$$\sum_{i=2}^{t} (i-1) d_i = \alpha \sum_{i=2}^{t} (i-1) d_i^0$$

and define

$$\Delta d = d - \alpha d^0.$$

Then it holds

$$\sum_{i=2}^{t} (i-1) \Delta d_i = \sum_{i=2}^{t} (i-1)(d_i - \alpha d_i^0) = 0.$$

By Lemma 2.2 it is now sufficient to show that there exists a period $p \in \{1, \ldots, t-1\}$ such that

$$p \sum_{i=p+1}^{t} d_i \geq p \sum_{i=p+1}^{t} \alpha d_i^0 \Leftrightarrow p \sum_{i=p+1}^{t} \Delta d_i \geq 0, \tag{2.3}$$

because then

$$\left(\sum_{i=1}^{t-1} \frac{1}{i} \right) p \sum_{i=p+1}^{t} d_i \geq \left(\sum_{i=1}^{t-1} \frac{1}{i} \right) p \sum_{i=p+1}^{t} \alpha d_i^0 = \alpha \sum_{i=2}^{t} (i-1) d_i^0 = \sum_{i=2}^{t} (i-1) d_i.$$

Assume that (2.3) does not hold, i.e., for all $p \in \{1, \ldots, t-1\}$

$$p \sum_{i=p+1}^{t} \Delta d_i < 0 \Leftrightarrow \sum_{i=p+1}^{t} \Delta d_i < 0.$$

But then

$$0 > \sum_{p=1}^{t-1} \sum_{i=p+1}^{t} \Delta d_i = \sum_{i=2}^{t} \sum_{p=1}^{i-1} \Delta d_i = \sum_{i=2}^{t} (i-1) \Delta d_i = 0,$$

which is a contradiction. □

We are now ready to state our main theorem.

Theorem 2.4 *Let periods* $1, \ldots, t$ *be an order interval in an optimal solution of a problem instance with demand sequence* $d_1, \ldots, d_t$*. Then it holds for the total holding cost*

$$H(t) = h \sum_{i=2}^{t} (i-1) d_i \leq K \sum_{i=1}^{t-1} \frac{1}{i}. \tag{2.4}$$

Proof By Lemma 2.3 we have that for any problem instance (2.2) holds. Applying Lemma 2.1 with $c = \sum_{i=1}^{t-1} \frac{1}{i}$ gives the result. □

To derive the bound on the holding cost in an optimal order interval we used the fact that one additional setup in some period can never decrease the total cost. A result by Van Hoesel and Wagelmans (2000) shows that adding more than one setup cannot lead to a cost reduction either. Namely, Van Hoesel and Wagelmans (2000) show that the function $z(n)$ is convex, where $z(n)$ is the optimal cost of the lot-sizing problem with exactly n setups.

A direct consequence of Lemma 2.2 is that there exists problem instances for which the ratio between holding cost and setup cost becomes arbitrarily large. Namely, for the problem instance with demand d^0, $K = 1$, $h = 1$ and T periods an optimal solution is to have only one setup in period 1. For this instance the ratio equals $\sum_{t=1}^{T-1} \frac{1}{t} \to \infty$ as $T \to \infty$. So we have the following theorem.

Theorem 2.5 *There exist problem instances for which the ratio between holding cost and setup cost in an optimal solution is arbitrarily large.*

Theorem 2.5 shows that the PPB criterium is not justified because holding cost and setup cost need not be balanced in an optimal solution.

Finally, note that the sum of the first $t-1$ terms of the harmonic series $\sum_{i=1}^{t-1} \frac{1}{i}$ increases relatively slowly. It is well known that

$$\lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{1}{i} - \log n \right) = \gamma,$$

where $\gamma = 0.577\ldots$ is the Euler-Mascheroni constant. Using this result it is not difficult to show that

$$\sum_{i=1}^{t-1} \frac{1}{i} \leq \gamma + \log t. \tag{2.5}$$

So using (2.4) and (2.5) the holding cost of any optimal order interval satisfies

$$H(t) \leq K(\gamma + \log t), \tag{2.6}$$

which means that the holding cost equals at most a quantity proportional to the setup cost and the logarithm of the number of periods in the order interval.

2.3 Application to heuristics

2.3.1 Heuristic H

Theorem 2.4 immediately suggests a new heuristic for the economic lot-sizing problem with time-invariant costs. The heuristic selects an order interval that covers periods $1, \ldots, t$ with t the largest period that satisfies

$$h \sum_{i=2}^{t} (i-1) d_t \leq K \sum_{i=1}^{t-1} \frac{1}{i}. \tag{2.7}$$

In this way the solution possesses a property that is satisfied by any optimal solution. We will call this heuristic H. Unfortunately, the following example shows that the worst case performance of H can be arbitrarily bad.

Example 2.6 Consider a problem instance with $d_1 > 0, d_T = \frac{K}{h(T-1)} \sum_{t=1}^{T-1} \frac{1}{t}$ and $d_t = 0$ for $t = 2, \ldots, T-1$. Then the heuristic H generates a solution with only a setup in period 1 having a total cost of $C^H = K + K \sum_{t=1}^{T-1} \frac{1}{t}$. However, the optimal solution is to order in periods 1 and T with a total cost of $C^* = 2K$. Clearly, the worst case ratio equals

$$\frac{C^H}{C^*} = \frac{K(1 + \sum_{t=1}^{T-1} \frac{1}{t})}{2K} \to \infty \text{ as } T \to \infty.$$

2.3.2 Heuristic H^*

In Example 2.6 we used $c = \sum_{i=1}^{t-1} \frac{1}{i}$, which is the smallest c that satisfies (2.1) for an arbitrary problem instance. However, this is not the best value of c given a specific problem instance. We can improve H by dynamically updating c. The implied heuristic works as follows. Assume that we arrive in some period t with the last setup in period 1. Calculate the smallest c that satisfies (2.1), say c_t, and check whether

$$h \sum_{i=2}^{t} (i-1) d_t \leq c_t K. \tag{2.8}$$

If the latter inequality holds, proceed with period $t+1$. Otherwise, make an order that covers periods $1, \ldots, t-1$, start a new order in period t and proceed with period $t+1$. We will call this heuristic H^*, the 'refined' version of heuristic H.

Heuristic H^* has a nice interpretation. Note that if we arrive in some period t that does not satisfy (2.8), then there exists some period $p \in \{2, \ldots, t\}$ such that an additional setup in period p leads to a cost reduction. So heuristic H^* chooses the order intervals as large as possible (except for possibly the last order interval) such that no additional setup may improve the solution. We will use this interpretation to determine the worst case performance of H^*.

Theorem 2.7 *The worst case performance of H^* is at most 2.*

Proof Consider a solution for some arbitrary instance d generated by H^* with cost C^{H^*}. Modify this solution by adding a setup in each setup period of the optimal solution (if none yet) and modify the order quantities accordingly. Denote the cost of this solution by $\overline{C}$. Assume that there are n^* setups in the optimal solution so that we add at most $n^* - 1$ setups (both solutions have a setup in period 1) to our heuristic solution. Then we have

$$C^{H^*} \leq \overline{C} \leq C^* + (n^* - 1)K \leq 2C^*$$

with C^* the cost of the optimal solution. The first inequality follows because by definition of our heuristic, adding setups to the heuristic solution cannot improve the solution. The second inequality holds because the new solution has at most $n^* - 1$ additional setups and less holding cost compared to the optimal solution. Furthermore, the last inequality holds because $C^* \geq Kn^*$. In conclusion, we have $\frac{C^{H^*}}{C^*} \leq 2$. □

The following example shows that this bound is tight.

Example 2.8 Consider a problem instance with $T = 2n$ ($n \in \mathbb{N}$), $K = h = 1$, $d_t = 2\varepsilon$ for t odd and $d_t = 1 - \varepsilon$ for t even. For ε sufficiently small heuristic H^* generates a solution with setups in periods $1, 3, \ldots, T-1$ with total cost

$$C^{H^*} = nK + nh(1-\varepsilon) = 2n - n\varepsilon.$$

However, an alternative solution with setup periods $1, 2, 4, \ldots, T$ has a total cost of

$$C^A = (n+1)K + 2h\varepsilon(n-1) = (n+1) + 2(n-1)\varepsilon.$$

If C^* is the cost of the optimal solution, then the worst case ratio satisfies

$$\frac{C^{H^*}}{C^*} \geq \frac{C^{H^*}}{C^A} = \frac{2n - n\varepsilon}{(n+1) + 2(n-1)\varepsilon} \to 2 \text{ for } \varepsilon = \frac{1}{n} \text{ and } n \to \infty.$$

Theorem 2.7 and Example 2.8 show that H^* is a heuristic with worst case ratio 2. This is the best possible worst case ratio for the class of heuristics in which H^* is contained (see Chapter 3). Note that H^* is not contained in the class of heuristics considered by Axsäter (1985) and hence the result cannot be derived from this paper. Theorems 2.9 and 2.10 show that H^* possesses some other nice properties.

Theorem 2.9 *The number of setup periods generated by H^* is at most the number of setup periods in any optimal solution.*

Proof Consider some order interval $r, \ldots, s-1$ of an optimal solution. It is sufficient to show that H^* will generate at most one setup in this interval. Assume this is not the case and let v and w be two consecutive setup periods of the heuristic solution such that $r \leq v < w < s$. By definition of H^* there exists a period $p \in \{v+1, \ldots, w\}$ such that

$$K + h \sum_{i=v+1}^{p-1} (i-v)d_i + K + h \sum_{i=p+1}^{w} (i-p)d_i < K + h \sum_{i=v+1}^{w} (i-v)d_i,$$

which implies

$$K + h \sum_{i=r+1}^{p-1} (i-r)d_i + K + h \sum_{i=p+1}^{w} (i-p)d_i < K + h \sum_{i=r+1}^{w} (i-r)d_i.$$

Now add a setup in period p in the optimal solution. Then the interval $r, \ldots, s-1$ has a cost of

$$\begin{aligned} K + h\sum_{i=r+1}^{p-1}(i-r)d_i + K + h\sum_{i=p+1}^{s-1}(i-p)d_i &= \\ K + h\sum_{i=r+1}^{p-1}(i-r)d_i + K + h\sum_{i=p+1}^{w}(i-p)d_i + h\sum_{i=w+1}^{s-1}(i-p)d_i &< \\ K + h\sum_{i=r+1}^{w}(i-r)d_i + h\sum_{i=w+1}^{s-1}(i-p)d_i &\leq K + h\sum_{i=r+1}^{s-1}(i-r)d_i, \end{aligned}$$

where the last inequality follows from $p > r$. This means we have found a better solution than the optimal solution, which is a contradiction. □

Theorem 2.10 *Let $r, \ldots, s-1$ be some order interval generated by H^*. Then an optimal solution with minimum number of setups has at most 2 setups in this interval.*

Proof Assume we have an optimal solution with consecutive setups in periods $u < v < w$ with $u \geq r$ and $w \leq s-1$ and minimum number of setups. By definition of optimality we have

$$(v-u)\sum_{i=v}^{w-1} d_i > K.$$

That is, the additional cost of removing the setup in period v is larger than the setup cost. Note that we have strict inequality because we assume it is an optimal solution with minimum number of setups. Because $r \leq u$

$$(v-r)\sum_{i=v+1}^{w-1} d_i \geq (v-u)\sum_{i=v+1}^{w-1} d_i > K.$$

This means that adding a setup in period v in the solution generated by H^* would lead to cost reduction and hence H^* should have a setup in period $w-1 < s-1$. But this is a contradiction with our initial assumption. □

Theorem 2.11 *Let n^* be the minimum number of setups in an optimal solution and let n be the number of setups generated by H^*. Then $n \leq n^* \leq 2n$ or $\frac{1}{2}n^* \leq n \leq n^*$.*

Proof Immediate from Theorems 2.9 and 2.10. □

A consequence of Theorem 2.11 is that heuristic H^* will not suffer from generating too many setups compared to the number of setups in an optimal solution. Furthermore, the number of setups is not less than half the number of setups of an optimal solution (with minimum number of setups).

We end this section with some remarks about H^*. First, heuristic H^* can also be adapted for the case of time-varying holding cost by using the interpretation that the heuristic generates order intervals covering as many demand periods as possible such that no additional setup may improve the solution. Similar arguments as used in the proof of Theorem 2.7 show that this heuristic also has a worst case performance of 2. Second, heuristic H^* can be implemented in a backward way having the same worst case performance. These additional properties are similar to the properties of the heuristic presented in Bitran et al. (1984). This heuristic starts a new order if total holding cost in the current order interval exceeds the setups cost. Finally, heuristic H^* can be implemented in linear time since there exists a linear time algorithm for the economic lot-sizing problem with time-invariant costs (see, for instance, Federgruen and Tzur (1991)).

2.3.3 An application of heuristic H

In this section we apply heuristic H to a problem instance presented in Axsäter (1982). This instance is used to show that the SM-heuristic has an infinite worst case performance. We construct a solution based on H and compare it with the SM solution. The problem instance has a demand $d_1 > 0$ and $d_t = \frac{K}{(t-1)h}$ for $t = 2, 3 \ldots, T$. The SM heuristic generates a solution with one setup, resulting in a cost of $C^{SM} = TK$, that is a cost of K per period.

We will now construct a solution for this problem instance based on heuristic H. Assume period p is the last setup period of the current order interval. Then H searches for the largest period t that satisfies

$$h \sum_{i=2}^{t} (i-1) d_{i+p-1} = K \sum_{i=2}^{t} \frac{i-1}{i+p-2} \leq K \sum_{i=1}^{t-1} \frac{1}{i}. \tag{2.9}$$

Note that (2.9) is certainly satisfied when the largest term at the left hand side of (2.9) is smaller than the smallest term at the right side, i.e., when

$$\frac{t-1}{t+p-2} \leq \frac{1}{t-1} \Leftrightarrow t^2 - 3t + 3 + p \leq 0. \tag{2.10}$$

The length of the largest order interval for which (2.10) holds equals

$$t = \left\lfloor \frac{3}{2} + \sqrt{p - \frac{3}{4}} \right\rfloor .$$

For example, when $p = 1$ we make an order for $t = 2$ periods and when $p = 7$ we make an order to cover $t = 4$ periods.

Using (2.6) we have that $C^H_{p,p+t}$, the cost (both setup and holding cost) in the interval $[p, p+t-1]$ of heuristic H, satisfies

$$C^H_{p,p+t} \leq K\left(1+\gamma+\log\left(\frac{3}{2}+\sqrt{p-\frac{3}{4}}\right)\right),$$

whereas $C^{SM}_{p,p+t}$, the cost generated by the SM-heuristic in the interval $[p, p+t-1]$, equals

$$C^{SM}_{p,p+t} = Kt = K\left\lfloor\frac{3}{2}+\sqrt{p-\frac{3}{4}}\right\rfloor.$$

Because

$$\frac{C^{SM}_{p,p+t}}{C^H_{p,p+t}} \geq \frac{\left\lfloor\frac{3}{2}+\sqrt{p-\frac{3}{4}}\right\rfloor}{1+\gamma+\log(\frac{3}{2}+\sqrt{p-\frac{3}{4}})} \rightarrow \infty \text{ as } p \rightarrow \infty,$$

we have that

$$\frac{C^{SM}}{C^H} \rightarrow \infty \text{ for } T \rightarrow \infty.$$

So using the solution based on heuristic H as a benchmark also shows that SM has an infinite worst case ratio as shown by Axsäter (1982).

2.3.4 Combining heuristic H^* with other heuristics

There are different ways to use heuristic H^* in combination with other heuristics. First, given the solution generated by some heuristic, check in each order interval whether there exists a period in which an additional setup leads to a cost improvement. In fact, in this way H^* is applied to every order interval and this may lead to a cost reduction.

Another approach is to combine two heuristics in such a way that a new order is started when one of the criteria of both heuristics is satisfied. Applying a combination of H^* and SM, H^* and LUC, or H^* and PPB (where we take the simple version of Axsäter (1982) where the holding cost are smaller than K in each order interval) to Example 2.8 shows that the worst case ratio of all combinations is still at least 2. Namely, all combinations of heuristics find the same solution as H^*. Moreover, the stopping criterion of PPB dominates the stopping criterion of H^*, which means that applying a combination of both heuristics is the same as just applying PPB.

2.4 The constant demand case with infinite horizon

It is known that SM and LUC generate optimal solutions for the constant demand case. In this section we will analyze the performance of H^* for the constant demand case. As in the EOQ-model we assume an infinite horizon and constant demand for all periods ($d_t = d$ for $t = 1, 2, \ldots$). In the infinite horizon case we want to minimize $C(T)$, the cost per period when the order intervals are of length T, i.e.,

$$C(T) = \frac{1}{T}\left(K + h\sum_{t=1}^{T} d\right) = \frac{K}{T} + \frac{1}{2}(T-1)hd. \tag{2.11}$$

(Note that in this section we use the notation T to specify the length of the order intervals instead of specifying the length of the model horizon.)

2.4.1 Specification of H^*

First, we show how c_t in (2.8) can be determined for the constant demand case. In period t our approach searches for the smallest c_t that satisfies

$$c_t ph \sum_{t=p+1}^{t} d \geq h\sum_{t=2}^{t}(t-1)d. \tag{2.12}$$

Note that

$$C(p) = c_t ph \sum_{t=p+1}^{t} d = c_t p(t-p)hd$$

is minimized for $p = \frac{1}{2}t$ if t is even with $C(\frac{1}{2}t) = \frac{1}{4}c_t hdt^2$ and for $p = \frac{1}{2}t - \frac{1}{2}$ if t is odd with $C(\frac{1}{2}t - \frac{1}{2}) = \frac{1}{4}c_t hd(t-1)(t+1)$. Furthermore, we have

$$H(t) = h\sum_{t=2}^{t}(t-1)d = \frac{1}{2}hdt(t-1).$$

So for the even case (2.12) reduces to

$$\frac{1}{4}c_t hdt^2 \geq \frac{1}{2}hdt(t-1) \Leftrightarrow c_t \geq 2\frac{t-1}{t}$$

and for the odd case

$$\frac{1}{4}c_t hd(t-1)(t+1) \geq \frac{1}{2}hdt(t-1) \Leftrightarrow c_t \geq 2\frac{t}{t+1}.$$

Thus, the smallest value c_t that satisfies (2.12) is $c_t = 2\frac{t-1}{t}$ if t is even and $c_t = 2\frac{t}{t+1}$ if t is odd.

By substituting the values of c_t in (2.12), it follows that heuristic H^* chooses order intervals of size T with T as large as possible and satisfying

$$T^2 \leq \frac{4K}{dh} \text{ and } T \text{ even}$$

or

$$T^2 - 1 \leq \frac{4K}{dh} \text{ and } T \text{ odd.}$$

However, this value of T is not the optimal one in general. Wemmerlöv (1983) shows that the optimal T, say T^*, that minimizes (2.11) is the largest T^* that satisfies

$$T^*(T^* - 1) \leq \frac{2K}{dh}. \tag{2.13}$$

This means that H^* may generate order intervals that cover too many periods. However, Section 2.4.2 will show that the solutions generated by H^* are reasonable. By Lemma 2.1 and because $c_t \leq 2$ for all t, H^* will generate order intervals with holding cost smaller than $2K$. This also follows from

$$H(T) = h\sum_{t=2}^{T}(t-1)d = \frac{1}{2}hdT(T-1) \leq \begin{cases} \frac{1}{2}hdT^2 \leq 2K \text{ if } T \text{ is even} \\ \frac{1}{2}hd(T^2-1) \leq 2K \text{ if } T \text{ is odd.} \end{cases} \tag{2.14}$$

As an example, for $K = 800$, $d = 100$ and $h = 1$, H^* generates order intervals of $T = 5$ periods. This is a solution with cost 2.86% above the optimal cost which is attained with order intervals of $T^* = 4$ periods.

2.4.2 Worst case analysis

As shown by the last small example, heuristic H^* does not necessarily lead to optimal solutions in the constant demand case with an infinite model horizon. In this section we will analyze the worst case performance for this special case. Equations (2.13) and (2.14) show that the order intervals are dependent on the ratio $\frac{K}{dh}$. Because we are interested in the relative performance, we may assume w.l.o.g. that $dh = 1$. Let $T^*(K)$ $(T(K))$ denote the order interval determined by the optimal (heuristic) procedure when the setup cost equals K. So we are performing a parametric analysis on K. As K defines any relevant problem instance, the worst case ratio of heuristic H^* equals

$$\sup_{K>0}\left\{\frac{C(T^*(K))}{C(T(K))}\right\}.$$

First, we will consider the cost function $C(T^*(K))$, the optimal cost per period with setup cost K. Note that for a fixed T the function $C(T)$ is linear and increasing in K. As $C(T^*(K))$ is the lower envelope of a number of linear functions, it follows that $C(T^*(K))$ is continuous and piecewise linear concave. Furthermore, the breakpoints will occur for those values of K for which (2.13) is satisfied with equality, i.e., for $K = \frac{1}{2}T^*(T^* - 1)$. For those values of K we have that $C(T^*) = C(T^* - 1) = T^* - 1$.

The function $C(T(K))$ is piecewise linear but not continuous. From (2.14) it follows that the discontinuities may occur at $4K = T^2 - 1$ with T odd and at $4K = T^2$ with T even. Furthermore, from (2.11) it follows that the slopes of the functions satisfy

$$\frac{\partial C(T^*(K))}{\partial K} = \frac{1}{T^*(K)} \geq \frac{1}{T(K)} = \frac{\partial C(T(K))}{\partial K}$$

as $T^*(K) \leq T(K)$. So for each value K the slope of $C(T^*(K))$ equals at least the slope of $C(T(K))$. This means that to determine the worst case ratio, the only interesting points are the discontinuity points of $C(T(K))$. This can also be seen from the graphs of the functions $C(T^*(K))$ and $C(T(K))$ in Figure 2.1. The thin line represents the function

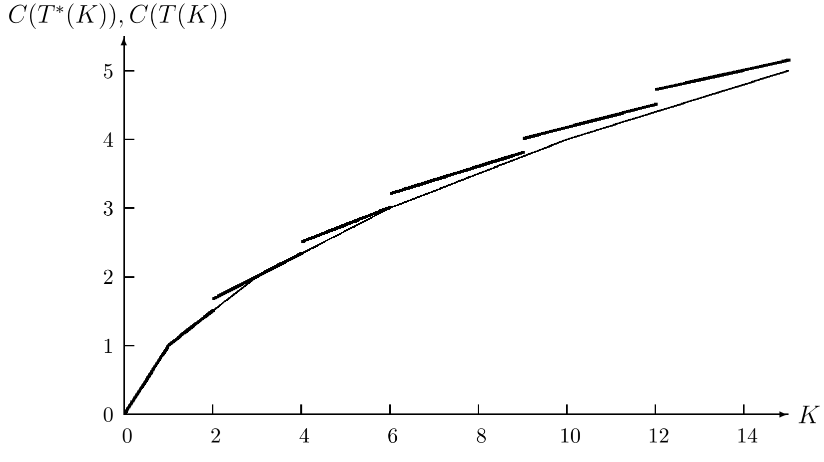


Figure 2.1: Graphs of the cost functions $C(T^*(K))$ and $C(T(K))$

$C(T^*(K))$ and the thick lines represent the function $C(T(K))$.

We will analyze the worst case ratio for T is odd, say $T = 2n+1$ with $n \in \mathbb{N}$. (Note that for $T = 1$ we have $K = 0$ and both the heuristic and optimal solution have cost zero.) In this case the discontinuities occur at $K = \frac{T^2-1}{4} = n^2 + n$. Expressing the cost function in terms of n we have

$$C(n) = \frac{n^2+n}{2n+1} + \frac{1}{2}((2n+1)-1) = \frac{n(n+1)}{2n+1} + n.$$

Let $C^*(n)$ denote the optimal cost for $T = 2n+1$ and $K = n^2+n$. From (2.13) it can be derived that

$$T^*(K) = \left\lfloor \frac{1}{2} + \sqrt{\frac{1}{4} + 2K} \right\rfloor \text{ or } T^*(n) = \left\lfloor \frac{1}{2} + \sqrt{\frac{1}{4} + 2(n^2+n)} \right\rfloor.$$

So we have

$$C^*(n) = \frac{n(n+1)}{\left\lfloor \frac{1}{2} + \sqrt{\frac{1}{4} + 2(n^2+n)} \right\rfloor} + \frac{1}{2}\left(\left\lfloor \frac{1}{2} + \sqrt{\frac{1}{4} + 2(n^2+n)} \right\rfloor - 1 \right).$$

For odd values of T the worst case ratio can now be expressed in terms of n as

$$\sup_{n\in\mathbb{N}} \left\{ \frac{C(n)}{C^*(n)} \right\}.$$

For $n = \{1, 2, 3, 4, \ldots\}$ we have $\frac{C(n)}{C^*(n)} = \{\frac{10}{9}, \frac{16}{15}, \frac{15}{14}, \frac{16}{15}, \ldots\}$. It can be shown that $\frac{C(n)}{C^*(n)}$ is maximal for $n = 1$. Furthermore, we have

$$\lim_{n\to\infty} \frac{C(n)}{n} = \frac{3}{2} \text{ and } \lim_{n\to\infty} \frac{C^*(n)}{n} = \sqrt{2},$$

implying that

$$\lim_{n\to\infty} \frac{C(n)}{C^*(n)} = \lim_{n\to\infty} \frac{C(n)/n}{C^*(n)/n} = \frac{3}{4}\sqrt{2}.$$

For even values of T a similar analysis as above shows that the worst case ratio is smaller than $\frac{15}{14}$ and the limiting value is $\frac{3}{4}\sqrt{2}$. In conclusion, for a given setup cost K the relative error lies in the interval $[0, \frac{1}{9}]$ and the relative error tends to $\frac{3}{4}\sqrt{2} - 1 \leq 6.1\%$ for K sufficiently large.

The graph of the relative error $\varepsilon(K)$ as a function of K can be found in Figure 2.2. We see from Figure 2.2 that H^* generates optimal solutions for the intervals $\langle 0, 2\rangle$ and $[3, 4\rangle$. Furthermore, we see that the maximum error is attained for $K = 2$. For this value of K and for all K in the interval $[2, 3\rangle$ heuristic H^* generates a solution with three periods per order interval, whereas it is optimal to order every two periods.

2.5 Conclusion

In this chapter we presented a new property for an optimal solution of the ELS problem. In any order interval the total holding cost is bounded from above by a quantity proportional to the setup cost and the logarithm of the number of periods in the order interval. Furthermore, we showed by an example that this bound is tight. This means that in an optimal solution the ratio between the holding cost and setup cost can be arbitrarily large. This is in contrast to the classical EOQ model where setup cost and holding cost are perfectly balanced in an optimal solution.

The property was used to construct a new heuristic. In this heuristic a new order is placed when the holding cost exceeds the upper bound. We showed that a refinement of this heuristic has worst case ratio 2. Furthermore, we also showed that a solution

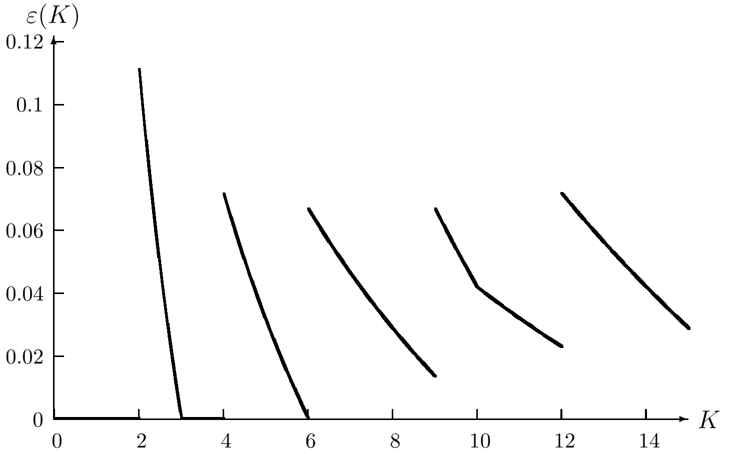


Figure 2.2: The relative error of heuristic H^*

generated by this heuristic has a nice theoretical property: the number of setups of the heuristic solution is at least half the number of setups generated by the optimal solution and at most the number of setups generated by the optimal solution. Finally, we showed that the worst case ratio tends to $\frac{3}{4}\sqrt{2}$ in case of time-invariant demand and sufficiently large setup cost.

Chapter 3

Performance bounds for a general class of on-line lot-sizing heuristics

Abstract

In this chapter we analyze the worst case performance of heuristics for the classical economic lot-sizing problem with time-invariant cost parameters. We consider a general class of on-line heuristics that is often applied in a rolling horizon environment. We develop procedures to systematically construct worst case instances for a fixed time horizon and use them to derive worst case problem instances for an infinite time horizon. Our analysis shows that the heuristics in our class have worst case ratio at least 2 and a worst case ratio at least $\frac{3}{2}$ if we relax some assumptions. Furthermore, we show how the results can be used to construct heuristics with optimal worst case performance for small model horizons.

3.1 Introduction

The economic lot-sizing (ELS) problem is a well-known problem in inventory management and is described as follows. Given the (deterministic) demand for a discrete and finite planning horizon, find a production plan that satisfies demand and minimizes total costs. Costs include setup cost for each time period production takes place and holding cost for each item carried over from a period to the next period.

Although the ELS problem can be solved in polynomial time, heuristics are often used to solve the problem. One reason is that exact algorithms (such as the algorithm by Wagner and Whitin (1958)) are difficult to understand and hence are often not used by practitioners. Furthermore, heuristics are often applied when the ELS problem needs to be solved in a rolling horizon environment. In that situation heuristics may perform better than the Wagner-Whitin-algorithm (see for example Stadtler (2000) and Van den Heuvel and Wagelmans (2005)). Note that in a rolling horizon environment, lot-sizing heuristics can be considered as on-line algorithms, because decisions have to be taken while not all future demand information is known.

Two methods are commonly used to measure the performance of heuristics. First, we have the empirical methods in which a simulation study is performed (see, e.g., Baker (1989), Fisher et al. (2001) and Simpson (2001)). The difficulty of a simulation study is to construct a representative testbed. Second, we have analytical methods which can be split into probabilistic and worst case analysis. Probabilistic methods analyze the expected performance of heuristics given the distribution of some problem parameters (see Axsäter (1988)). In worst case analysis one searches for a bound on the relative performance of heuristics for any problem instance (see Axsäter (1982), Bitran et al. (1984), Axsäter (1985) and Vachani (1992)).

In this chapter we are interested in the worst case performance of heuristics for the ELS problem. As mentioned above several papers on this subject have appeared in the literature. Axsäter (1982) and Bitran et al. (1984) analyze the worst case performance of some specific lot-sizing rules. Vachani (1992) analyzes the worst case performance of seven heuristics, where also data dependence, such as the length of the time horizon and demand properties (constant and bounded demand), is taken into account. The paper that is closest to our research is Axsäter (1985). He shows that all on-line heuristics which use a specific type of decision rule have worst case ratio at least 2. A nice aspect of this result is that it applies to almost all popular heuristics.

Our research was motivated by the following natural questions. First, do there exist on-line heuristics with worst case performance smaller than 2? Second, can we construct problem instances with large performance ratio for a broader class of on-line heuristics than Axsäter (1985)? In this chapter we will provide a positive answer to the last question by showing that a general class of on-line heuristics has worst case ratio at least 2. Although this means that we generalize the result of Axsäter (1985), we would like to emphasize that our approach is (necessarily) completely different than his. In fact, we believe that the actual contribution of this chapter lies not only in the fact that we provide a worst case problem instance, but also in the description of the systematic way in which we have searched for this instance.

This chapter is organized as follows. In Section 3.2 we formally introduce the economic lot-sizing problem and we define our class of on-line heuristics by three properties. In Section 3.3 we show that heuristics satisfying the first property have worst case ratio at least $\frac{3}{2}$, whereas Section 3.4 shows that the heuristics satisfying all three properties have worst case ratio at least 2. In Section 3.5 we show how the analysis of Section 3.3 can be used to construct new heuristics for small time horizons with optimal worst case performance. The chapter is completed with the conclusion in Section 3.6.

3.2 Definitions, problem formulation and observations

We start this section by describing the ELS problem mathematically. If we use the following notation

T : model horizon
d_t : demand in period t
K : setup cost
h : unit holding cost
x_t : production quantity in period t
I_t : ending inventory in period t,

then the ELS problem can be modeled as

$$\begin{array}{rll} C^*(d,T) = & \min & \sum_{t=1}^{T} (K\delta(x_t) + hI_t) \\ & \text{s.t.} & I_t = I_{t-1} - d_t + x_t \qquad t = 1,\dots,T \\ & & x_t, I_t \geq 0 \qquad t = 1,\dots,T \\ & & I_0 = 0, \end{array}$$

where

$$\delta(x) = \begin{cases} 0 & \text{for } x = 0 \\ 1 & \text{for } x > 0. \end{cases}$$

First, note that we may assume w.l.o.g. that $K = 1$ as the objective function only depends on the ratio K/h. Furthermore, we may assume w.l.o.g. that $h = 1$. Namely, defining the variables $x'_t = hx_t$, $I'_t = hI_t$ and $d'_t = hd_t$ leads to the model

$$\begin{array}{rll} C^*(d',T) = & \min & \sum_{t=1}^{T} (\delta(x'_t) + I'_t) \\ & \text{s.t.} & I'_t = I'_{t-1} - d'_t + x'_t \qquad t = 1,\dots,T \\ & & x'_t, I'_t \geq 0 \qquad t = 1,\dots,T \\ & & I'_0 = 0. \end{array}$$

This shows that, when considering the worst case performance of heuristics, it suffices to consider only problem instances with $K = h = 1$. This means that a problem instance is completely defined by a demand sequence $d = d_1, \dots, d_T$. Finally, we may also assume w.l.o.g. that $d_1 > 0$ since otherwise this period can be ignored.

Let d be a problem instance and let $C^H(d)$ be the cost of a solution generated by some heuristic H on instance d. We define the performance ratio $r(d)$ of H for instance d as $r(d) = C^H(d)/C^*(d)$, where $C^*(d)$ is the optimal cost for this instance. Furthermore, the worst case ratio of H is defined as

$$\sup_{d \in I} r(d),$$

where I is the set of all problem instances. From the definitions it follows that the performance ratio is a measure for a particular problem instance d and the worst ratio is a measure for a set of instances.

Axsäter (1985) considers a class of on-line heuristics where a setup is made in period $n+1$ (with the previous setup in period 1) if

$$\sum_{t=1}^{k} a_{tk} d_t \leq 1 \text{ for } k = 2, \ldots, n \text{ and}$$

$$\sum_{t=1}^{n+1} a_{t,n+1} d_t > 1,$$

where a_{tk} $(1 \leq t \leq k \leq T)$ are constants. After the setup assignment to period $n+1$, this period becomes period 1 and the procedure starts again. Axsäter (1985) proves that this class of heuristics has a worst case ratio of at least 2 (and this bound is tight for some heuristics) by considering nine different cases dependent on the properties of the constants a_{tk}.

As in Axsäter (1985) we consider a complete class of on-line heuristics. Our general class of heuristics is defined by the following properties:

Property 1 *Decisions are made period by period (so previously made decisions are fixed and cannot be changed) and the decision whether to have a setup or not in a period does not depend on future demand.*

Property 2 *The decision whether to have a setup or not only depends on the cost of the current lot-size.*

Property 3 *The heuristics are deterministic, i.e., applying the heuristic to the same problem instance leads to the same outcome.*

Property 1 states that the decisions are made starting in period 1 and in every next period we decide whether to make a setup or not irrespective of future demand. Property 2 is a natural assumption. If period s is a setup period, then the decision in period $t > s$ does not affect the cost before period s and hence these costs are not taken into account when making the decision. Property 3 essentially states that the heuristic is consistent. As far as we know no randomized on-line algorithms are known for the ELS problem. Therefore, Property 3 does not seem to exclude any heuristic proposed in the literature. It is clear that the class of Axsäter (1985) is contained in the class of heuristics we consider. This implies that the best worst case ratio of any heuristic in our class is at most 2. The heuristic we proposed in Chapter 2 is a heuristic with worst case ratio 2 and included in our class of heuristics but not in the class considered by Axsäter (1985).

Because we are interested in the best heuristics within our class, some heuristics can immediately be eliminated from the analysis. Assume there is a demand instance with $d_t > 1$ in some period $t > 1$ and assume there is some heuristic H that generates no setup in period t. Let s be the setup period preceding period t. Then the holding cost of demand in period t equals $(t-s)d_t > 1$. So having a setup in period t is less expensive and by this additional setup the holding cost for demands after period t will also decrease. In other words, any heuristics H' that generates the same setups as H including setups in periods with $d_t > 1$ is better than H. Therefore, H can be left out of consideration when analyzing our class of heuristics.

Using this observation, the Lemma 3.1 shows that only problem instances with $d_t \leq 1$ are of interest when we are looking for worst case examples. For this reason we will assume that $d_t \leq 1$ in the remainder of this chapter.

Lemma 3.1 *If there exists a problem instance $d = d_1, \ldots, d_T$ with $d_t > 1$ for some $t > 1$ and a heuristic H satisfying Properties 1-3 and having performance ratio r for this instance, then there also exists a problem instance d' with $d'_t \leq 1$ and performance ratio at least r.*

Proof Let $t > 1$ the smallest period with $d_t > 1$. First, both the optimal solution and any heuristic H will have a setup in period t and assume that they have cost C^* and C^H, respectively. Furthermore, let $C^*_{1,t-1}$ and $C^*_{t,T}$ ($C^H_{1,t-1}$ and $C^H_{t,T}$) be the cost of the optimal (heuristic) solution for periods $1, \ldots, t-1$ and periods $t, \ldots, T$, respectively. Now consider the instances $d^1 = d_1, \ldots, d_{t-1}$ and $d^2 = d_t, \ldots, d_T$ with the modification $d_t = 1$. Because of the properties of our class of heuristics, H will generate a solution with cost $C^H_{1,t-1}$ for d^1 and a solution with cost $C^H_{t,T}$ for d^2. Because

$$C^*/C^H = (C^H_{1,t-1} + C^H_{t,T})/(C^*_{1,t-1} + C^*_{t,T}) = r \Leftrightarrow C^H_{1,t-1} + C^H_{t,T} = rC^*_{1,t-1} + rC^*_{t,T},$$

either

$$C^H_{1,t-1} \geq rC^*_{1,t-1} \Leftrightarrow C^H_{1,t-1}/C^*_{1,t-1} \geq r,$$

or

$$C^H_{t,T} \geq rC^*_{t,T} \Leftrightarrow C^H_{t,T}/C^*_{t,T} \geq r.$$

So in one of the cases we have an instance with performance ratio at least r. For instance d^1 we have that demands equal at most 1. If instance d^2 has performance ratio at least r and it has another period with demand strictly larger than 1, then repeating the above argument will lead to a problem instance with performance ratio at least r and demands at most 1. $\square$

The observation that $K = h = 1$ w.l.o.g. and that $d_t \leq 1$ leads to some interesting insights. First, it is clear that every problem instance has cost at most T: the cost of the trivial lot-for-lot (L4L) heuristic which has a setup in each period. Because the optimal solution has cost at least 1, the worst case ratio of L4L is at most T. Furthermore, if $d_t \geq p > 0$ for all $t = 1, \ldots, T$, then the optimal solution has cost at least p in each period and the worst case ratio of L4L is at most $\frac{T}{Tp} = \frac{1}{p}$. Now look at Table 3.1 where we reproduced the summary of the worst case analysis on the seven heuristics by Vachani (1992, p. 805, Table 2). When we look at instances with a finite time horizon

Heuristic	T	$d_t = d$	$d_t \leq p, p > 0$	$d_t \geq p, p \leq 1$
EOQ	∞	1.059	∞	∞
POQ	T	1.059	∞	∞
SM	$\sqrt{T}/2\sqrt{2} \leq w \leq T$	1	∞	$\frac{1}{p}$
LUC	∞	1	∞	∞
PPB	$3T/(T+2)$	$\frac{3}{2}$	3	$3-2p$
BMY	$2T/(T+1)$	1	2	$2-p$
FC	$\sqrt{T}/2\sqrt{2} \leq w \leq T$	∞	∞	$\frac{1}{p}$

Table 3.1: Data dependent worst case ratios (w) of some heuristics

(column 2) and demand bounded from below (column 5), then it follows that in the worst case only PPB and BMY perform strictly better than the L4L heuristic. The other five heuristics perform as bad or even worse than the L4L heuristic on one of the two problem characteristics. So from a worst case analysis point of view these heuristics perform badly.

3.3 Constructing worst case examples for heuristics satisfying Property 1

In this section we will assume that heuristics satisfy Property 1. So in each period the heuristic 'decides' to start a new order a to add the demand to the current order. By this property worst case performance can be interpreted as a game between a heuristic and an adversary. In each period t the heuristic 'receives' some demand d_t from the adversary and the heuristic has to 'decide' whether to add demand d_t to the current production run (incurring holding cost) or to start a new one (incurring setup cost). Whereas the heuristic wants to minimize the performance ratio, the adversary tries to maximize the performance ratio.

3.3.1 A relaxed mathematical formulation of the problem

It is well known that given a demand sequence $d = d_1, \ldots, d_T$, a solution for the ELS problem is completely determined by its setup periods (the zero inventory property). A production plan (consisting of all setup periods) can be represented by a vector $P \in \{0,1\}^T$ with $P_t = 1$ if t is a setup period and $P_t = 0$ otherwise. As we may assume w.l.o.g. that demand in period 1 is positive, $P_1 = 1$. Let $P(T)$ be the set of all production plans of T periods. Let $d = d_1, \ldots, d_T$ be a demand sequence and $P \in P(T)$ a production plan. Let $C(d, P, t)$ be the cost of the first t periods for demand sequence d in production plan P, i.e.,

$$C(d, P, t) = \sum_{i=1}^{t} \left(P_i + (i - p(i))d_t\right), \tag{3.1}$$

where $p(i)$ is the setup period preceding period i (or period i itself if i is a setup period). Then the performance ratio for instance d and plan P is defined as

$$\max_{t=1,\ldots,T} \frac{C(d, P, t)}{C^*(d, t)}.$$

Note that we take the maximum over all periods as every sequence $d_1, \ldots, d_t$ represents a problem instance for the ELS problem (the adversary can stop at any moment or the demand beyond period t can be set equal to zero).

Now consider a binary tree of depth T representing the set $P(T)$ (see Figure 3.1). In

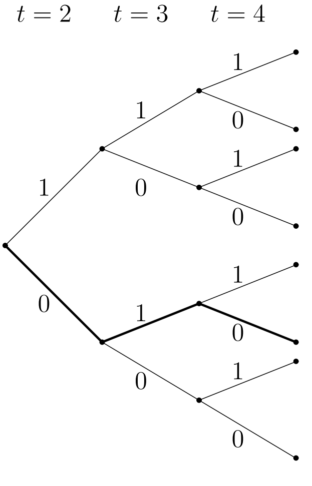


Figure 3.1: Representing production plans by a binary tree

each node of depth t one branch represents a new setup in period $t+1$ and the other branch represents a non-setup period. For example, the path in Figure 3.1 represents the plan $P = \{1, 0, 1, 0\}$. Note that given a demand sequence, every heuristic has to choose a path (corresponding to a production plan) through the binary tree. So the tree reflects that decisions are made according to Property 1. Hence the performance ratio $R(d, T)$ of any heuristic on demand sequence d of length T equals at least

$$R^*(d,T) = \min_{P \in P(T)} \max_{t=1,\ldots,T} \frac{C(d,P,t)}{C^*(d,t)}$$

and the worst case ratio of any heuristic equals at least

$$W^*(T) = \max_{d \in [0,1]^T} R^*(d,T) = \max_{d \in [0,1]^T} \min_{P \in P(T)} \max_{t=1,\ldots,T} \frac{C(d,P,t)}{C^*(d,t)} \tag{3.2}$$

as the worst case ratio is the worst performance ratio over all problem instances. Again we only consider problem instances with demand $d_t \leq 1$. If we have an instance d with some demands strictly larger than 1 and if d' is the instance with demand 1 for these periods, then it is not difficult to see that $R^*(d,T) = R^*(d',T)$ and hence d can be ignored when evaluating (3.2). (Note that Lemma 3.1 cannot be applied because Properties 2 and 3 do not hold.)

The above formulation is not complete. This can be seen as follows. Assume that we have two demand sequences $d^1 = [d_1, d_2, d_3^1]$ and $d^2 = [d_1, d_2, d_3^2]$. Furthermore, assume that the performance ratios for each (partial) production plan are as shown in Figure 3.2, where (as in Figure 3.1) an upper branch represents a setup period. It follows from

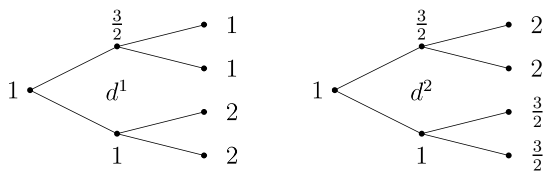


Figure 3.2: Performance ratio for d^1 and d^2

the figure that $R^*(d^1, 3) = R^*(d^2, 3) = \frac{3}{2}$. If d^1 and d^2 are the only possible problem instances, then from (3.2) it follows $W^*(3) = \frac{3}{2}$. However, the worst case performance of any heuristic H is at least 2. Namely, if H generates no setup in period 2, then we give d_3^1 in period 3 leading to performance ratio 2. On the other hand, if H generates a setup in period 2, then we give d_3^2 in period 3 again leading to performance ratio 2. Hence the worst case ratio equals 2. The problem of the mathematical formulation is that it allows not for different demand sequences which are dependent on the decision of the heuristic. In other words, it is not possible that two branches (arising from the same node) have different remaining demand sequences. However, it is possible in the formulation to have zero demands as the remaining demand sequence, because we evaluate the performance ratio for all t-period production plans. This means that the problem formulation leads to lower bounds on the worst case performance of any heuristic.

We have plotted the graph of the function $R^*(d, 5)$ with $d_2 = 0$ and $d_5 = \frac{1}{4}$ in Figure 3.3. It is clear that finding the demand sequence d that optimizes $R^*(d, 5)$ is not a nice concave maximization problem.

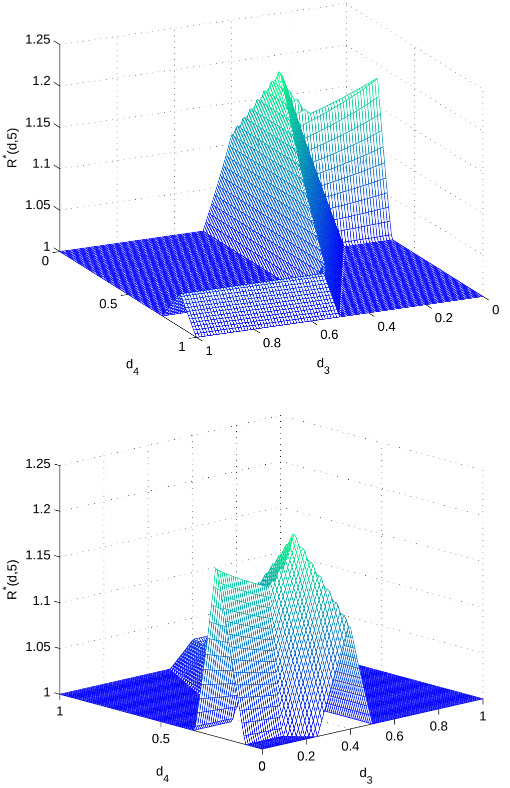


Figure 3.3: The graph of the function $R^*(d, 5)$ with $d_2 = 0$ and $d_5 = \frac{1}{4}$

3.3.2 A special class of production plans

Because equation (3.2) is hard to analyze, we will consider a further relaxation of the problem. First we will derive a lower bound on the value $R^*(d,T)$. Define the set of production plans P^i $(i = 1,\dots,T)$ as follows

$$P_t^1 = \begin{cases} 1 & \text{for } t = 1, \\ 0 & \text{for } t = 2,\dots,T, \end{cases}$$

$$P_t^i = \begin{cases} 1 & \text{for } t = 1 \text{ and } t = i, \\ 0 & \text{for } t = 2,\dots,i-1, \end{cases} \quad \text{for } i = 2,\dots,T.$$

Note that P^i $(i = 2,\dots,T)$ is a production plan for i periods and P^1 is a plan for T periods. In figure 3.4 production plans P^i $(i = 1,\dots,4)$ are the paths from the root to the leafs in the tree. Define

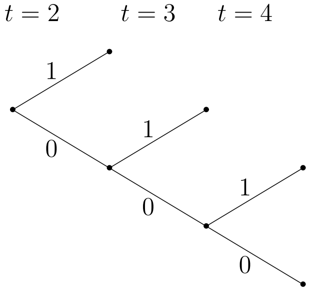


Figure 3.4: Production plans P^i $(i = 1,\dots,4)$ represented by paths in a tree

$$r(d,1) = \frac{C(d,P^1,T)}{C^*(d,T)} \text{ and } r(d,i) = \frac{C(d,P^i,i)}{C^*(d,i)} \text{ for } i = 2,\dots,T$$

and let

$$R(d,T) = \min_{i=1,\dots,T} r(d,i).$$

So the $r(d,i)$ represent the performance ratios of the leaf nodes in Figure 3.4 and $R(d,T)$ is the minimum performance ratio over these nodes.

Lemma 3.2 *For any instance $d_1,\dots,d_T$ it holds*

$$R(d,T) \le R^*(d,T).$$

Proof Let $P \in P(T)$. Then there exists a j for which P^j is a subpath of P. But then

$$r(d,j) \leq \max_{i=1,\ldots,T} \frac{C(d,P,i)}{C^*(d,i)}$$

as the term at the left hand side is contained in the maximum at the right hand side. □

Lemma 3.2 shows that, when using the special set of production plans, we have a lower bound on the performance ratio for d. The motivation for taking P^i $(i = 2, \ldots, T)$ is that one expects that these plans lead to high costs because in general it is not profitable to have a setup in the last period. Plan P^1 is needed, because with this plan included, any production plan P has a plan P^i as subplan (and so without P^1 Lemma 3.2 does not hold). It is clear that for a fixed d the value $R(d,T)$ is a lower bound on $W^*(T)$. Furthermore, we define the lower bound $W(T)$ on $W^*(T)$ as

$$W(T) = \max_{d \in [0,1]^T} R(d,T) = \max_{d \in [0,1]^T} \min_{t=1,\ldots,T} r(d,t). \tag{3.3}$$

Note that problem (3.3) is more tractable then problem (3.2) because the 'min max'-part is replaced by a 'min'-part. We will now derive some properties for a demand sequence that maximizes (3.3).

Lemma 3.3 *Let d be an instance that maximizes (3.3). Then the value of $d_T = \frac{1}{T-1}$.*

Proof First note that d_T only occurs in the calculation of $r(d,1)$ and $r(d,T)$ because they contain the terms $C(d,P^1,T)$ and $C^*(d,T)$. The holding costs for d_T equal $(T-1)d_T$ in plan P^1. If $p \geq 2$ is the setup period preceding period T in the optimal plan P^*, then the holding cost in this plan equals $(T-p)d_T$ (if $p = 1$, then $r(d,1) = 1$ and hence $W(T) = 1$, which cannot optimal). Increasing d_T will increase the performance ratio $r(d,1) = \frac{C(d,P^1,T)}{C^*(d,T)}$. However, the performance ratio $r(d,T) = \frac{C(d,P^T,T)}{C^*(d,T)}$ is decreasing in d_T. Therefore, $\min\{r(d,1), r(d,T)\}$ is maximized when $C(d,P^1,T) = C(d,P^T,T)$, i.e., when $(T-1)d_T = 1$ or $d_T = \frac{1}{T-1}$. □

In the remainder of this chapter we will assume that $d_T = \frac{1}{T-1}$ so that $r(d,1) = r(d,T)$ and hence

$$W(T) = \max_{d \in [0,1]^T} \min_{i=2,\ldots,T} r(d,i). \tag{3.4}$$

Another useful property of an optimal demand sequence can be found in the following lemma.

Lemma 3.4 *Let d be an instance with $d_{j-1} > 0$, $d_j = 0$ and $3 \leq j \leq T$. Then there exists an instance d' with $R(d',T) \geq R(d,T)$ and $d'_j > 0$.*

Proof Define an instance d' with $d'_2 = 0$ and $d'_{t+1} = \frac{t-1}{t} d_t$ for all $t < j$ and $d'_t = d_t$ for all $t > j$. So demand before period j is shifted one period and scaled. Clearly, $d'_j = \frac{j-2}{j-1} d_{j-1} > 0$. Let $i \leq j$. Then the holding cost for demand d'_{t+1} $(t < i)$ in P^{i+1} equals $t d'_{t+1} = t \frac{t-1}{t} d_t = (t-1) d_t$, which is the holding cost for demand d_t in P^i. Therefore, $C(d', P^{i+1}, i+1) = C(d, P^i, i)$ for $i \leq j$. Furthermore, because demand beyond period j is unchanged we also have $C(d', P^i, i) = C(d, P^i, i)$ for $i > j$.

Let P be the optimal plan for some i-period problem for instance d. Now shift all setup periods before period j (except for period 1) one period further. We will use this plan for the $(i+1)$-period problem with demand d' if $i < j$ and for the i-period problem with demand d' if $i > j$. Clearly, the setup costs for both plans are equal. Furthermore, let $t < j$ and let p be the setup period before period t in plan P. Then the holding cost for demand d_t equals $(t-p)d_t$ and the holding cost for demand d'_{t+1} equals $((t+1) - (p+1)) d'_t = (t-p) \frac{t-1}{t} d_t \leq (t - p + 1) d_t$. Using similar arguments one can show that holding cost for d'_t equals at most the holding cost for d_t for $t > j$. Therefore, $C^*(d, i) \geq C^*(d', i+1)$ for $i < j$ and $C^*(d, i) \geq C^*(d', i)$ for $i > j$.

Using the above (in)equalities it follows that

$$\begin{aligned} r(d,i) &= \tfrac{C(d,P^i,i)}{C^*(d,i)} \leq \tfrac{C(d',P^{i+1},i+1)}{C^*(d',i+1)} = r(d',i+1) \quad && \text{for } i < j \\ r(d,i) &= \tfrac{C(d,P^i,i)}{C^*(d,i)} \leq \tfrac{C(d',P^i,i)}{C^*(d',i)} = r(d',i) && \text{for } i > j. \end{aligned}$$

Furthermore, $r(d,j) \geq r(d,j-1)$ because $C(d, P^j, j) \geq C(d, P^{j-1}, j-1)$ and $C^*(d,j) = C^*(d, j-1)$ since $d_j = 0$. Now the lemma follows because

$$R(d,T) = \min_{i=2,\ldots,T} r(d,i) \leq \min_{i=2,\ldots,T} r(d',i) = R(d',T).$$

□

The previous lemma shows that if we have an instance with a positive demand period followed by a zero demand period, then we can find an instance with larger performance ratio by shifting and scaling all the demand before this zero demand period by one period. Therefore, there exists a solution that maximizes (3.4) and has no positive demands followed by zero demands (except for period 1). Let d be a problem instance with $d_t = 0$ for $t = 2, \ldots, n-1$. For this instance we have $r(d,i) = 2$ for $i = 2, \ldots, n-1$ and hence $R(d,T)$ is equal to

$$R(d,T) = \min_{i=n,\ldots,T} r(d,i).$$

Let $W(T,n)$ be the maximum of (3.4) with $d_t = 0$ for $t = 2, \ldots, n-1$, i.e.,

$$W(T,n) = \max_{d_t \in [0,1], t=n,\ldots,T-1} \min_{i=n,\ldots,T} r(d,i). \tag{3.5}$$

Then the following corollaries follow immediately from Lemma 3.4.

Corollary 3.5 *For any model horizon T it holds*

$$W(T) = \max_{1<n<T} W(T, n). \tag{3.6}$$

Corollary 3.6 *For $1 < n < T$ we have $W(T, n) \leq W(T+1, n+1)$.*

Proof A T-period instance can be considered as a $(T+1)$-period instance with $d_{T+1} = 0$. □

The following lemma shows another property of an optimal demand sequence.

Lemma 3.7 *Let d^* be an optimal solution of (3.5) with $n > 1$ the first period with $d_n > 0$ and $d_t > 0$ for $t = n, \ldots, T$. Then $r(d^*, i) = r(d^*, i+1)$ for $i = n, \ldots, T-1$.*

Proof Assume that the lemma does not hold and let

$$r(d^*, u) = \min_{i=n,\ldots,T} \{r(d^*, i)\} < \max_{i=n,\ldots,T} \{r(d^*, i)\} = r(d, v).$$

Assume that $u < v$ (the case with $u > v$ can be proven analogously). We will construct an alternative solution with demand sequence $d' = d^* + \varepsilon$ such that $r(d^*, u) < r(d', u) \leq r(d', v)$ and $r(d', i) = r(d^*, i)$ for $i \in \{n, \ldots, T\} \backslash \{u, v\}$. This means we have found a better solution for (3.5) which is a contradiction.

To achieve this, let $r_i = r(d^*, i)$ for $i = n, \ldots, T$, keep the production plans fixed and consider the equations $r(d, i) = r_i$ in the variables $d_n, \ldots, d_{T-1}$. So the function $r(d, i)$ is the ratio of two linear functions (see equation (3.1)), $r(d, i)$ is defined on the variables $d_n, \ldots, d_i$ for $i = n, \ldots, T-1$ and $r(d, T)$ is defined on the variables $d_n, \ldots, d_{T-1}$. First, note that the optimal plan P^{*i} will not have a setup in period i, because moving the setup from period i to period $i-1$ will not increase the cost. Therefore, $r(d, i)$ is either strictly increasing or strictly decreasing in d_i, because d_i appears in the denominator and it does not appear in the nominator. Similarly, $r(d, T)$ is either strictly increasing or strictly decreasing in d_{T-1}, because d_{T-1} appears in the nominator.

Now let $d'_i = d^*_i$ for $i = n, \ldots, u-1$ and let $d'_u = d^*_u + \varepsilon_u$ with $\varepsilon_u \in \mathbb{R}$ such that $r(d', u) > r(d^*, u)$. Note that $r(d', i) = r_i$ for $i = n, \ldots, u-1$ and $r(d', i)$ may be changed for $i = u, \ldots, T$. Now choose ε_i with $d'_i = d^*_i + \varepsilon_i$ such that $r(d', i) = r_i$ for $i = u+1, \ldots, v-1$. Because the function $r(d, i)$ is strictly in/decreasing in d_i, the value $d'_i = d^*_i + \varepsilon_i$ is uniquely defined by the equation $r(d', i) = r_i$ for given $d'_n, \ldots, d'_{i-1}$ and hence the values ε_i $(i = u+1, \ldots, v-1)$ exist. In a similar way we choose $d'_i = d^*_i + \varepsilon_i$ for $i = v, \ldots, T-1$ such that $r(d', i) = r_i$ for $i = v+1, \ldots, T$. Again such values ε_i $(i = v, \ldots, T-1)$ exist. Namely, start with an arbitrary value of ε_v. Then the values ε_i $(i = v+1, \ldots, T-1)$ are uniquely determined by the equations $r(d', i) = r_i$ for $i = v+1, \ldots, T-1$. Now it is possible that $r(d', T) \neq r_T$. This means that the choice of ε_v

was not right and the right value of ε_v can be found by binary search since $r(d,i)$ is strictly in/decreasing in d_i. So summarizing, given an ε_u, the values ε_i for $i = u+1, \ldots, T-1$ are uniquely determined by the equations $r(d', i) = r_i$ for $i \in \{u+1, \ldots, T\} \backslash \{v\}$.

Finally, consider the value $r(d', v)$. If $r(d', v) \geq r(d^*, v)$, then we have proved the lemma as we have found a strictly better solution for (3.5) which is a contradiction. If $r(d', v) < r(d^*, v)$, then we choose ε_u sufficiently small such that $r(d', v) \geq r(d', u) > r(d^*, u)$ and again we have found a better solution.

We end the proof with some remarks. First, if period u is not unique, then we can repeat the above procedure. Second, if there is some $d'_i < 0$ (which is not feasible), then ε_u must be chosen sufficiently small such that $d'_i \geq 0$. Third, it is possible that by the change from d^* to d' the optimal production plans will also change. In this case the denominator of $r(d', i)$ will be smaller and hence $r(d', i)$ will be larger which means that the proof still holds. □

3.3.3 Finding the optimal demand sequence given the production plans

If we can find the values $W(T, n)$, then by Corollary 3.5 we can find the value $W(T)$. The difficulty in evaluating $W(T, n)$ is that the optimal plans and the optimal demand sequences have to be determined simultaneously. For example, a change in a demand sequence may cause a change in the optimal production plans. In this section we will derive an approach to calculate the optimal demand sequence assuming that the optimal production plans are known.

Assume that we have a demand sequence with $d_t = 0$ for $t = 2, \ldots, n-1$. If the optimal plans are known, then by Lemma 3.7 the optimal demand sequence (with respect to (3.5)) can be found by solving the system

$$r(d, i) = r(d, i+1) \text{ for } i = n, \ldots, T-1. \tag{3.7}$$

Given the plans P^i $(i = n, \ldots, T)$ and the optimal production plans P^{*i} for each horizon $i = n, \ldots, T$, it follows from (3.1) that both the nominator and the denominator of $r(d, i) = \frac{C(d, P^i, i)}{C(d, P^{*i}, i)}$ are linear functions in the variables $d_n, \ldots, d_{T-1}$. Now by 'cross-multiplying', system (3.7) is a system of multivariate quadratic equations. This is in general a hard problem, because by the method of repeated substitution one has to find the roots of univariate polynomials of high degree. Example 3.8 illustrates this.

Example 3.8 Consider a problem with $T = 5$ and $n = 3$ so that $d_2 = 0$ and $d_5 = \frac{1}{4}$. In Table 3.2 the production plans, corresponding costs and performance ratios are shown. From $r(d, 3) = r(d, 4)$ it follows that $d_4 = 2d_3^2 + 3d_3 - 1$. Substituting this in $r(d, 3) = r(d, 5)$ we have $12d_3^3 + 24d_3^2 + 3d_3 - 4 = 0$. Solving this equation we have $d_3^* \approx 0.328$,

i	P^i	$C(d, P^i, i)$	P^{*i}	$C(d, P^{*i}, i)$	$r(d, i)$
3	$\{1, 0, 1\}$	2	$\{1, 0, 0\}$	$1 + 2d_3$	$\frac{2}{1+2d_3}$
4	$\{1, 0, 0, 1\}$	$2 + 2d_3$	$\{1, 0, 1, 0\}$	$2 + d_4$	$\frac{2+2d_3}{2+d_4}$
5	$\{1, 0, 0, 0, 1\}$	$2 + 2d_3 + 3d_4$	$\{1, 0, 1, 0, 0\}$	$2 + d_4 + \frac{1}{2}$	$\frac{2+2d_3+3d_4}{\frac{5}{2}+d_4}$

Table 3.2: Performance ratios for $T = 5$ and $n = 3$

$d_4^* \approx 0.201$ and $r^* \approx 1.207$. Note that this problem can be solved exactly because there exists a closed formula for finding the root of a polynomial of degree 3. However, no closed formulas exist for polynomials of degree at least 5 and hence another approach is required.

We will derive an approach to find the values d_t that maximize (3.5) numerically. Assume that the optimal ratio equals r^*. This means there exists an instance d^* with $r(d^*, i) = r^*$ for $i = n, \ldots, T$. Furthermore, assume for the moment that the optimal plans corresponding to the i-period problem of d^*, say P^{*i}, are known (note that $C(d^*, P^{*i}, i) = C^*(d^*, i)$). If the value r^* is not known, we can start with an initial guess r. Then for fixed r the system

$$\frac{C(d, P^i, i)}{C(d, P^{*i}, i)} = r \text{ for } i = n, \ldots, T$$

is a system of $m + 1$ linear equations in m variables $(d_n, \ldots, d_{T-1})$, which means it is overdefined. Define the residual of equation i by

$$e_i(r, d) = rC(d, P^i, i) - C(d, P^{*i}, i)$$

and the sum of squared residuals by

$$S(r, d) = \sum_{i=n}^{T} e_i(r, d)^2. \tag{3.8}$$

Because

$$e_i(r^*, d^*) = r^* C(d^*, P^i, i) - C(d^*, P^{*i}, i) = 0,$$

it follows $S(r^*, d^*) = 0$. Therefore, given the optimal production plans, the demand sequence d^* that maximizes (3.5) with corresponding ratio r^* is a solution of the problem of minimizing (3.8).

Clearly, given a fixed r, minimizing (3.8) is nothing but least squares fitting which is a relatively easy problem. Because by definition the ratio r^* is at least 1 and by the result of Axsäter (1985) r^* is at most 2, the value r^* can be found by a search procedure on the interval $[1, 2]$ given that the optimal production plans are known. (Note that we have not proved the existence of a unique solution in this interval.) We will call the above method to find the optimal demand sequence the least squares procedure (LSP).

3.3.4 An initial guess for the optimal plans

The procedure of the previous section cannot directly be applied, because the set of optimal production plans is not known. Therefore, we will construct a set P'^i ($i = n, \ldots, T$) that serves as an 'approximation' for the set of optimal production plans. For ease of notation let

$$r'(d,i) = \frac{C(d,P^i,i)}{C(d,P'^i,i)} \text{ for } i = n, \ldots, T \text{ and } R'(d,T) = \min_{i=n,\ldots,T} r'(d,i).$$

Lemma 3.9 *Given an instance with $d_t = 0$ ($t = 2, \ldots, n-1$), $d_t > 0$ ($t = n, \ldots, T-1$) and arbitrary plans P'^i ($i = n, \ldots, T$). Then*

$$R'(d,T) \leq R(d,T).$$

Proof From the optimality of $C^*(d,i)$ it follows $C(d,P'^i,i) \geq C^*(d,i)$. Therefore

$$r'(d,i) = \frac{C(d,P^i,i)}{C(d,P'^i,i)} \leq \frac{C(d,P^i,i)}{C^*(d,i)} = r(d,i)$$

and the lemma follows from Lemma 3.2. □

Note that starting with plans P'^i that are worse than P^i leads to $r'(d,i) < 1$. Therefore, we have to start with a reasonable guess. Let k be a fixed integer with $n < k \leq T$ and consider the set of production plans P'^i ($i = n, \ldots, T$) defined as follows (for ease of notation we do not show the dependence on k of this set)

$$P'^i_t = \begin{cases} 1 & \text{for } t = 1 \\ 0 & \text{for } t = 2, \ldots, i, \end{cases} \quad \text{for } i = n, \ldots, k-1 \tag{3.9}$$

$$P'^i_t = \begin{cases} 1 & \text{for } t = 1 \text{ and } t = n \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } i = k, \ldots, T. \tag{3.10}$$

As for the plans P^i, the plan P'^i represents a plan for an i-period problem instance. The value k indicates that plans consisting of at least k periods have an additional setup in period n. We will come back on the choice of k in the next section. The motivation to take plans P'^i is that for small horizons ($t \leq k-1$) it seems reasonable to have only a setup in period 1 and for larger horizons ($t \geq k$) it seems reasonable to have an additional setup to reduce the holding costs. Using Lemmas 3.2 and 3.9 a lower bound $W'(T,n)$ on $W(T,n)$ can be found by solving the optimization problem

$$W'(T,n) = \max_{d_t \in [0,1], t=n,\ldots,T-1} \ \min_{i=n,\ldots,T} r'(d,i). \tag{3.11}$$

Example 3.10 To illustrate the use of the sets P^i and P'^i consider a problem instance for $T = 3$. In this case $P^2 = \{1, 1\}$, $P^3 = \{1, 0, 1\}$ and with $k = 3$ we have $P'^2 = \{1, 0\}$, $P'^3 = \{1, 1, 0\}$. From Lemma 3.3 it follows that $d_3 = \frac{1}{2}$ and

$$W'(3,2) = \max_{d_2 \in [0,1]} \min \left\{ \frac{2}{1+d_2}, \frac{2+d_2}{5/2} \right\}.$$

As the first term in the minimization is decreasing in d_2 and the second term is increasing in d_2, we have

$$\frac{2}{1+d_2} = \frac{2+d_2}{5/2}$$

in the optimal solution. Solving this quadratic equation (note that we do not need the procedure of Section 3.3.3) we have $d_2 = \frac{1}{2}(\sqrt{21} - 3) \approx 0.79$ and $W'(3,2) = \frac{1}{5}(1 + \sqrt{21}) \approx 1.12$. So the instance $d_1 = 1, d_2 = \frac{1}{2}(\sqrt{21} - 3), d_3 = \frac{1}{2}$ is a problem instance with performance ratio $\frac{1}{5}(1 + \sqrt{21})$ and hence a lower bound on the worst case ratio for $T = 3$.

3.3.5 An iterative procedure to find worst case examples

In this section we will describe an iterative procedure in which the plans P'^i are updated in each iteration. We start with some initial guess for the optimal plans and calculate the optimal demand sequence using the least squares procedure. Now given this demand sequence, we can determine the 'real' optimal plans corresponding to this demand sequence. If these plans are different from our initial guess, a new iteration is performed starting with these new plans. The iterative procedure is schematically illustrated in Table 3.3.

In Step 1 we start with the initial guess P'^i_{old} $(i = n, \ldots, T)$ for the optimal production plans. Given these plans, we calculate the optimal demand sequence d^* and the corresponding performance ratio r^* using the LSP of Section 3.3.3. In Step 3 we check whether the guess was right by calculating the optimal plans P'^i_{new} corresponding to d^*. If yes, the procedure is terminated. If not, then we go back to Step 2 and start with these plans. Note that by Lemma 3.9 in every iteration the performance ratio will increase. Because the number of plans is finite, the iterative procedure will terminate.

3.3.6 Some numerical results

The iterative procedure (IP) of Table 3.3 was implemented in Visual Basic. When starting the IP, we have multiple initial guesses for P'^i_{old} as k in (3.9) and (3.10) can range from $n + 1$ to T. Given some value of T and n we started the IP with all possible values of k and it turned out that the IP always terminated with the same plans P'^i_{new}. The generated values for $W'(T, n)$ with $T = 3, \ldots, 20$ and $n = 2, \ldots, T - 1$ can be found in Table 3.4. Below each performance ratio the minimum number of iterations needed before

Iterative procedure to calculate $W'(T,n)$	
Step 1:	Start with some initial guess P'^{i}_{old} $(i = n, \dots, T)$
Step 2:	Calculate r^* and d^* given P'^{i}_{old} $(i = n, \dots, T)$ using the LSP

Step 3:	Calculate P'^{i}_{new} $(i = n, \dots, T)$ given d^*
	If $P'^{i}_{new} = P'^{i}_{old}$ $(i = n, \dots, T)$ Then
	    Output: $W'(T,n) = r^*$ and d^*
	    Stop
	Else
	    $P'^{i}_{old} = P'^{i}_{new}$ $(i = n, \dots, T)$
	    Go to Step 2
	End if

Table 3.3: Iterative procedure to calculate $W'(T,n)$

termination for all initial plans is shown. Note that the performance ratio is 1 if $T = 2$ or if $n = T$. In the latter case we only have one strictly positive demand (beside the demand in period 1) which is similar to the case $T = 2$.

$T \setminus n$	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**
3	**1.117**																	
	1																	
4	1.150	**1.165**																
	1	1																
5	1.175	**1.207**	1.191															
	2	1	1															
6	1.197	1.232	**1.235**	1.207														
	3	1	1	1														
7	1.215	1.252	**1.274**	1.259	1.219													
	4	3	1	1	1													
8	1.226	1.270	1.293	**1.301**	1.276	1.227												
	4	2	1	1	1	1												
9	1.235	1.282	1.304	**1.320**	1.318	1.289	1.234											
	3	4	2	1	1	1	1											
10	1.241	1.291	1.317	1.330	**1.336**	1.331	1.299	1.239										
	5	4	3	1	1	1	1	1										
11	1.247	1.298	1.329	1.343	**1.353**	1.350	1.340	1.307	1.243									
	4	4	3	2	1	1	1	1	1									
12	1.252	1.305	1.337	1.354	1.364	**1.369**	1.363	1.346	1.314	1.246								
	5	4	3	3	1	1	1	1	1	1								
13	1.255	1.309	1.342	1.362	1.372	1.379	**1.381**	1.372	1.352	1.320	1.249							
	5	4	3	3	2	1	1	1	1	1	1							
14	1.258	1.312	1.347	1.367	1.378	1.385	**1.390**	1.389	1.380	1.356	1.324	1.251						
	5	4	4	4	2	1	1	1	1	1	1	1						
15	1.260	1.315	1.350	1.372	1.386	1.393	**1.398**	**1.398**	1.396	1.386	1.359	1.329	1.253					
	4	5	4	3	3	2	1	1	1	1	1	1	1					
16	1.262	1.317	1.352	1.375	1.391	1.400	1.404	**1.408**	1.407	1.401	1.391	1.362	1.332	1.255				
	5	5	6	4	3	2	1	1	1	1	1	1	1	1				
17	1.263	1.319	1.355	1.379	1.394	1.404	1.410	1.414	**1.416**	1.414	1.405	1.395	1.364	1.335	1.257			
	5	5	5	4	4	3	2	1	1	1	1	1	1	1	1			
18	1.264	1.320	1.357	1.381	1.397	1.407	1.414	1.418	1.421	**1.422**	1.419	1.408	1.399	1.367	1.338	1.258		
	5	6	5	5	4	3	2	1	1	1	1	1	1	1	1	1		
19	1.265	1.321	1.358	1.383	1.399	1.411	1.418	1.423	1.425	**1.427**	**1.427**	1.424	1.413	1.402	1.371	1.341	1.259	
	5	6	5	5	4	4	3	2	1	1	1	1	1	1	1	1	1	
20	1.266	1.322	1.360	1.384	1.401	1.413	1.422	1.427	1.430	1.432	**1.433**	1.430	1.427	1.417	1.405	1.374	1.343	1.260
	5	6	5	4	4	4	3	2	2	1	1	1	1	1	1	1	1	1

Table 3.4: Performance ratios and number of iterations for different model horizons

Table 3.4 shows some interesting results. First, it follows that $W'(T,n) \leq W'(T+1,n+1)$. As this property holds for $W(T,n)$ (see Lemma 3.4) and because the IP converges to the same solutions when starting with different initial guesses, it suggests that $W(T,n)$ and $W'(T,n)$ are equal. Second, we see that for a fixed T the value of n that maximizes $W'(T,n)$ (the performance ratios in bold), say $n(T)$, is increasing in T. Furthermore, we see that for $n < n(T)$, $W'(T,n)$ is increasing in n, and for $n > n(T)$, $W'(T,n)$ is decreasing in n. Third, the minimum number of iterations shows that the initial guesses are reasonable. For n close to T we see that one of the initial guesses is the optimal one. Finally, we note that for large values of T we can find performance ratios close to $\frac{3}{2}$. For example, $W'(80,100) = 1.494$ and $W'(480,500) = 1.499$.

Again look at the graph of $R^*(d,5)$ in Figure 3.3. The function $R^*(d,5)$ has two local optima: $d_3^1 \approx 0.328$, $d_4^1 \approx 0.201$ with $R^*(d^1,5) \approx 1.207$ and $d_3^2 = 0$, $d_4^2 \approx 0.226$ with $R^*(d^2,5) \approx 1.191$. The performance ratios of these two solutions are equal to the ratios found by the IP ($W'(5,3)$ and $W'(5,4)$ in Table 3.4), which shows that the IP leads to the optimal solutions for $T = 5$ with $d_2 = 0$.

3.3.7 Two convergence results

In this section we will give two convergence results. First, we will give a lower bound on the worst case ratio of heuristics applied to instances with only positive demands in the last two periods (and in period 1). Then we will give a problem instance for which any heuristic satisfying Property 1 has worst case ratio at least $\frac{3}{2}$.

Lemma 3.11 *For the value $W(T+1,T)$ it holds*

$$\lim_{T\to\infty} W(T+1,T) = \frac{1}{4}(\sqrt{17}+1) \approx 1.281.$$

Proof Assume that demand in period T equals $d_T = \frac{c}{T-1}$ (for ease of notation we use $T-1$ in the denominator). Furthermore, by Lemma 3.3 we have $d_{T+1} = \frac{1}{T}$. It is not difficult to see that in the T-period problem it is optimal to have a setup only in period 1 for appropriate c, whereas plan P^T has setups in periods 1 and T. So we have $r(c,T) = \frac{2}{1+c}$. For the $(T+1)$-period problem it is optimal to have setups in periods 1 and T, whereas P^{T+1} has setups in periods 1 and $T+1$ implying that

$$r(c,T+1) = \frac{2+c}{2+\frac{1}{T}}.$$

Now the maximum of

$$W(T+1,T) = \max_c \min\left\{\frac{2}{1+c}, \frac{2+c}{2+\frac{1}{T}}\right\},$$

is attained for $c_T = \frac{1}{2}(\sqrt{17+8/T}-3)$ with

$$W(T+1,T) = \frac{\frac{1}{2}(\sqrt{17+8/T}+1)}{2+1/T} \to \frac{1}{4}(\sqrt{17}+1) \approx 1.281 \text{ as } T \to \infty.$$

$\square$

Note that for $T+1=3$, we have that $W(3,2) = \frac{1}{5}(\sqrt{21}+1)$ which corresponds with the value found in Example 3.10. Lemma 3.11 shows that any heuristic satisfying Property 1 has at least worst case ratio $\frac{1}{4}(\sqrt{17}+1)$ in the case of two non-zero demands (except period 1). This also shows that the values on the diagonal of Table 3.4 tend to $\frac{1}{4}(\sqrt{17}+1)$.

The numerical results of the IP led to the construction of a problem instance with performance ratio $\frac{3}{2}$. In Table 3.5 we have presented the output of the IP for a 100-period problem instance with $d_{80}, \ldots, d_{100} > 0$. This problem instance has a performance ratio 1.494. Note from the table that the ratio between the holding cost of two consecutive

t	80	81	82	83	84	85	86	87	88	89
$(t-1)d_t$	0.3391	0.2271	0.1520	0.1018	0.0682	0.0456	0.0306	0.0205	0.0137	0.0092
$\frac{(t-1)d_t}{(t-2)d_{t-1}}$	-	0.670	0.670	0.670	0.670	0.670	0.670	0.670	0.670	0.670
t	90	91	92	93	94	95	96	97	98	99
$(t-1)d_t$	0.0061	0.0041	0.0028	0.0018	0.0012	0.0016	0.0064	0.0242	0.0874	0.3017
$\frac{(t-1)d_t}{(t-2)d_{t-1}}$	0.670	0.670	0.670	0.670	0.670	1.305	3.976	3.781	3.608	3.454

Table 3.5: Problem instance generated from the IP with performance ratio 1.494

periods is approximately $\frac{2}{3}$ (for $t = 80, \ldots, 94$). We will present a problem instance with this property and we will show that the performance ratio of this instance tends to $\frac{3}{2}$ for T large. Define the sequence $x_k = \frac{1}{3}(\frac{2}{3})^k$ (note that $\frac{x_k}{x_{k-1}} = \frac{2}{3}$). In the proof we will use the following property of the sequence x_k.

Lemma 3.12 *Let $x_t = \frac{1}{3}(\frac{2}{3})^t$ for $t = 0, 1, \ldots$. Then for all $i = 1, 2, \ldots$*

$$\frac{2+\sum_{t=0}^{i-1} x_t}{1+\sum_{t=0}^{i} x_t} = \frac{3}{2}.$$

Proof First, note that the lemma holds for $i=1$ as $\frac{2}{1+1/3} = \frac{3}{2}$. Assume that the lemma holds for some $i \geq 1$ so that $4 + 2\sum_{t=0}^{i-1} x_t = 3 + 3\sum_{t=0}^{i} x_t$. Since $2x_i = 3x_{i+1}$ we have

$$4+2\sum_{t=0}^{i-1} x_t + 2x_i = 3+3\sum_{t=0}^{i} x_t + 3x_{i+1} \Leftrightarrow 4+2\sum_{t=0}^{i} x_t = 3+3\sum_{t=0}^{i+1} x_t \Leftrightarrow \frac{2+\sum_{t=0}^{i} x_t}{1+\sum_{t=0}^{i+1} x_t} = \frac{3}{2}.$$

$\square$

Theorem 3.13 *Any heuristic satisfying Property 1 has worst case ratio at least* $\frac{3}{2}$.

Proof We will prove the theorem by showing that there exists a problem instance d with

$$\lim_{T\to\infty} R'(d,T) = \frac{3}{2}.$$

Define the demand sequence d with time horizon T^2+T+1 as follows: $d_1 = 1$, $d_t = 0$ for $t = 2,\dots,T^2-1$), $d_t = \frac{x_{t-T^2}}{t-1}$ for $t = T^2,\dots,T^2+T$ with x_t as in Lemma 3.12 and $d_{T^2+T+1} = \frac{1}{T^2+T}$. (Using the notation of the previous sections we have set T to T^2+T+1 and n to T^2.) First, for $i = T^2,\dots,T^2+T+1$ we have

$$C(d,P^i,i) = 2 + \sum_{t=T^2}^{i-1}(t-1)d_t = 2 + \sum_{t=0}^{i-T^2-1} x_t$$

and

$$C(d,P^1,T^2+T+1) = 1 + \sum_{t=T^2}^{T^2+T+1}(t-1)d_t = 1 + \sum_{t=0}^{T} x_t + 1.$$

Now let P'^i be a production plan for the i-period problem ($i = T^2,\dots,T^2+T$) with only a setup in period 1 and let P'^{T^2+T+1} be a production plan with setups in periods 1 and T^2 (so using the notation of Section 3.3.4 we have set k to T^2+T+1). Then we have

$$C(d,P'^i,i) = 1 + \sum_{t=T^2}^{i}(t-1)d_t = 1 + \sum_{t=0}^{i-T^2} x_0$$

and

$$\begin{aligned}
C(d,P'^{T^2+T+1},T^2+T+1) &= 2 + \sum_{t=T^2+1}^{T^2+T+1}(t-T^2)d_t = 2 + \sum_{t=T^2+1}^{T^2+T}\frac{t-T^2}{t-1}x_{t-T^2} + \frac{T+1}{T^2+T} \\
&\leq 2 + \sum_{t=T^2+1}^{T^2+T}\frac{T}{T^2}x_{t-T^2} + \frac{1}{T} = 2 + \frac{1}{T}\sum_{t=1}^{T}\frac{1}{3}\left(\frac{2}{3}\right)^t + \frac{1}{T} \leq 2 + \frac{2}{T}.
\end{aligned}$$

By Lemma 3.12 we have that

$$r'(d,i) = \frac{2+\sum_{t=0}^{i-T^2-1}x_t}{1+\sum_{t=0}^{i-T^2}x_t} = \frac{3}{2} \text{ for } i = T^2,\dots,T^2+T.$$

Furthermore, because $\lim_{T\to\infty}\sum_{t=0}^{T}x_t = 1$ we have that $C(d,P^{T^2+T+1},T^2+T+1) \to 3$, $C(d,P^1,T^2+T+1) \to 3$ and $C(d,P'^{T^2+T+1}) \to 2$ as $T\to\infty$ and hence $r'(d,T^2+T+1) \to \frac{3}{2}$ and $r'(d,1) \to \frac{3}{2}$ as $T\to\infty$. In conclusion, demand sequence d is an instance with performance ratio $R'(d,T^2+T+1) = \frac{3}{2}$ for $T\to\infty$. □

Clearly, for showing that each heuristic satisfying Property 1 has worst case ratio at least $\frac{3}{2}$, it is sufficient to give the problem instance and hence the sections before this section are not needed. However, these sections give insight in the properties of worst case examples. Moreover, the IP shows how to find problem instances with high performance ratio for a fixed T. For example, it follows from Table 3.4 that each heuristic satisfying Property 1 has a worst case ratio at least 1.433 for a 20-period problem instance.

3.4 Analysis of heuristics satisfying Properties 1-3

In this section we will consider heuristics satisfying Properties 1-3. First we will show how we can use the additional properties to construct better worst case examples. Next we will give the problem instance with performance ratio 2.

3.4.1 A procedure to construct worst case examples

We start this section with a lemma that shows how we can use the additional properties to construct worst case examples.

Lemma 3.14 *Given a demand instance $d = [d_0, \ldots, d_T]$ and a heuristic H satisfying Properties 1-3 which has a setup in periods 0 and T. Let d' be an instance with $d'_t = d_t$ for $t = 0, \ldots, T$ and $d'_t = d_{t-T}$ for $t > T$. Then H has a setup in period t for d if and only if H has a setup in period $iT + t$ for $i > 1$.*

Proof We will prove the lemma for a problem instance $d = [d_0, \ldots, d_T]$ and a problem instance $d' = d'_0, \ldots, d'_{2T}$ with $d'_t = d_t$ for $t = 0, \ldots, T$ and $d'_t = d_{t-T}$ for $t = T + 1, \ldots, 2T$ (for ease of notation we start in period 0). Then the lemma easily follows for the more general case. Assume some heuristic H in our class generates setups in periods 0 and T (among possible other periods) for problem instance d. First, it is clear that H will generate the same setup periods in periods $0, \ldots, T$ for d'. Now consider period $T + 1$. As the decision only depends on the cost of the current lot-size (Property 2), H is in the same situation as in period 1. Furthermore, because we assume consistent behavior (Property 3), the same decision will be made as in period 1. Repeating this argument for the next periods shows that p $(1 \leq p \leq T)$ is a setup period in the solution for problem instance d if and only if period $p + T$ is a setup period for problem instance d'. $\square$

Lemma 3.14 will be used to construct a worst case proplem instance. We start with a problem instance d similar to the one in the previous section (for convenience we start in period 0), i.e., $d_0 > 0$, $d_t = 0$ $(t = 2, \ldots, T^2 - 1)$, $d_t = \frac{x_{t-T^2}}{t}$ $(t = T^2, \ldots, T^2 + T)$, $d_{T^2+T+1} = \frac{1}{T^2+T+1}$, where x_t is unknown. Assume that we have some heuristic H that has the first setup in period $k = T^2 + p$ with $0 \leq p \leq T + 1$. Then the holding cost for demand in period $T^2 + t$ ($t < p$) equals x_t and the total holding cost up to period $k - 1$ equals $y_{p-1} = \sum_{t=0}^{p-1} x_t$.

Instead of calculating the performance ratio for the k-period instance, we duplicate the problem instance with blocks of k periods, obtaining the instance d' with $d'_t = d_t$ for $t = 0, \ldots, k$ and $d'_t = d_{t-k}$ for $t > k$. From Lemma 3.14 it follows that H makes the same decision in period $t + k$ as in period t. Therefore, heuristic H will generate setups in periods ik for $i = 0, 1, 2, \ldots$ having cost $C^H = 1 + y_{p-1}$ every k periods. However, it may be better to have setups in period 1 and periods $T^2 + iqk$ for $i = 0, 1, 2, \ldots$ and some $q \in \mathbb{N}$. The situation with $q = 2$ is illustrated in Figure 3.5. The x-signs represent the setup periods of H and the alternative solution A. It can be shown that for large T the

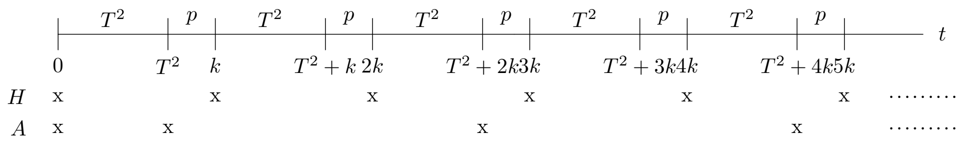


Figure 3.5: Setup periods of H and the alternative solution A

costs tend to $1 + \frac{1}{2}q(q-1)y_p$ in every qk periods (starting in period T^2), which equals $C^* = 1/q + \frac{1}{2}(q-1)y_p$ in every k periods (the formal derivation will be given in the next section). This means that the worst case ratio equals at least

$$\frac{C^H}{C^*} = \frac{1 + y_{p-1}}{1/q + \frac{1}{2}(q-1)y_p}.$$

It is not difficult to verify that it is optimal to choose $q = 2$ if $\frac{1}{3} \leq y_p \leq 1$ and $q = 3$ if $y_p \leq \frac{1}{3}$. For smaller values of y_p it is even better to take $q > 3$, but the worst case ratio is already larger than 2 in this case if we choose $q = 3$ and $y_{p-1} \approx y_p$.

If H generates the first setup in period $k = T^2 + T + 1$, then the heuristic has cost $C^H = 1 + y_T$ in every k periods. However, it is better to have setups in period 1 and periods $T^2 + ik$ for $i = 0, 1, 2, \ldots$. For this solution the holding costs are negligible for large T (again the formal derivation will be given in the next section) and we have cost $C^* = 1$ every k periods so that the performance ratio equals $1 + y_T$. This means that H will have a performance ratio larger than 2 if $1 + y_T > 2$. The case that H makes no setup at all also leads to solutions with performance ratio at least r and will be dealt with in the next section.

From the above analysis it follows that we have an expression for the performance ratio of H, given any decision for the first setup period t. As a result we can formulate the problem of finding a worst case example as an optimization problem. Let r be the performance ratio. Then we want to solve the model

$$\begin{array}{lll} \max & r & \\ \text{s.t.} & r \leq \frac{1+y_{t-1}}{a(y_t)} & t=0,\ldots,T \\ & r \leq 1+y_T & \\ & y_t \leq y_{t+1} & t=0,\ldots,T-1 \\ & y_0 \geq 0, & \end{array}$$

where

$$a(y_t) = \begin{cases} \frac{1}{2}+\frac{1}{2}y_t & \text{if } y_t \geq \frac{1}{3} \\ \frac{1}{3}+y_t & \text{if } y_t \leq \frac{1}{3}. \end{cases}$$

The first set of constraints deals with the case that H generates its first setup in some period $t+T^2$ with $0 \leq t \leq T$ having performance ratio $(1+y_{t-1})/a(y_t)$. The second constraint handles the case with the first setup occurring in period T^2+T+1 having performance ratio $1+y_t$. The last two sets of constraints ensure that $x_t \geq 0$. Namely, if we solve the model, then the variables x_t (used to construct the problem instance d) can be found by letting $x_0 = y_0$ and $x_t = y_t - y_{t-1}$ for $t > 0$.

The model is a nonlinear optimization problem and hence difficult to solve in general. However, we are able to use the model to find worst case examples. If we fix r, then there exists a problem instance with performance ratio r if there exists a sequence y_k that satisfies the following constraints.

$$\begin{array}{ll} ra_t \leq 1+y_{t-1} & t=0,\ldots,T \\ r \leq 1+y_T & \\ y_t \leq y_{t+1} & t=0,\ldots,T-1 \\ y_0 \geq 0 & \\ a_t = \frac{1}{3}z_t^1+\frac{2}{3}z_t^2+1z_t^3 & t=0,\ldots,T \\ y_t = 0z_t^1+\frac{1}{3}z_t^2+1z_t^3 & t=0,\ldots,T \\ z_t^1+z_t^2+z_t^3=1 & t=0,\ldots,T \\ z_t^1 \leq 1-b_t & t=0,\ldots,T \\ z_t^3 \leq b_t & t=0,\ldots,T \\ b_t \in \{0,1\} & t=0,\ldots,T. \end{array}$$

Note that the piecewise linear function $a(y_t)$ is modelled by the last six sets of constraints and the variables $a_t = a(y_t)$, z_t^1, z_t^2, z_t^3 and the binary variables b_t. If $0 \leq y_t \leq \frac{1}{3}$, then y_t can be written as $0z_t^1+\frac{1}{3}z_t^2$ with $z_t^1+z_t^2=1$ and $a_t = \frac{1}{3}z_t^1+\frac{2}{3}z^2$. Similarly, if $\frac{1}{3} \leq y_t \leq 1$, then y_t can be written as $\frac{1}{3}z_t^2+1z_t^3$ with $z_t^2+z_t^3=1$ and $a_t = \frac{2}{3}z_t^2+1z^2$. When $y_t \leq \frac{1}{3}$ the binary variable $b_t = 0$ implying $z_t^3 = 0$, and when $y_t \geq \frac{1}{3}$ we have $b_t = 1$ implying $z_t^1 = 0$.

Thus for a fixed r, finding a problem instance reduces to finding a feasible solution to a set of linear constraints consisting of binary and continuous variables. The largest value of r that satisfies these constraints can be found by performing a binary search. Table 3.6 shows a feasible solution for $r = 1.95$ and $T = 19$. In the next section we will

t	0	1	2	3	4	5	6	7	8	9
x_t	0.180	0.092	0.047	0.015	0.034	0.035	0.036	0.037	0.038	0.039
x_t/x_{t-1}	-	0.51	0.51	0.32	2.26	1.03	1.03	1.03	1.02	1.03
t	10	11	12	13	14	15	16	17	18	19
x_t	0.040	0.041	0.042	0.043	0.044	0.045	0.046	0.048	0.049	0.050
x_t/x_{t-1}	1.03	1.03	1.03	1.03	1.03	1.02	1.02	1.03	1.03	1.02

Table 3.6: Feasible solution for $T = 19$ with $r = 1.95$

give a problem instance with $r = 2$ which is based on the values x_t of Table 3.6.

3.4.2 The worst case problem instance

In this section we will prove that the worst case performance of each heuristic in our special class is at least 2. In the proof we will use a demand sequence based on Table 3.6 and we will derive some properties for this sequence. Let $n \in \mathbb{N}$ be fixed and let $x_k = \frac{1}{6}(\frac{1}{2})^k$ for $k = 0, \dots, n-1$ and $x_k = \frac{2}{3n}$ for $k = n, \dots, 2n-1$. The terms x_k are based on the pattern observes in Table 3.6. For the first terms in this table it holds $x_t/x_{t-1} \approx \frac{1}{2}$ and for the last terms it holds $x_t/x_{t-1} \approx 1$. These are exactly the properties for x_k as $x_k/x_{k-1} = \frac{1}{2}$ for $k \leq n-1$ and $x_k/x_{k-1} = 1$ for $k \geq n+1$. Note that for $t = 3$ and $t = 4$ in Table 3.6 there is some deviation from this behavior. Finally, define the partial sum $y_k = \sum_{i=0}^{k} x_k$.

Lemma 3.15 *For $0 \leq k \leq n-1$ we have*

$$\frac{1 + y_{k-1}}{\frac{1}{3} + y_k} = 2. \tag{3.12}$$

Proof First, note that (3.12) holds for $k = 0$ as $\frac{1}{1/3+1/6} = 2$. Assume that (3.12) holds for some $k \geq 0$ so that $\frac{2}{3} + 2y_k = 1 + y_{k-1}$. Since $2x_{k+1} = x_k$ we have

$$\frac{2}{3} + 2y_k + 2x_{k+1} = 1 + y_{k-1} + x_k \Leftrightarrow \frac{2}{3} + 2y_{k+1} = 1 + y_k \Leftrightarrow \frac{1 + y_k}{\frac{1}{3} + y_{k+1}} = 2.$$

□

Lemma 3.16 *For $n \leq k \leq 2n-1$ we have*

$$\frac{1 + y_{k-1}}{\frac{1}{2} + \frac{1}{2}y_k} \to 2 \text{ for } n \to \infty.$$

Proof Clearly,

$$\frac{1+y_{k-1}}{\frac{1}{2}+\frac{1}{2}y_k} = \frac{1+y_k}{\frac{1}{2}+\frac{1}{2}y_k} - \frac{x_k}{\frac{1}{2}+\frac{1}{2}y_k} = 2 - \frac{x_k}{\frac{1}{2}+\frac{1}{2}y_k}$$

and the lemma follows because $x_k = \frac{2}{3n} \to 0$ for $n \to \infty$. □

Lemma 3.17 *For $n \to \infty$ we have $y_{2n-1} \to 1$.*

Proof

$$\lim_{n\to\infty} y_{2n-1} = \lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{1}{6}\left(\frac{1}{2}\right)^k + \lim_{n\to\infty} \sum_{k=n}^{2n-1} \frac{2}{3n} = \frac{1}{3} + \frac{2}{3} = 1.$$

□

Theorem 3.18 *Every heuristic satisfying Properties 1-3 has worst case ratio at least 2.*

Proof Consider a problem instance defined as follows: $d_0 > 0$, $d_t = 0$ $(t = 2, \dots, T^2-1)$, $d_t = \frac{x_{t-T^2}}{t}$ $(t = T^2, \dots, T^2+T)$, $d_{T^2+T+1} = \frac{1}{T^2+T+1}$ and $d_{T^2+T+2} = \frac{1}{T^2+T+2}$ and $T = 2n$. Consider any heuristic H and let $k > 0$ be the first period where H makes a setup. Dependent on the period k we will define an (possibly infinite) problem instance where we not necessarily use the complete instance d. We will show that the performance ratio of H for all problem instances tends to 2.

- $k \in \{1, \dots, T^2-1\}$:
  Because it is optimal to have only a setup in period 0, the cost of the optimal solution equals $C^* = 1$. As the cost of H equals $C^H = 2$, H has performance ratio $C^H/C^* = 2$.

- $k \in \{T^2, \dots, T^2+n\}$:
  Extend the k-period problem with $d_t = d_{t-k}$ for $t > k$. Because of Lemma 3.14, H will have setups in periods ik for $i = 0, 1, 2, \dots$ with cost

$$C^H = 1 + \sum_{i=1}^{k-1} i d_i = 1 + \sum_{i=T^2}^{k-1} x_i = 1 + y_{k-1-T^2}$$

in every k periods. Now consider a solution with setups in period 1 and periods $T^2 + i3k$ for $i = 0, 1, 2, \ldots$. The holding cost of this solution for every $3k$ periods equals

$$\begin{aligned}
\sum_{i=T^2}^{3k+T^2-1} (i-T^2)d_i &= \sum_{i=T^2}^{k}(i-T^2)d_i + \sum_{i=T^2+k}^{2k}(i-T^2)d_i + \sum_{i=T^2+2k}^{3k}(i-T^2)d_i \\
&= \sum_{i=T^2}^{k}\frac{i-T^2}{i}x_{i-T^2} + \sum_{i=T^2+k}^{2k}\frac{i-T^2}{i-k}x_{i-T^2-k} + \sum_{i=T^2+2k}^{3k}\frac{i-T^2}{i-2k}x_{i-T^2-2k} \\
&\le \sum_{i=T^2}^{k}\frac{T}{T^2}x_{i-T^2} + \sum_{i=T^2+k}^{2k}\frac{T^2+2T}{T^2}x_{i-T^2-k} + \sum_{i=T^2+2k}^{3k}\frac{2T^2+3T}{T^2}x_{i-T^2-2k} \\
&= \left(3+\frac{6}{T}\right)\sum_{i=0}^{k-T^2}x_i = \left(3+\frac{6}{T}\right)y_{k-T^2},
\end{aligned}$$

where the inequality follows from $k - T^2 \le T$. So the average cost for every k periods equals

$$C^* \le \frac{1}{3} + \left(1+\frac{2}{T}\right)y_{k-T^2}.$$

As $k - 1 - T^2 \le n - 1$, by Lemma 3.15 we have that

$$\frac{C^H}{C^*} \ge \frac{1 + y_{k-1-T^2}}{\frac{1}{3} + y_{k-T^2} + \frac{2}{T}y_{k-T^2}} \to 2 \text{ for } T \to \infty.$$

- $k \in \{T^2 + n + 1, \ldots, T^2 + T\}$:
  Extend the k-period problem with $d_t = d_{t-k}$ for $t > k$. As in the previous case H will have setups in periods ik for $i = 0, 1, 2, \ldots$ with cost $C^H = 1 + y_{k-1-T^2}$ in every k periods. Now consider a solution with setups in period 1 and periods $T^2 + i2k$ for $i = 0, 1, 2, \ldots$. By a similar argument as in the previous case the holding cost of this solution for every $2k$ periods equals at most $\left(1 + \frac{3}{T}\right) y_{k-T^2}$. So the average cost for every k periods equals

$$C^* \le \frac{1}{2} + \left(\frac{1}{2}+\frac{3}{2T}\right)y_{k-T^2}.$$

  As $n \le k - 1 - T^2 \le 2n - 1$, by Lemma 3.16 we have that

$$\frac{C^H}{C^*} \ge \frac{1 + y_{k-1-T^2}}{\frac{1}{2} + \frac{1}{2}y_{k-T^2} + \frac{3}{2T}y_{k-T^2}} \to 2 \text{ for } T \to \infty.$$

- $k = T^2 + T + 1$:
  Again we extend the k-period problem with $d_t = d_{t-k}$ for $t > k$. Heuristic H

will generate setups in periods ik for $i = 0, 1, 2, \ldots$ with cost $C^H = 1 + y_T$ in every k periods. Consider a solution with setups in period 1 and periods $T^2 + ik$ for $i = 0, 1, 2, \ldots$. This solution has cost $C^* = 1 + \sum_{i=T^2}^{T^2+T}(i - T^2)d_i + \frac{T+1}{T^2+T+1} \leq 1 + \frac{1}{T}y_T + \frac{1}{T} \leq 1 + \frac{2}{T}$ as $y_T \leq 1$. By Lemma 3.17 we have

$$\frac{C^H}{C^*} \geq \frac{1 + y_T}{1 + \frac{2}{T}} \to 2 \text{ for } T \to \infty.$$

- $k = T^2 + T + 2$ or there is no setup:
  In both cases the total costs of the heuristic equal $C^H = 3 + y_T$ for the $(T^2 + T + 2)$-period problem. By Lemma 3.17 we have $C^H \to 4$ for $T \to \infty$. Consider a solution with setups in periods 1 and T^2. Then the cost of this solution $C^* \leq 2 + \frac{1}{T}y_T + \frac{T+1}{T^2+T+1} + \frac{T+2}{T^2+T+2} \to 2$ as $T \to \infty$ concluding that

$$\frac{C^H}{C^*} \to 2 \text{ for } T \to \infty.$$

In conclusion, no matter where the first setup (after period 0) occurs, we can always construct a problem instance with performance ratio 2 and hence the worst case performance of any heuristic in our class is at least 2. □

3.4.3 Implications

Look-ahead look-back heuristics

The heuristics in our class are myopic in the sense that they do not take into account future demand. However, there is a broader class of heuristics which has a so-called look ahead-look back feature. When the decision is to make a setup in period t or not, there is an option to look back and look ahead a number of periods and to move the setup to one of those periods if an improvement can be made. Wemmerlöv (1983) proposes a variant of the PPB where it is allowed to look ahead and look back one period in order to improve the current solution. Heuristics possessing the look ahead-look back feature can be considered as a compromise between the class of myopic heuristics and the heuristics using the complete model horizon.

Consider a heuristic satisfying Properties 1-3 with the additional option to look ahead and look back l perriods. A slightly modified version of the worst case example of Section 3.4.2 shows that heuristics with the look ahead-look back feature also have worst case ratio at least two. Let $d = d_0, \ldots, d_{T^2+T+2}$ be the demand sequence used in the proof of Theorem 3.18. Now define the sequence d' with $d'_{(l+1)t} = d_t/(l+1)$ for $t = 0, \ldots, T^2+T+2$ and let the remaining demands be equal to zero. So we have added l zero-demand periods between every two positive demand periods.

Assume that some heuristic generates the first setup in some period $q = p(l+1)$ with $p \in \mathbb{N}$. When looking back or looking ahead l periods, there are only zero-demand periods and hence cost will not decrease when moving the setup to one of these periods. Furthermore, the holding cost for some period $r(l+1)$ with $r < p$ equals $r(l+1)d'_{r(l+1)} = r(l+1)d_r/(l+1) = rd_r$, which is the holding cost for demand d_r in problem instance d when it is produced in period 0. So the holding costs up to period q for instance d' with the first setup in period 0 are equal to the holding costs up to period p for instance d with the first setup in period 0. Therefore, starting with instance d' and applying similar arguments as in the proof of Theorem 3.18 shows that our class of heuristics with an additional look ahead-look back feature also has worst case ratio at least two. Thus the following corollary follows from the above discussion.

Corollary 3.19 *Let H be a heuristic satisfying Properties 1-3 with the additional property to look ahead and look back l periods for some fixed $l > 0$. Then H has worst case ratio at least 2.*

Rolling horizon environment

Often the demand for the complete horizon T is not known, but the demand for the first n periods is known. In this case the lot-sizing problem for n periods is solved, the first lot-size decision is implemented and the horizon is rolled forward to the period where the next lot-size starts. Again it is assumed that the next n periods are known and the procedure is repeated. This is known as lot-sizing in a rolling horizon environment, where n is called the planning horizon. As the heuristics within our class use no future demand information, they are suitable to be applied in a rolling horizon environment.

Consider a rolling horizon environment with a planning horizon of n periods. We can easily construct a problem instance with worst case performance arbitrary large. Take the instance with $d_{tn} = \varepsilon$ for $t = 0, 1, 2, \ldots$ and zero demands elsewhere. In period 0 any algorithm faces zero demands in all periods except for period 0 and hence a lot-size of ε is made in period 0. Now the horizon is rolled forward to period n and we are in the same situation as in period 0. So any heuristic will generate a solution with setups in periods tn for $t = 0, 1, 2, \ldots$. Clearly, the worst case performance becomes arbitrary large, as it is optimal to have only a setup in period 0 for ε sufficiently small.

The deficiency of any algorithm is that no solution with a lot-size covering more than n periods can be constructed, whereas it may be optimal to have lot-sizes that cover more than n periods. In the latter case the optimal solution can never be constructed by any algorithm in a rolling horizon environment with planning horizon n. This is in contrast with the situation where the planning horizon is not bounded. In this case the heuristics make the setups in the wrong periods, while it is possible to construct a solution with the same setups as in the optimal solution.

In a rolling horizon environment it seems not fair to measure worst case performance by comparing the rolling horizon solution with the optimal solution over T periods. Therefore, Simpson (2001) proposes to measure the heuristic performance by comparing the heuristic solution with the optimal solution for which no lot size covers more than n periods. Call this the n-optimal solution. Clearly, the worst case performance now depends on the length of the planning horizon n.

Consider the extreme case that $n = 1$. In this case both any heuristic and the n-optimal solution have a setup in each period. Using the alternative performance measure, each heuristic has worst case performance one. Furthermore, consider the case $n = 2$ and the simple heuristic that makes a setup in each period t with $d_t > \frac{1}{2}$. It is not difficult to verify that the ratio of the cost of any 2-period lot-size in the 2-optimal solution is at most $\frac{3}{2}$ smaller than the cost of the same two periods in the heuristic solution. Therefore, the worst case performance of this simple heuristic is at most $\frac{3}{2}$. So for planning horizon $n = 1$ and $n = 2$ there are heuristics with worst case performance smaller than two when using the alternative performance measure.

On the other hand, consider the case that n is relatively large. Now we can use our problem instance of Section 3.4.2. Let T be as large as possible such that $T^2+T+2 \leq n$. Furthermore, let $k \leq T^2+T+2 \leq n$ be the first setup period generated by some heuristic. As in the proof of Section 3.4.2, duplicating the k-period instance leads to a solution with setups in periods ik $(i = 0, 1, 2, \dots)$. However, an alternative solution is to have setups in period 0 and periods $T^2 + ik$ $(i = 0, 1, 2, \dots)$. Note that the alternative solution only has lot-sizes that cover no more than $k \leq n$ periods. So it follows from the proof in Section 3.4.2 that our class of heuristics has worst case ratio at least two when using the alternative performance measure and n sufficiently large (and hence T sufficiently large).

At first sight it seems counterintuitive that the larger the planning horizon (i.e., the more information available), the larger the worst case ratio. However, when using the alternative performance measure for small planning horizons, it is rather that the n-optimal solutions are relatively bad than that the heuristics generate good solutions.

Capacitated lot-sizing heuristics

Assume that we have a heuristic for the capacitated lot-sizing problem satisfying Properties 1-3. First note that the uncapacitated lot-sizing problem is a special case of the capacitated lot-sizing problem, because the uncapacitated lot-sizing problem is a capacitated lot-sizing problem with arbitrary large capacities. Therefore, it follows immediately that any heuristic for the capacitated lot-sizing problem satisfying Properties 1-3 has worst case ratio at least two.

3.5 Constructing new heuristics

In this section we will show that we can use the results from the IP (see Section 3.3.5) to construct new heuristics. It is clear that we can construct an optimal heuristic for the case $T = 2$, because it is optimal to have a setup in period 2 if and only if $d_2 > 1$. This result can be generalized as follows.

Observation 3.20 *Assume that we have a T-period instance and a plan generated for the first $T - 1$ periods with the last setup in period p. Then it is optimal to make a new setup in period T if and only if $d_T > \frac{1}{T-p}$.*

3.5.1 An optimal heuristic for $T = 3$

Example 3.10 shows that $\overline{d}_2 = \frac{1}{5}(\sqrt{21} - 3)$ might be a threshold value for the 3-period problem. Therefore, we construct a heuristic as follows.

Heuristic for $T = 3$ (H_3)
$P_1 = 1$
If $d_2 < \frac{1}{2}(\sqrt{21} - 3)$ Then $P_2 = 0$
If $d_3 < \frac{1}{2}$ Then $P_3 = 0$
Else $P_3 = 1$
Else $P_2 = 1$
If $d_3 < 1$ Then $P_3 = 0$
Else $P_3 = 1$

Table 3.7: Heuristic H_3 for $T = 3$

Proposition 3.21 *Heuristic H_3 has a worst case ratio of at most $\frac{1}{5}(1 + \sqrt{21}) \approx 1.117$.*

Proof First, by Observation 3.20 instances have a performance ratio larger than 1 if there is a non-optimal decision in period 2.

- Assume we have an instance d with $d_2 < \overline{d}_2$ and the optimal solution has a setup in period 2. Then one can show that an instance with $d_3 = \frac{1}{2}$ will give the largest performance ratio. But then the performance ratio of this instance equals

$$\frac{2 + d_2}{5/2} < \frac{2 + \overline{d}_2}{5/2} = \frac{1}{5}(1 + \sqrt{21}).$$

- Assume we have an instance d with $d_2 \geq \overline{d}_2$ and the optimal solution has no setup in period 2. Then one can show that an instance with $d_3 = 0$ will give the largest performance ratio. But then the performance ratio of this instance equals

$$\frac{2}{1+d_2} \leq \frac{2}{1+\overline{d}_2} = \frac{1}{5}(1+\sqrt{21}).$$

So the worst case ratio of H_3 is at most $\frac{1}{5}(1+\sqrt{21})$. □

Example 3.10 and Proposition 3.21 show that the worst case ratio of Heuristic H_3 equals $\frac{1}{5}(1+\sqrt{21})$ and this bound is tight.

In the literature there has also been some research on the worst case performance for lot-sizing heuristics with a finite time horizon. Vachani (1992) analyzed the performance bounds of several algorithms (not necessarily in the class of the heuristics we consider). In Table 3.8 we summarize the results for the case $T = 3$. It follows from Table 3.8 that our simple heuristic outperforms all other heuristics. For the notations we refer to Vachani

Heuristic	EOQ	POQ	SM	LUC	PPB	BMY	FC	H_3
Performance bound	∞	3	$\frac{4}{3}$	∞	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{6}{5}$	$\frac{1}{5}(1+\sqrt{21})$

Table 3.8: A heuristic for $T = 3$

(1992). All performance bounds can be derived from (the references to) the examples in Vachani (1992) except for SM. The performance bound for SM is derived from the following example.

Example 3.22 Consider an instance with $d_1 = 1$, $d_2 = 0$ and $d_3 = \frac{1}{4} + \varepsilon$ with $\varepsilon > 0$. Let $AC(t)$ be the average cost for the first t periods with only a setup in period 1. Because $AC(1) = 1$, $AC(2) = \frac{1}{2}$ and $AC(3) = \frac{3/2+2\varepsilon}{3} > AC(2)$, SM has a setup in periods 1 and 3 with total cost $C^H = 2$. However, it is optimal to have only a setup in period 1 with cost $C^* = \frac{3}{2} + 2\varepsilon$. Therefore, the performance ratio of this instance equals

$$\frac{C^H}{C^*} = \frac{2}{\frac{3}{2}+2\varepsilon} \rightarrow \frac{4}{3} \text{ as } \varepsilon \rightarrow 0.$$

Finally, we note that the bound for PPB is smaller than the bound in Vachani (1992) which is $\frac{3T}{T+2} = \frac{9}{5} > \frac{3}{2}$. The claim in Vachani (1992) that the example in Bitran et al. (1984) yields a tight bound is not correct. Namely the example yields a bound of $\frac{3T}{T+3}$ when T is a multiple of 3. The example from Bitran et al. (1984) and the instance $d_1 = 1$, $d_2 = 1 - \varepsilon$, $d_3 = 2\varepsilon$ have a performance ratio of $\frac{3}{2}$ for $\varepsilon \rightarrow 0$.

3.5.2 An optimal heuristic for $T = 4$

Similar to the case $T = 3$ we can construct a heuristic for the case $T = 4$ with worst case ratio 1.165. The heuristic can be found in Table 3.9. The construction of this heuristic

Heuristic for $T = 4$ (H_4)
$P_1 = 1$
$\overline{d}_2 = 0.740$
$\overline{d}_3^1 = 0.657$
$\overline{d}_3^2 = 0.359$
If $d_2 \leq \overline{d}_2$ Then $P_2 = 0$
If $d_3 \leq \overline{d}_3^1$ Then $P_3 = 0$
If $d_4 \leq \frac{1}{3}$ Then $P_4 = 0$
Else $P_4 = 1$
Else $P_3 = 1$
If $d_4 \leq 1$ Then $P_4 = 0$
Else $P_4 = 1$
Else $P_2 = 1$
If $d_3 \leq \overline{d}_3^2$ Then $P_3 = 0$
If $d_4 \leq \frac{1}{2}$ Then $P_4 = 0$
Else $P_4 = 1$
Else $P_3 = 1$
If $d_4 \leq 1$ Then $P_4 = 0$
Else $P_4 = 1$

Table 3.9: Heuristic H_4 for $T = 4$

is not as straightforward as the case $T = 3$. The value $\overline{d}_2 \approx 0.740$[1] maximizes $W(4,2) \approx 1.150$, the value $\overline{d}_3^2 = \frac{1}{22}(\sqrt{177} - 9) \approx 0.359$ maximizes $W(4,3) = \frac{1}{14}(3 + \sqrt{177}) \approx 1.165$ and the value $\overline{d}_3^1 \approx 0.657$ maximizes $\min\{3/(2 + d_3), (3 + d_3)/(5/2 + \overline{d}_2)\}$ at a value of approximately 1.129.

Proposition 3.23 *Heuristic H_4 has a worst case ratio of at most $\frac{1}{14}(3 + \sqrt{177}) \approx 1.165$.*

Proof First note that the heuristic can be represented by a decision tree as in Figure 3.6. Within each node one can find a node number and above the (relevant) nodes one can

[1] Using a similar approach as in Example 3.8, it can be shown that $\overline{d}_2$ is the positive root of the cubic equation $3x^3 + 12x^2 + 3x - 10 = 0$.

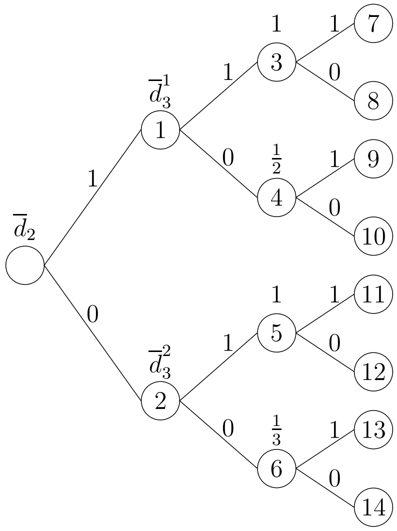


Figure 3.6: Heuristic H_4 represented by a decision tree

find the demand levels at which the decision whether or not to have a new setup are made. The proof consists of calculating the performance ratio at all nodes of the decision tree. That is, for each node we will consider all (relevant) optimal production plans and we will show that for each node the performance ratio will be at most $\frac{1}{14}(3+\sqrt{177}) \approx 1.165$.

- Node 1: $P^H = \{1,1\}$, $P^* = \{1,0\}$, $C^H/C^* = \frac{2}{1+d_2} \le \frac{2}{1+\overline{d}_2} \approx 1.150$
- Node 2: $P^H = \{1,0\}$, $P^* = \{1,0\}$, $C^H/C^* = 1$
- Node 3: $P^H = \{1,1,1\}$

- $P^* = \{1,0,1\}$, $C^H/C^* = \frac{3}{2+d_2} \le \frac{3}{2+\overline{d}_2} \approx 1.095$
- $P^* = \{1,1,0\}$, $C^H/C^* = \frac{3}{2+d_3} \le \frac{3}{2+\overline{d}_3^1} \approx 1.129$

- Node 4: $P^H = \{1,1,0\}$

- $P^* = \{1,0,0\}$, $C^H/C^* = \frac{2+d_3}{1+d_2+2d_3} \le \frac{2}{1+\overline{d}_2} \approx 1.150$
- $P^* = \{1,0,1\}$, $C^H/C^* = \frac{2+d_3}{2+d_2} \le \frac{2+\overline{d}_3^1}{2+\overline{d}_2} \approx 0.970$ (So $P^* = \{1,0,1\}$ cannot be an optimal plan.)

- Node 5: $P^H = \{1,0,1\}$

- $P^* = \{1,0,0\}$, $C^H/C^* = \frac{2+d_2}{1+d_2+2d_3} \leq \frac{2}{1+2\overline{d}_3^2} \approx 1.165$

- $P^* = \{1,1,0\}$, $C^H/C^* = \frac{2+d_2}{2+d_3} \leq \frac{2+\overline{d}_2}{2+\overline{d}_3^2} \approx 1.162$

- Node 6: $P^H = \{1,0,0\}$

- $P^* = \{1,0,1\}$, $C^H/C^* = \frac{1+d_2+2d_3}{2+d_2} \leq \frac{1+2\overline{d}_3^2}{2} \approx 0.859$ (So $P^* = \{1,0,1\}$ cannot be an optimal plan.)

- $P^* = \{1,1,0\}$, $C^H/C^* = \frac{1+d_2+2d_3}{2+d_3} \leq \frac{1+\overline{d}_2+2\overline{d}_3^2}{2+\overline{d}_3^2} \approx 1.042$

For nodes 7–14 we note that the case with $P_4^* = 1$ is not of interest. Because the cost in period 4 of the optimal solution is always at least equal to the cost in period 4 of the heuristic solution, the performance ratios will be at most equal to the performance ratios of the 3-period problems (that is, the performance ratios corresponding to nodes 3–6).

- Nodes 7 and 11: Because both the heuristic and the optimal solution have a setup in period 4 (as $d_4 > 1$), the performance ratios will be smaller than the performance ratios of nodes 3 and 5, respectively.
- Node 8: $P^H = \{1,1,1,0\}$, $P^* = \{1,0,1,0\}$, $C^H/C^* = \frac{3+d_4}{2+d_2+d_4} \leq \frac{3}{2+\overline{d}_2} \approx 1.095$
- Node 9: $P^H = \{1,1,0,1\}$, $P^* = \{1,0,1,0\}$, $C^H/C^* = \frac{3+d_3}{2+d_2+d_4} \leq \frac{3+\overline{d}_3^1}{2+\overline{d}_2+\frac{1}{2}} \approx 1.129$
- Node 10: $P^H = \{1,1,0,0\}$

- $P^* = \{1,0,0,0\}$, $C^H/C^* = \frac{2+d_3+2d_4}{1+d_2+2d_3+3d_4} \leq \frac{2}{1+\overline{d}_2} \approx 1.150$

- $P^* = \{1,0,1,0\}$, $C^H/C^* = \frac{2+d_3+2d_4}{2+d_2+d_4} \leq \frac{2+\overline{d}_3^1+2\cdot\frac{1}{2}}{2+\overline{d}_2+\frac{1}{2}} \approx 1.129$

- Node 12: $P^H = \{1,0,1,0\}$

- $P^* = \{1,0,0,0\}$, $C^H/C^* = \frac{2+d_2+d_4}{1+d_2+2d_3+3d_4} \leq \frac{2}{1+2\overline{d}_3^2} \approx 1.165$

- $P^* = \{1,1,0,0\}$, $C^H/C^* = \frac{2+d_2+d_4}{2+d_3+2d_4} \leq \frac{2+\overline{d}_2}{2+\overline{d}_3^2} \approx 1.162$

- Node 13: $P^H = \{1,0,0,1\}$

- $P^* = \{1,1,0,0\}$, $C^H/C^* = \frac{2+d_2+2d_3}{2+d_3+2d_4} \leq \frac{2+\overline{d}_2+2\overline{d}_3^2}{2+\overline{d}_3^2+2\cdot\frac{1}{3}} \approx 1.143$

- $P^* = \{1,0,1,0\}$, $C^H/C^* = \frac{2+d_2+2d_3}{2+d_2+d_4} \leq \frac{2+2\overline{d}_3^2}{2+\frac{1}{3}} \approx 1.165$

- Node 14: $P^H = \{1,0,0,0\}$

- $P^* = \{1,1,0,0\}$, $C^H/C^* = \frac{1+d_2+2d_3+3d_4}{2+d_3+2d_4} \leq \frac{1+\overline{d}_2+2\overline{d}_3^2+3\cdot\frac{1}{3}}{2+\overline{d}_3^2+2\cdot\frac{1}{3}} \approx 1.143$
- $P^* = \{1,0,1,0\}$, $C^H/C^* = \frac{1+d_2+2d_3+3d_4}{2+d_2+d_4} \leq \frac{1+2\overline{d}_3^2+3\cdot\frac{1}{3}}{2+\frac{1}{3}} \approx 1.165$

$\square$

Note that if $d_2 > \overline{d}_2$, then we already know in the second period that the performance ratio will be smaller or equal than 1.150. This is because there is a relatively high cost in period 2 (at least $\overline{d}_2$), which causes a relatively small performance ratio.

3.6 Conclusion & Future research

In this chapter we studied the worst case performance for a general class of on-line lot-sizing heuristics. We showed that heuristics whick make lot-sizing decisions on a period-by-period basis have a worst case ratio of at least $\frac{3}{2}$, even if the optimal solution and the heuristic solution have at most 2 setups (including the setup in the first period). Using the analysis to construct the worst case examples enabled us to find heuristics with optimal worst case performance for three- and four-period problem instances.

Furthermore, deterministic heuristics that make decisions only based on the current lot-size on a period-by-period basis have worst case ratio at least 2. This result generalizes the result of Axsäter (1985), who shows that a more restrictive class of heuristics has worst case ratio at least 2. The problem instance with this performance behavior was found by formulating the problem of finding worst case examples as a MIP.

We conclude this chapter with some issues for further research. First, the question is still open whether there exist on-line heuristics with a worst case performance strictly smaller than 2. We showed that the worst case performance of such heuristics is at least $\frac{3}{2}$, but no heuristics are known in the literature with worst case performance smaller than 2. Clearly, there is a gap to be closed here. Second, for the 3- and 4-period problem we constructed heuristics with optimal worst case performance. The question is: can we generalize this to arbitrary model horizons? In the proofs for the 3- and 4-period case we considered all possible production plans, which grow at an exponential rate. Therefore, to answer the last question one will need another proof strategy.

Part II

Economic lot-sizing and pricing

Introduction

The demand that a manufacturer has to satisfy is usually created by activities of its marketing department. Instead of taking the marketing and production decisions more or less independently, it may be beneficial to integrate these decisions. This leads to models that are more complex than when we are only concerned with optimal production decisions. As an example, suppose that the selling price of the item still has to be set and that the demand functions are given for the periods under consideration. Then the planning problem consists of deciding simultaneously how high to set the price and how much to produce in each period such that the total profit is maximized.

In this part of the thesis we consider the economic lot-sizing (ELS) problem where demand can be affected by pricing. So instead of assuming that demand is deterministic as in the classical ELS problem, we assume that demand is a deterministic function of the price (assuming some market power of the manufacturer). We will call this the joint pricing and lot-sizing problem.

We see in the literature that two pricing assumptions are made. First, one can assume that a different price can be set in each period. For instance, this is the case for a monopolist. Thomas (1970) showed that this problem can be solved in a similar way as the well-known algorithm proposed by Wagner and Whitin (1958). For instance, if demand is assumed to be linear in the price, then the algorithm requires $\mathcal{O}(T^2)$ computation time. This means that this problem can still be solved efficiently.

Another assumption is that a single price must be set for all periods. For instance, this is the case if firms rely on sales from catalogues. In this case it may be rather expensive to communicate price changes to the customers. Kunreuther and Schrage (1973) propose a heuristic algorithm to solve this problem. However, they do not give any approximation and complexity results of their algorithm. Gilbert (1999) proposes an exact algorithm to solve this problem in $\mathcal{O}(T^3)$ time, but imposes some additional restrictions on the demand and cost parameters. In a more recent paper Gilbert (2000) considers the case with multiple items and production capacities. In that paper it is assumed that setup costs are negligible. The problem is solved iteratively by using the dual problem of the production scheduling problem.

Recently, two papers appeared on a joint pricing and lot-sizing model with the generalization to account for time-invariant production capacities. Geunes et al. (2006) consider a model where the revenue functions are assumed to be piecewise linear concave in the demand level. They show that the problem can be solved in polynomial time. In Geunes et al. (2005) it is shown that this result still holds for arbitrary concave revenue functions, which generalizes the result of Thomas (1970). Furthermore, Geunes et al. (2005) also generalize the work of Kunreuther and Schrage (1973), Gilbert (1999) and Van den Heuvel and Wagelmans (2006b) by showing that their models with the inclusion of time-invariant capacities can still be solved in polynomial time.

Besides the assumption on pricing, we can distinguish other differences between marketing-production models. For example, demand may depend on other marketing instruments than price such as promotion. Sogomonian and Tang (1993) consider a T-period discrete model where the promotion periods and promotion levels have to be determined. A solution of the model is found by solving a number of nested longest path problems. Another distinction in models is that between discrete time and continuous time models. Whitin (1955) considers a pricing problem in the EOQ framework. Finally, there is also literature on marketing-production models in a stochastic environment. For an overview of literature on marketing-production decision-making models we refer to Eliashberg and Steinberg (1993).

In Chapters 4 and 5 we will consider a joint pricing and lot-sizing model. In Chapter 4 we consider the problem where prices may vary over time. In particular, we will focus on a problem with a time-invariant demand function and time-invariant cost parameters. Although Bhattacharjee and Ramesh (2000) proposed a heuristic approach, we will show that the problem can be solved to optimality in a practically efficient way. In Chapter 5 we consider the joint pricing and lot-sizing problem where the price has to be constant over time. By refining the algorithm of Kunreuther and Schrage (1973) we will show that the problem can be solved in polynomial time.

Chapter 4

A joint pricing and lot-sizing model with general prices over time*

Abstract

In this chapter we consider a joint pricing and lot-sizing model. It is assumed that demand is a (deterministic) function of price, where prices may vary over time. We focus on a special case with time-invariant cost parameters considered by Bhattacharjee and Ramesh (2000). While Bhattacharjee and Ramesh (2000) proposed two heuristics, we show that the problem can be solved by a (practically) efficient algorithm proposed by Thomas (1970). Moreover, the problem can be solved even faster by applying the results on a special partition problem derived by Orlin (1985).

4.1 Introduction

In this chapter we consider a joint pricing and lot-sizing model for a manufacturer who is dealing in a single product. It is assumed that the manufacturer has some monopolistic power and he can affect his demand by pricing. For a given planning horizon the manufacturer wants to maximize his profit considering revenue and all relevant costs. The problem is to simultaneously make pricing and lot-sizing decisions such that profit is maximized.

In this chapter we focus on a special case of the problem considered by Bhattacharjee and Ramesh (2000) with time-invariant cost and demand parameters. Bhattacharjee and Ramesh (2000) proposed two heuristic algorithms to solve this special case of the problem. In this chapter we improve on their paper by showing that the problem can be solved to optimality in a (practically) efficient way. We do this by applying (a slightly modified

*This chapter is based on Van den Heuvel and Wagelmans (2003).

version of) the approach of Thomas (1970). Moreover, we show that an even faster method exists when we apply the results on a special partition problem derived by Orlin (1985).

This chapter is organized as follows. In Section 4.2 we describe the joint pricing and lot-sizing model considered by Bhattacharjee and Ramesh (2000) and give a mathematical formulation. In Section 4.3 we present the exact method proposed by Thomas (1970) and in Section 4.4 we apply this method to the Bhattacharjee and Ramesh (2000) case. In Section 4.5 we introduce a special case of a partition problem and in Section 4.6 we show how to apply the results for this problem to solve the Bhattacharjee and Ramesh (2000) case even faster. In Section 4.7 we point out some concerns about the main results presented by Bhattacharjee and Ramesh (2000). The chapter is ended with the conclusion in Section 4.8.

4.2 Problem description

We consider the following joint pricing and lot-sizing model of Bhattacharjee and Ramesh (2000). There is a monopolistic manufacturer dealing in a single product over a finite time horizon. At the beginning of each period lot-sizing and pricing decisions are made, where in each period a different price can be set. It is assumed that demand satisfies

$$d(p) = \beta p^{-\alpha}, \tag{4.1}$$

where $\alpha > 1$ is the demand elasticity, $\beta > 0$ is a constant and p is the price, where price in each period t may be bounded from above or below, i.e., $p_{\min} \leq p_t \leq p_{\max}$. The objective of the manufacturer is to maximize his profit, i.e., to find the best price for which revenue minus cost is maximized. Costs include a fixed setup cost for each period with positive production and a unit production cost for each item produced. Furthermore, holding cost is incurred for carrying inventory from a period to the next period. Bhattacharjee and Ramesh (2000) assume that the cost parameters are time-invariant.

Assuming that all demand must be satisfied (i.e., loss of demand is not allowed) and using the following notation,

T	=	model horizon
K	=	fixed setup cost
c	=	per unit production cost
h	=	holding cost per unit per period
p_t	=	price set in period t
$d(p_t)$	=	demand in period t when price equals p_t
q_t	=	produced quantity in period t
I_t	=	ending inventory in period t,

the problem can be formulated as follows

$$\begin{array}{lll} \max & \sum_{t=1}^{T} d(p_t)p_t - C(D(p)) & \\ \text{s.t.} & p_{\min} \leq p_t \leq p_{\max} & t = 1,\dots,T, \end{array} \tag{4.2}$$

where

$$\begin{array}{rlll} C(D(p)) = & \min & \sum_{t=1}^{T} (K\delta(q_t) + cq_t + hI_t) & \\ & \text{s.t.} & I_t = I_{t-1} - d(p_t) + q_t & t = 1,\dots,T \\ & & q_t, I_t \geq 0 & t = 1,\dots,T \\ & & I_0 = 0, & \end{array}$$

with

$$\delta(x) = \begin{cases} 0 & \text{for } x = 0 \\ 1 & \text{for } x > 0 \end{cases}$$

and $D(p)$ is the demand vector $D(p) = [d(p_1),\dots,d(p_T)]$.

In problem (4.2) we maximize the total revenue minus total cost over all periods, where there may be bounds on the price. If price is not restricted, then we set $p_{\min} = 0$ and $p_{\max} = \infty$ in the model. The total cost is represented by $C(D(p))$, which is the cost of a 'standard' economic lot-sizing (ELS) problem (Wagner and Whitin, 1958). We minimize setup, unit production and holding costs, such that demand is satisfied and production quantity and ending inventory are non-negative in each period. Furthermore, we may assume without loss of generality that starting inventory is zero.

4.3 Solution approach by Thomas (1970)

In this section we give a brief description of the exact algorithm by Thomas (1970) to solve the joint pricing and lot-sizing problem. Note that Thomas (1970) presented the model as a minimization problem, whereas we present it as a maximization model (in line with the formulation of (4.2)). Furthermore, Thomas (1970) assumed general cost parameters and no bounds on the prices in each period.

Let a subplan be a consecutive number of periods for which demand is satisfied. Thomas (1970) shows that the optimal solution consists of a series of consecutive subplans as in the ELS problem. For a subplan consisting of periods $j,\dots,t$ ($1 \leq j \leq t \leq T$) define p_{jt} as the price vector $p_{jt} = [p_j,\dots,p_t]$ and define $\pi_{jt}(p_{jt})$ as the total profit if production takes place in period j to satisfy demand in periods $j,\dots,t$, i.e.,

$$\pi_{jt}(p_{jt}) = \sum_{k=j}^{t} [p_k - c_j - \sum_{i=j}^{k-1} h_i] d_k(p_k) - K_j. \tag{4.3}$$

Furthermore, define π_{jt} as the maximum profit for a subplan consisting of periods $j,\dots,t$, i.e.,

$$\pi_{jt} = \max_{p_{jt}} \pi_{jt}(p_{jt}). \tag{4.4}$$

Thomas (1970) shows that if a setup takes place in period j and the next setup in period $t+1$, then the optimal price for period $k = j, \ldots, t$ must be set at the value p_k^* that maximizes

$$\pi_k(p_k) = (p_k - c_j - \sum_{i=j}^{k-1} h_i) d_k(p_k)$$

and hence

$$\pi_{jt} = \sum_{k=j}^{t} \pi_k(p_k^*) - K_j. \tag{4.5}$$

Dependent on the structure of $d_t(p_t)$ we can calculate this optimal price in an analytical way or, if necessary, by enumeration. Throughout this chapter, we assume that this problem is tractable.

Let $F(t)$ be the optimal profit up to period t. As there exists an optimal solution consisting of a series of consecutive subplans, the following forward recursion enables us to find the optimal profit for the whole model horizon:

$$F(t) = \max_{j=1,\ldots,t} \{F(j-1) + \pi_{jt}\} \text{ for } t = 1, \ldots, T \text{ with } F(0) = 0. \tag{4.6}$$

It is not difficult to see that, given the values π_{jt}, this algorithm runs in $\mathcal{O}(T^2)$ time.

4.4 Applying Thomas' approach to the Bhattacharjee and Ramesh (2000) case

In this section we apply Thomas' approach to the Bhattacharjee and Ramesh (2000) case. With their assumptions we can find the optimum of (4.3) in an analytical way. Substituting demand function (4.1) and the time-invariant cost parameters (i.e., $K_t = K$, $c_t = c$ and $h_t = h$ for $t = 1, \ldots, T$) in (4.3) we have that

$$\pi_{jt}(p_{jt}) = \sum_{k=j}^{t} [p_k - c - \sum_{i=j}^{k-1} h] \beta p_k^{-\alpha} - K = \sum_{k=j}^{t} [p_k - c - (k-j)h] \beta p_k^{-\alpha} - K. \tag{4.7}$$

Calculating the first order conditions for the subplan consisting of periods $i = j, \ldots, t$ we have

$$\frac{\partial \pi_{jt}(p_{jt})}{\partial p_i} = 0 \Rightarrow p_i^* = \frac{\alpha(c + (i-j)h)}{\alpha - 1} \geq 0 \text{ as } \alpha > 1,\ i \geq j \text{ and } c, h \geq 0. \tag{4.8}$$

Note that p_i^* is not dependent on the prices set in the other periods of the subplan. Furthermore, note that p_i^* does only depend on period j and not on period t, which implies that the optimal price for a single period is only dependent on the starting period of the subplan and independent of the number of periods in the subplan. Finally, one can verify that

$$\left.\frac{\partial \pi_{jt}(p_{jt})}{\partial p_i}\right|_{p_i} > 0 \text{ for } p_i < p_i^* \text{ and } \left.\frac{\partial \pi_{jt}(p_{jt})}{\partial p_i}\right|_{p_i} < 0 \text{ for } p_i > p_i^*.$$

This implies that the maximum profit function for a single period in a subplan is unimodal and that it has a unique optimum at price p_i^*.

If we analyze the second order partial derivative we find

$$\frac{\partial^2 \pi_{jt}(p_{jt})}{\partial {p_i}^2} = 0 \Rightarrow \widehat{p}_i = \frac{(\alpha+1)(c+(i-j)h)}{\alpha-1} > p_i^*.$$

It is not difficult to verify that the second order partial derivative is smaller than zero for $p_i < \widehat{p}_i$ and larger than zero for $p_i > \widehat{p}_i$. This means that the maximum profit function for a single period in a subplan is concave for $p_i < \widehat{p}_i$ and convex for $p_i > \widehat{p}_i$.

Because Bhattacharjee and Ramesh (2000) assume a time-invariant demand function and time-invariant cost parameters, it follows from (4.3) and (4.8) that

$$\pi_{jt} = \pi_{1,t-j+1} \text{ for all } 1 \le j \le t \le T. \tag{4.9}$$

This means that it is only necessary to evaluate π_{1t} for $t = 1, \ldots, T$. The following recursion formulas can be used to calculate π_{1t} for $t = 1, \ldots, T$ in $\mathcal{O}(T)$ time:

$$p_{t+1}^* = p_t^* + \frac{\alpha h}{\alpha - 1} \tag{4.10}$$

$$\pi_{1,t+1} = \pi_{1,t} + (p_{t+1}^* - c - th)\beta \left(p_{t+1}^*\right)^{-\alpha} \tag{4.11}$$

with

$$p_1^* = \frac{\alpha c}{\alpha - 1} \text{ and } \pi_{11} = (p_1^* - c)\beta \left(p_1^*\right)^{-\alpha} - K.$$

By applying recursion formula (4.6) and using (4.10) and (4.11), we can find the optimal total profit, the optimal subplans and the optimal prices.

It follows from (4.10) that prices in consecutive periods within a subplan have an interesting property: the difference between consecutive prices is constant as $p_{i+1}^* - p_i^* = \frac{\alpha h}{\alpha - 1}$. This means that even in the case of time-invariant demand functions and time-invariant demand parameters, prices are not constant over time (as also noted in Geunes et al. (2005)). The above property does not only hold for production cost functions with a setup and unit production cost, but for arbitrary production cost functions.

Theorem 4.1 *Consider a subplan $1, \ldots, t$ with aggregate demand D. Then, given any production cost function f, the prices that maximize the profit of the subplan satisfy $p_{i+1}^* - p_i^* = \frac{\alpha h}{\alpha - 1}$.*

Proof Let f be an arbitrary production cost function and consider the problem

$$\begin{aligned} \max \quad & \sum_{k=1}^{t} [p_k - (k-1)h] \beta {p_k}^{-\alpha} - f(D) \\ \text{s.t.} \quad & \sum_{k=1}^{t} \beta {p_k}^{-\alpha} = D, \end{aligned}$$

which maximizes the total profit of the subplan. Introducing the Lagrangian multiplier λ, the following set of equations solves the optimization problem:

$$\begin{gathered} \tfrac{\partial}{\partial p_i}[\textstyle\sum_{k=1}^{t} (p_k - (k-1)h) \beta {p_k}^{-\alpha} - f(D)] + \lambda \tfrac{\partial}{\partial p_i}[D - \textstyle\sum_{k=1}^{t} \beta {p_k}^{-\alpha}] = 0 \quad \text{for } i = 1, \dots, t, \\ \textstyle\sum_{k=1}^{t} \beta {p_k}^{-\alpha} = D. \end{gathered}$$

After some algebra the first set of equations reduces to

$$p_i^* = \frac{\alpha h (i-1)}{\alpha - 1} + \lambda \frac{\alpha}{\alpha - 1} \text{ for } i = 1, \dots, t$$

and it follows that $p_{i+1}^* - p_i^* = \frac{\alpha h}{\alpha - 1}$. □

Given some aggregate demand level D, there is a straightforward way to find the optimal prices. Because all prices can be expressed in p_1, i.e., $p_k = \frac{\alpha(k-1)h}{\alpha-1} + p_1$, solving $\sum_{k=1}^{t} \beta {p_k}^{-\alpha} = D$ reduces to solving

$$\sum_{k=1}^{t} \beta \left(\frac{\alpha(k-1)h}{\alpha - 1} + p_1 \right)^{-\alpha} = D,$$

which is an equation in the single variable p_1. Furthermore, because the left hand side is decreasing in p_1, the price p_1 that satisfies the equation can be found by binary search. This gives us a simple procedure to calculate the optimal prices given some aggregate demand level D.

As Theorem 4.1 holds for any aggregate demand level D of a subplan, it also holds for the optimal demand level. However, this does not imply that the price property also holds for optimal solutions of the Bhattacharjee and Ramesh (2000) case with arbitrary production cost functions. Namely, in the case of arbitrary production cost functions an optimal solution does not necessarily consist of a series of subplans and hence Theorem 4.1 cannot be applied. However, in case of concave production cost functions, an optimal solution consists of a series of consecutive subplans (Zangwill, 1968) and hence the price property still holds.

It follows from (4.10) and (4.11) that π_{1t} can be determined in $\mathcal{O}(t)$ time for a fixed t, and it follows from (4.6) that it takes $\mathcal{O}(T^2)$ time to evaluate $F(T)$. So the method proposed by Thomas (1970) is better than the heuristics proposed by Bhattacharjee and Ramesh (2000) in two ways. First, it is an exact algorithm instead of a heuristic approach. Second, the method appears to require a much lower running time. We implemented the algorithm in C++ and it took less than a second to solve a 1000-period problem instance, whereas Bhattacharjee and Ramesh (2000) only report results for their heuristics for problem instances with a maximum of 15 periods. They used a complete enumeration to calculate the optimal values for some 5-period and 10-period problems. This took more than one hour for a 10-period problem. Note that Thomas (1970) proved a planning horizon theorem that can be used to further speed up computations.

Thomas' approach does not take into account the constraint $p_{\min} \leq p_t \leq p_{\max}$. However, this constraint does not make the problem harder to solve. Including this constraint, the price that maximizes (4.7) for each period i must be equal to $p_{\min}$, $p_{\max}$ or p_i^* as the profit function for a single period is unimodal. This means that we have to check a constant number of prices for determining the optimal profit in a single period of a subplan. Hence, the (theoretical) running time of the algorithm is not affected by this constraint.

Although the running time of Thomas' approach is $\mathcal{O}(T^2)$, this does not imply that the problem can be solved in polynomial time. This is because the input size of the problem is $\mathcal{O}(\log(\max\{T, K, c, h, \alpha, \beta\}))$ and therefore the DP algorithm is pseudopolynomial in the input size. In Section 4.6 we will propose a solution approach for which the running time is lower than Thomas' approach. To this end, we will need some results on a special partition problem derived by Orlin (1985).

4.5 Intermezzo: A partition problem

4.5.1 Introduction

Given a set of T identical objects $\mathcal{T} = \{1, \ldots, T\}$ which can be partitioned into subgroups and a cost of $C(i)$ associated with a subgroup of size i (where the size is the number of items in the subgroup). The objective is to find a partition of $\mathcal{T}$ that minimizes the total cost of the subgroups. We will call this problem the partition problem. Note that we may assume that items within a subgroup have consecutive indices, because the costs only depend on the sizes of the subgroups. Furthermore, we assume that the function $c(i) = \frac{C(i)}{i}$ is quasiconvex[1], where $c(i)$ represents the average cost per object in a subgroup of size i. Finally, as in Orlin (1985) we assume that there is some 'oracle function' which calculates the value $C(i)$ for each i in constant time.

[1]A function $f : \mathbb{R} \to \mathbb{R}$ is quasiconvex if and only if there exists a $x_0 \in \mathbb{R}$ (or $x_0 = \pm\infty$) such that $f(x)$ is non-increasing for all $x \leq x_0$ and $f(x)$ is non-decreasing for all $x \geq x_0$.

In the case of general cost functions $C(i)$ there is a simple dynamic programming (DP) to find the partition with minimum cost. Let $G(i)$ be the minimum cost when the first i items are partitioned. Then the following recursion finds the optimal partition:

$$G(i) = \min_{j=0,\dots,i-1} \{G(j) + C(i-j)\} \text{ for } i = 1, \dots, T \tag{4.12}$$

with $G(0) = 0$. It is not difficult to see that the DP algorithm runs in $\mathcal{O}(T^2)$ time. The following sections will deal with the cases that $c(i)$ is quasiconvex and $C(i)$ is convex.

In Section 4.5.2 we will show that there exists an efficient algorithm to solve the partition problem when $c(i)$ is quasiconvex. This extends the work of Orlin (1985), who considers the same problem with $C(i)$ a convex function, which is a special case of our problem. We derive properties of an optimal solution and use them to develop an efficient algorithm. In Section 4.5.3 we use the proofs of Section 4.5.2 to show that the case with $C(i)$ convex has only two candidate solutions. The proofs in this section are alternative proofs for the proofs in Orlin (1985).

4.5.2 Subgroups with quasiconvex average cost

Let q be the size of a subgroup with minimum cost per object, i.e., $q = \arg\min_{i=1,\dots,T}\{c(i)\}$. We assume that $q < T$. If not, then the optimal solution is not to partition $\mathcal{T}$ at all. Furthermore, for the moment we assume that q is known. We will come back on this assumption later. If T is an integer multiple of q, say $T = nq$ with $n \in \mathbb{N}$, then it is optimal to partition $\mathcal{T}$ in n subgroups of size q. Clearly, this solution has the lowest average cost per object. If T is not an integer multiple of q, then the optimal solution may consist of subgroups with sizes both smaller and larger than q. Such a problem instance is shown in Example 4.2.

Example 4.2 Let $T = 9$ and let $C(i)$ be defined as in Table 4.1. Note that $c(i)$ is

i	1	2	3	4	5	6	7	8	9
$C(i)$	120	220	300	396	425	516	840	1040	1260
$c(i)$	120	110	100	99	85	86	120	130	140

Table 4.1: Costs of Example 4.2

quasiconvex. In this example the subgroup of size 5 has the lowest cost per object. Taking a solution with subgroups of size 4 and 5 leads to a cost of $396 + 425 = 821$. However, it is optimal to partition the group into subgroups of sizes 3 and 6 with a total cost of $300+516 = 816$. So the optimal solution does not include a subgroup of size $q = 5$.

Intuitively one may expect that the sizes of the subgroups do not deviate too much from the optimal size q. The following two lemmas confirm this. We denote L_i as the size of subgroup i so that $L_1, \ldots, L_n$ with $\sum_{i=1}^{n} L_i = T$ represents a partition of $\mathcal{T}$ in n subgroups.

Lemma 4.3 *Let $L_1, \ldots, L_n$ be a partition of $\mathcal{T}$ and let I be the set with subgroups larger than q, i.e., $I = \{i : L_i > q\}$. Then $\sum_{i \in I}(L_i - q) \leq q - 1$.*

Proof Assume $\sum_{i \in I}(L_i - q) \geq q$. Then we can decrease the sizes of the subgroups in I such that each subgroup is of at least size q and we create a new subgroup of size q. Clearly, we have a feasible solution and we have a decrease in average cost per object because the sizes of the subgroups decrease (but are still larger than q) and $c(i)$ is quasiconvex. □

Lemma 4.4 *Let $L_1, \ldots, L_m$ be a partition of N and let I be the set with subgroups smaller than q, i.e., $I = \{i : L_i < q\}$. Then $\sum_{i \in I}(q - L_i) \leq q - 1$.*

Proof Assume $\sum_{i \in I}(q - L_i) \geq q$. Let L_j be the smallest subgroup with $j \in I$. Construct a new solution by adding the objects of L_j to the subgroups L_i, such that $L_i \leq q$ $(i \in I \backslash \{j\})$. This is possible because

$$\sum_{i \in I}(q - L_i) \geq q \Leftrightarrow q - L_j + \sum_{i \in I \backslash \{j\}} (q - L_i) \geq q \Leftrightarrow L_j \leq \sum_{i \in I \backslash \{j\}} (q - L_i).$$

Again, we have a decrease in cost because the average cost per object decreases and $c(i)$ is quasiconvex. □

Lemmas 4.3 and 4.4 show that the sizes of the subgroups cannot deviate too much from q. This helps us to develop an efficient procedure to find the optimal solution of the partition problem.

Theorem 4.5 *Let q be the size of subgroup with lowest cost per item. Then an optimal solution of the partition problem can be found in a time that is polynomial in q.*

Proof We only give the proof in case $L_i \geq q$ for all $i = 1, \ldots, n$. Similar arguments can be used to prove the general case. Let I be the set of subgroups not equal to q, i.e., $I = \{i : L_i > q\}$. We are interested in the total number of objects that may be contained in subgroups larger than q. By definition of I this amount equals $\sum_{i \in I} L_i$. It can be shown that the set I and the values of L_i $(i \in I)$ that maximize $\sum_{i \in I} L_i$ and satisfy Lemma 4.3, are $L_i = q + 1$ and $|I| = q - 1$. So for an arbitrary solution it holds

$$\sum_{i \in I} L_i \leq (q - 1)(q + 1).$$

This implies that at most $(q-1)(q+1)$ objects may be partitioned into subgroups of sizes larger than q and the remaining objects are partitioned into subgroups of size q. Our solution procedure is now as follows. Let $m = \min\{i : T - qi \leq (q-1)(q+1)\}$ (if $(q-1)(q+1) \geq T$ we set $m = 0$). Then we have at least m subgroups of size q in the optimal solution. Now the remaining part of the problem, consisting of partitioning $T - mq \leq (q-1)(q+1)$ objects, can be solved by the DP algorithm of Section 4.5.1 and this takes no more than $\mathcal{O}(q^4)$ time. □

Note that the procedure in the proof is at least as fast as applying recursion (4.6). This is because we have to apply recursion (4.6) to at most $(q-1)(q+1) \leq T$ periods.

4.5.3 Subgroups with convex cost

Orlin (1985) considers a special case of the problem and he assumes that $C(i)$ is convex. This is a stronger assumption, since $C(i)$ is convex implies that $c(i)$ is quasiconvex. In the convex case we can even further speed up computations. In fact, we will show that the optimal number of subgroups n can take on at most two values and we do not need the DP approach. To show this we need the following lemmas.

Lemma 4.6 *There exists an optimal partition for which $|L_i - L_j| \leq 1$ $(i, j = 1, \ldots, m)$.*

Proof Let L_i and L_j be subgroups with $L_i - L_j \geq 2$. Then moving one object from L_i to L_j does not lead to a cost increase since $C(i)$ is convex. Repeating this procedure for all L_i and L_j with $L_i - L_j \geq 2$ leads to a solution with $|L_i - L_j| \leq 1$ and costs not larger than the initial solution. □

A direct consequence of Lemma 4.6 is that there exists an optimal solution where either all subgroups have size at most q or size at least q. This is not necessarily true for the case when $c(i)$ is quasiconvex (see Example 4.2). The following theorem shows that in case $L_i \geq q$ for $i = 1, \ldots, n$, the number of subgroups n is uniquely determined.

Theorem 4.7 *If $L_i \geq q$ for $i = 1, \ldots, n$, then $n = \left\lfloor \frac{T}{q} \right\rfloor$.*

Proof First, we have

$$T = \sum_{i=1}^{n} L_i \geq nq \Leftrightarrow n \leq \frac{T}{q} \Leftrightarrow n \leq \left\lfloor \frac{T}{q} \right\rfloor,$$

where the last equivalence follows from the integrality of n. Second, from Lemma 4.3 we have

$$\begin{aligned}\sum_{i=1}^{n}(L_i - q) \leq q - 1 \quad &\Leftrightarrow \quad T - nq \leq q - 1 \\ &\Leftrightarrow \quad n \geq \frac{T - (q-1)}{q} \\ &\Leftrightarrow \quad n \geq \left\lceil \frac{T - (q-1)}{q} \right\rceil \\ &\Leftrightarrow \quad n \geq \left\lfloor \frac{T}{q} \right\rfloor .\end{aligned}$$

This implies that $n = \left\lfloor \frac{T}{q} \right\rfloor$. □

Similarly to the proof of Theorem 4.7 we can prove Theorem 4.8.

Theorem 4.8 *If $L_i \leq q$ for $i = 1, \ldots, n$, then $n = \left\lceil \frac{n}{q} \right\rceil$.*

It follows from Theorems 4.7 and 4.8 that

$$\left\lfloor \frac{T}{q} \right\rfloor \leq n \leq \left\lceil \frac{T}{q} \right\rceil .$$

Furthermore, given n, the sizes of the subgroups are uniquely determined by Lemma 4.6. Some algebra shows that we can only have subgroups of size $t = \left\lfloor \frac{T}{n} \right\rfloor$ and size $t + 1$. To be more precise, we have exactly $n(t + 1) - T$ subgroups of size t and $T - tn$ subgroups of size $t + 1$. Note that if T is an integer multiple of q, then we have immediately found the optimal partition.

As a final remark, we have assumed throughout this section that q is known (or can be found by some analytical approach). If this is not the case, then q can be found by searching for the smallest i that satisfies $c(i + 1) - c(i) \geq 0$, which takes $\mathcal{O}(\log T)$ time.

4.6 Applying Orlin's approach to the Bhattacharjee and Ramesh (2000) case

Instead of solving the time-invariant joint pricing and lot-sizing problem with Thomas' approach of Section 4.3, we can use the approach of the previous section (which we will refer to as Orlin's approach). Namely, given the values π_{1t}, the pricing problem is equivalent to the problem of partitioning the model horizon T into n subplans with t_i periods ($i = 1, \ldots, n$) each with a profit π_{1t_i}, such that the total profit of the subplans $\sum_{i=1}^{n} \pi_{1t_i}$ is maximized and $\sum_{i=1}^{n} t_i = T$. To apply the results of the previous section, we need the following theorem.

Theorem 4.9 *The function π_{1t} is strictly increasing and concave in t.*

Proof First, from (4.11) it follows that

$$\Delta\pi_t = \pi_{1t} - \pi_{1,t-1} = (p_t^* - c - (t-1)h)\beta p_t^{*-\alpha} = \beta\left(\frac{\alpha}{\alpha-1}\right)^{-\alpha}(c+(t-1)h)^{1-\alpha}.$$

It follows immediately that $\Delta\pi_t > 0$, which means that π_t is increasing in t. Furthermore, because $\Delta\pi_t$ is decreasing in t, π_{1t} is concave in t. □

From Theorem 4.9 it follows that the pricing problem is equivalent to the partition problem of Section 4.5.3. (As we formulated the pricing problem as a maximization problem, we need concavity instead of convexity to apply the results of Section 4.5.) Furthermore, the analysis of Section 4.5.3 shows that if q is the subplan with maximum average profit per period, i.e., $q = \arg\max\{\frac{\pi_{1t}}{t} : t = 1, 2, \ldots, T\}$, then the horizon is partitioned into either $n = \lfloor \frac{T}{q} \rfloor$ or $n = \lceil \frac{T}{q} \rceil$ subplans. Furthermore, given n, the number of periods in each subplan is either $t = \lfloor \frac{T}{n} \rfloor$ or $t+1$. To be more precise, there are $n(t+1)-T$ subplans consisting of t periods and $T-tn$ subplans consisting of $t+1$ periods. We illustrate Orlin's approach in Example 4.10.

Example 4.10 Consider a problem instance with $K = 20$, $c = 2$, $h = 1$, $\alpha = 2$, $\beta = 80$ and $T = 10$, so that the demand function in each period equals $d(p) = 80p^{-2}$. Using recursion formulas (4.10) and (4.11) the optimal prices and profit for a subplan of t periods can be determined and are presented in Table 4.2. It follows from Table 4.2 that

t	1	2	3	4	5	6	7	8	9	10
p_t^*	4	6	8	10	12	14	16	18	20	22
π_{1t}	0	6.67	11.67	15.67	19.00	21.86	24.36	26.58	28.58	30.40
$\frac{\pi_{1t}}{t}$	0	3.33	3.89	3.92	3.80	3.64	3.48	3.32	3.18	3.04

Table 4.2: Costs corresponding to Example 4.10

the subplan with optimal average profit consists of 4 periods, i.e., $q = 4$. Using Orlin's approach the optimal solution consists of partitioning the horizon in either $n = \lfloor \frac{T}{q} \rfloor = 2$ or $n = \lceil \frac{T}{q} \rceil = 3$ subplans.

- $n = 2$:
  There are only subplans consisting of $t = \lfloor \frac{T}{n} \rfloor = 5$ periods (as 2 exactly divides 10). The total revenu in this case equals $2\pi_{1,5} = 2 \cdot 19 = 38$.

- $n = 3$:
  There are subplans consisting of $t = \lfloor \frac{T}{n} \rfloor = 3$ periods and $t+1 = 4$ periods. That is, we have two subplans consisting of 3 periods and one subplan consisting of 4 periods with total profit $2\pi_{1,3} + \pi_{1,4} = 2 \cdot 11.67 + 15.67 \approx 39$.

So the optimal solution is to partition the horizon into 2 subplans consisting of 3 periods and 1 subplan consisting of 4 periods and the total profit equals approximately 39. Note that it is sufficient to calculate Table 4.2 up to $t = 5$ to find the optimal value q.

It is not immediately clear that π_{1t} is concave when the additional constraint $p_{\min} \leq p_t \leq p_{\max}$ holds. The following theorem shows that the concavity property still holds in this case and again Orlin's approach can be applied.

Theorem 4.11 *The function π_{1t} is concave in t when the constraint $p_{\min} \leq p_t \leq p_{\max}$ holds.*

Proof Let $\pi_i(p) = (p - c - (i-1)h)\beta p^{-\alpha}$ be the profit in period i when the price equals p (where setup cost is excluded). As already shown in Section 4.4, the price that maximizes $\pi_i(p)$ in the unconstrained case equals $p_i^* = \frac{\alpha(c+(i-1)h)}{\alpha-1}$ and $p_i^* < p_{i+1}^*$. When the constraint $p_{\min} \leq p_t \leq p_{\max}$ holds, the price that maximizes $\pi_i(p)$ equals $\hat{p}_i = p_{\min}$ in case $p_i^* < p_{\min}$ and the price that maximizes $\pi_i(p)$ equals $\hat{p}_i = p_{\max}$ in case $p_i^* > p_{\max}$, because $\pi_i(p)$ is unimodal in p. Summarizing, the prices $\hat{p}_i$ that maximize π_{1t} are

$$\hat{p}_i = \begin{cases} p_{\min} & \text{if } p_i^* < p_{\min} \\ p_i^* & \text{if } p_{\min} \leq p_i^* \leq p_{\max} \\ p_{\max} & \text{if } p_i^* > p_{\max} \end{cases} \text{ for } i = 1, \ldots, t.$$

To show that π_{1t} is concave we have to show that $\pi_i(\hat{p}_i)$ is decreasing in i. From Theorem 4.9 it follows that $\pi_i(p_i^*) \geq \pi_{i+1}(p_{i+1}^*)$. Furthermore, because $\pi_i(p) - \pi_{i+1}(p) = h\beta p^{-\alpha} > 0$, it immediately follows that $\pi_i(p_{\min}) > \pi_{i+1}(p_{\min})$ and $\pi_i(p_{\max}) > \pi_{i+1}(p_{\max})$. Thus, it remains to show that $\pi_i(\hat{p}_i) - \pi_{i+1}(\hat{p}_{i+1}) \geq 0$ for $p_i^* < p_{\min} \leq p_{i+1}^*$ and $p_i^* \leq p_{\max} < p_{i+1}^*$.

- $p_i^* < p_{\min} \leq p_{i+1}^*$:
  In this case $\hat{p}_i = p_{\min}$, $\hat{p}_{i+1} = p_i^*$ and hence

$$\pi_i(p_{\min}) \geq \pi_i(p_{i+1}^*) \geq \pi_{i+1}(p_{i+1}^*).$$

- $p_i^* \leq p_{\max} < p_{i+1}^*$: In this case $\hat{p}_i = p_i^*$, $\hat{p}_{i+1} = p_{\max}$ and hence

$$\pi_i(p_i^*) \geq \pi_{i+1}(p_{i+1}^*) \geq \pi_{i+1}(p_{\max}).$$

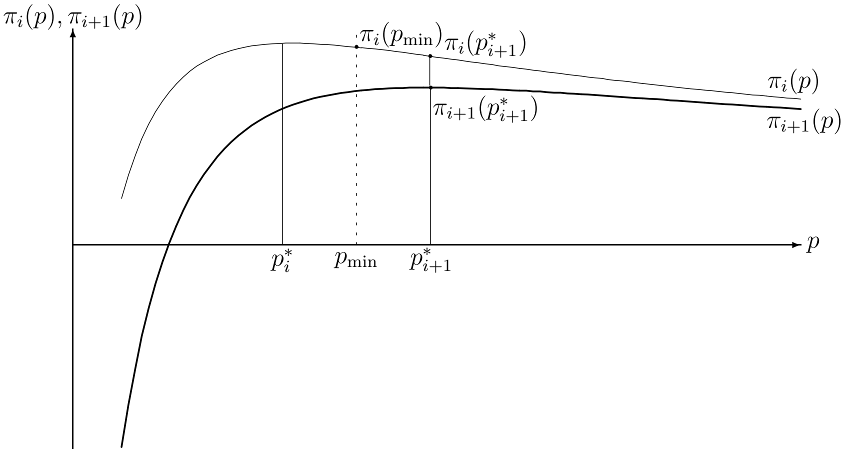


Figure 4.1: Case $p^*_i < p_{\min} \leq p^*_{i+1}$

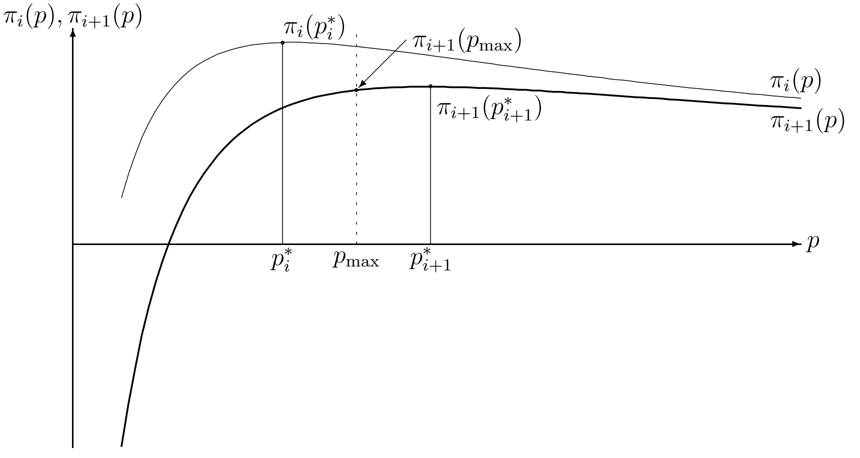


Figure 4.2: Case $p^*_i \leq p_{\max} < p^*_{i+1}$

Both cases are illustrated in Figures 4.1 and 4.2. □

From Theorems 4.9 and 4.11 it follows that Orlin's approach can be applied to the Bhattacharjee and Ramesh (2000) case both with unbounded and bounded prices. Unfortunately, this does not imply that the problem can be solved in polynomial time as the optimal value of q has to be determined. From (4.10) and (4.11) it follows that there exists no closed-form solution to determine π_{1t} and it takes $\mathcal{O}(t)$ time to calculate π_{1t}. Therefore, finding q may take $\mathcal{O}(T)$ time and hence Orlin's approach has a pseudopolynomial running time.

However, Orlin's approach will be much faster in practice in case $q \ll T$. Furthermore, in case of other demand functions Orlin's approach may be polynomial, whereas Thomas' approach will be pseudopolynomial. For example, if we consider a simple linear demand function, i.e.,

$$d(p) = \beta - \alpha p,$$

then some algebra shows that the optimal price in period i of subplan $1, \ldots, t$ equals

$$p_i^* = \frac{1}{2}\left(\frac{\beta}{\alpha} + c + (i-1)h\right)$$

and the optimal profit in period i equals

$$\pi_i(p_i^*) = \frac{1}{2}\left(\frac{\beta}{\alpha} - (c + (i-1)h)\right)^2.$$

This means that there exists a closed-form formula for $\pi_{1t} = \sum_{i=1}^{t} \pi_i(p_i^*) - K$ (since $\sum_{i=1}^{t} i^2 = \frac{1}{6}t(t+1)(2t+1)$). Furthermore, it can be shown that π_{1t} is concave in t. Therefore, the optimal value q can be found by binary search and takes $\mathcal{O}(\log T)$ time. This means that Orlin's approach leads to a polynomial time algorithm (while Thomas' approach does not).

In case the profit function π_{1t} is quasiconcave, the value q can still be found by binary search. If π_{1t} can be calculated in constant time, this means that it takes again $\mathcal{O}(\log T)$ time to find q. Therefore, applying the solution approach in the proof of Theorem 4.5 will be faster than applying Thomas' approach. Finally, we note that Orlin's approach and the approach for the quasiconcave case become constant time algorithms when q is fixed and given (in contrast to Thomas' approach).

Bhattacharjee and Ramesh (2000) also consider the problem with perishable goods. They assume that goods perish after m periods and hence they cannot be used anymore to satisfy demand. Because of the time-invariant cost parameters the optimal solution still consists of a series of consecutive subplans. Therefore, it is not optimal to have subplans of more than m periods. In this case, again Orlin's approach can be applied by setting $q := \min\{q, m\}$. (Note that subplans larger than m periods are not feasible in a solution. The latter may occur when $q = m$ and we have $n = \lfloor \frac{T}{q} \rfloor$ subplans consisting of $t = \lceil \frac{T}{n} \rceil$ and $t+1$ periods. In this case the solution with $n = \lceil \frac{T}{q} \rceil$ is optimal.)

Finally, we note that Orlin (1985) generalized a result by Chand (1982). Chand (1982) also considers a partition problem in an 'economic lot-sizing environment'. More specifically, he considers the ELS problem with time-invariant demand and cost parameters. In this problem the costs of a subplan only depend on the number of periods in the subplan.

Chand (1982) shows that there exists an optimal solution for which the number of periods in any two subplans differ at most one period. Therefore, a similar approach as Orlin (1985) can be applied to solve the problem. (The generalization of Orlin (1985) is that lower and upper bounds are set on the number of periods in the subplan.)

4.7 Concerns about the results presented in Bhattacharjee and Ramesh (2000)

Bhattacharjee and Ramesh (2000) proposed two heuristic algorithms to solve the joint pricing and planning problem. To justify the application of heuristics, they refer to the exponential nature of the problem and the characteristics of the maximum profit function. Indeed there are 2^{T-1} possible production plans (assuming positive demand in period 1). However, Thomas' approach (see Section 4.3) circumvents this 'exponentiality' and the problem can be solved to optimality by an efficient method. Note that the classical Wagner-Whitin problem also has an exponential number of possible production plans, but it can still be solved in polynomial time (Wagner and Whitin, 1958).

Furthermore, in both heuristics there is a predetermined value r, which defines the maximum number of periods in a subplan to be considered. For such a subplan all (2^{r-1}) production plans are generated at the start of the algorithms. This does not only mean that the heuristics have a running time which is exponential in r, but the heuristics may also perform poorly if the optimal size of a subplan is larger than r. Bhattacharjee and Ramesh (2000) report worst case deviations of more than 28% and 18% for the two heuristics.

Besides the fact that the problem can be efficiently solved to optimality, Bhattacharjee and Ramesh (2000) make several incorrect statements about the problem. Bhattacharjee and Ramesh (2000, Theorem 1, p. 587) claim that for a profit-maximizing firm it is always profitable to meet total demand. This means that shortage cost can be ignored and only the model with no loss of demand needs to be considered. In the proof of this result the authors use the fact that by increasing price in case of a shortage, there is an increase in revenue and a saving in shortage cost. However, later they assume that $p_{\min} \leq p_t \leq p_{\max}$, which is a contradiction with the proof where it is assumed that price can always be increased. In Example 4.12 we show that it can be optimal to have loss of demand.

Example 4.12 Consider a T-period example with $K = 9$, $p_{\max} - c = 3$, $p_{\min} - c > 0$, $s = 1$ and $h = 1$, where s is the per unit shortage cost. Furthermore, let the demand function be such that $d(p_{\max}) = 2$ and $(p_{\max} - c)d(p_{\max}) > (p - c)d(p)$ for all $p < p_{\max}$. For example, the function $d(p) = \frac{72}{p^2}$ with $p_{\min} = 4$, $p_{\max} = 6$ and $c = 3$ satisfies this property. Observe that

- it is not optimal to have both $p_t < p_{\max}$ and unsatisfied demand in that period t (see the argument used by Bhattacharjee and Ramesh (2000) in the "proof" of their Theorem 1),
- it is not optimal to have $p_t < p_{\max}$ and satisfy all demand in that period t (because it is better to set the price to $p_{\max}$ and then satisfy all demand),
- it is not optimal to have $p_t = p_{\max}$ and satisfy demand only partially in that period t (because also satisfying the remainder of the demand will increase profit and lower shortage costs).

It follows that in an optimal solution the price is set to $p_{\max}$ in every period and a period's demand is either completely satisfied or completely unsatisfied. One can easily verify that for the parameter values given above only production runs that cover the complete demand of two consecutive periods are profitable. A production run to cover only one period's demand is more expensive than leaving the demand unsatisfied. In case of three or more periods, the holding costs of the third period are higher than the shortage costs. Hence, the optimal solution is as follows. When T is even, there is a setup in every odd-numbered period to satisfy completely the 4 units of demand of that and the next period. In case T is odd, then there is exactly one (take any) odd-numbered period for which the 2 units of demand are not satisfied. For the remaining periods the solution has the same structure as in the case of an even number of periods.

4.8 Conclusion

In this chapter we considered a joint pricing and lot-sizing problem, where prices were allowed to vary over time. We focused on a special case of this problem considered by Bhattacharjee and Ramesh (2000) with a time-invariant demand function and time-invariant cost parameters. We showed that a direct application of a method proposed by Thomas (1970) led to an algorithm with a running time quadratic in the model horizon. Moreover, applying the results on a partition problem derived by Orlin (1985) led to an improvement in running time (although the running time was still pseudopolynomial). By slightly generalizing the result of Orlin (1985), we showed that this result still holds in case the profit function is quasiconcave in the number of periods of a subplan.

Chapter 5

A joint pricing and lot-sizing problem with constant prices over time*

Abstract

In this chapter we consider the economic lot-size model, where demand is a deterministic function of price. In the model a single price needs to be set for all periods. The objective is to find an optimal price and ordering decisions simultaneously. Kunreuther and Schrage (1973) proposed a heuristic algorithm to solve this problem. The contribution of this chapter is twofold. First, we derive an exact algorithm to determine the optimal price and lot-sizing decisions. Moreover, we show that our algorithm boils down to solving a number of lot-sizing problems that is quadratic in the number of periods, i.e., the problem can be solved in polynomial time.

5.1 Introduction

In Chapter 4 we considered a joint pricing model where prices were allowed to vary over time. There are several reasons why this may not be desirable. First, from Example 4.10 it can be seen that prices within a subplan (a consecutive number of periods for which demand is covered by production in a single period) are strictly increasing over time. If this happens in practice, customers may react on this and postpone purchases until the price drops. In this case it may be desirable to set a constant price over all periods. Second, as also mentioned in the general introduction, some firms rely on on sales from

*This chapter is based on Van den Heuvel and Wagelmans (2006b).

catalogues. It may be expensive to communicate price changes to customers and hence a constant price over time is also desirable in this case.

In this chapter we consider a similar model as in Chapter 4 except that we assume a constant price over time. In this chapter we build on the work of Kunreuther and Schrage (1973) and Gilbert (1999). We propose an exact algorithm to solve the economic lot-sizing (ELS) problem with deterministic demand functions and a constant price for all periods. We propose an $\mathcal{O}(T^3 \log T)$ algorithm that solves the general problem as introduced by Kunreuther and Schrage (1973). Furthermore, we show that our algorithm runs in $\mathcal{O}(T^2)$ for the special case considered in Gilbert (1999).

The remainder of this chapter is organized as follows. In Section 5.2 we give a mathematical formulation of the problem. In the next section we describe the heuristic algorithm proposed by Kunreuther and Schrage (1973). In Section 5.4 we introduce an exact algorithm which is an extension of the heuristic and we illustrate the algorithm by an example. In Section 5.5 we analyze the time complexity of our algorithm and show that the running time is polynomial in the number of periods. This chapter is ended up with a conclusion and we give some suggestions for further research.

5.2 Problem description

First, we will describe the classical ELS problem in short. If we use the following notation

T : model horizon
D_t : demand in period t
K_t : setup costs in period t
c_t : unit production costs in period t
h_t : unit holding costs in period t
x_t : production quantity in period t
I_t : ending inventory in period t,

then the problem can be formulated as

$$\begin{array}{rlll} C(D) = & \min & \sum_{t=1}^{T} (K_t \delta(x_t) + c_t x_t + I_t h_t) & \\ & \text{s.t.} & I_t = I_{t-1} - D_t + x_t & t = 1, \ldots, T \\ & & x_t, I_t \geq 0, & t = 1, \ldots, T \\ & & I_0 = 0 & \end{array}$$

where

$$\delta(x) = \begin{cases} 0 & \text{for } x = 0 \\ 1 & \text{for } x > 0. \end{cases}$$

and D is the demand vector of length T. The first set of constraints models that the ending inventory in period t equals the ending inventory in period $t-1$ plus the amount produced in period t minus demand in period t. The second set of constraints models that production and ending inventory are non-negative in each period. The above problem can be solved efficiently by dynamic programming. Wagner and Whitin (1958) proposed an $\mathcal{O}(T^2)$ algorithm to solve the problem. Federgruen and Tzur (1991), Wagelmans et al. (1992) and Aggarwal and Park (1993) independently improved this result to $\mathcal{O}(T \log T)$ for the general case and to $\mathcal{O}(T)$ for certain interesting cases, such as time-invariant cost parameters.

Assume now that demand is not deterministic as in the ELS problem, but demand is a deterministic function of the price and a single price for all periods needs to be set. Kunreuther and Schrage (1973) propose the demand function

$$D_t(p) = \alpha_t + \beta_t d(p),$$

where $d(p)$ is a differentiable non-increasing function and $\alpha_t, \beta_t \geq 0$. This means that demand will not increase, if the price increases. Note that the function $d(p)$ is independent of period t. If $d(p) = -p$ we have a linear demand curve. Now the objective is not to minimize the total costs as in the classical ELS problem, but to maximize total profit. If we introduce the additional notation

$$R_t(p) = pD_t(p), \text{ the revenue in period } t,$$

then the objective can be formulated as

$$\Pi(D(p)) = \max_{p>0:D(p)\geq 0} \left\{ \sum_{t=1}^{T} R_t(p) - C(D(p)) \right\},$$

where $D(p)$ corresponds to the vector $[D_1(p), \ldots, D_T(p)]$.

Kunreuther and Schrage (1973) show that $C(D(p))$ is a concave piecewise linear function of the 'demand effect' $d(p)$. This can be seen in the following way. Let S be the set of production periods (in increasing order) of a production plan. Furthermore, let $s(i)$ be the first production period after period i and if period i has no successor, then $s(i) = T+1$ by convention. Because of the zero-inventory property (Wagner and Whitin, 1958), a production plan is completely determined by its production periods. Now the cost $S(d(p))$ of a production plan S equals

$$\begin{aligned} S(d(p)) &= \sum_{i \in S} K_i + \sum_{i \in S} c_i \sum_{t=i}^{s(i)-1} D_t(p) + \sum_{i=1}^{T} \sum_{t=i+1}^{s(i)-1} h_{t-1} D_i(p) \\ &= \sum_{i \in S} K_i + \sum_{i \in S} c_i \sum_{t=i}^{s(i)-1} (\alpha_t + \beta_t d(p)) + \sum_{i=1}^{T} \sum_{t=i+1}^{s(i)-1} h_{t-1}(\alpha_i + \beta_i d(p)), \end{aligned}$$

which shows that $S(d(p))$ is a linear function of $d(p)$ for a fixed production plan S. Because $C(D(p))$ is the lower envelope of linear functions and the number of possible production plans is finite, $C(D(p))$ is a concave piecewise linear function of $d(p)$. This also means that each line piece corresponds to a specific production plan S.

5.3 Heuristic algorithm

In this section we present the heuristic algorithm proposed by Kunreuther and Schrage (1973). The algorithm works as follows:

1. Set $i := 0$, start with some initial price p_i and set $\Pi_i := -\infty$.
2. Set $i := i + 1$.
3. Calculate the demand vector $D(p_{i-1})$ and let S_i be the optimal production plan corresponding to problem $C(D(p_{i-1}))$
4. Calculate the price p_i which maximizes the net profit $\Pi_i := \sum_{t=1}^{T} R_t(p_i) - S_i(d(p_i))$
5. If $\Pi_i - \Pi_{i-1} > 0$, then go to step 2.
6. The terminating price is p_i with corresponding net profit Π_i.

In the remainder of this chapter we will call this the KS-procedure. In step 3 of the algorithm, we are looking for an optimal production plan given the demand vector $D(p_{i-1})$, which boils down to solving an instance of the ELS problem. In step 4 we are looking for an optimal price corresponding to a fixed production plan. If we have a relatively easy function $d(p)$ (for instance, $d(p) = -p$), the optimization problem may be relatively easy to solve (for instance, $\sum_{t=1}^{T} R_t(p)$ is quadratic for $d(p) = -p$). As in Kunreuther and Schrage (1973) we assume that the optimization problem in step 3 can be solved (either analytically or by a search procedure).

Kunreuther and Schrage (1973) show that the above algorithm does not skip over any optimum on its way to the terminating price, i.e., if p_{i-1} and p_i are two successive prices generated by the KS-procedure, then $\sum_{t=1}^{T} R_t(p_i) - C(D(p_i)) \geq \sum_{t=1}^{T} R_t(p) - S(d(p))$ for all prices p between p_{i-1} and p_i and all plans S. Thus, if we have a lower bound p_L and an upper bound p_U on the optimal price p^* and we apply the KS-procedure with these values as initial prices resulting in terminating prices p_L^* and p_U^*, respectively, then the optimal price p^* satisfies the inequality $p_L^* \leq p^* \leq p_U^*$. So, if the terminating prices are equal, we have found the optimal price.

A graphical representation of the algorithm is shown in figure 5.1. In the figure it is assumed that $d(p) = -p$, so that $\sum_{t=1}^{T} R_t(p)$ is quadratic and $C(D(p))$ is a non-increasing concave piecewise linear function of p. The derivatives of the revenue and cost function are plotted against the price p. Because $d(p) = -p$, it follows immediately that the

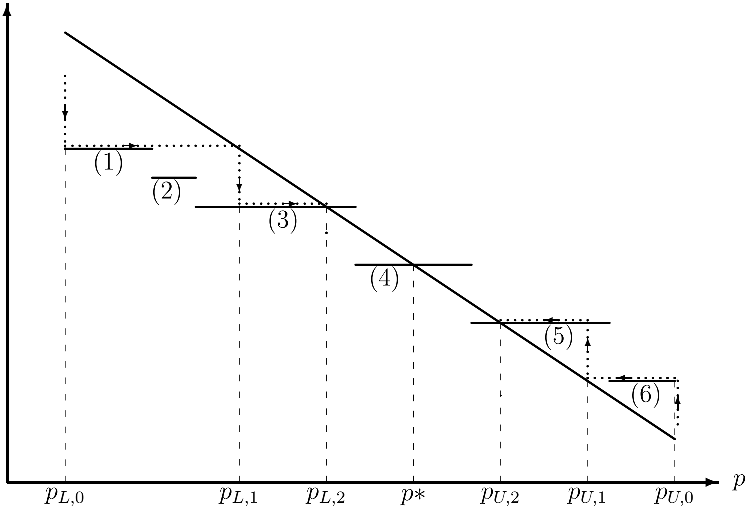


Figure 5.1: Heuristic algorithm

derivatives are a linear function and a piecewise constant function, respectively. The algorithm starts with some initial price $p_{L,0}$. The marginal cost of the production plan corresponding to this price is line (1). In step 4 of the algorithm we are looking for the optimal price corresponding to this production plan. This optimal price occurs in the point where marginal revenue equals marginal cost ($p = p_{L,1}$). Now we are looking for the optimal production plan corresponding to the price $p = p_{L,1}$, which is line (3). Note that the algorithm skips over line (2). It may be clear that there is no optimal solution in this part, because marginal revenue is larger than marginal cost. This means that it is not necessary to calculate all pieces of $C(D(p))$ explicitly. Now the optimal price corresponding to line (3) is $p_{L,2}$. At this point the algorithm terminates, because the optimal production plan corresponding to this price is again line (3).

In an analogous way we can find a terminating price starting with an upper bound for the optimal price p^*. Starting with price $p = p_{U,0}$, we see that the algorithm also terminates in two iterations at price $p = p_{U,2}$. So, essentially the KS-procedure consists of a local search with multiple starting points. In this example there is another local optimal price p^* which satisfies $p_{L,2} < p^* < p_{U,2}$. If the net profit in p^* is larger than in $p_{L,2}$ and $p_{U,2}$, then the KS-procedure will not find the optimal solution. Kunreuther and Schrage

(1973) give an example for which the algorithm does not converge. A slight modification of that example shows that this may have major consequences for the net profit.

Example 5.1 Consider a three-period problem with $K_1 = 13.2$, $K_2 = K_3 = 10$, $h_t = h = 1$, $c_t = c = 0$ and $D_t(p) = \alpha_t - \beta_t d(p)$ with $\alpha_1 = 25$, $\alpha_2 = \alpha_3 = 22$, $\beta_t = \beta = 12$ and $d(p) = -p$. If we start with $p_{L,0} = 0$, the algorithm terminates after one iteration at $p_{L,1} = 0.96$ with net profit $\Pi_L = -0.14$. Starting with $p_{U,0} = 2.08$ (the largest price for which demand is positive), the algorithm terminates after one iteration at $p_{U,1} = 1.46$ with net profit $\Pi_U = -2.64$. However, the optimal price equals $p^* = 1.13$ with net profit $\Pi_U = 0.36 > \Pi_L > \Pi_U$. This example shows that the KS-procedure may fail to find a profitable solution if one does exist. In this case a firm may decide to not produce at all, while it is profitable to produce. We will consider this problem instance in more detail later. (Note: In this example all prices and profits have been rounded to nearest cent.)

5.4 Exact algorithm

In this section we present an exact algorithm to solve the ELS problem with deterministic demand functions. The main idea of the exact algorithm is that the KS-procedure is restarted. Assume that the heuristic algorithm starts with $p_L < p_U$ and terminates with prices p_L^* and p_U^* which satisfy $p_L \leq p_L^* < p_U^* \leq p_U$. To restart the algorithm we need a suitable price $\overline{p}$ that satisfies $p_L^* < \overline{p} < p_U^*$. In the exact algorithm we will find such a price by applying an idea often attributed to Eisner and Severance (Eisner and Severance, 1976) on the function $C(D(p))$.

We will first explain how the Eisner Severance(ES)-method works. Assume we would like to perform a parametric analysis on an optimization problem

$$\begin{aligned} z(\lambda) &= \min(c + \lambda d)x \\ & \quad x \in X, \end{aligned}$$

where X is a finite set. Here the vector of objective coefficients is a linear function of a parameter λ. Assume we have some algorithm A to solve this optimization problem for a fixed λ. It is well-known that the objective function $z(\lambda)$ is a piecewise linear concave function of λ. Assume that we want to find all breakpoints in the interval $[\lambda_L, \lambda_U]$. This can be done in the following way. Apply algorithm A on the optimization problem with $\lambda = \lambda_L$ and $\lambda = \lambda_U$. Calculate the intersection point $\overline{\lambda}$ of the lines corresponding to λ_L and λ_U, say $l_L(\lambda)$ and $l_U(\lambda)$. Perform algorithm A on the optimization problem with $\lambda = \overline{\lambda}$. If the objective value is equal to $l_L(\overline{\lambda})$ and $l_U(\overline{\lambda})$, there is no new line piece between λ_L and λ_U. Otherwise, apply the above procedure recursively with $\lambda_L := \lambda_L$ and $\lambda_U := \overline{\lambda}$ and with $\lambda_L := \overline{\lambda}$ and $\lambda_U := \lambda_U$.

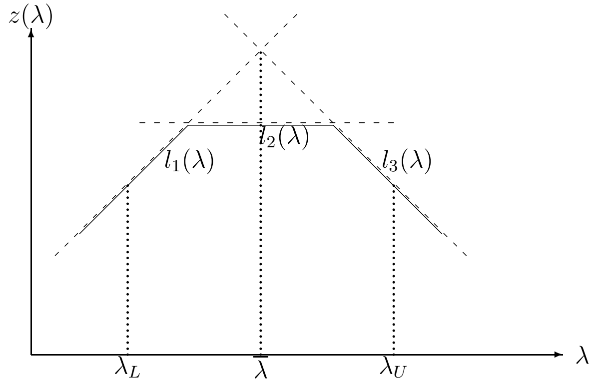


Figure 5.2: The Eisner Severance method

An example of the ES-method is shown in figure 5.2. We see that the piecewise linear concave function consists of three line pieces: $l_1(\lambda)$, $l_2(\lambda)$ and $l_3(\lambda)$. If we apply algorithm A with $\lambda = \lambda_L$ and $\lambda = \lambda_U$, the corresponding lines are $l_1(\lambda)$ and $l_3(\lambda)$, respectively. These lines intersect at $\lambda = \overline{\lambda}$. Applying algorithm A on the optimization problem with $\lambda = \overline{\lambda}$, we find line $l_3(\lambda)$. If we repeat this procedure with $\lambda_L := \lambda_L$ and $\lambda_U := \overline{\lambda}$ and with $\lambda_L := \overline{\lambda}$ and $\lambda_U := \lambda_U$, we will not find new line pieces and the ES-method terminates. If B is the total number of breakpoints it is easy to show that algorithm A needs to be applied $2B - 1$ times. For more details we refer to Eisner and Severance (1976).

As already mentioned before, $C(D(p))$ is a piecewise linear function of $d(p)$ and each piece corresponds to a production plan. Assume now that the KS-procedure terminates with prices $p_L^* < p_U^*$. It follows from Section 5.3 that in every iteration a new production plan is generated except for the last iteration. Because the interval $[p_L^*, p_U^*]$ has not been searched, it is possible that there are still production plans in $[p_L^*, p_U^*]$ that have a local or global maximum (see also figure 5.1). If S_L^* and S_U^* are the optimal production plans corresponding to p_L^* and p_U^*, respectively, then by applying one iteration of the ES-method we can find the intersection point of the cost functions $S_L^*(d(p))$ and $S_U^*(d(p))$ in terms of $d(p)$, say in the point $d(p) = v$. Furthermore, because $d(p)$ is a continuous non-increasing function of p, there exists a p with $p_L^* < p < p_U^*$ for which $d(p) = v$. Therefore we propose the following recursive algorithm.

```
Start the KS-procedure with p_L and p_U (p_L < p_U)
Let p_L^* and p_U^* be the terminating prices
If p_L^* = p_U^* then
    Stop; the optimal price is p_L^*
Else
    Call Optim(p_L^*, p_U^*)
End if

Function Optim(p_L, p_U)
    p̄ =ES(S_L, S_U)
    If S̄(d(p̄)) = S_L(d(p̄)) then
        End function
    Else
        Perform the KS-procedure with p = p̄
        Let p̄^* be the terminating price
        CheckOptima
        If p̄^* > p̄, then
            Call Optim(p_L, p̄); Call Optim(p̄^*, p_U)
        Else
            Call Optim(p_L, p̄^*); Call Optim(p̄, p_U)
        End if
    End if
End function
```

In the algorithm the function CheckOptima checks if the found optimum is larger than the previous found optimum (if any) and stores the corresponding price and production plan. The production plans S_L, S_U and $\overline{S}$ correspond to the prices p_L, p_U and $\overline{p}$, respectively. The function ES applies one iteration of the ES-method and finds a price $\overline{p}$ with the property $p_L < \overline{p} < p_U$ as described before. If there are no new production plans between S_L and S_U, we can terminate the function Optim. If not, we apply the KS-procedure on the price $\overline{p}$. It is not clear beforehand, whether the terminating price $\overline{p}^*$ is larger of smaller than $\overline{p}$. If it is larger, then there may still be production plans between p_L and $\overline{p}$ and between $\overline{p}^*$ and p_U. Therefore, we repeat the same procedure with starting prices p_L and $\overline{p}$ and with prices $\overline{p}^*$ and p_U. If it is smaller, the procedure is repeated the other way around. In the following example we illustrate the exact algorithm.

Example 5.2 Consider the same problem instance as in example 5.1. For this problem instance we have four possible production plans $S_1 = \{1, 2, 3\}$, $S_2 = \{1, 2\}$, $S_3 = \{1, 3\}$

and $S_4 = \{1\}$ with corresponding costs

$$\begin{array}{ccc} S_1(p) & = & 33 \\ S_2(p) = S_3(p) & = & 45 - 12p \\ S_4(p) & = & 79 - 36p. \end{array}$$

The total revenue equals $R(p) = \sum_{t=1}^{T} R_t(p) = p(69 - 36p)$. From example 5.1 we know that starting with $p_L = 0$ the KS-procedure terminates at $p_L^* = 0.96$ and starting with $p_U = 2.08$ the KS-procedure terminates at $p_U^* = 1.46$. The production plans corresponding to these prices are S_1 and S_4, respectively. For the marginal cost and marginal revenue functions we refer to figure 5.3.

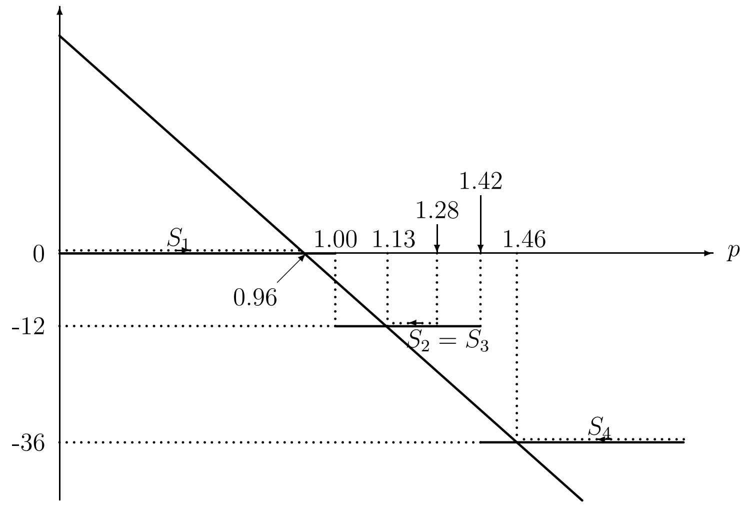


Figure 5.3: Marginal cost and marginal revenue functions

Whereas the KS-procedure stops, the exact algorithm continues and performs one step of the ES-method, i.e., the intersection point of $S_1(p)$ and $S_4(p)$ is calculated. These plans intersect in the point $p_{ES} = 1.28$. Because $S_1(1.28) \neq S_4(1.28)$, the algorithm applies the KS-procedure with starting price $p_{ES} = 1.28$. The optimal production plan corresponding to this price is S_2. The optimal price corresponding to S_2 is $p_{ES}^* = 1.13$ and the KS-procedure terminates, because the optimal plan for this price is again S_2. (See also figure 5.3.)

Now it is possible that there are still optima in the interval $[p_L^*, p_{ES}^*]$ and $[p_{ES}, p_U^*]$. Therefore the procedure Optim is called again with prices p_L^*, p_{ES}^* and p_{ES}, p_U^*. Starting this procedure with $p_L^* = 0.96$ and $p_{ES}^* = 1.13$, we calculate the intersection point of plans $S_1(p)$ and $S_2(p)$, which occurs at $p = 1.00$. Because $S_1(1.00) = S_2(1.00)$, there are no new production plans between p_L^* and p_{ES}^* and the procedure terminates. Analogously, starting with $p_{ES} = 1.28$ and $p_U^* = 1.46$, the procedure terminates at price $p = 1.42$.

Because the KS-procedure does not skip over any local optimum (Kunreuther and Schrage, 1973), it must hold that $p_L \leq \overline{p}^* \leq p_U$. Furthermore, because the number of possible production plans is finite, the algorithm will always terminate. Moreover, because the whole interval $[p_L, p_U]$ is searched, the global optimum will be found. It is clear that the running time of the algorithm depends on the number of production plans, i.e., the number of line pieces of the function $C(D(p))$. In the following section we will give an upper bound on the number of production plans that may be optimal for some value of $d(p)$.

5.5 Time complexity of the algorithm

In this section we will show that the time complexity of our algorithm is polynomial. To perform the ES-method we need to solve an ELS problem and in every iteration of the KS-procedure we also need to solve an ELS problem. We assume that step 4 of the KS-procedure requires less time than step 3, so that solving the ELS problem takes the most time. Solving an ELS problem requires $\mathcal{O}(T \log T)$ time in general. Because in every iteration of the KS-procedure a new production plan is generated (except for the last iteration) and because the ES-method is applied to find a new production plan between two existing plans and terminates if such a plan is not found, the number of ELS problems to be solved is at most twice the number of production plans. Thus, if we are able to find a polynomial bound on the number of optimal production plans, the algorithm runs in polynomial time. Because every optimal production plan corresponds to a line piece in $C(D(p))$, it is sufficient to find a polynomial bound on the number of line pieces in $C(D(p))$ as a function of $d(p)$.

5.5.1 Improvement on the result of Gilbert (1999)

Before deriving the time complexity of our algorithm for the general case, we first show how our algorithm performs in the special case considered by Gilbert (1999). In his paper it is assumed that demand satisfies $D_t(p) = \beta_t d(p)$ for $t = 1, \ldots, T$. Furthermore, he assumes that cost parameters are time-invariant, i.e., $K_t = K$, $c_t = c$ and $h_t = h$ for $t = 1, \ldots, T$.

Gilbert (1999) shows that the number of line pieces of the function $C(D(p))$ is at most T. First we note that this could have been derived from a result by Van Hoesel and

Wagelmans (2000). Namely, for a fixed value of $d(p)$, Gilbert (1999) considers the model

$$\begin{array}{lll} \min & \sum_{t=1}^{T} \left(K\delta(x_t) + cx_t + hI_t\right) & \\ \text{s.t.} & I_t = I_{t-1} + x_t - \beta_t d(p) & t = 1,\ldots,T \\ & x_t, I_t \geq 0 & t = 1,\ldots,T. \end{array}$$

Dividing the objective function and the constraints by $d(p)$ and substituting $x'_t := x_t/d(p)$ and $I'_t := I_t/d(p)$, we obtain the model

$$\begin{array}{lll} \min & \sum_{t=1}^{T} \left(\frac{K}{d(p)}\delta(x'_t) + cx'_t + hI'_t\right) & \\ \text{s.t.} & I'_t = I'_{t-1} + x'_t - \beta_t & t = 1,\ldots,T \\ & x'_t, I'_t \geq 0 & t = 1,\ldots,T. \end{array}$$

By varying the value of $d(p)$ this results in a parametric analysis on the setup cost of an ELS problem with demand β_t in period t. Van Hoesel and Wagelmans (2000) show that by varying the setup cost, the interval $[0, K]$ can be partitioned into $q \leq T$ intervals $[k_i, k_{i+1}]$ $(i = 1,\ldots,q)$ where each interval corresponds to an optimal production plan. Moreover, the optimal production plan corresponding to the interval $[k_{i-1}, k_i]$ has at least one setup more than the the optimal production plan corresponding to the interval $[k_i, k_{i+1}]$. But this implies that $C(D(p))$ consists of at most T line pieces.

So by applying our algorithm to this special case, it follows that we have to solve at most $2T$ ELS problems with time-invariant cost parameters. Furthermore, because the ELS problem without speculative motives (i.e., $c_t + h_t \geq c_{t+1}$ for $t = 1,\ldots,T-1$) can be solved in linear time, our algorithm runs in $\mathcal{O}(T^2)$ time for this special case. This result improves upon the result of Gilbert (1999), who proposes an $\mathcal{O}(T^3)$ algorithm. But we have a stronger result. By the analysis of Van Hoesel and Wagelmans (2000), it follows that the function $C(D(p))$ still consists of at most T line pieces in the case of non-stationary production and holding costs but no speculative motives. This implies that we have the following proposition:

Proposition 5.3 *The generalization of Gilbert's model with non-stationary production and holding costs without speculative motives can be solved in $\mathcal{O}(T^2)$ time.*

5.5.2 Time complexity of the general problem

In this section we will show that for the general case the total number of production plans that may be optimal for some price p is still polynomial in T. Note that the total number of different production plans is equal to 2^{T-1}. Namely, we assume that in period 1 production takes place and in the remaining $T-1$ periods one may produce or not. In the proof we will show that a number of production plans will not occur in $C(D(p))$. Before we start the proof we give an example for model horizon $T = 4$. If we denote the intercept of a line by a and the slope by $-b$, then a line $l(p)$ can be represented by

$$l(p) = a - bp$$

and the intersection point of $l_1(p)$ and $l_2(p)$ occurs in

$$p_{1,2} = \frac{a_1 - a_2}{b_1 - b_2}.$$

It may be clear that if we have four lines $l_1(p), \dots, l_4(p)$, then a necessary condition to have three breakpoints in the first quadrant is that if $a_1 > a_2 > a_3 > a_4$ then $b_1 > b_2 > b_3 > b_4$. If we have three breakpoints, then a picture of such a situation is as follows.

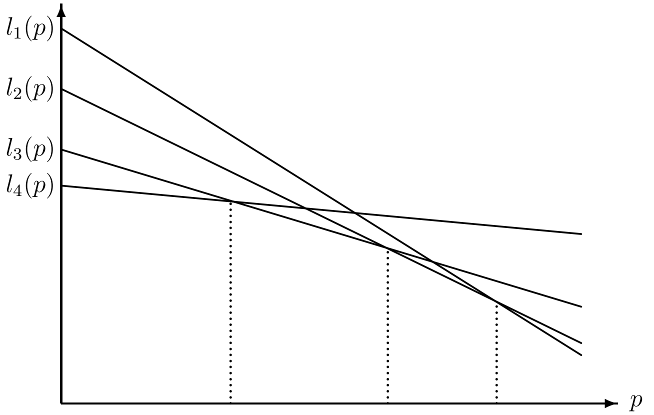


Figure 5.4: Example of 3 breakpoints

Example 5.4 Consider the ELS problem with $T = 4$ periods with demand in period t according to $D_t(p) = \alpha_t - p\beta_t$. In this section a production plan is represented by a binary number. For instance the number 1010 corresponds to a production plan with production in period 1 and period 3. Now consider the following four production plans: 1010, 1011, 1110 and 1111 with the corresponding lines

$$\begin{aligned}
1010:&\quad l_1(p) = K_1 + (\alpha_1+\alpha_2)c_1 + K_3 + (\alpha_3+\alpha_4)c_3 - p((\beta_1+\beta_2)c_1 + (\beta_3+\beta_4)c_3)\\
1011:&\quad l_2(p) = K_1 + (\alpha_1+\alpha_2)c_1 + K_3 + \alpha_3c_3 + K_4 + \alpha_4c_4 - p((\beta_1+\beta_2)c_1 + \beta_3c_3 + \beta_4c_4)\\
1110:&\quad l_3(p) = K_1 + \alpha_1c_1 + K_2 + \alpha_2c_2 + K_3 + (\alpha_3+\alpha_4)c_3 - p(\beta_1c_1 + \beta_2c_2 + (\beta_3+\beta_4)c_3)\\
1111:&\quad l_4(p) = K_1 + \alpha_1c_1 + K_2 + \alpha_2c_2 + K_3 + \alpha_3c_3 + K_4 + \alpha_4c_4 - p(\beta_1c_1 + \beta_2c_2 + \beta_3c_3 + \beta_4c_4).
\end{aligned}$$

Note that

$$\begin{aligned}
a_1 - a_2 = \quad a_3 - a_4 = \quad &\alpha_4(c_3 - c_4) - K_4\\
b_1 - b_2 = \quad b_3 - b_4 = \quad &\beta_4(c_3 - c_4).
\end{aligned}$$

So we have $p_{1,2} = p_{3,4}$. Such a situation is illustrated in figure 5.5.

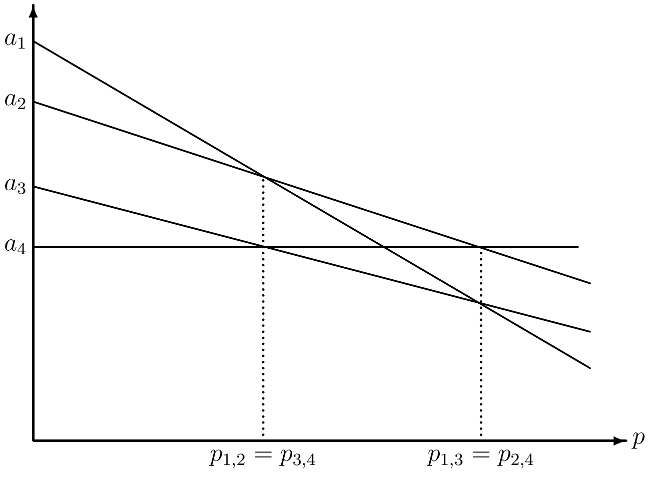


Figure 5.5: Example with $a_1 > a_2 > a_3$

It is not difficult to verify (and it can be seen from the picture) that if we have four lines with the above properties, then there can be at most two breakpoints and we 'lose' at least one breakpoint. We can generalize this to the case where we have n pairs of lines with this nice property.

Lemma 5.5 *If we have $n \geq 2$ pairs of lines $l_1, \ldots, l_{2n}$ with the properties $a_{2i-1} - a_{2i} = \triangle a$ and $b_{2i-1} - b_{2i} = \triangle b$ with $\triangle b \neq 0$ for $i = 1, \ldots, n$, then we lose at least $n-1$ breakpoints.*

Proof Assume that we have n pairs of lines with the above property. The lines l_{2i-1} and l_{2i} correspond to pair i. Furthermore, we may assume without loss of generality that $\triangle b > 0$, so that $l_{2i-1}(p) > l_{2i}(p)$ for $p < p^*$ and $l_{2i-1}(p) < l_{2i}(p)$ for $p > p^*$, where p^* is the point for which $l_{2i-1}(p^*) = l_{2i}(p^*)$. Denote the lower envelope of the lines by $F_n(p)$, i.e.,

$$F_n(p) = \min_{i=1,\ldots,2n} l_i(p).$$

It is clear that $F_n(p)$ has a breakpoint in $p^* = \triangle a / \triangle b$. Now we can distinguish the following two cases.

- There is exactly one pair that has an intersection point in $F_n(p^*)$. That is, there exists a unique pair i for which

$$i = \arg \min_{j=1,\ldots,n} l_{2j-1}(p^*).$$

We will now show that every pair $j \neq i$ will contribute at most one line to $F_n(p)$. Assume that $p_{2j,2i} < p^*$. Then we have that $b_{2j} < b_{2i}$ and the line $l_{2j}(p)$ may contribute to $F_n(p)$. But then $l_{2j-1}(p)$ will not contribute to $F_n(p)$, because

$$b_{2j-1} = \triangle b + b_{2j} = b_{2i-1} - b_{2i} + b_{2j} < b_{2i-1}$$

and

$$l_{2j-1}(p^*) > l_{2i-1}(p^*),$$

which implies that $l_{2j-1}(p) > F_n(p)$ for $p > p^*$. The above situation is illustrated in figure 5.6.

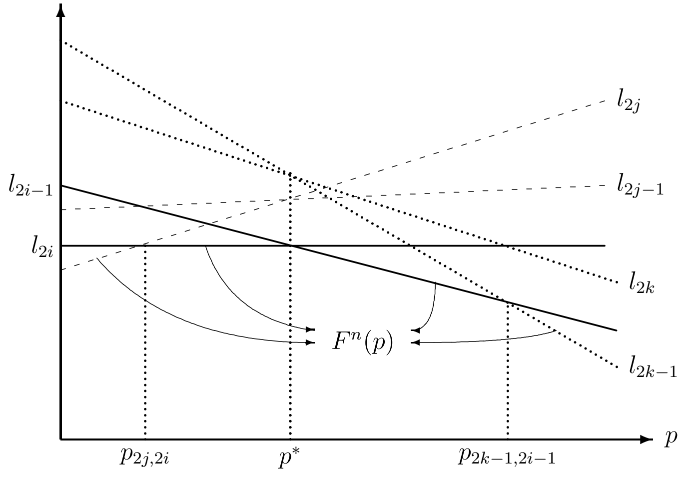


Figure 5.6: Example with $p_{2j,2i} < p^*$ and $p_{2k-1,2i-1} > p^*$

In a similar way one can show that if there is a pair k with $p_{2k-1,2i-1} > p^*$, then the line $l_{2k-1}(p)$ may contribute to $F_n(p)$ and $l_{2k}(p)$ will not contribute to $F_n(p)$. This situation is also illustrated in figure 5.6. To summarize, there is exactly one pair (namely pair i) that contributes two line pieces to $F_n(p)$ and there are $n-1$ pairs that contribute at most one line piece to $F_n(p)$, so that at least $n-1$ breakpoints will not occur.

- There are two or more pairs of lines that intersect in the point $F_n(p^*)$. In this case there are exactly two of these pairs that contribute one line piece to $F_n(p)$ (see figure 5.7). One can prove in an analogous way as in the previous case that the pairs which do not intersect in $F_n(p^*)$, contribute at most one line piece to $F_n(p)$. Thus, if we have $k \geq 2$ pairs that intersect in $F_n(p^*)$, then $F_n(p)$ consists of at most $n-k+2$ line pieces, so that we lose at least $n+k-2 > n-1$ breakpoints.

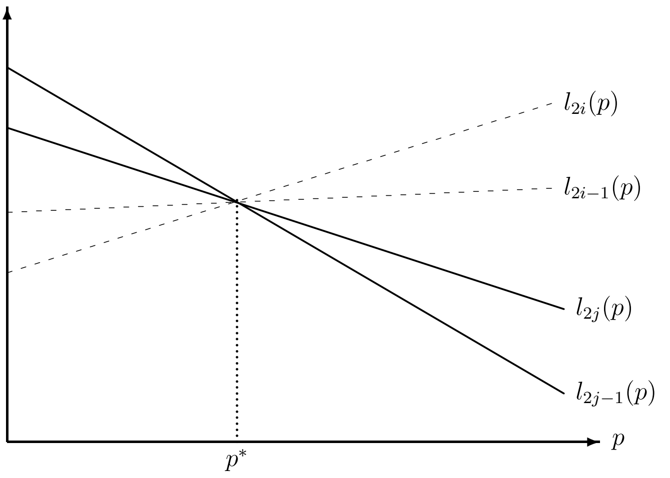


Figure 5.7: Two pairs of lines intersect in p^*

Thus, in both cases we lose at least $n-1$ breakpoints. $\square$

Theorem 5.6 *The number of breakpoints of $C(D(p))$ as a function of $d(p)$ is at most $\frac{1}{2}T(T-1)$.*

Proof In the proof again a production plan is represented by a binary number of length T. We assume that demand in period 1 is positive, so that a production plan is completely determined by a T-digit binary number starting with a 1. Now consider the following pairs of production plans

$$1 \bullet \cdots \bullet \quad 1 \quad 0 \cdots 0 \tag{5.1}$$

$$1 \underbrace{\bullet \cdots \bullet}_{T-n-2} \quad 1 \quad \underbrace{\circ \cdots \circ}_{n}. \tag{5.2}$$

A $\bullet$-sign in a production represents a period where production takes place in plan (5.1) if and only if production takes place in plan (5.2). Furthermore, in exactly one of the periods with a $\circ$-sign production takes place. For instance, if $T = 5$, then the sequences

$$1 \quad \bullet \quad 1 \quad 00$$
$$1 \quad \bullet \quad 1 \quad \circ\circ$$

represent two times two pairs of production plans

10100	10100
10101	10110
11100	11100
11101	11110.

We will now show that production pairs with identical ‘tails’ satisfy the properties of Lemma 5.5. Let S be the set of production periods (in increasing order) for a production plan. For instance with $T = 5$ the production plan 10101 corresponds to $S = \{1, 3, 5\}$. Furthermore, if i is an production period, then $s(i)$ is the successive production period and the successor of the last production period equals $T + 1$ by convention. Now the cost $S(d(p))$ of a production plan S as a function of $d(p)$ equals

$$S(d(p)) = \sum_{i\in S} \left(K_i + c_i \sum_{t=i}^{s(i)-1} D_t(p) \right) = \sum_{i\in S} \left(K_i + c_i \sum_{t=i}^{s(i)-1} \alpha_t \right) - d(p) \sum_{i\in S} c_i \sum_{t=i}^{s(i)-1} \beta_t. \tag{5.3}$$

Note that there are no holding costs in (5.3). It can be shown (see for example Wagelmans et al. (1992)) that every ELS problem can be rewritten as an ELS problem without holding cost by making the substitution $c_t := c_t + \sum_{i=t}^{T} h_t$.

Assume now that we have a pair of production plans S_1 and S_2 according to (5.1) and (5.2), respectively, with period m and n ($m < n$) corresponding to the one but last and last production period in S_2, i.e., $S_1 = \{i_1, \ldots, i_k, m\}$ and $S_2 = \{i_1, \ldots, i_k, m, n\}$. It follows from (5.3) that

$$S_2(d(p)) - S_1(d(p)) = K_n + (c_n - c_m) \sum_{t=T-n+1}^{T} \alpha_t + p(c_m - c_n) \sum_{t=T-n+1}^{T} \beta_t. \tag{5.4}$$

It follows immediately from (5.4) that the difference in intercept and slope is independent of the sequence $i_1, \ldots, i_k$.

Now for a fixed $n \geq 1$ and a fixed production period in the last n periods, there are $T - n - 2$ positions for the $\bullet$-sign, implying that we have 2^{T-n-2} pairs with the properties of Lemma 5.5. It follows from this lemma that we lose $2^{T-n-2} - 1$ breakpoints. Because we have n possible positions for the $\circ$-sign and n ranges from 1 to $T - 3$, we lose $\sum_{n=1}^{T-3} n(2^{T-n-2} - 1)$ breakpoints in total. Furthermore, because the maximum number of breakpoints for model horizon T equals $2^{T-1} - 1$, the maximum number of breakpoints equals

$$
\begin{aligned}
&2^{T-1} - 1 - \sum_{n=1}^{T-3} n(2^{T-n-2} - 1) = \\
&2^{T-1} - 1 + \frac{1}{2}(T-3)(T-2) - \sum_{n=1}^{T-3} n2^{T-n-2} = \\
&2^{T-1} - 1 + \frac{1}{2}(T-3)(T-2) - (-2(T-1) + 2^{T-1}) = \\
&\frac{1}{2}T(T-1),
\end{aligned}
$$

where the second equality follows from Lemma 5.7 (see appendix 5.A).

As it follows from example 5.4 and figure 5.5, if $l_i(p), l_j(p)$ and $l_m(p), l_n(p)$ are pairs of lines with the properties of Lemma 5.5, then the pairs $l_i(p), l_m(p)$ and $l_j(p), l_j(p)$ also have these properties. Call these pairs the complement pairs. Thus, to complete the proof we have to check that none of the pairs defined in (5.1) and (5.2) are complement pairs of each other. To see that this holds, note that the tails of the complement pairs (i.e., the production periods of the last $n+1$ periods) are identical for pairs constructed according to (5.1) and (5.2) with n fixed. However, the tails of the production pairs defined in (5.1) and (5.2) are different. This means that the pairs can never be complement pairs of each other. □

So using the result of Theorem 5.6 and noting that the ELS problem can be solved in $\mathcal{O}(T \log T)$ time, it follows that our algorithm has a time complexity of $\mathcal{O}(T^3 \log T)$.

5.5.3 An $\Omega(T^2)$ example

In this section we show that the bound on the number of optimal production plans in $C(D(p))$ in Section 5.5.2 is tight. Consider the following problem instance. Let

$$
\begin{array}{lcll}
D_t(p) & = & D - p & \text{for } t = 1, \ldots, T \\
K_t & = & (T-t+1)K & \text{for } t = 1, \ldots, T \\
c_t & = & T^{T-t+1} & \text{for } t = 1, \ldots, T \\
h_t & = & 0 & \text{for } t = 1, \ldots, T.
\end{array}
$$

Again production plans are represented by binary numbers. We construct production plans in the following way. We start with a production plan with only a setup in period 1. The next production plan is constructed by adding a setup in period T and the following production plans are constructed by moving this setup one period earlier. If this is not possible anymore, we start the procedure again by adding a setup in period T. Formally, if

$$S_i = 1 \bullet \cdots \bullet 010 \cdots 0$$

is a production plan, then

$$S_{i+1} = 1 \bullet \cdots \bullet 100 \cdots 0$$

is the next production plan, where $\bullet \cdots \bullet$ is the same sequence in both plans, and if

$$S_j = 1 \cdots 10 \cdots 00$$

is a production plan, then

$$S_{j+1} = 1 \cdots 10 \cdots 01$$

is the successive production plan. We will call this the 'in'-plans. For instance, the in-plans corresponding to $T = 4$ are $1000, 1001, 1010, 1100, 1101, 1110, 1111$.

The main idea of the proof is as follows. If we only consider the in-plans, we will show that these plans all contribute to the lower envelope. Then, if we consider the remaining production plans, the 'out'-plans (for instance, 1011 is an out-plan for $T = 4$), we will show that they will not contribute to the lower envelope. Therefore, we will first show some properties of the production plans. Note that we can construct all production plans by binary counting, starting with the number $10 \cdots 0$ and ending with $11 \cdots 1$.

The first property is that the slopes of all plans are decreasing. Let S_1 be a production plan and S_2 be the successive production plan. Furthermore, let $k \leq T$ be the last non-production period and n be the first production period before k of plan S_1 (see the following picture).

$$\begin{array}{rcccc} & & n & k \\ S_1 = & 1 \bullet \cdots \bullet & 10 \cdots 0 & 01 \cdots 1 \\ S_2 = & 1 \bullet \cdots \bullet & 10 \cdots 0 & 10 \cdots 0 \end{array}$$

If we denote the (negative) slope $b(S_i)$ of plan S_i by

$$b(S_i) = \sum_{i \in S_i} c_i \sum_{t=i}^{s(i)-1} 1,$$

then we have that

$$b(S_1) - b(S_2) = \tag{5.5}$$

$$\sum_{i \in S_1} c_i \sum_{t=i}^{s(i)-1} 1 - \sum_{i \in S_2} c_i \sum_{t=i}^{s(i)-1} 1 = \tag{5.6}$$

$$c_n + \sum_{t=k+1}^{T} c_t - c_k(T - k + 1) \geq \tag{5.7}$$

$$T \cdot T^{T-k+1} + \sum_{t=k+1}^{T} T^{T-k+1} - (T - k + 1)T^{T-k+1} \geq \tag{5.8}$$

$$\sum_{t=k+1}^{T} T^{T-k+1} > 0. \tag{5.9}$$

Note that the third inequality holds, because $n \leq k-1$ which implies that $c_n \geq c_{k-1} = Tc_k$ and the last inequality holds because $k \geq 2$ so that $T-1 \geq T-k+1$.

If K is sufficiently small, then we can show that the intercepts of the production plans (constructed by binary counting) are also decreasing. Recall that the intercept $a(S_i)$ of production plan equals

$$a(S_i) = \sum_{i \in S_i} \left(K_i + c_i \sum_{t=i}^{s(i)-1} \alpha_t \right).$$

If we neglect the term $\sum_{i \in S_i} K_i$ and because demand is constant, we have

$$a(S_i) = \sum_{i \in S_i} c_i \sum_{t=i}^{s(i)-1} D = Db(S_i).$$

Because $b(S_i)$ is decreasing for consecutive plans S_i, it is clear that the intercepts $a(S_i)$ are also decreasing.

For the intercepts of the in-plans we can prove a stronger result. Namely, the difference between the slopes of consecutive plans are decreasing, i.e., $b(S_i) - b(S_{i+1}) > b(S_{i+1}) - b(S_{i+2})$. If S_i, S_{i+1} and S_{i+2} are of the following structure

$$\begin{array}{llll} & & n & k \\ S_i & = 1 \cdots 1 & 0 \cdots 000 & 10 \cdots 0 \\ S_{i+1} & = 1 \cdots 1 & 0 \cdots 001 & 00 \cdots 0 \\ S_{i+2} & = 1 \cdots 1 & 0 \cdots 010 & 00 \cdots 0, \end{array} \tag{5.10}$$

then it holds that

$$b(S_i) - b(S_{i+1}) = c_n - (T-k+1)c_k + (T-k+2)c_{k-1}$$

and

$$b(S_i) - 2b(S_{i+1}) + b(S_{i+2}) = (T-k+3)T^2 - 2(T-k+2)T + (T-k+1) > 0$$

for $T \geq 1$ and $T-k+1 \geq 0$, so that $b(S_i) - b(S_{i+1}) > b(S_{i+1}) - b(S_{i+2})$. For other consecutive in-plans one can obtain the same result.

Now we consider the intersection points of two plans S_1 and S_2. It follows from 5.3 that if $S_1(p) = S_2(p)$, then

$$p_{S_1,S_2} = D + \frac{\sum_{i \in S_1} K_i - \sum_{i \in S_2} K_i}{b(S_1) - b(S_2)}. \tag{5.11}$$

Consider two consecutive in-plans S_i and S_{i+1}. If we have a transition from plan S_i to S_{i+1} according to (5.10), then we have

$$\sum_{j \in S_i} K_j - \sum_{j \in S_{i+1}} K_j = K_k - K_{k-1} = (T-k+1)K - (T-(k-1)+1)K = -K$$

and if we have a transition according to

$$\begin{aligned} S_i &= 1\cdots 10\cdots 00 \\ S_{i+1} &= 1\cdots 10\cdots 01, \end{aligned} \tag{5.12}$$

then

$$\sum_{j\in S_i} K_j - \sum_{j\in S_{i+1}} K_j = -K_T = -K.$$

This means that $\sum_{j\in S_i} K_j - \sum_{j\in S_{i+1}} K_j$ is constant. Because $b(S_i) - b(S_{i+1})$ is decreasing, it follows from (5.11) that the intersection points are decreasing, which implies that all in-plans contribute to the lower envelope.

We will now show that the out-plans will not contribute to the lower envelope. Therefore, consider the following production plans

$$\begin{array}{llll} & \quad\;\; n & & k \\ S_i & = 1\cdots 1 & 0\cdots 00 & 10\cdots 0 \\ S_{i+1} & = 1\cdots 1 & 0\cdots 00 & 1\circ\cdots\circ \\ S_{i+2} & = 1\cdots 1 & 0\cdots 01 & 00\cdots 0, \end{array} \tag{5.13}$$

where S_{i+1} is a plan with at least one setup in the period(s) with a $\circ$-sign. Thus, S_i is an in-plan with S_{i+2} as a successor and S_{i+1} is an out-plan. To show that S_{i+1} does not contribute to the lower envelop, it is sufficient to show that $p_{S_{i+1},S_{i+2}} > p_{S_i,S_{i+2}}$. We have that

$$\sum_{i\in S_{i+1}} K_i \geq \sum_{i\in S_i} K_i + K_m \geq \sum_{i\in S_{i+2}} K_i,$$

because $K_k + K_m \geq K_{k-1}$, where m ($k < m \leq T$) is a setup period in S_{i+1}. But then it follows from (5.11) that

$$p_{S_i,S_{i+2}} < D \leq p_{S_{i+1},S_{i+2}}.$$

Because all out-plans have the same structure as S_{i+1} in (5.13), only the in-plans contribute to the lower envelope.

To count the number of breakpoints of the lower envelope, we have to count the number of in-plans. We have exactly one in-plan with a setup in period 1, $T-1$ in-plans with 2 setups, $T-2$ in-plans with 3 setups, etc. This means that the total number of line pieces equals

$$1 + \sum_{t=1}^{T-1} t = 1 + \frac{1}{2}T(T-1)$$

and the number of breakpoints equals $\frac{1}{2}T(T-1)$.

5.6 Concluding remarks

In this chapter we have presented an exact algorithm to solve the ELS problem with deterministic demand functions. We improved the heuristic algorithm proposed by Kunreuther and Schrage (1973) by showing that the problem is a parametric analysis on the demand vector of the ELS problem. Whereas the Kunreuther-Schrage algorithm terminates in cases of no convergence, we applied one step of the Eisner-Severance method to restart the algorithm. Furthermore, we have shown that the running time of our algorithm is polynomial in the number of periods.

Although the model itself is quite simple, the assumptions made are less restrictive than one may initially think. First, consider the assumption that price is constant over all periods. As mentioned in the introduction, this is essentially the case when we are dealing with catalog goods, because it could be very expensive to communicate price changes to the customers. Other cases mentioned by Kunreuther and Schrage (1973) are price planning by retail stores and the case that economic conditions dictate that future price changes are undesirable. Gilbert (2000) gives other reasons for limited price flexibility in the short term. Suppliers to original equipment manufacturers may be stuck to long-term contracts with predetermined prices. Other firms try to discourage customers from postponing purchases by setting a constant price.

Second, consider the assumption that demand depends on price according to $D_t(p) = \alpha_t + \beta_t d(p)$. Because α_t and β_t may depend on t, this allows us to model demand functions that vary over time. Of course, good estimates of the function $d(p)$ and the parameters α_t, β_t are needed. These estimates could be derived from historical data or obtained from the marketing department. If good estimates are not available or if there is a lot of uncertainty on the dependency of price on demand, then other models should be applied (for example a stochastic model).

As also mentioned in the introduction of Part II of this thesis, Geunes et al. (2005) generalized the model in this chapter to account for time-invariant production capacities. They prove that this problem can be solved in $\mathcal{O}(T^9)$ time. Their proof is based on the ideas of Section 5.5.2 and they are able to identify pairs of production plans for which the corresponding cost functions differ in intercept and slope by the same amount. Geunes et al. (2005) show that there exist $\mathcal{O}(T^3)$ intervals consisting of piecewise linear concave functions of $\mathcal{O}(T^3)$ line segments. Each line segment can be found in $\mathcal{O}(T^3)$ by solving a capacitated ELS problem (Van Hoesel and Wagelmans, 1996), which results in the $\mathcal{O}(T^9)$ algorithm.

Furthermore, Federgruen and Meissner (2005) use our results in a more game theoretical framework. They consider a setting with a number of producers who all consider an ELS problem. The demand of a single producer does not only depend on the price set by the producer himself, but also on the prices set by the other producers. Given the prices set by the other producers, each producer has to find his optimal price, the so-called best response problem. It turns out the the best response problem is equivalent to the model in this chapter. Therefore, Federgruen and Meissner (2005) can use the efficient algorithm in this chapter to determine a Nash-equilibrium for their model in an efficient way.

5.A Appendix

Lemma 5.7 *For $n \geq 1$ it holds that*

$$\sum_{i=1}^{n} i2^{n+1-i} = 2^{n+2} - 2(n+2). \tag{5.14}$$

Proof We will prove Lemma 5.7 by induction. It is easy to verify that both the left and right hand side of (5.14) equal 2 for $n = 1$. Assume now that Lemma 5.7 holds for some $n \geq 1$. We will show that it also holds for $n + 1$. We have

$$\begin{aligned}
\sum_{i=1}^{n+1} i2^{n+2-i} \quad &= \sum_{i=1}^{n} i2^{n+2-i} + (n+1)2 \\
&= 2\sum_{i=1}^{n} i2^{n+1-i} + 2n + 2 \\
&= 2(2^{n+2} - 2n - 4) + 2n + 2 \\
&= 2^{n+3} - 2(n+3),
\end{aligned}$$

which completes the proof. □

Part III

Economic lot-sizing with a remanufacturing option

Introduction

Reverse logistics is a field that has emerged during the last decades. The main idea of reversed logistics is that products and materials are reused in production processes. This is in contrast to traditional 'forward' logistics, where there is a 'one-way' product flow towards the customer. There are several reasons why firms have an increased interest in reversed logistics. First, by environmental legislation the government forces companies to take back used products. The second reason is of economical nature. It may be cheaper to remanufacture used products than to produce new items from scratch. Third, the reuse of products can be seen as a marketing element. By the environmental consciousness of the society companies get some kind of 'green' image and may gain a competitive advantage in this way.

In this part of the thesis we extend the classical economic lot-sizing (ELS) problem with product returns (or returns for short) and a remanufacturing option. We will call this the ELS problem with a remanufacturing option (ELSR). A known quantity of returns comes back from the customer each period. Each return can be remanufactured (which includes for instance disassembly, cleaning, testing, repair and reassembly) in that period or in a later period after which the remanufactured item is assumed to be as-good-as-new. This distinguishes remanufacturing from other recovery types such as material and energy recycling. For a comparison of recovery types we refer to Thierry et al. (1995).

We will consider two different models which include remanufacturing. In the first model we assume that there is a joint setup cost when manufacturing or remanufacturing takes place in a period. For instance, this may be the case when remanufacturing and manufacturing are performed on the same production line. In the second model we assume that there is separate setup cost. That is, in a period where both manufacturing and remanufacturing occurs there is both a manufacturing setup cost and a remanufacturing setup cost. The latter can be the case when manufacturing and remanufacturing are performed on different production lines.

The literature on remanufacturing in an economic lot-sizing context is relatively scarce. In fact, all models in the literature consider the model with separate setup cost. We will now discuss the literature that is most related to our work. For a more general overview of quantitative models for reversed logistics we refer to Fleischmann et al. (1997) and Dekker et al. (2004).

Richter and Sombrutzki (2000) and Richter and Weber (2001) consider two restricted versions of the model described above. In the first model they assume that there are sufficient returns to satisfy all demand and there is no manufacturing. Furthermore, costs are time-invariant in this model. In the second model it assumed that the number of returns in the first period is sufficient to satisfy demand for the whole horizon. Again the manufacturing option need not be used, but it may be economically attractive to use the manufacturing option. Richter and Sombrutzki (2000) and Richter and Weber (2001) show that both problems can be solved by a Wagner-Whitin-like recursion.

Li et al. (2006) consider a similar model as Richter and Sombrutzki (2000) and Richter and Weber (2001) and also assume that sufficient returns are available to satisfy demand. Furthermore, the model is generalized to the multiple product case, where a product of 'higher grade' can be used to satisfy demand for 'lower grade' products. A heuristic procedure is developed to compute near-optimal solutions for the problem. We note that the assumption of sufficient returns may not be realistic in practice. For example, in a situation where demand is stationary over time and a fraction of this demand comes back each period, there will not be sufficient returns to satisfy all demand in the long run.

Beltrán and Krass (2002) assume that returns can be immediately used (without remanufacturing) to satisfy demand. For example, this is the case for the retailing industry where returned clothes can be immediately sold. Therefore, in their model the net demand (actual demand minus returns) can be negative. By proving a generalization of the well-known zero-inventory property, Beltrán and Krass (2002) show that the problem can be solved in cubic time for general concave cost functions and in quadratic time for the case of non-speculative motives.

Golany et al. (2001) consider an extension of the ELSR model where returns can be disposed. They show that this problem is $\mathcal{NP}$-hard for setup-linear cost (and hence for general concave cost functions). Furthermore, they show that the problem can be solved in polynomial time in the case of linear cost by representing the model as a network flow model. Yang et al. (2005) consider the same model as Golany et al. (2001). They show that the problem with disposals is $\mathcal{NP}$-hard even in the case of time-invariant cost parameters and develop a polynomial time dynamic programming heuristic. First, the flow patterns corresponding to the extreme point solutions of the network flow representation are examined. Then the maximum number of flow patterns is selected for which a (feasible) sequence of flow patterns can still be constructed in polynomial time.

The remainder of this part is organized as follows. In Chapter 6 we will consider the ELSR with a joint setup cost. We will show that this problem can be solved in polynomial time for time-invariant cost parameters. Furthermore, we will show the equivalence between the ELSR problem with sufficient returns to satisfy demand and the ELS problem with a capacity bound on inventory. In Chapter 7 we will consider the ELSR with separate setup cost. We will show that this problem is $\mathcal{NP}$-hard and we will derive some complexity results for related models. Furthermore, we will propose a genetic algorithm for the problem.

Chapter 6

The economic lot-sizing problem with a remanufacturing option and joint setup cost*

Abstract

In this chapter we consider the economic lot-sizing problem with a remanufacturing option. We assume that returned products come back from the customer and can be remanufactured to satisfy demand. Furthermore, there is setup cost when production (manufacturing or remanufacturing) occurs, which we will call a joint setup cost. This can be the case when both manufacturing and remanufacturing are performed on the same production line. We consider two special cases: (i) costs are constant over time, (ii) there are sufficient returned items to satisfy demand and there is no manufacturing. We show that both problems can be solved in polynomial time by a dynamic programming approach.

6.1 Problem description and mathematical model

The economic lot-sizing problem with a remanufacturing option (ELSR) is formally described as follows. As in the classical economic lot-sizing (ELS) model, there is a known demand in each time period of a finite time horizon. Furthermore, in each period there is a known number of returned items (or returns for short). These returns can be remanufactured and used to satisfy demand. We assume that customers cannot distinguish between remanufactured returns and newly remanufactured products. We will refer to remanufactured returns and newly remanufactured products as serviceables. It is assumed that returns that arrive in a period can be immediately remanufactured and used to satisfy the

*This chapter is based on Teunter, Bayındır and Van den Heuvel (2006) .

demand in the same period. We assume that we have a fixed setup cost when production occurs, where a production period is a period with manufacturing or remanufacturing. Furthermore, there is a unit cost for each item manufactured and remanufactured. Finally, we have a holding cost for each time period a serviceable or a return is kept in stock.

If we use the following notation:

Parameters:
- T : model horizon,
- d_t : demand in period t,
- r_t : returns in period t,
- K_t : setup cost for production in period t,
- p_t^m : unit manufacturing cost in period t,
- h_t^s : unit cost for holding serviceables in period t,
- p_t^r : unit remanufacturing cost in period t,
- h_t^r : unit cost for holding returns in period t,

Decision variables:
- x_t^m : amount of items manufactured in period t,
- I_t^s : amount of serviceables in stock at the end of period t,
- x_t^r : amount of items remanufactured in period t,
- I_t^r : amount of returns in stock at the end of period t,

then the ELSR problem can be modeled as follows.

$$
\begin{array}{lll}
[\text{ELSR}] \quad \min & \sum_{t=1}^{T} \left(K_t \delta(x_t^m + x_t^r) + p_t^m x_t^m + h_t^s I_t^s + p_t^r x_t^r + h_t^r I_t^r \right) & \\
\text{s.t.} & I_t^s = I_{t-1}^s + x_t^m + x_t^r - d_t & t = 1, \ldots, T \\
& I_t^r = I_{t-1}^r + r_t - x_t^r & t = 1, \ldots, T \\
& x_t^m, x_t^r, I_t^s, I_t^r \geq 0 & t = 1, \ldots, T \\
& I_0^s = I_0^r = 0, &
\end{array}
$$

where

$$
\delta(y) = \begin{cases} 0 & \text{for } y = 0 \\ 1 & \text{for } y > 0. \end{cases}
$$

In the objective function we have the setup cost, unit manufacturing and remanufacturing cost and the holding costs for returns and serviceables. Note that we have a setup cost if the manufacturing or remanufacturing variable is strictly positive. The first two sets of constraints are the inventory balance constraints for serviceables and returns, respectively. The third set of constraints imposes non-negativity on the variables and the last constraint shows that we start with no inventory of serviceables and returns.

The mathematical formulation can be represented as a network flow problem which is shown in Figure 6.1. The flow on the upper upper arcs represents the manufacturing periods and the flow on the middle vertical arcs the remanufacturing periods. Furthermore, the flow on the upper horizontal arcs represents the inventory of serviceables and the flow on the lower horizontal arcs the inventory of returns. Finally, we have the incoming returns represented by the flow on the lower vertical arcs and the demands represented by the flow on the outgoing diagonal arcs.

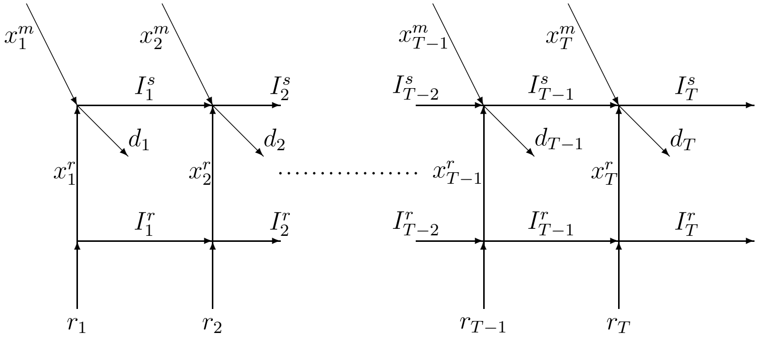


Figure 6.1: Network flow representation of the ELSR problem

6.2 The time-invariant cost case

In this section we consider a special case of the ELSR problem where we assume that all cost parameters are time-invariant, i.e., $K_t = K$, $p_t^m = p^m$, $p_t^r = p^r$, $h_t^s = h^s$ and $h_t^r = h^r$ for $t = 1, \ldots, T$. Furthermore, we make the additional assumptions that $h^r \leq h^s$ and $p^r \leq p^m$. The first assumption is a practical assumption as remanufacturing adds value to an item. The second assumption also seems reasonable in the case that remanufacturing is motivated economically. In this section we will derive some structural properties of an optimal solution, which will be used to construct a polynomial time dynamic programming algorithm.

6.2.1 Structural properties of an optimal solution

The following two lemmas state some structural properties of an optimal solution.

Lemma 6.1 *There exists an optimal solution that satisfies the zero-inventory property for serviceables: for any period with a setup it holds that the stock of serviceables at the beginning of the period is zero, i.e., $I^s_{t-1}(x^m_t + x^r_t) = 0$ for $t = 1, 2, \ldots, T$.*

Proof Consider any solution $\pi = (x^m, x^r, I^s, I^r)$. Since the initial stock (at the end of period 0) of serviceables is zero, the property obviously holds for the first period (in which there is a setup since demand in period 1 can w.l.o.g. be assumed to be positive). Now consider any other period $t \geq 2$ with a setup under solution π, and let s denote the preceding period with a setup. So, s and t are successive setup periods under solution π. We shall complete the proof by showing that if the stock of serviceables is positive at the beginning of period t, then an alternative optimal solution $\pi' = (x'^m, x'^r, I'^s, I'^r)$ can be constructed which satisfies the properties of the lemma. We consider two cases.

- π only remanufactures in period s:
  Then π' remanufactures one less product in s and one more in t, i.e., $x'^r_s = x^r_s - 1$ and $x'^r_t = x^r_t + 1$. Clearly, this solution is feasible because $x^r_s > 0$ and it has the same number of setups. Furthermore, we have $I'^r_i = I^r_i + 1$ and $I'^s_i = I^s_i - 1$ for $i = s, \ldots, t-1$. Therefore, we have no change in unit remanufacturing cost and a reduction in holding cost of $(t-s)(h^s - h^r) \geq 0$.

- π manufactures in period s:
  Then π' manufactures one less product in s and one more in t, i.e., $x'^m_s = x^m_s - 1$ and $x'^m_t = x^m_t + 1$. Again this solution is feasible as $x^m_s > 0$ and has the same setups. Furthermore, we have $I'^s_i = I^s_i - 1$ for $i = s, \ldots, t-1$ and hence we have a reduction in holding cost of $h^s(t-s) \geq 0$ and no change in unit manufacturing cost.

In both cases we have constructed an alternative solution with cost at most equal to the optimal solution. If the alternative solution is strictly smaller, we have a contradiction with the optimality of solution π. If not, we get a solution π' with $I'^s_{t-1} = 0$ by repeating the argument. □

Lemma 6.2 *There exists an optimal solution that satisfies the following property: in every manufacturing period, the stock of returns at the end of that period is zero, i.e., $I^r_t x^m_t = 0$ for $t = 1, 2, \ldots, T$.*

Proof Consider any solution $\pi = (x^m, x^r, I^s, I^r)$ that does not satisfy the lemma. Then there must be some period t, $1 \leq t \leq T$, with manufacturing and with a positive stock of returns at the end, i.e., $I^r_t x^m_t > 0$. Again we will construct an alternative solution $\pi' = (x'^m, x'^r, I'^s, I'^r)$ with cost at most equal to the optimal solution. Let v be the remanufacturing period following t. We consider two cases.

- period v does not exist:
  Consider an alternative solution π' which manufactures one less product and remanufactures one more product in period t, i.e., $x_t'^m = x_t^m - 1$ and $x_t'^r = x_t^r + 1$. Clearly, solution π' is feasible as $x_t^m > 0$ and $I_t^r > 0$. Furthermore, we have $I_i'^r = I_i^r - 1$ and $I_i'^s = I_i^s$ for $i = t, \ldots, T$. This means that we have a reduction in cost of $(p^m - p^r) + h^r(T - t + 1) \geq 0$.

- period v exists:
  An alternative solution π' is to manufacture one less product and remanufacture one more product in period t, and to manufacture one more product and remanufacture one less product in period v, i.e., $x_t'^m = x_t^m - 1$, $x_t'^r = x_t^r + 1$, $x_v'^m = x_v^m + 1$ and $x_v'^r = x_v^r - 1$. Again, solution π' is feasible as $x_t^m > 0$ and $x_v^r > 0$ and we have $I_i'^r = I_i^r - 1$ and $I_i'^s = I_i^s$ for $i = t, \ldots, v-1$. For solution π' the setup costs, the unit (re)manufacturing and holding costs for serviceables remain the same. Furthermore, there is a reduction in holding cost for returns of $h^r(v - t) \geq 0$.

Again, if we have a strict cost reduction, there is a contradiction with the optimality of solution π and, otherwise, by repeating the argument, we get a solution which satisfies the lemma. □

In the classical ELS problem the zero-inventory property says that there is no production if there is still inventory left. For the ELSR problem with time-invariant cost there is also no production (remanufacturing or manufacturing) if the inventory of serviceables is still positive. Therefore, Lemma 6.1 is a generalization of the well-known zero-inventory property. Lemma 6.2 shows that priority is given to the remanufacturing option. This means that in any optimal solution there is only manufacturing in a certain period if the initial stock of returns at the beginning of that period is insufficient for remanufacturing the entire lot.

6.2.2 A dynamic programming algorithm

To visualize the structure of candidate solutions that satisfy Lemmas 6.1 and 6.2, we construct a graph associated with the candidate solutions. This graph consists of the arcs of the decision variables with positive flow (this representation is also used in Zangwill (1968)). It follows from Lemmas 6.1 and 6.2 that the graph of a candidate solution consists of the 'block'-structure as illustrated in Figure 6.2. For example, Lemma 6.1 is violated in case $I_{u-1}^s > 0$ or $I_w^s > 0$ and Lemma 6.2 is violated in case $I_{t-1}^r > 0$ or $I_v^r > 0$.

This block-structure will be used for the construction of an efficient DP algorithm. A block is defined by the periods $1 \leq t \leq u \leq v \leq w \leq T$, where demands in periods $v, \ldots, w$ are satisfied by returns from periods $t, \ldots, v$. For ease of notation, as in the

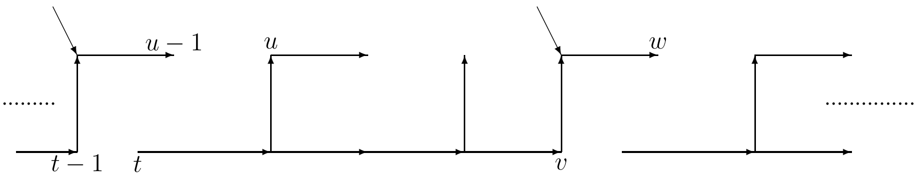


Figure 6.2: The 'block'-structure of an optimal solution

ELS model, we can rewrite model [ELSR] as a model without holding cost by using the substitutions $I_t^s = \sum_{i=1}^t (x_i^m + x_i^r - d_i)$ and $I_t^r = \sum_{i=1}^t (r_i - x_i^r)$, and redefining the unit cost for manufacturing and remanufacturing as $c_t^m = p_t^m + \sum_{i=t}^T h_i^s$ and $c_t^r = p_t^r + \sum_{i=t}^T (h_i^s - h_i^r)$, respectively. Furthermore, for ease of notation we let $D_{s,t} = \sum_{i=s}^t d_i$ and $R_{s,t} = \sum_{i=s}^t r_i$.

In the DP algorithm we will use the following variables:

$f_{t,v}^{u,w}$: minimum cost to satisfy demands in periods $u, \ldots, w$ when using returns from period t onwards and only one manufacturing period in v (∞ if no feasible solution exists),

f_v^w: minimum cost to satisfy demands in periods $1, \ldots, w$ with the last manufacturing period in v.

Note that there only exists a feasible block if $R_{t,i} \geq D_{u,i}$ for $i = u, \ldots, v-1$ and $R_{t,v} \leq D_{u,w}$. Given the values $f_{t,v}^{u,w}$, the value f_v^w $(1 \leq v \leq w \leq T)$ can be recursively determined by the equation

$$f_v^w = \min_{1 \leq t \leq u \leq v} \{f_{t-1}^{u-1} + f_{t,v}^{u,w}\}, \tag{6.1}$$

where the recursion is initialized by $f_0^0 = 0$ and $f_0^t = \infty$ for $t = 1, \ldots, T$. It is not difficult to see that it takes $\mathcal{O}(T^4)$ to calculate all values f_v^w by using (6.1).

In the remainder of this section we will show how to calculate the cost of the blocks $f_{t,v}^{u,w}$. To this end we define the partial block

$f_t^{u,v-1}$: minimum cost to satisfy demand in periods $u, \ldots, v-1$ by pure remanufacturing with returns available from period t (∞ if infeasible).

Note that there exists a feasible partial block if $R_{t,i} \geq D_{u,i}$ for $i = u, \ldots, v-1$. The partial block is illustrated by the dashed box in Figure 6.3. Richter and Sombrutzki (2000) show that the ELSR problem with only remanufacturing and time-invariant costs can be solved in $\mathcal{O}(T^2)$. This means that given the periods t, u, $v-1$ $(t \leq u \leq v-1)$, the value $f_t^{u,v-1}$ can be determined in $\mathcal{O}(T^2)$ and hence all values $f_t^{u,v-1}$ $(1 \leq t \leq u \leq v-1 \leq T)$ can

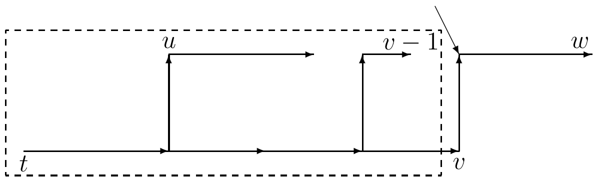


Figure 6.3: Partial block $f_t^{u,v-1}$

be determined in $\mathcal{O}(T^5)$ time. However, we can speed up computations by using the recursive equation

$$f_t^{u,v-1} = \min_{\substack{u \le i \le v-1: \\ R_{t,i} \ge D_{u,v-1}}} \{f_t^{u,i-1} + K_i + c_i^r D_{i,v-1}\} \tag{6.2}$$

with $f_t^{u,u-1} = 0$ for $1 \le t \le u \le T$. Using (6.2) all values $f_t^{u,v-1}$ can be found in $\mathcal{O}(T^4)$ time.

Given the minimum cost of the partial blocks $f_t^{u,v-1}$, the minimum cost of the blocks $f_{t,v}^{u,w}$ can be easily calculated. First, the ending inventory at the end of period $v-1$ equals $I_{v-1}^r = r_{t,v-1} - d_{u,v-1}$ since $d_{u,v-1}$ returns are remanufactured. By Lemma 6.2 all the returns available in period v are used to satisfy (part) of the demand in periods $v, \ldots, w$ and the remaining demand is satisfied by manufacturing. Thus, the amount of items remanufactured equals $R_{t,v} - D_{u,v-1}$ and the amount manufactured equals $D_{v,w} - (R_{1,v} - D_{u,v-1}) = D_{u,w} - R_{t,v}$. Therefore, we have

$$f_{t,v}^{u,w} = f_t^{u,v-1} + K_v + c_v^r(R_{t,v} - D_{u,v-1}) + c_v^m(D_{u,w} - R_{t,v}). \tag{6.3}$$

From (6.3) it follows that all all values $f_{t,v}^{u,w}$ $(1 \le t \le u \le v \le w \le T)$ can be calculated in $\mathcal{O}(T^4)$ time.

To find the optimal solution of the overall, problem we have two possibilities for the last block of a candidate solution. Either the last production period is a manufacturing period or the last period is a pure remanufacturing period. Both situations are illustrated in Figures 6.4 and 6.5. This means that the optimal solution f^T is found by the recursive equation

$$f^T = \min\left\{ \min_{1 \le t \le T}\{f_t^T\}, \min_{1 \le v \le w \le T}\{f_{v-1}^{w-1} + f_v^{w,T}\} \right\}. \tag{6.4}$$

Clearly, equation (6.4) can be determined in $\mathcal{O}(T^2)$ time. Because equations (6.1), (6.2), (6.3) and (6.4) can be determined in $\mathcal{O}(T^4)$, $\mathcal{O}(T^4)$, $\mathcal{O}(T^4)$ and $\mathcal{O}(T^2)$ time, respectively, the total computation time to find the optimal solution equals $\mathcal{O}(T^4)$. Therefore, we have the following theorem.

Theorem 6.3 *The ELSR problem with time-invariant cost can be solved in $\mathcal{O}(T^4)$ time.*

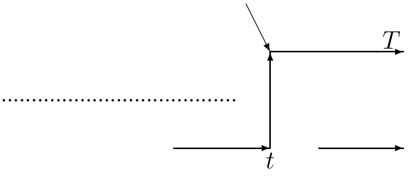


Figure 6.4: The last production period is a manufacturing period

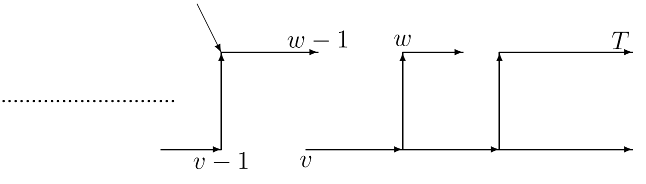


Figure 6.5: The last production period is a pure remanufacturing period

6.3 The pure remanufacturing case

As also mentioned in the introduction of this part of the thesis, Richter and Sombrutzki (2000) consider a special case of the ELSR problem. They assume that sufficient returns enter into the system to satisfy the demand in all periods and hence manufacturing is not necessary to satisfy demand. Note that it may be economical to use the manufacturing option, but we assume in this section that it will not be used. We will call this special case of the ELSR problem the 'pure' ELSR problem. Richter and Sombrutzki (2000) show that the problem with time-invariant cost can be solved by a Wagner-Whitin type of algorithm. In this section we will show that the problem with non-stationary cost can also be solved in polynomial time. Moreover, we will show that the problem is equivalent to the ELS problem with bounded inventory.

6.3.1 Model formulation

By removing the manufacturing option in model [ELSR] the mathematical formulation of the pure ELSR problem becomes

$$
\begin{array}{lll}
\text{[PR]} \quad \min & \sum_{t=1}^{T} \left(K_t \delta(x_t^r) + p_t^r x_t^r + h_t^s I_t^s + h_t^r I_t^r \right) & \\
\text{s.t.} & I_t^s = I_{t-1}^s + x_t^r - d_t & t = 1, \ldots, T \\
& I_t^r = I_{t-1}^r + r_t - x_t^r & t = 1, \ldots, T \\
& x_t^r, I_t^s, I_t^r \geq 0 & t = 1, \ldots, T \\
& I_0^s = I_0^r = 0. &
\end{array}
$$

So this model is equivalent to model [ELSR] with $x_t^m = 0$. It is not difficult to see that there exists a feasible solution to [PR] if and only if $\sum_{i=1}^t r_i \geq \sum_{i=1}^t d_i$ for $t = 1, \ldots, T$.

As in Section 6.1 we can rewrite model [ELSR] to a model without holding cost by making the substitutions $I_t^s = \sum_{i=1}^t (x_i^r - d_i)$ and $I_t^r = \sum_{i=1}^t (r_i - x_i^r)$ in the objective function. Then the formulation becomes

$$
\begin{array}{lll}
[\mathrm{PR}'] & \min & \sum_{t=1}^T \left(K_t \delta(x_t^r) + c_t^r x_t^r - h_t^s \sum_{i=1}^t d_i + h_t^r \sum_{i=1}^t r_i\right) & \\
& \text{s.t.} & I_t^s = I_{t-1}^s + x_t^r - d_t & t = 1, \ldots, T \\
& & I_t^r = I_{t-1}^r + r_t - x_t^r & t = 1, \ldots, T \\
& & x_t^r, I_t^s, I_t^r \geq 0 & t = 1, \ldots, T \\
& & I_0^s = I_0^r = 0, &
\end{array}
$$

with

$$c_t^r = p_t^r + \sum_{i=t}^T (h_i^s - h_i^r).$$

There are no holding cost in the new objective function and the last two terms are constants and hence can be ignored in the optimization of the model. Note that the term c_t^r is negative in case the unit cost of remanufacturing and holding a serviceable in inventory up to period T is lower than the unit cost of holding a return in stock up to period T. In practice this will not be likely as one might expect that holding a return in stock is cheaper than holding a serviceable in stock. Again the mathematical formulation can be represented as a network flow model with concave cost (see Figure 6.6). Note that

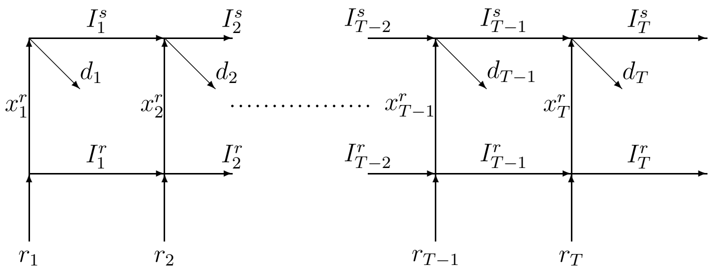


Figure 6.6: Network flow representation of the pure ELSR problem

there are no manufacturing arcs in the network flow representation.

6.3.2 A dynamic programming algorithm

We will use the network flow formulation to construct a dynamic programming algorithm to solve the problem. The following lemma is sufficient to derive a polynomial time algorithm.

Lemma 6.4 *Let periods u and v be two consecutive remanufacturing periods of an optimal solution. Then there exists an optimal solution with either $I_u^r = 0$ or $I_{v-1}^s = 0$.*

Proof Assume that the lemma does not hold, which implies $I_t^r > 0$ and $I_t^s > 0$ for $t = u, \ldots, v-1$. Construct an alternative solution by increasing remanufacturing in period u and decreasing remanufacturing in period v by some small amount ε. This alternative solution has a change in cost of $\Delta c = \varepsilon(p_u^r - p_v^r + \sum_{t=u}^{v-1}(h_t^s - h_t^r))$. If $\Delta c < 0$ (which means that there is a cost reduction), then we can increase remanufacturing in period u and increase remanufacturing in period u until either $I_u^r = 0$ or $x^v = 0$. If $\Delta c \geq 0$, then we can increase remanufacturing in period v and decrease remanufacturing in period u until either $I_{v-1}^r = 0$ or $x_u^r = 0$. In all cases we have a solution which satisfies the property of the lemma and has no larger cost than the original solution. $\square$

In fact, in the proof we use a property of network flow problems with concave costs which was shown by Zangwill (1968) (we used a direct proof for illustrative reasons). Again consider the graph associated with an optimal solution consisting of the arcs of the decision variables with positive flow. Zangwill (1968) shows that this graph forms a tree in the original network. In other words, there exists an optimal solution for which the associated graph has no cycles. Clearly, an associated graph with a cycle corresponds to a solution that does not satisfy the property of Lemma 6.4. In Figure 6.7 we have illustrated a candidate solution that satisfies the property of Lemma 6.4. That is, either

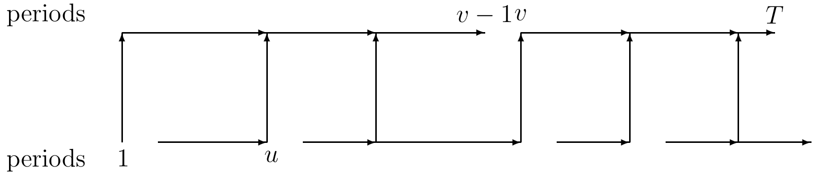


Figure 6.7: Candidate solution

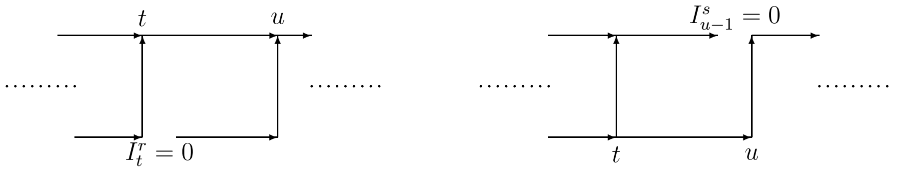
the inventory of returns is zero at the end of a remanufacturing period (see for example period u with $I_u^r = 0$) or the inventory of serviceables is zero when remanufacturing starts (see for example period v with $I_{v-1}^s = 0$).

These properties lead to the construction of a dynamic programming algorithm. We define the following variables:

f_u^r: minimum cost up to period u with u the last remanufacturing and no returns in stock at the end of period u, i.e., $I_u^r = 0$,

f_w^s: minimum cost to satisfy demand up to period w with no serviceables in stock at the end of period w, i.e., $I_w^r = 0$.

First, we will derive the recursive equation for f_u^r $(1 \leq u \leq T)$. Let t be the last remanufacturing period before period u. By Lemma 6.4 we may restrict ourselves to solutions with either $I_t^r = 0$ or $I_{u-1}^s = 0$. Both situations are illustrated in Figure 6.8.



Case $I_t^r = 0$ Case $I_{u-1}^s = 0$

Figure 6.8: The two cases to calculate f_u^r

- $I_t^r = 0$:
  In this case we remanufacture $R_{t+1,u}$ items in period u. This solution is feasible if there are sufficient returns up to period t to satisfy demand up to period $u-1$, i.e., $R_{1,t} \geq D_{1,u-1}$.

- $I_{u-1}^s = 0$:
  In this case we remanufacture all returns available up to period u which are not used to satisfy the demands up to period $u-1$, i.e., the remanufacturing quantity equals $R_{1,u} - D_{1,u-1}$.

It follows from above that we have the following recursive equation for f_u^r:

$$f_u^r = \min \left\{ \min_{\substack{1 \leq t < u: \\ R_{1,t} \geq D_{1,u-1}}} \{f_t^r + K_u + c_u^r R_{t+1,u}\}, f_{u-1}^s + K_u + c_u^r (R_{1,u} - D_{1,u-1}) \right\}. \quad (6.5)$$

In a similar way as above we can derive a recursive equation for f_w^s $(1 \leq w \leq T)$. Let v be the last remanufacturing period before w. First, note that there exists a feasible solution with the last remanufacturing in period v that satisfies demand up to period w if and only if $R_{1,v} \geq D_{1,w}$. Let t be the remanufacturing period before v. Again we have to consider the two different cases that follow from Lemma 6.4. That is, either $I_t^r = 0$ or $I_{v-1}^s = 0$. Both cases are illustrated in Figure 6.9.

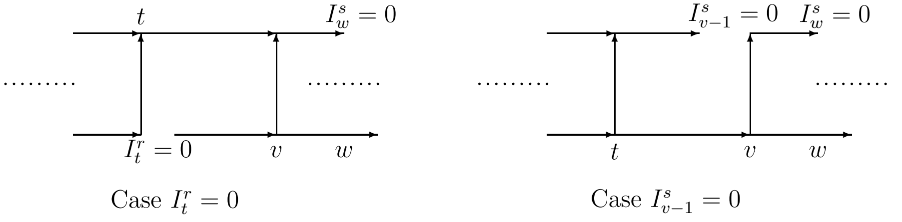


Figure 6.9: The two cases to calculate f_w^s

- $I_t^r = 0$:
  In this case the returns up to period t must be sufficient to satisfy the demand up to period $v-1$. Furthermore, as there is also remanufacturing in period v, the amount remanufactured up to period t cannot be larger than the demand up to period w. Therefore we have a feasible solution of the desired type if $D_{1,v-1} \leq R_{1,t} \leq D_{1,w}$ and the amount remanufactured in period v equals $D_{1,w} - R_{1,t}$.

- $I_{v-1}^s = 0$:
  In this case the amount of remanufactured returns equals $D_{v,w}$.

It follows from the two cases that the following expression can be used to determine f_w^s:

$$f_w^s = \min_{\substack{1 \leq v \leq w: \\ R_{1,v} \geq D_{1,w}}} \left\{ \min_{\substack{1 \leq t < v: \\ D_{1,v-1} \leq R_{1,t} \leq D_{1,w}}} \{f_t^r + K_v + c_v^r(D_{1,w} - R_{1,t})\}, f_{v-1}^s + K_v + c_v^r D_{v,w} \right\}. \tag{6.6}$$

By using the convention that $\min\{\emptyset\} = \infty$, it follows from (6.6) that $f_w^s = \infty$ if there is no v with $R_{1,v} < D_{1,w}$, that is, there is no feasible solution.

Recursion formulas (6.5) and (6.6) are initialized by $f_0^r = f_0^s = 0$. Furthermore, the cost of the optimal solution f^* is found by

$$f^* = \min \left\{ \min_{\substack{1 \leq t \leq T: \\ R_{1,t} \geq D_{1,T}}} \{f_t^r\}, f^T \right\}. \tag{6.7}$$

The first term in (6.7) corresponds to the case that there are no returns in stock at the end of the last remanufacturing period. To assure feasibility, there must be sufficient returns in period t to satisfy all demand, i.e., $R_{1,t} \geq D_{1,T}$. The second term in (6.7) corresponds to the case that we have no serviceables in stock at the end of period T.

It is not difficult to verify that a straightforward implementation of (6.5), (6.6) and (6.7) leads to a running time of $\mathcal{O}(T^2)$, $\mathcal{O}(T^3)$ and $\mathcal{O}(T)$ time, respectively. Therefore, there is an algorithm with a running time of $\mathcal{O}(T^3)$ to solve the pure ELSR. In the next section we show how to improve the running time.

6.3.3 Improving the running time of the DP algorithm

It follows from Section 6.3.2 that equation (6.6) is the bottle neck in the running time. In particular, it is the first term of (6.6) that causes the cubic running time. Therefore, if we are able to reduce the running time to calculate this term, then the total running time of the algorithm will decrease. We can rewrite the first term as

$$\min_{\substack{1 \leq v \leq w: \\ R_{1,v} \geq D_{1,w}}} \left\{ \min_{\substack{1 \leq t < v: \\ D_{1,v-1} \leq R_{1,t} \leq D_{1,w}}} \{f_t^r + K_v + c_v^r(D_{1,w} - R_{1,t})\} \right\} = \tag{6.8}$$

$$\min_{\substack{1 \leq t < w: \\ R_{1,t} \leq D_{1,w}}} \left\{ f_t^r + \min_{\substack{t < v \leq w: \\ R_{1,v} \geq D_{1,w}, \\ R_{1,t} \geq D_{1,v-1}}} \{K_v + c_v^r(D_{1,w} - R_{1,t})\} \right\}. \tag{6.9}$$

We will now focus on the second term in this minimization problem. That is, given t we want to solve

$$\min_{\substack{t \leq v \leq w: \\ R_{1,v} \geq D_{1,w}, \\ R_{1,t-1} \geq D_{1,v-1}}} \{K_v + c_v^r(D_{1,w} - R_{1,t-1})\}.$$

(Note that for notational convenience we have shifted period t one period.) First, let $b(t) = \max\{w \leq T+1 : R_{1,t-1} \geq D_{1,w-1}\}$, that is, $b(t)$ is the first period for which demand cannot be satisfied when all returns up to period $t-1$ are remanufactured. For $w = b(t), \ldots, T$ define the index set

$$J_{t,w} = \{v \leq w : R_{1,v} \geq D_{1,w}, R_{1,t-1} \geq D_{1,v-1}\} = \{v \leq b(t) : R_{1,v} \geq D_{1,w}\}.$$

and for $w < b(t)$ let $J_{t,w} = \emptyset$. By this definition, the set $J_{t,w}$ contains all periods for which a feasible solution of the type as in Figure 6.10 exists. Note that $J_{t,w+1} \subseteq J_{t,w}$ for

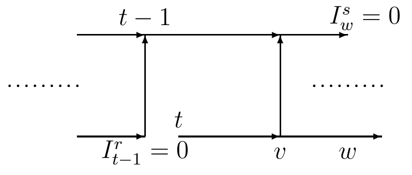


Figure 6.10: Example of a feasible period v

$w \geq b(t)$ as $R_{1,v} \geq D_{1,w+1}$ implies that $R_{1,v} \geq D_{1,w}$. Furthermore, let

$$C^r_{t,w} = \min_{v \in J_{t,w}} \{K_v + c^r_v(D_{1,w} - R_{1,t-1})\},$$

so that (6.9) can be written as

$$\min_{1 \leq t < w} \{f^r_t + C^r_{t+1,w}\}.$$

We will show that for fixed t all values $C^r_{t,w}$ $(w = t, \ldots, T)$ can be calculated in $\mathcal{O}(T \log T)$ time. To this end, define the function

$$g_{t,w}(x) = \min_{v \in J_{t,w}} \{K_v + c^r_v x\}.$$

Note that $C^r_{t,w} = g_{t,w}(D_{1,w} - R_{1,t-1})$. Furthermore, note that $g_{t,w}(x)$ is the lower envelope of a number of linear line segments. Given a period t, we can use this property to construct the functions $g_{t,w}(x)$ $(w = b(t), \ldots, T)$ efficiently. Following a similar approach as in Wagelmans et al. (1992) we start with $g_{t,T}(x)$ (note that $J_{t,T}$ may be empty). Then we can construct $g_{t,w}(x)$ from $g_{t,w+1}(x)$ for all $w = T-1, \ldots, t$ by adding linear line segments (if any) to a lower envelope of linear line segments. Note that updating a lower envelope of $\mathcal{O}(T)$ linear line segments by a single linear segment as well as determining the value $g_{t,w}(x)$ for some fixed x takes $\mathcal{O}(\log T)$. Because the index set satisfies $J_{t,w+1} \subseteq J_{t,w}$ and $|J_{t,w}| = \mathcal{O}(T)$, we have to add at most $\mathcal{O}(T)$ line segments. Therefore, we have the following lemma.

Lemma 6.5 *For a given $t \geq 2$ all values $C^r_{t,w}$ $(w = t, \ldots, T)$ can be determined in $\mathcal{O}(T \log T)$.*

Lemma 6.5 implies that it takes $\mathcal{O}(T^2 \log T)$ to calculate all values $C^r_{t,w}$ $(2 \leq t \leq w \leq T)$. This means that (6.9) and hence (6.6) can be performed in $\mathcal{O}(T^2 \log T)$ time. Therefore, we have the following theorem.

Theorem 6.6 *The pure ELSR problem can be solved in $\mathcal{O}(T^2 \log T)$ time.*

6.3.4 Equivalence with the ELS problem with bounded inventory

In the pure ELSR problem the cumulative amount remanufactured up to some period is bounded by the number of returns available up to that period. Now consider the ELS problem with bounded inventory (ELSB), which is equivalent to the standard ELS problem except that inventory is bounded in each period. For example, this is the case when a warehouse has a finite capacity. This bound on inventory causes that cumulative production is bounded. In this section we will show that the pure ELSR and the ELSB problem are equivalent.

First, we start with a formulation of the ELSB problem. Let M_t be the maximum inventory at the end of period t (after demand is satisfied in that period). Then the ELSB can be formulated as

$$
\begin{array}{lllr}
\text{[ELSB]} & \min & \sum_{t=1}^{T} \left(K_t\delta(x_t) + p_t x_t + h_t I_t\right) & \\
& \text{s.t.} & I_t = I_{t-1} + x_t - d_t & t = 1,\dots,T \\
& & I_t \leq M_t & t = 1,\dots,T \\
& & x_t, I_t \geq 0 & t = 1,\dots,T \\
& & I_0 = 0. &
\end{array}
$$

By applying the substitution $I_t = \sum_{i=1}^{t}(x_i - d_i)$ we get the model

$$
\begin{array}{lllr}
\text{[ELSB}'\text{]} & \min & \sum_{t=1}^{T} \left(K_t\delta(x_t) + c_t x_t - h_t \sum_{i=t}^{T} d_i\right) & \\
& \text{s.t.} & \sum_{i=1}^{t} x_t \geq \sum_{i=1}^{t} d_t & t = 1,\dots,T \\
& & \sum_{i=1}^{t} x_i \leq M_t + \sum_{i=1}^{t} d_i & t = 1,\dots,T \\
& & x_t \geq 0 & t = 1,\dots,T.
\end{array}
$$

with $c_t = p_t + \sum_{i=t}^{T} h_i$.

Again consider model [PR′] and apply the substitutions $I_t^s = \sum_{i=1}^{t}(x_i^r - d_i)$ and $I_t^r = \sum_{i=1}^{t}(r_i - x_i^r)$ to the constraints. Then the model reduces to

$$
\begin{array}{lllr}
\text{[PR}''\text{]} & \min & \sum_{t=1}^{T} \left(K_t\delta(x_t^r) + c_t^r x_t^r - h_t^s \sum_{i=1}^{t} d_i + h_t^r \sum_{i=1}^{t} r_i\right) & \\
& \text{s.t.} & \sum_{i=1}^{t} x_i^r \geq \sum_{i=1}^{t} d_i & t = 1,\dots,T \\
& & \sum_{i=1}^{t} x_i^r \leq \sum_{i=1}^{t} r_i & t = 1,\dots,T \\
& & x_t^r \geq 0 & t = 1,\dots,T.
\end{array}
$$

It follows from the mathematical formulations that models [PR″] and [ELSB′] are equivalent. Namely, the parameters and variables d_t, $\sum_{i=1}^{t} r_i$, K_t, c_t^r and x_t^r of model [PR″] correspond to the parameters and variables d_t, $M_t + \sum_{i=1}^{t} d_i$, K_t, c_t and x_t of model [ELSB′], respectively.

Thus, if we have a pure ELSR problem, we can solve an equivalent ELSB problem by setting the appropriate cost parameters and letting the inventory bounds equal to

$M_t = \sum_{i=1}^{t}(r_t - d_t) \geq 0$. Similarly, if we have an ELSB problem, we can solve an equivalent pure ELSB problem by setting the appropriate cost parameters and letting the returns equal to $r_t = d_t + (M_t - M_{t-1}) \geq 0$. The last inequality follows from the fact that we may assume w.l.o.g. that $M_{t-1} \leq M_t + d_t$ (as also shown in Atamtürk and Küçükyavuz (2005)). Namely, if $M_{t-1} > M_t + d_t$ and production in period $t-1$ is such that $I_{t-1} = M_{t-1}$, then the inventory level in period t equals $I_t = I_{t-1} + x_t - d_t \geq M_{t-1} - d_t > M_t$ which is not feasible. Therefore, we can redefine the inventory bounds to $M'_{t-1} = \min\{M_{t-1}, M'_t + d_t\}$ with $M'_T = M_T$ without changing the feasible region. Clearly, these bounds satisfy $M'_{t-1} \leq M'_t + d_t$.

To the best of our knowledge the running time of the fastest algorithm to solve the ELSB problem is $\mathcal{O}(T^3)$ (Love, 1973). This means that reformulating an ELSB problem to an equivalent pure ELSR problem and solving it with the dynamic programming algorithm of Section 6.3.2 leads to an algorithm with running time $\mathcal{O}(T^2 \log T)$, which is an improvement over the algorithm of Love (1973). This result is formally stated in Theorem 6.7.

Theorem 6.7 *The ELSB problem can be solved in $\mathcal{O}(T^2 \log T)$ time.*

Love (1973) uses the notion of inventory regeneration periods and inventory periods to construct his algorithm. An inventory regeneration period t is a period with $I_t = 0$ and an inventory period is a period with $I_t = 0$ or $I_t = M_t$ (by this definition an inventory regeneration period is a special case of an inventory period). Love (1973) shows that if u and v are two consecutive production periods in an extreme solution, then there is an inventory period t with $u \leq t < v$. When using the one-to-one correspondence between the ELSB and pure ELSR problem and noting that $I_t = I^s_t$ in models [ELSB] and [PR], this property is equivalent to the property of Lemma 6.4 and can also be seen in Figure 6.7. A period u with $I^r_u = 0$ corresponds to an inventory period as $I^r_u = 0$ implies $\sum_{i=1}^{u} x^r_i = \sum_{i=1}^{u} r_i$ and hence $I_u = I^s_u = \sum_{i=1}^{u}(r_i - d_i) = M_u$. Furthermore, a period with $I^s_{v-1} = 0$ corresponds to an inventory regeneration period.

6.4 Open problem: the general case

For the ELSR problem with general cost parameters Lemmas 6.1 and 6.2 do not hold anymore. This is shown in Examples 6.8 and 6.9, respectively.

Example 6.8 Consider the 2-period problem instance with $h^r_1 = h^s_t = 0$ and the remaining parameters as in Table 6.1. The (unique) optimal solution for this problem instance is to remanufacture 15 items in period 1 ($x^r_1 = 15$) and to manufacture 5 items in period 2 ($x^m_2 = 5$) with a total cost of 45. This means that the stock of serviceables at the end of

t	1	2
d_t	10	10
r_t	15	0
K_t	10	10
p_t^m	5	2
p_t^r	1	5

Table 6.1: Problem instance of Example 6.8

period 1 equals 5 ($I_1^s = 5$). Therefore, Lemma 6.1 (the zero-inventory property for serviceables) is violated because there is manufacturing in period 2 while there is a positive starting inventory of serviceables.

Example 6.9 Consider the 2-period problem instance with $h_1^r = h_t^s = 0$ and the remaining parameters as in Table 6.2. The (unique) optimal solution for this problem instance

t	1	2
d_t	10	10
r_t	10	5
K_t	1	1
p_t^m	4	3
p_t^r	3	1

Table 6.2: Problem instance of Example 6.9

is to manufacture 5 items in period 1 ($x_1^m = 5$), to remanufacture 5 items in period 1 ($x_1^r = 5$) and to remanufacture 10 items in period 2 ($x_2^r = 10$) with a total cost of 47. This means that the stock of returns at the end of period 1 is strictly positive ($I_1^r = 5$). Therefore, Lemma 6.2 (priority is given to remanufacturing) is violated because there is manufacturing while there is a positive inventory of returns. Note that $p_t^r \leq p_t^m$ for $t = 1, 2$, which is a reasonable assumption if remanufacturing is motivated economically.

It follows from the examples that the dynamic programming approach of Section 6.3.2 cannot be used to solve the general ELSR problem. In Figure 6.11 we have illustrated part of the graph associated with a candidate solution for the general problem. At first sight, it seems that we can construct a polynomial time dynamic programming algorithm by defining the variable

$f_{t,v}^{uw}$: minimum cost to satisfy demand in periods $u, \ldots, w$ with returns available from periods $t, \ldots, v$ with $I_i^r > 0$ for $i = t, \ldots, v-1$.

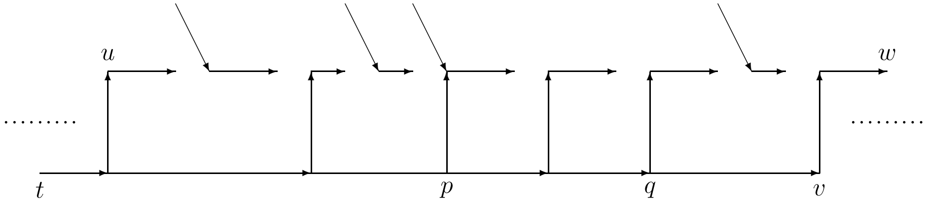


Figure 6.11: Partial solution for the general ELSR problem

However, the problem is that we can have both pure manufacturing and pure remanufacturing periods in a 'block' defined as above and hence there is not just a single manufacturing period at the end of a block as for the time-invariant cost case. As a result, the inventory of returns is not uniquely determined for the periods in a partial block. For example, the level I^r_{p-1} cannot simply be determined by subtracting cumulative demand from cumulative returns, but it depends on the demand periods that are satisfied by pure manufacturing periods. In fact, we have an exponential number of combinations for which demands can be satisfied by pure manufacturing and hence the number of stock levels I^r_{p-1} can be exponential. Therefore, the above approach does not help to construct a polynomial time dynamic programming algorithm. In fact, we do not know whether the ELSR problem can be solved in polynomial time or whether it belongs to the class of $\mathcal{NP}$-hard problems.

6.5 Conclusion

In this chapter we considered the ELS problem with a remanufacturing option. We showed that the problem with time-invariant cost parameters can be solved in $\mathcal{O}(T^4)$ time. We developed a dynamic programming algorithm where we utilized the 'block' structure of the graph associated with an extreme point solution. Furthermore, we showed that the problem without the manufacturing option, sufficient returns and non-stationary cost can be solved in $\mathcal{O}(T^2 \log T)$. This generalizes the work of Richter and Sombrutzki (2000), who considered the model with time-invariant cost parameters. Furthermore, we showed that this problem is equivalent to the ELS problem with bounded inventory and hence this problem can be solved in $\mathcal{O}(T^2 \log T)$. This improves on Love (1973), who proposed an $\mathcal{O}(T^3)$ algorithm. Finally, the ELSR problem with general cost and no restrictions on the returns is not known to be polynomially solvable or $\mathcal{NP}$-hard. This is an issue for future research.

Chapter 7

The economic lot-sizing problem with a remanufacturing option and separate setup cost*

Abstract

In this chapter we consider the economic lot-sizing problem with a remanufacturing option. We assume that products come back from the customers and those returned products can be remanufactured to satisfy demand. In this chapter we assume that there is separate setup cost when remanufacturing or manufacturing occurs. For instance, this can be the case when both manufacturing and remanufacturing are performed on different production lines. We will show that this problem is $\mathcal{NP}$-hard even in the case of time-invariant cost. Furthermore, we will develop a genetic algorithm for the problem.

7.1 Problem description and mathematical model

The economic lot-sizing problem with a remanufacturing option and separate setup cost (ELSR) is equivalent to the problem of Section 6.1 except that there is a setup cost for both manufacturing and remanufacturing. For example, this is the case when the manufacturing process and remanufacturing process are performed on different production lines. In this chapter we will again use the abbreviation ELSR as we also did for the joint setup cost case in the previous chapter. However, no ambiguity will occur as we will only consider the separate setup cost case in this chapter. Using the additional notation

K_t^m: setup cost for manufacturing in period t,

*This chapter is based on Van den Heuvel (2004) and Teunter, Bayındır and Van den Heuvel (2006).

K_t^r: setup cost for remanufacturing in period t,

the ELSR problem can be modeled as follows:

$$
\begin{array}{lll}
\text{[ELSR]} & \min & \sum_{t=1}^{T} \left(K_t^m \delta(x_t^m) + p_t^m x_t^m + h_t^s I_t^s + K_t^r \delta(x_t^r) + p_t^r x_t^r + h_t^r I_t^r\right) \\
 & \text{s.t.} & I_t^s = I_{t-1}^s + x_t^m + x_t^r - d_t \qquad t = 1, \ldots, T \\
 & & I_t^r = I_{t-1}^r + r_t - x_t^r \qquad t = 1, \ldots, T \\
 & & x_t^m, x_t^r, I_t^s, I_t^r \geq 0 \qquad t = 1, \ldots, T \\
 & & I_0^s = I_0^r = 0,
\end{array}
$$

where

$$
\delta(y) = \begin{cases} 0 & \text{for } y = 0 \\ 1 & \text{for } y > 0. \end{cases}
$$

Note that the mathematical model is equivalent to the mathematical model of Section 6.1 except that the term $K_t^m \delta(x_t^m) + K_t^r \delta(x_t^r)$ occurs in the objective function in stead of the term $K_t \delta(x_t^m + x_t^r)$.

In the remainder of this chapter we will analyze this problem. In Section 7.2 we will show that the problem is $\mathcal{NP}$-hard and we will derive complexity results for related problems. In the next section we will derive structural properties of an optimal solution. In Section 7.4 we will consider the ELSR problem where the production periods are fixed and the optimal production quantities have to be determined. This problem will be the basis for the genetic algorithm developed in Section 7.5. The chapter is ended with the conclusion in Section 7.6.

7.2 Complexity results

7.2.1 The general ELSR problem

As mentioned in the introduction Richter and Sombrutzki (2000) and Richter and Weber (2001) show that some special cases of the ELSR problem can be solved in polynomial time. However, Richter and Sombrutzki (2000, p. 311) mention that "There are probably no simple algorithms to solve that general model ...". In this section we will show that the ELSR problem is $\mathcal{NP}$-hard in general. In the proof we will use a reduction from the well-known $\mathcal{NP}$-complete PARTITION problem. For complexity theory we refer to Garey and Johnson (1979) and Papadimitriou and Steiglitz (1982).

Problem PARTITION (problem [SP12] in Garey and Johnson (1979)): Given n positive integer $a_1, \ldots, a_n$. Does there exist a set $S \subset N = \{1, \ldots, n\}$ such that $\sum_{i \in S} a_i = \sum_{i \in N \setminus S} a_i = A$? (Note that we may assume without loss of generality that $a_i < A$ for $i = 1, \ldots, n$.)

Theorem 7.1 *The ELSR problem with separate setup cost is $\mathcal{NP}$-hard for time-invariant cost parameters.*

Proof Given an instance of PARTITION, define an instance for the ELSR problem as follows:
$T = n$,
$d_t = a_t$ for $t = 1, \ldots, T$,
$r_1 = A$, $r_t = 0$ for $t = 2, \ldots, T$,
$K_t^m = K_t^r = 1$ for $t = 1, \ldots, T$,
$p_t^m = 1$ for $t = 1, \ldots, T$,
$p_t^r = 0$ for $t = 1, \ldots, T$,
$h_t^s = 3$ for $t = 1, \ldots, T$,
$h_t^r = 0$ for $t = 1, \ldots, T$.
Clearly, this reduction can be done in polynomial time. We will show that the answer to PARTITION is positive if and only if the ELSR instance has a solution with cost at most $T + A$.

- if part:
  Assume that we have a solution for the ELSR instance with cost at most $T + A$. First, if serviceables are held in stock, then a solution with strictly less cost can be constructed. Assume that such a period exists. Then there must be a (re)manufacturing period t with at least one item in stock at the end of period t. Decreasing the number of items being (re)manufactured by one in period t and increasing the number of items being (re)manufactured by one in period $t + 1$ will reduce the total cost by at least 1. By repeating the above procedure we get a solution without serviceables in stock and hence we may restrict ourselves to a solution with cost at most $T + A$ where no serviceables are held in stock.

  Furthermore, because at most A items can be remanufactured and all demand has to be satisfied, we incur at least variable cost A for manufactured items and we incur exactly cost A if all returns are remanufactured. Finally, if there is both remanufacturing and manufacturing in at least one period, then the total setup costs will exceed T. Because the total cost is at most $T + A$, demand in each period is satisfied by either manufacturing or remanufacturing and the total amount remanufactured equals A. Therefore, the remanufacturing periods (or the manufacturing periods) form the set S.

- only if part:
  Let S be the set for which $\sum_{i \in S} a_i = \sum_{i \in N \setminus S} a_i = A$. It is easy to verify that by remanufacturing a_t items in each period $t \in S$ and manufacturing a_t items in each period $t \in N \setminus S$ all demand is satisfied and total costs equal $T + A$.

□

Note that the ELSR problem instance in the proof has reasonable assumptions on the cost parameters. As already mentioned in Chapter 6, remanufacturing adds value to an item and hence it is reasonable to assume that holding serviceables is at least as costly as holding returns (i.e., $h_t^s \geq h_t^r$). Furthermore, if remanufacturing is motivated economically, then the assumption that the unit remanufacturing cost equals at most the unit manufacturing cost (i.e., $p_t^m \geq p_t^r$) is also reasonable. Finally, in practice it is likely that the total amount of demand will be larger than the total amount of returns (i.e., $\sum_{t=1}^{T} d_t \geq \sum_{t=1}^{T} r_t$).

Note that the solution for the PARTITION instance and the optimal cost of the ELSR instance are independent of the ordering of $a_1, \ldots, a_n$ (as in the $\mathcal{NP}$-completeness proof for the capacitated lot-sizing problem (Florian et al., 1980)). This shows that the ELSR problem is also $\mathcal{NP}$-hard in the case of increasing (or decreasing) demand over time and time-invariant cost parameters.

In the literature there are some models related to the ELSR model. In the following two sections we will derive alternative proofs, stronger results and new results for those related models.

7.2.2 The ELSR problem with a disposal option

Golany et al. (2001) and Yang et al. (2005) consider the ELSR problem with an additional disposal option (ELSRD for short). If we introduce the additional notation

x_t^d: the number of items disposed of in period t,

K_t^d: setup cost for disposing items in period t,

p_t^d: unit cost for disposing an item in period t,

then the ELSRD problem can be formulated as

$$
\begin{array}{lll}
\text{[ELSRD]} & \min & \sum_{t=1}^{T}(K_t^m \delta(x_t^m) + p_t^m x_t^m + h_t^s I_t^s K_t^r \delta(x_t^r) + p_t^r x_t^r + h_t^r I_t^r \\
& & +K_t^d \delta(x_t^d) + p_t^d x_t^d) \\
& \text{s.t.} & I_t^s = I_{t-1}^s + x_t^m + x_t^r - d_t \qquad t = 1, \ldots, T \\
& & I_t^r = I_{t-1}^r + r_t - x_t^r - x_t^d \qquad t = 1, \ldots, T \\
& & x_t^m, x_t^r, x_t^d, I_t^s, I_t^r \geq 0 \qquad t = 1, \ldots, T \\
& & I_0^s = I_0^r = 0,
\end{array}
$$

where

$$
\delta(y) = \begin{cases} 0 & \text{for } y = 0 \\ 1 & \text{for } y > 0. \end{cases}
$$

Model [ELSRD] is equivalent to model [ELSR] except that there is an additional term $K_t^d\delta(x_t^d) + p_t^d x_t^d$ in the objective function and a term x_t^d in the return inventory balance constraints, which shows that the inventory of returns will decrease if items are disposed of. Golany et al. (2001) and Yang et al. (2005) present a more general model where all cost functions are assumed to be general concave functions instead of (setup-)linear functions. However, for the complexity proofs it is sufficient to consider the formulation with (setup-)linear costs as a (setup-)linear function is a special case of a concave function. Golany et al. (2001) show that the ELSRD problem is $\mathcal{NP}$-hard for non-stationary cost parameters and Yang et al. (2005) improve this result by showing that it is also $\mathcal{NP}$-hard for the time-invariant cost case.

Yang et al. (2005) argue that there are some similarities between the ELSRD problem and the capacitated lot-sizing (CLS) problem. In this section we will show that every CLS problem is equivalent to a special case of the ELSRD without manufacturing. As the CLS problem is $\mathcal{NP}$-hard (Florian et al., 1980), this immediately implies that the ELSRD problem is $\mathcal{NP}$-hard.

The CLS problem is equivalent to the ELS problem except that there is a production capacity in each period. If we let

C_t: the production capacity in period t,

and we use the same notation as for the ELS problem, then the CLS problem can be formulated as

$$\begin{array}{lll} [\mathrm{CLS}] & \min & \sum_{t=1}^{T} (K_t\delta(x_t) + p_t x_t + h_t I_t) & \\ & \text{s.t.} & I_t = I_{t-1} + x_t - d_t & t = 1,\dots,T \\ & & x_t + u_t = C_t & t = 1,\dots,T \\ & & x_t, u_t, I_t \geq 0 & t = 1,\dots,T \\ & & I_0 = 0, & \end{array}$$

where u_t is the quantity of unused capacity in period t.

Consider the special case of the ELSRD problem with sufficient returns to satisfy demand, no manufacturing, no option to hold returns in stock and no disposal cost. The latter is forced when we impose the conditions $R_{1,t} \geq D_{1,t}$, $K_t^m = h_t^r = \infty$ and $K_t^d = p_t^d = 0$ for $t = 1,\dots,T$ on model [ELSRD]. In an optimal solution for this special case no manufacturing will occur ($x_t^m = 0$) and no returns will be held in stock ($I_t^r = 0$). Hence, the model can be rewritten to

$$\begin{array}{llll} [\mathrm{ELSRD}'] & \min & \sum_{t=1}^{T} (K_t^r\delta(x_t^r) + p_t^r x_t^r + h_t^s I_t^s) & \\ & \text{s.t.} & I_t^s = I_{t-1}^s + x_t^r - d_t & t = 1,\dots,T \\ & & x_t^r + x_t^d = r_t & t = 1,\dots,T \\ & & x_t^r, x_t^d, I_t^s \geq 0 & t = 1,\dots,T \\ & & I_0^s = 0. & \end{array}$$

It follows immediately from the mathematical formulation that models [CLS] and [ELSRD′] are equivalent. That is, producing, holding items and not using capacity in [CLS] correspond to remanufacturing, holding serviceables and disposing in [ELSRD′], respectively. Therefore, we have the following theorem.

Theorem 7.2 *The ELSRD problem is a generalization of the CLS problem.*

Bitran and Yanasse (1982) show that the CLS problem with time-invariant cost parameters (among other families of problem instances) is $\mathcal{NP}$-hard. This implies that the ELSRD problem with time-invariant cost is $\mathcal{NP}$-hard. This means that we have an alternative proof for the complexity result in Yang et al. (2005). Because the ELSRD problem is a generalization of the ELSR problem and the ELSR problem with time-invariant cost is $\mathcal{NP}$-hard, the result is also immediately obtained from Theorem 7.1. Thus, the following corollary follows both from Theorem 7.1 and Theorem 7.2.

Corollary 7.3 *The ELSRD problem with time-invariant cost is $\mathcal{NP}$-hard.*

7.2.3 The ELSRD problem with fixed ending inventories

Yang et al. (2005) actually consider a model which is slightly different from model [ELSDR]. They assume that the ending inventories of returns and serviceables (I_T^r and I_T^s) are not decision variables but fixed parameters. They show that every T-period ELSRD problem with variable ending inventories can be converted into a $(T+1)$-period ELSRD problem with fixed ending inventories equal to zero. This is done by constructing an artificial period $T+1$ with demand $d_{T+1} = R_{1,T}$ (note that there will never be more than $R_{1,T}$ serviceables in stock if the costs are positive), returns $r_{T+1} = 0$, zero manufacturing and disposal cost ($K_{T+1}^m = p_{T+1}^m = K_{T+1}^d = p_{T+1}^d = 0$) and infinitely large remanufacturing cost ($K_{T+1}^r = p_{T+1}^r = \infty$). For this modified instance, returns left in stock in period T can be disposed of at zero cost in period $T+1$ and demand not satisfied by serviceables in stock at the end of period T can be manufactured at zero cost in period $T+1$. This means that there exists an optimal solution with zero cost in period $T+1$ and zero stock left in period $T+1$. Moreover, this solution also gives an optimal solution for the T-period problem with variable ending inventories. Therefore, Yang et al. (2005) restrict themselves to the problem with fixed ending inventories.

We can perform a similar 'trick' to the ELSR problem (without disposals). We add an additional period $T+1$ with the same parameters as above except that we set the remanufacturing cost to zero, i.e., $K_{T+1}^r = p_{T+1}^r = 0$. Again, solving the $(T+1)$-period problem with the restriction $I_{T+1}^r = I_{T+1}^s = 0$ gives an optimal solution for the T-period problem with variable ending inventories. In the remainder of this section we will consider the complexity of the ELSR problem with zero ending inventories (denoted by ELSR_0).

We will show that we can derive stronger complexity results for the ELSR$_0$ problem than for the ELSR problem. Note that if the T-period problem has time-invariant cost, then this property does not necessarily hold anymore for the converted $(T+1)$-period instance. Therefore, complexity results derived for the ELSR problem with time-invariant cost do not necessarily hold for the ELSR$_0$ problem with time-invariant cost. The following theorem shows that the ELSR$_0$ problem is $\mathcal{NP}$-hard.

Theorem 7.4 *The ELSR$_0$ problem with time-invariant cost and zero unit manufacturing, remanufacturing and holding cost for returns is $\mathcal{NP}$-hard.*

Proof We will use a reduction from problem PARTITION. Define a problem instance for the ELSR$_0$ problem as follows:

$T = n$,
$d_t = a_t$ for $t = 1, \ldots, T$,
$r_1 = A$, $r_t = 0$ for $t = 2, \ldots, T$,
$K^m = K^r = 1$,
$h^s = 2$,
$p^m = p^r = h^r = 0$.

We will show that the answer to PARTITION is yes if and only if the ELSR$_0$ problem has a solution with cost at most T. First, the cost to satisfy demand d_t is at least 1, because this demand is satisfied by manufacturing, remanufacturing or serviceables in stock. More precisely, we have a solution with cost at most T if in each period demand is satisfied by either pure manufacturing or pure remanufacturing (and not by serviceables in stock). Because the stock of returns must be zero at the end of period T, there exists a solution with cost at most T if and only there exists a subset of demand periods for which the sum of demands equals the returns in period 1, i.e., there exists an $S \subset N = \{1, \ldots, n\}$ such that $\sum_{i \in S} a_i = A$. □

The problem instance in the proof has a trivial optimal solution with cost T for the ELSR problem with variable ending inventories. Namely, manufacturing an amount of d_t in each period $t = 1, \ldots, T$ is a solution with a manufacturing setup cost of 1 in each period and hence total cost T. In the following theorem we derive a stronger result for the ELSR$_0$ problem.

Theorem 7.5 *Unless $\mathcal{P} = \mathcal{NP}$, there does not exist a polynomial time ε-approximation algorithm with $\varepsilon < \frac{1}{T}$ for the ELSR$_0$ problem.*

Proof Again consider the problem instance of the proof of Theorem 7.4. First, we construct a simple polynomial time heuristic for such instances. Let p be the (unique) period that satisfies $\sum_{i=1}^{p-1} a_i < A \leq \sum_{i=1}^{p} a_i$. Then $x_t^r = a_t$ $(t = 1, \ldots, p-1)$, $x_p^r =$

$A - \sum_{i=1}^{p-1} a_i$, $x_p^m = a_p - x_p^r$ and $x_t^m = a_t$ $(t = p+1, \ldots, T)$ is a feasible solution with cost at most $T+1$ (the cost equals T if $\sum_{i=1}^{p} a_i = A$). So the cost of the heuristic solution C^H satisfies $C^H \leq T+1$. Furthermore, from the the proof of Theorem 7.4 and the simple heuristic it follows that the cost of the optimal solution C^* satisfies $C^* \in \{T, T+1\}$. This means that

$$\frac{C^H - C^*}{C^*} \leq \frac{1}{T}$$

and hence the simple heuristic is an $\frac{1}{T}$-approximation algorithm for this class of problem instances.

Now assume that we have a polynomial time ε-approximation algorithm with $\varepsilon < \frac{1}{T}$ and denote the cost of a solution by C^A. Consider the following cases:

- $C^* = T$:
  The solution found by the ε-approximation algorithm satisfies
  $$C^A \leq (1+\varepsilon)C^* = (1+\varepsilon)T < (1+\frac{1}{T})T = T+1.$$
  Because the cost must be integral, this implies that $C^A = T = C^*$ and hence we have found the optimal solution in polynomial time.

- $C^* = T+1$:
  In this case the ε-approximation algorithm finds a solution with $C^A \geq T+1$ and hence we know that $C^* > T$ implying $C^* = T+1$. Now the optimal solution can be found in polynomial time by applying the simple heuristic.

From both cases it follows that given a polynomial time ε-approximation algorithm with $\varepsilon < \frac{1}{T}$ we can solve the ELSR$_0$ instance in polynomial time. But then problem PARTITION can also be solved in polynomial time, which would imply that $\mathcal{P} = \mathcal{NP}$. □

The following corollary follows from Theorem 7.5.

Corollary 7.6 *Unless $\mathcal{P} = \mathcal{NP}$, there does not exist a fully polynomial time approximation scheme (FPTAS) for the ELSR$_0$ problem.*

Proof Assume there exists a FPTAS that runs in $\mathcal{O}(p(|I|)\frac{1}{\varepsilon^n})$ time, where $p(x)$ is a polynomial, $|I|$ the size of the problem instance and n some fixed integer. Consider the problem instance in the proof of Theorem 7.4 and let $\varepsilon = \frac{1}{2T}$. It follows from the proof of Theorem 7.5 that for $\varepsilon = \frac{1}{2T}$ the problem instance is solved to optimality. But then the problem instance is solved in $\mathcal{O}(p(|I|)T^n)$ which would imply that problem PARTITION can be solved in polynomial time. This is a contradiction, unless $\mathcal{P} = \mathcal{NP}$. □

Because the ELSR$_0$ problem is a special case of the ELRSD problem with fixed ending inventories, Theorem 7.5 and Corollary 7.6 also hold for this problem. These results are stronger than the results in Yang et al. (2005).

7.3 Properties of an optimal solution

Although the ELSR problem is $\mathcal{NP}$-hard, we can find some properties of an optimal solution. To describe these properties we will need some definitions. As in Chapter 6 we will use the notion of regeneration periods. As we have two types of inventory, we also have two types of regeneration periods. We call period t a serviceable regeneration period if $I_t^s = 0$ and (u, v) is called a serviceable regeneration interval if $I_{u-1}^s = I_v^s = 0$ and $I_t^s > 0$ for $t \in [u, v-1]$ (we use $[s, t]$ as a shorthand notation for the set $\{s, \dots, t\}$). Similarly, period t is called a return regeneration period if $I_t^r = 0$ and (u, v) is called a return regeneration interval if $I_{u-1}^r = I_v^r = 0$, $I_t^s > 0$ and $x_v^r > 0$ for $t \in [u, v-1]$ (we add the condition $x_v^r > 0$ to exclude a series of zero return periods to be a regeneration interval).

Furthermore, let u and v $(u \leq v)$ be consecutive production periods with u a remanufacturing period and v a manufacturing period (or opposite). We call (u, v) a connected pair if $I_t^s > 0$ for $t \in [u, v-1]$. In case u is a manufacturing period and v a remanufacturing period, we will call (u, v) a MR connected pair and in case u is a remanufacturing period and v a manufacturing period, we will call (u, v) a RM connected pair. From the definition above it follows that if t is both a manufacturing and remanufacturing period, then (t, t) is also a connected pair. We will call this special case a connected period.

If we use the network formulation of Section 6.1 and the results of Zangwill (1968), then the following lemmas can be derived.

Lemma 7.7 *Let (s, t) be a serviceable regeneration interval in an optimal solution. Then there exists an optimal solution with at most one manufacturing period between periods s and t.*

Lemma 7.8 *Let (s, t) be a return regeneration interval in an optimal solution. Then there exists an optimal solution with at most one connected pair between periods s and t.*

Lemma 7.9 *Let s be the last remanufacturing period that satisfies $I_s^r = 0$ ($s = 0$ if this period does not exist) and let t be the first remanufacturing period after s (if any). Then there exists an optimal solution for which periods $[t, T]$ are not part of a connected pair.*

If we have an ELSR problem with no returns ($r_t = 0$ for $t = 1, \dots, T$), then Lemma 7.7 is equivalent to the ZI property of the classical ELS problem. The last lemma says that the periods after the last return regeneration period satisfy the ZI property for serviceables (there is no period with both serviceables at the beginning of the period and production). We will call this a ZIS interval.

Again, for ease of notation we rewrite model [ELSR] as a model without holding cost by redefining the unit cost for manufacturing and remanufacturing as $c_t^m = p_t^m + \sum_{i=t}^{T} h_i^s$ and

$c_t^r = p_t^r + \sum_{i=t}^{T}(h_i^s - h_i^r)$. Furthermore, for the sake of simplicity we make the assumption that $c_t^r \geq c_{t+1}^r$ for $t = 1, \ldots, T-1$. (We will come back on this assumption later.) For example, this assumption holds when the unit remanufacturing cost is time-invariant and the unit holding cost for returns equals at most the unit holding cost for serviceables, i.e., $p_t^r = p^r$ and $h_t^r \leq h_t^s$ for $t = 1, \ldots, T$. From this assumption we can derive the following lemma.

Lemma 7.10 *Let (s,t) be a serviceable regeneration interval in an optimal solution of an ELSR problem instance with $c_t^r \geq c_{t+1}^r$. Then there exists an optimal solution with at most one remanufacturing period between periods s and t.*

Proof Suppose that p and q are remanufacturing periods such that $s \leq p < q \leq t$. Then by decreasing remanufacturing in period p and increasing remanufacturing in period q by a sufficiently small quantity a feasible solution with lower cost is obtained. $\square$

Using Lemmas 7.7, 7.8 and 7.10 and the representation of a solution by the arcs with positive flow in the network flow formulation, a typical candidate return regeneration interval looks like the graph in Figure 7.1. We see in Figure 7.1 that there is at most one

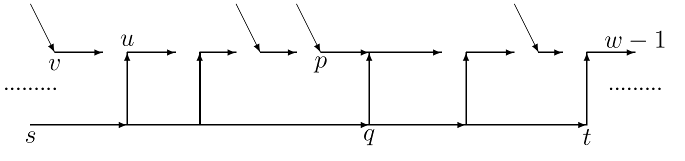


Figure 7.1: Candidate optimal return regeneration interval

manufacturing period or remanufacturing period in every serviceable regeneration interval and there is only one connected pair (periods p and q) in the return regeneration interval. This means that this candidate return regeneration interval satisfies the properties stated in Lemmas 7.7, 7.8 and 7.10.

7.4 A dynamic programming algorithm for the ELSR problem with given production periods

7.4.1 Introduction

Consider the ELSR problem for which only the manufacturing and remanufacturing periods (the periods with strictly positive production) are given and the manufacturing and remanufacturing quantities have to be determined such that the total costs are minimized (we denote this problem by ELSRG). In Section 7.5 we will use the solution to this problem to develop a genetic algorithm for the ELSR problem. For the ELS problem (which is equivalent to the ELSR problem with no returns), the problem with given production periods is trivial as, by the ZI property, the production quantity in any production period covers the demands up to the next production period. The following example shows that there is no trivial solution for determining the production quantities in the ELSRG problem.

Example 7.11 Consider a three period problem instance with demand and returns according to Table 7.1. Assume that periods 1 and 3 are manufacturing periods and

t	1	2	3
d_t	10	10	10
r_t	6	6	0

Table 7.1: Problem instance of Example 7.11

period 2 is a remanufacturing period. In case of the time-invariant cost parameters $K^m = K^r = 10$, $h^s = 2$, $p^m = h^r = 1$ and $p^r = 0$, the (unique) optimal solution is $x_1^m = 10$, $x_2^r = 12$ and $x_3^m = 8$ with a total cost of 58. However, in case $h^s = 4$ (and the other cost parameters remain the same), the (unique) optimal solution is $x_1^m = x_2^r = x_3^m = 10$ with a total cost of 60. So in the first case all available returns are remanufactured, while in the second case only sufficient returns are remanufactured to satisfy demand in period 2. It follows from this example that the optimal production quantities depend on the specific cost parameters.

From model [ELSR] it follows immediately that for given production periods, the remaining problem is a linear programming (LP) problem. Hence, the ELRSG problem can be simply solved by any LP solver. However, in this section we will derive a dynamic programming (DP) to solve this problem. There are three reasons to prefer a DP algorithm over using an LP solver. First, a dedicated algorithm is in general faster than a general LP solver. Second, for the development of a DP algorithm structural properties of an optimal solution are often needed, which gives more insight in the problem. Third, no (expensive) commercial software (such as Cplex) is needed.

In the development of the DP algorithm we will use the properties of an optimal solution derived in the previous section. From Lemmas 7.7-7.10 the optimal solution consists of a series of return regeneration intervals, possibly followed by a ZIS interval. In the remainder of this section we will first identify the different types of return regeneration intervals, where the types depend on the periods for which demand is satisfied. Subsequently, we will show how to calculate the costs for these intervals and for the ZIS intervals. Finally, we will give the recursion formulas of the DP algorithm.

7.4.2 The different types of regeneration intervals

First, we will start with identifying the different types of return regeneration intervals. Therefore, we will need some additional notation:

P^M: the set of manufacturing periods,

P^R: the set of remanufacturing periods including an artificial remanufacturing period 0,

$P^R(t)$: the remanufacturing periods before period t, i.e., $P^R(t) = \{s \in P^R : s < t\}$,

$\sigma^R(t)$: the remanufacturing period succeeding period t or period t itself if it is a remanufacturing period ($T+1$ if the period does not exist),

$\sigma(t)$: the production period succeeding period t or period t itself if it is a production period ($T+1$ if the period does not exist).

By adding the artificial remanufacturing period 0, period 1 is by definition a return regeneration period. Furthermore, we add a subscript "+" to the definitions $\sigma^R(t)$, $\sigma(t)$ when we exclude period t. (So $\sigma^R(t) = \min\{s \in P^R : s \geq t\}$ and $\sigma^R_+(t) = \min\{s \in P^R : s > t\}$.)

Given the set P^R, we will now identify different types of candidate regeneration intervals. Let r and t ($r < t$) be two remanufacturing periods. Then (s,t) with $s = r+1$ is a candidate return regeneration interval. By definition of a return regeneration interval, the returns in periods $[s,t]$ ($R_{s,t}$) are completely remanufactured to satisfy demand. To calculate the cost corresponding to a return regeneration interval, we have to determine which demands are satisfied by remanufacturing and by manufacturing. Furthermore, to construct a feasible (forward) DP algorithm, we have to include the demands that are satisfied by manufacturing in the return regeneration intervals in such a way that a series of return regeneration intervals covers a consecutive series of demands (starting at period 1). Therefore, we define the following periods corresponding to a return regeneration interval (s,t) (see also Figure 7.1):

u: the next remanufacturing period after period s, i.e., $u = \sigma^R(s)$,

v: the next production period after period s, i.e., $v = \sigma(s)$,

w: the next production period after t, i.e., $w = \sigma_+(t)$.

Note that for given remanufacturing periods s and t, the periods u, v and w are uniquely specified and satisfy $s \leq v \leq u \leq t < w$. For notational convenience we do not show the dependency of the periods u, v and w on periods s and t. We define the following variables corresponding to the different types of return regeneration intervals, which will be used in the DP algorithm:

$f^{00}_{s,t}$: minimum cost for regeneration interval (s,t), where demands in periods $[v, w-1]$ are satisfied and $I^s_{v-1} = I^s_{w-1} = 0$ (∞ if infeasible),

$f^{01}_{s,t}$: minimum cost for regeneration interval (s,t), where demands in periods $[v, \sigma_+(w)-1]$ are satisfied and $I^s_{v-1} = 0$ and $I^s_w > 0$ (∞ if infeasible),

$f^{10}_{s,t}$: minimum cost for regeneration interval (s,t), where demands in periods $[\sigma_+(v), w-1]$ are satisfied and $I^s_{v-1} > 0$ and $I^s_{w-1} = 0$ (∞ if infeasible),

$f^{11}_{s,t}$: minimum cost for regeneration interval (s,t), where demands in periods $[\sigma_+(v), \sigma_+(w)-1]$ are satisfied and $I^s_{v-1} > 0$ and $I^s_w > 0$ (∞ if infeasible).

From the definitions it follows that we have four types (00, 01, 10 and 11) of return regeneration intervals. Note that the candidate regeneration interval of Figure 7.1 is of type 00. From the definition of type 00 it follows that the cost of manufacturing between periods s and u (if any) are also included. Furthermore, for a type 00 regeneration interval period v is the start of a serviceable regeneration interval and period $w-1$ is the end of a serviceable regeneration interval. By Lemma 7.8 it follows that there is at most one connected pair within periods $[v, w-1]$.

The place where the connected pair occurs in the (previous) regeneration interval determines the type of regeneration interval. In Figure 7.2 all types of regeneration intervals are illustrated. Note that for a given regeneration interval (s,t) not all types may be defined. For example, type 01 does not exist if the production period after w is a remanufacturing period. In this case we set the cost of such a regeneration interval to infinity.

Furthermore, given the type of regeneration interval, the demand periods to be covered are known. This information can be used to determine whether there is an RM or MR connection. Assume for the moment that the regeneration interval satisfies the ZI property for serviceables. Then the total quantity of items being remanufactured to satisfy demand equals

$$x = \sum_{i \in P^R : s \leq i \leq t} D_{i,\sigma_+(i)-1}. \tag{7.1}$$

If $R_{s,t} > x$, then there are some returns left that can be used to (partially) satisfy demand periods which are satisfied by manufacturing. This means that we must have a RM

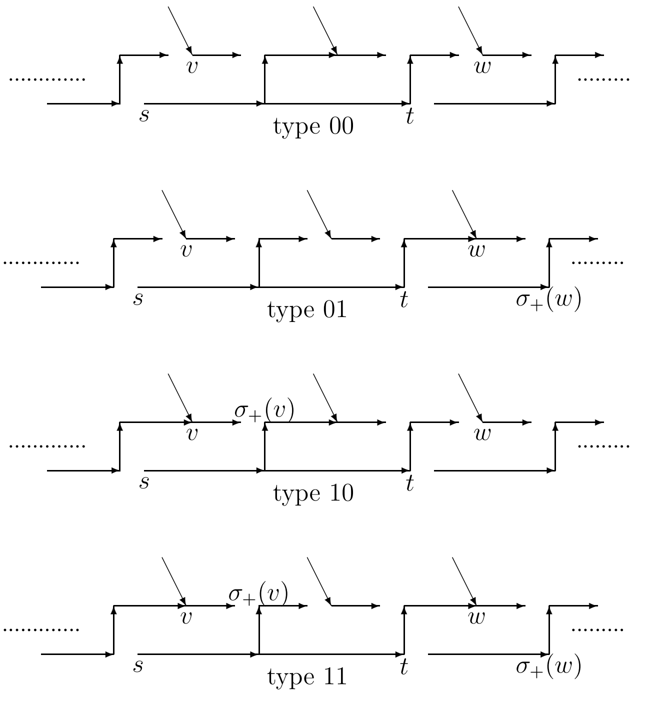


Figure 7.2: The four types of regeneration intervals

connection. Note that by this observation a double period can be considered as a special case of a RM connection. If $R_{s,t} < x$, then there are insufficient returns to satisfy x units of demand by remanufacturing and hence we must have a MR connection. In case $R_{s,t} = x$ we have a regeneration interval that satisfies the ZI property for serviceables. This case can be considered as a special case of both an RM and an MR connection.

7.4.3 Calculating the cost for different types of regeneration intervals

Once the type of connection is known, the connected pair with the lowest cost has to be determined. For example, in Figure 7.1 there are three candidate MR connections: all remanufacturing periods immediately preceded by a manufacturing period. We will show how to calculate the cost for a type 00 regeneration interval. The costs for the other types can be calculated in a similar way. The following lemma shows how to find the connected pair with the lowest cost.

Lemma 7.12 *Consider an optimal solution where* (i, j) *is a connected pair in a return regeneration interval* (s, t) *with an MR connection. Then* $c_i^m - c_j^r \leq c_p^m - c_q^r$ *for all candidate connected pairs* (p, q)*.*

Proof Assume there is a connected pair (p, q) with $c_p^m - c_q^r < c_i^m - c_j^r$ and $q < i$ (the case with $j < p$ can be proven analogously). Increasing manufacturing in period p, decreasing remanufacturing in period q, decreasing manufacturing in period i and increasing remanufacturing in period j by $\varepsilon > 0$ units (see Figure 7.3) leads to a cost change of $\varepsilon((c_p^m - c_q^r) - (c_i^m - c_j^r)) < 0$, which is a strict cost reduction. We can choose ε as large as

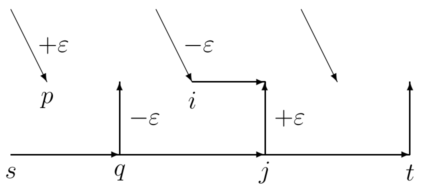


Figure 7.3: Illustration of the proof of Lemma 7.12

possible such that either (i) $I_{j-1}^s = 0$, (ii) $I_j^r = 0$, (iii) $x_q^r = 0$ or (iv) $x_i^m = 0$. In all cases we have found a feasible solution with strictly less cost, which contradicts the optimality. Note that in case (i) we have found a better solution where (i, j) is no connected pair. In case (ii) there is a better solution where regeneration interval (s, t) is split into two regeneration intervals (s, j) and $(j + 1, t)$. In cases (iii) and (iv) it turns out that either remanufacturing in period q or manufacturing in period i is not needed and so we can save an additional setup cost. $\square$

In a similar way we can prove the following lemma.

Lemma 7.13 *Consider an optimal solution where* (i, j) *is a connected pair in a return regeneration interval* (s, t) *with an RM connection. Then* $c_i^r - c_j^m \leq c_p^r - c_q^m$ *for all candidate connected pairs* (p, q)*.*

Given a type of regeneration interval, the type of connection and the optimal connected pair, the manufacturing and remanufacturing quantities can be determined. Assume that we have a regeneration interval (s, t) with a MR connection, (i, j) is the optimal connected pair and demand for periods $[v, w - 1]$ have to be satisfied. Then in each manufacturing or remanufacturing period not included in the connected pair production occurs for an integral number of periods up to the next production period as these periods satisfy the ZI property for serviceables. Furthermore, in period i, besides manufacturing for an integral number of periods up to remanufacturing period j, manufacturing an additional quantity $y = x - R_{s,t}$ (with x defined in (7.1)) is necessary to compensate for the shortage of returns and we can remanufacture y items less in period j. So the production quantities become

$$x_t^m = \begin{cases} D_{t,\sigma_+(t)-1} & \text{if } t \in P^M \backslash \{i\} \cap [v, w-1] \\ D_{t,\sigma_+(t)-1} + y & \text{if } t = i, \end{cases}$$

$$x_t^r = \begin{cases} D_{t,\sigma_+(t)-1} & \text{if } t \in P^R \backslash \{j\} \cap [v, w-1] \\ D_{t,\sigma_+(t)-1} - y & \text{if } t = j. \end{cases}$$

Using similar arguments, the production quantities for a RM connection with a connected pair (i, j) are

$$x_t^m = \begin{cases} D_{t,\sigma_+(t)-1} & \text{if } t \in P^M \backslash \{j\} \cap [v, w-1] \\ D_{t,\sigma_+(t)-1} - y & \text{if } t = j, \end{cases}$$

$$x_t^r = \begin{cases} D_{t,\sigma_+(t)-1} & \text{if } t \in P^R \backslash \{i\} \cap [v, w-1] \\ D_{t,\sigma_+(t)-1} + y & \text{if } t = i \end{cases}$$

with $y = R_{s,t} - x$.

Given a type of regeneration interval, the type of connection, the optimal connected pair and the manufacturing and remanufacturing quantities, the optimal cost for a regeneration interval can now be calculated. However, first we have to check whether we have a feasible solution for the regeneration interval. That is, we have to check whether $x_t^m \geq 0$, $x_t^r \geq 0$, $I_t^r \geq 0$ and $I_t^s \geq 0$ for $t \in [v, w-1]$. By allowing that $I_t^r \geq 0$ (instead of $I_t^r > 0$ according to the definition of a return regeneration interval), it may be that a regeneration interval consists of multiple regeneration intervals where at most one regeneration interval has a connected pair and the other regeneration intervals satisfy the ZI property for serviceables. Therefore, we do not need additional variables representing the costs of return regeneration intervals satisfying the ZI property for serviceables. If no feasible solution exists, we set the cost to infinity. Otherwise the cost is calculated by

$$f_{s,t}^{00} = \sum_{i=s}^{t} \left(K_t^m \delta(x_t^m) + c_t^m x_t^m + K_t^r \delta(x_t^r) + c_t^r x_t^r \right).$$

For a given regeneration interval it takes $\mathcal{O}(T)$ time to find the optimal connected pair, to perform the feasibility check and to calculate the cost. As there are $\mathcal{O}(T^2)$ possible regeneration intervals, the computation time for determining all values $f_{s,t}^{\alpha,\beta}$ ($s, t \in P^R$ and $\alpha, \beta \in \{0, 1\}$) takes $\mathcal{O}(T^3)$ time.

We end this section with some remarks. If we have a regeneration interval with a RM (MR) connection and there is no manufacturing period immediately preceding (succeeding) a remanufacturing period, then we set the cost to infinity. Furthermore, if we have a regeneration interval of type 01 or 11, then the connected pair is immediately known and does not have to be searched for. Similarly, if the regeneration interval includes a double period, then it is also not necessary to search for the connected pair and the type 01 and 11 regeneration interval do not exist as a return regeneration interval has at most one connected pair. Finally, we come back to the assumption that $c_t^r \geq c_{t+1}^r$. By this property it followed from Lemma 7.10 that for any two consecutive remanufacturing periods p and q it always holds that $I_{q-1}^r = 0$ in an optimal solution. If we relax this assumption we either have $I_p^r = 0$ or $I_{q-1}^r = 0$ (see also Section 6.3). This case can be easily incorporated into the DP algorithm. However, to avoid additional notation and to keep things concise we have not included this case.

7.4.4 The interval that satisfies the ZI property for serviceables

By Lemma 7.9 we know that an optimal solution may end with an ZIS interval. In this section we show how to calculate the cost for such an interval. Let $s \in P^R$, $t = s + 1$, $v = \sigma(t)$ and

g_t^0: minimum cost for ZIS interval where returns in $[t, T]$ are used to satisfy demand in periods $[v, T]$ (∞ if infeasible),

g_t^1: minimum cost for ZIS interval where returns in $[t, T]$ are used to satisfy demand in periods $[\sigma_+(v), T]$ (∞ if infeasible).

Again we need two types of intervals as an ZIS interval may be preceded by either a type 00 or 10 regeneration interval or a type 01 or 11 regeneration interval (see also Section 7.4.5). For a type 0 ZIS interval the production quantities are determined by

$$\begin{aligned} x_i^m &= D_{i,\sigma+(i)-1} \quad \text{if } i \in P^M \cap [v, T], \\ x_i^r &= D_{i,\sigma+(i)-1} \quad \text{if } i \in P^R \cap [v, T], \end{aligned}$$

because (by definition) the interval satisfies the ZI for serviceables. After checking feasibility the cost is calculated by

$$g_t^0 = \sum_{i=v}^{T} \left(K_t^m \delta(x_t^m) + c_t^m x_t^m + K_t^r \delta(x_t^r) + c_t^r x_t^r \right).$$

In a similar way the production quantities and cost for a type 1 ZIS interval can be calculated. (Note that $g_t^0 = g_t^1 = 0$ if $v = T + 1$.) For a given $t = s + 1$ with $s \in P^R$ it takes $\mathcal{O}(T)$ time to calculate the cost and to perform the feasibility check. As there are $\mathcal{O}(T)$ possible values of t, it takes $\mathcal{O}(T^2)$ time to calculate all values g_t^α ($t = s + 1$ with $s \in P^R$ and $\alpha \in \{0, 1\}$).

7.4.5 The recursion formulas

From Lemmas 7.7-7.9 it follows that an optimal solution consists of a series of regeneration intervals (possibly followed by a ZIS interval). Therefore, we define the variables f_t^α with $t \in P^R$ and $\alpha \in \{0,1\}$ as follows:

f_t^0: minimum cost to satisfy demand in periods $[1, w-1]$ with $w = \sigma_+(t)$,

f_t^1: minimum cost to satisfy demand in periods $[1, \sigma_+(w)-1]$ with $w = \sigma_+(t)$.

So f_t^0 is the cost of a partial solution consisting of a series of return regeneration intervals ending with a type 00 or 10 regeneration interval and f_t^1 is the cost of a partial solution ending with a type 01 or 11 regeneration interval. Starting with $f_0^0 = 0$ and $f_0^1 = \infty$ the values of f_t^α can be calculated by

$$f_t^\alpha = \min_{s \in P^R(t)} \left\{ \min_{\beta \in \{0,1\}} \left\{ f_s^\beta + f_{s+1,t}^{\beta,\alpha} \right\} \right\} \text{ for all } t \in P^R \backslash \{0\}, \alpha \in \{0,1\}. \tag{7.2}$$

Note that a type 0 (1) partial solution must be followed by a type 00 or 01 (10 or 11) regeneration interval. Because the series of regeneration intervals may be followed by a ZIS interval the optimal cost f^* is found by

$$f^* = \min_{s \in P^R} \left\{ \min_{\alpha \in \{0,1\}} \left\{ f_s^\alpha + g_{s+1}^\alpha \right\} \right\}. \tag{7.3}$$

Again a type 0 (1) partial solution has to be followed by a type 0 (1) ZIS interval. It is not difficult to see from (7.2) and (7.3) that it takes $\mathcal{O}(T^2)$ and $\mathcal{O}(T)$ and time to calculate the values f_t^α ($t \in P^R$, $\alpha \in \{0,1\}$) and f^*, respectively. Furthermore, as it takes $\mathcal{O}(T^3)$ and $\mathcal{O}(T^2)$ to calculate $f_{s+1,t}^{\beta,\alpha}$ and g_{s+1}^α, respectively (see Sections 7.4.3 and 7.4.4), we have the following theorem.

Theorem 7.14 *The ELSRG problem can be solved in $\mathcal{O}(T^3)$ time.*

7.5 Application of the DP algorithm to a genetic algorithm

In Section 7.2 we have shown that the ELSR problem is $\mathcal{NP}$-hard and hence it is very unlikely that there exists an efficient algorithm to solve the problem. In this section we will propose a genetic algorithm (GA) to find good solutions. First, we will describe the algorithm and then we will present some numerical results. We propose a straightforward GA such that only a few parameters have to be specified. In this way we try to avoid the need to 'fine-tune' the algorithm in such a way that it performs well on our specific data set.

7.5.1 Description of the GA algorithm

Representation

We represent a candidate solution by two binary strings (y_t^m, y_t^r) $(t = 1, \ldots, T)$, where $y_t^m = 1$ $(y_t^r = 1)$ if period t is a manufacturing (remanufacturing) period and zero otherwise. From Section 7.4 it follows that this is not a complete description of a solution, because in general there are multiple feasible production quantities given a set of production periods. We use the DP algorithm of Section 7.4.5 to find the production quantities with minimum cost. Hence, there is a cost (fitness value) associated with each candidate solution.

Reproduction and Selection

In the algorithm we only use crossover (recombination) and no mutation. Assume that we have two candidate solutions (y^{m1}, y^{r1}) and (y^{m2}, y^{r2}) and we want to construct a new candidate solution (y^{m3}, y^{r3}). Table 7.2 shows the simple crossover rule that is used for recombining two strings y^{m1} and y^{m2}. The same rule holds for the recombination of two

$$\begin{array}{llcl}
y_t^{m1} = 0, & y_t^{m2} = 0 & \Rightarrow & y_t^{m3} = 0 \\
y_t^{m1} = 0, & y_t^{m2} = 1 & \Rightarrow & y_t^{m3} = \begin{cases} 0 & \text{with probability } \frac{1}{2} \\ 1 & \text{with probability } \frac{1}{2} \end{cases} \\
y_t^{m1} = 1, & y_t^{m2} = 0 & \Rightarrow & y_t^{m3} = \begin{cases} 0 & \text{with probability } \frac{1}{2} \\ 1 & \text{with probability } \frac{1}{2} \end{cases} \\
y_t^{m1} = 1, & y_t^{m2} = 1 & \Rightarrow & y_t^{m3} = 1
\end{array}$$

Table 7.2: Crossover rule of the GA

strings y^{r1} and y^{r2}. In each iteration of the GA algorithm we recombine all candidate solutions. Thus, if we have a start pool of n solutions, then we have $\frac{1}{2}n(n-1)$ additional solutions. From the total of $n+\frac{1}{2}n(n-1)$ we select the best (in terms of cost) n solutions.

Initialization

Assume we start with a pool of n solutions. First, we construct $n-2$ start solutions as follows. Let i be the index of a solution with $i \in \{1, \ldots, n-2\}$. This solution gets $n_i^P = \left\lceil \frac{2Ti}{n-2} \right\rceil$ production periods, implying that the number of production periods are equally distributed over the $n-2$ solutions. We assign $n_i^M = \lceil n_i^P/2 \rceil$ manufacturing periods and $n_i^R = \lfloor n_i^P/2 \rfloor$ remanufacturing periods (so that $n_i^M + n_i^R = n_i^P$) to solution i, where the manufacturing and remanufacturing are randomly and uniformly distributed over the T periods. Furthermore, we make two additional solutions: one with only manufacturing periods ($y_t^m = 1$ and $y_t^r = 0$ for $t = 1, \ldots, T$) and one with with only remanufacturing periods ($y_t^m = 0$ $y_t^r = 1$ for $t = 1, \ldots, T$). With this two solutions included in the start pool all possible solutions may be generated after recombination.

Termination

The GA algorithm is terminated if either one of the following two stopping criteria is satisfied: (i) a fixed number of iterations m is reached, (ii) the cost (fitness value) of the best n solutions are equal. In each of the at most m iterations we have to calculate the cost of $\frac{1}{2}n(n-1)$ solutions by the (T^3) DP algorithm. Therefore, the total running time of the GA is $\mathcal{O}(mn^2T^3)$.

7.5.2 Numerical results

We have tested the GA algorithm on the same data set as used in Teunter et al. (2006), so that a comparison can be made with the heuristics presented in that paper. Four different types of demand and return patterns are considered: stationary, linearly increasing, linearly decreasing and seasonal. The return ratio, i.e., the mean return rate as a percentage of the mean demand rate, is set to either 30%, 50%, or 70%. The total number of demand and return patterns considered are 10 and 22, respectively. For each pattern, four series of realizations are generated, so that the total number of demand and return series are 40 and 88, respectively.

All costs are time-invariant and unit manufacturing and remanufacturing are set to zero. The serviceable holding cost per period is normalized at 1. The remanufacturing holding cost is relatively small (0.2), moderate (0.5), or large (0.8). For both the manufacturing and remanufacturing setup costs, 3 values are considered. We remark that based on some preliminary investigations, these cost values are chosen such that during the planning horizon, which is fixed at 12 periods, the number of periods with a setup for the optimal solution varies between 2 and 6. The details on the demand and return patterns, and on the cost parameter values are summarized in Table 7.3. A full factorial design is applied, so that the total number of problem instances is $40 \times 88 \times 3 \times 3 \times 3 = 95,040$.

We have tested the GA algorithm against the optimal solutions and against the SM, LUC and PPB heuristic from Teunter et al. (2006). The SM, LUC and PPB heuristic are straightforward extensions of the Silver-Meal, Least Unit Cost and Part Period Balancing heuristic, respectively. For details on the heuristics we refer to Teunter et al. (2006). We have applied the GA algorithm with start pools of size $n = 2T$, $n = 4T$ and $n = 6T$ (denoted by GA2, GA4 and GA6, respectively) and the fixed number of iterations is set to $m = T$. This means that the running time of the three GAs is $\mathcal{O}(T^6)$.

Demand pattern

μ	σ	τ	a	c	d
Stationary					
100	10	0	0	na	na
100	20	0	0	na	na
Positive trend					
100	10	10	0	na	na
100	10	20	0	na	na
Negative trend					
210	10	-10	0	na	na
320	10	-20	0	na	na
Seasonal (peak in middle)					
100	10	0	20	12	1
100	10	0	40	12	1
Seasonal (valley in middle)					
100	10	0	20	12	3
100	10	0	40	12	3

Cost parameters

Parameter	Values
K^r, K^m	200, 500, 2000
h^r	0.2, 0.5, 0.8
h^s	1

Return pattern

μ	σ	τ	a	c	d
Stationary					
30	3	0	0	na	na
30	6	0	0	na	na
50	5	0	0	na	na
50	10	0	0	na	na
70	7	0	0	na	na
70	14	0	0	na	na
Positive trend					
30	3	3	0	na	na
30	3	6	0	na	na
70	7	7	0	na	na
70	7	14	0	na	na
Negative trend					
63	3	-3	0	na	na
96	3	-6	0	na	na
147	7	-7	0	na	na
224	7	-14	0	na	na
Seasonal (peak in middle)					
30	3	0	6	12	1
30	3	0	12	12	1
70	7	0	14	12	1
70	7	0	28	12	1
Seasonal (valley in middle)					
30	3	0	6	12	3
30	3	0	12	12	3
70	7	0	14	12	3
70	7	0	28	12	3

Table 7.3: Experimental Setting. The demand (and return) patterns are generated according to $d_t = \mu + \tau(t-1) + a\sin\left(\frac{2\pi t}{c} + d\frac{\pi}{2}\right) + \varepsilon_t$ for $t = 1, \ldots, T$, where μ is the starting level of the pattern, τ is the trend level, a is the amplitude of the cycle, c is the cycle length, d is the location of the peak of the cycle and ε_t $(t = 1, \ldots, T)$ are independently normally distributed random variables with standard deviation σ.

For each problem instance the optimal solution is determined with Cplex. For each heuristic and each problem instance we have calculated the deviation from optimality (in %), which we refer to as the error. Different performance measures are presented in

Heuristic	SM	LUC	PPB	GA2	GA4	GA6
Average error (%)	8.3	9.0	19.8	1.78	0.75	0.51
Standard deviation (%)	8.7	8.7	19.6	2.29	1.47	1.24
Maximum error (%)	82.5	98.4	221.4	49.4	22.0	19.5
Percentage within 1%	-	-	-	50.6	76.0	83.8
Average running time (s)	0.00	0.00	0.00	0.08	0.25	0.51

Table 7.4: Performance of the heuristics

Table 7.4 (the percentage of problem instances with error within 1% of optimality for SM, LUC and PPB are not reported in Teunter et al. (2006)). It follows that the GAs perform much better than the SM, LUC and PPB heuristic. This may be expected as the GAs take more computational effort ($\mathcal{O}(T^6)$) than the simple SM, LUC and PPB heuristic ($\mathcal{O}(T)$). The average error of GA4 and GA6 is even below 1%. Furthermore, when comparing SM, LUC and PPB with GA2, GA4, GA6, the maximum error have decreased dramatically. Although the maximum error of GA6 is still quite large, we see that only a fraction of the problem instances have a large error, as more than 80% of the solutions have an error within 1%. Finally, we see that the average running time of SM, LUC and PPB is negligible, while the average running time of GA2-GA6 is between 0.08 and 0.51 s (which is still acceptable).

7.6 Conclusion

In this chapter we considered the economic lot-sizing (ELS) problem with a remanufacturing option and separate setup costs (ELSR). We showed that the ELSR problem is $\mathcal{NP}$-hard, even in the case of time-invariant cost parameters. Furthermore, we derived complexity results for some related problems. First, no fully polynomial time approximation scheme exists for the ELSR problem with fixed ending inventories. Second, the ELSR problem with an additional option to dispose of returns is also $\mathcal{NP}$-hard. This result also followed from the fact that the ELSR problem with a disposal option is a generalization of the capacitated ELS problem.

Furthermore, we derived structural properties for an optimal solution of the ELSR problem. We used the properties to derive an $\mathcal{O}(T^3)$ dynamic programming (DP) algorithm for the ELSR problem with given production periods. In turn, we used this DP algorithm to develop a genetic algorithm (GA) for solving the ELSR problem (heuristically). Computational tests showed that the GA can generate solutions with a cost within 1% of the optimal cost on average. However, some problem instances showed a cost deviation of more than 20%. This means that still improvements can be made. Unfortunately, the GA algorithm does not provide a performance guarantee on the quality of the solutions. Therefore, an issue for future research is to find heuristics with such a performance guarantee.

Chapter 8

Summary of the main results

In Chapter 2 we derived a new property for an optimal solution of the economic lot-sizing (ELS) problem with time-invariant cost parameters. In this chapter we were interested in the proportion of order cost and holding cost in an optimal solution. Let an order interval be defined as the consecutive number of periods for which demand is satisfied by a single order. We showed that the ratio between the total holding cost and order cost in an order interval can be arbitrarily large. However, the ratio only increases proportionally to the logarithm of the number of periods in the order interval. Furthermore, we showed by an example that this bound is tight. This result is in contrast with the classical economic order quantity (EOQ) model, where setup cost and holding cost are perfectly balanced in an optimal solution.

The analysis resulted in the construction of a new heuristic. The main idea of the heuristic is that a new order is started when adding an additional order to the current order interval decreases the cost. We showed that this heuristic has worst case ratio 2. Furthermore, we also showed that a solution generated by this heuristic has a nice theoretical property: the number of setups of the heuristic solution equals at most the number of setups generated by the optimal solution and equals at least half of this number. Finally, we showed that the worst case ratio tends to $\frac{3}{4}\sqrt{2}$ in case of time-invariant demand and sufficiently large setup cost.

In Chapter 3 we analyzed the worst case performance for a general class of on-line lot-sizing heuristics. The class of heuristics satisfied the following properties: (i) decisions are made on a period-by-period basis, (ii) decisions only depend on the cost of the current lot-size, (iii) the decision rule is deterministic. We showed that heuristics satisfying (i) have a worst case ratio of at least $\frac{3}{2}$, even if the optimal solution and the heuristic solution have at most 2 setups (including the setup in the first period). Using the analysis to construct the worst case examples enabled us to find heuristics with optimal worst case performance for three- and four-period problem instances.

Furthermore, we showed that heuristics satisfying (i)–(iii) have worst case ratio at least 2. This generalizes the work of Axsäter (1985), who proved this result for a more restrictive class of heuristics. The problem instance with this performance ratio was found by formulating the problem of finding a worst case example as a mixed inter programming problem. As far as we know no heuristics satisfying (i) and having worst case ratio smaller than 2 exist. This leaves an issue for further research.

In Chapter 4 we extended the ELS problem to account for pricing decisions. We considered a joint pricing and lot-sizing problem, where prices were allowed to vary over time. We focused on a special case with a time-invariant demand function and time-invariant cost parameters considered by Bhattacharjee and Ramesh (2000). Bhattacharjee and Ramesh (2000) proposed two heuristics for the problem. However, we showed that by applying existing results in the literature, the problem can be solved to optimality. Application of (a slight modification of) the approach by Thomas (1970) led to the development of a (practically) efficient algorithm with a running time quadratic in the model horizon. Moreover, applying the results on a partition problem derived by Orlin (1985) led to an improvement in running time (although the running time was still pseudopolynomial).

In Chapter 5 we developed an exact algorithm for another type of pricing and lot-sizing problem. In the model of Chapter 5 prices were assumed to be constant over time, in contrast to the model of Chapter 4. We generalized the work of Kunreuther and Schrage (1973) who proposed a heuristic for the problem. We did not only show that the problem can be solved to optimality, but we also show that this can be done in $\mathcal{O}(T^3 \log T)$ time. Moreover, we improved on the the result of Gilbert (1999), who proposed an $\mathcal{O}(T^3)$ for a special case of the problem. We showed that applying our approach to this special case leads to an $\mathcal{O}(T^2)$ algorithm.

In Chapter 6 we considered the ELS problem with a remanufacturing option (ELSR). In this model we assumed that a (known) quantity of products comes back from the customer in each period and these returned products can be remanufactured to satisfy demand (besides regular manufacturing). In this chapter we assumed that there is a joint setup cost for manufacturing and remanufacturing. We showed that this problem can be solved in $\mathcal{O}(T^4)$ time under the assumption of time-invariant cost parameters. We developed a dynamic programming algorithm where we utilized the 'block' structure associated with an extreme point solution.

Furthermore, we showed that the problem without the manufacturing option, sufficient returns and non-stationary cost (the 'pure' remanufacturing problem) can be solved in $\mathcal{O}(T^2 \log T)$. Moreover, we showed that this problem is equivalent to the ELS problem with bounded inventory. This immediately provided an $\mathcal{O}(T^2 \log T)$ algorithm for this problem, which is an improvement on the $\mathcal{O}(T^3)$ algorithm of Love (1973). Finally, the ELSR problem with general cost parameters and no restrictions on the returns is not known to be polynomially solvable or $\mathcal{NP}$-hard. So this is an issue for future research.

In Chapter 7 again we considered the ELSR problem. However, in the model of this chapter we assumed a separate setup cost for both manufacturing and remanufacturing, in contrast to the model of Chapter 6. We showed that this ELSR problem is $\mathcal{NP}$-hard, even in the case of time-invariant cost parameters. This immediately proved that the ELSR problem with an additional option to dispose of returns is $\mathcal{NP}$-hard (this problem was considered by Golany et al. (2001) and Yang et al. (2005)). Furthermore, we showed that the ELSR problem with a disposal option is a generalization of the capacitated ELS problem, which also proved the above result. Finally, we showed that no fully polynomial time approximation scheme exists for the ELSR problem with fixed ending inventories, unless $\mathcal{P} = \mathcal{NP}$.

In the remainder of Chapter 7 we derived structural properties for an optimal solution of the ELSR problem with separate setup costs. We used the properties to derive an $\mathcal{O}(T^3)$ dynamic programming (DP) algorithm for the ELSR problem with given production periods. In turn, we used this DP algorithm to develop a genetic algorithm (GA) for solving the ELSR problem (heuristically). Computational tests showed that the GA can generate solutions with a cost within 1% of the optimal cost on average. However, some problem instances showed a cost deviation of more than 20%. This means that still improvements can be made. Unfortunately, the GA algorithm does not provide a performance guarantee on the quality of the solutions. Therefore, an issue for future research is to find heuristics with such a performance guarantee.

Nederlandse samenvatting (Summary in Dutch)

Een manier waarop bedrijven kosten kunnen reduceren is efficiënte productieplanning. Het centrale thema in dit proefschrift is een klassiek productieplanningsprobleem: het economische lot-sizing (ELS) probleem (Wagner and Whitin, 1958). Het probleem kan als volgt worden beschreven. Er is een producent met een (bekend verondersteld) vraag naar een bepaald product voor een eindige, discrete planningshorizon. Er moet een productieschema worden gevonden die in de vraag voorziet en waarbij de totale kosten worden geminimaliseerd. Deze kosten bestaan uit: setupkosten (voor elke periode dat er wordt geproduceerd), productiekosten (voor elk product dat wordt geproduceerd) en voorraadkosten (voor elk product dat in voorraad wordt gehouden).

In dit proefschrift bekijken we zowel aspecten rondom het klassieke probleem als uitbreidingen op het probleem. In Deel I houden we ons bezig met aspecten rondom het klassieke probleem en in Deel II and III bekijken we een aantal uitbreidingen op het ELS model.

In Hoofdstuk 2 van dit proefschrift zijn we geïnteresseerd in de eigenschappen van een optimale oplossing voor het ELS probleem met constante kostenparameters. In het bijzonder analyseren we de verhouding tussen de voorraadkosten en setupkosten. Deze kosten zijn namelijk gelijk in een optimale oplossing voor het *economic order quantity* (EOQ) model, een model dat nauw samenhangt met het ELS model. De vraag is nu of deze eigenschap ook in zekere mate geldt voor het ELS probleem. Ons onderzoek laat zien dat de verhouding tussen de setup- en voorraadkosten willekeurig groot kan zijn in een productie-interval. (Een productie-interval is gedefinieerd als een geheel aantal periodes waarvoor de vraag wordt voldaan door een enkele productieperiode.) De voorraadkosten nemen echter slechts toe met een factor die proportioneel is met de setupkosten en de logaritme van het aantal periodes in het productie-interval.

Onze analyse resulteerde in de ontwikkeling van een nieuwe heuristiek voor het ELS probleem. In de heuristiek wordt een productie-interval zo gekozen dat het aantal periodes in het interval zo groot mogelijk is en toevoegen van een extra productieperiode niet tot een kostenreductie leidt. Verder heeft de heuristiek de volgende theoretische eigenschap:

het aantal productieperiodes is ten hoogste gelijk aan het aantal productieperiodes in een optimale oplossing en ten minste gelijk aan de helft van dit aantal. Helaas genereert de heuristiek geen optimale oplossingen in het geval van constante vraag. In het slechtste geval heeft de heuristiek een relatieve afwijking van $11\frac{1}{9}\%$ en de relatieve fout convergeert naar $\frac{3}{4}\sqrt{2} - 1 (\approx 6.1\%)$ wanneer de setupkosten willekeurig groot worden.

In Hoofdstuk 3 beschouwen we een klasse heuristieken voor het ELS probleem met constante kostenparameters. Deze klasse bezit de volgende eigenschappen:

(i) Beslissingen worden periode voor periode genomen.

(ii) Beslissingen hangen alleen af van de kosten in het huidige productie-interval.

(iii) De beslissingsregels zijn deterministisch.

Door eigenschap (i) kan de klasse beschouwd worden als een klasse van on-line heuristieken. Veel heuristieken voor het ELS probleem in de literatuur vallen binnen deze klasse. In dit hoofdstuk zijn we geïnteresseerd in de worst-case performance van deze klasse, i.e., de maximale afwijking van de heuristische kosten ten opzichte van de optimale kosten (die we uitdrukken als ratio van deze twee kosten).

Ons onderzoek laat zien dat elke heuristiek binnen de bovengenoemde klasse een worst-case ratio van ten minste 2 heeft. Dit generaliseert het resultaat van Axsäter (1982), die laat zien dat deze eigenschap geldt voor een beperktere klasse van heuristieken. Verder laten we zien dat heuristieken die alleen aan eigenschap (i) voldoen, een worst-case ratio van ten minste $\frac{3}{2}$ hebben. Voor zover bekend bestaat er geen heuristiek die aan eigenschap (i) voldoet en een worst-case ratio kleiner dan 2 heeft. Dit is dus een probleem voor verder onderzoek.

Omdat het ELS model relatief eenvoudig is, bekijken we in Deel II en III een tweetal uitbreidingen van het klassieke ELS model. In Deel II, bestaande uit Hoofdstuk 4 en 5, bekijken we een model waarbij het mogelijk is om de vraag te beïnvloeden door middel van het zetten van de verkoopprijs (in tegenstelling tot het klassieke model waarbij de vraag gegeven wordt verondersteld). In Deel III, bestaande uit Hoofdstuk 6 en 7, breiden we het model uit met een optie tot herproductie (naast reguliere productie).

In Hoofdstuk 4 bekijken we het ELS model met een mogelijkheid tot prijszetting, waarbij de verkoopprijzen mogen variëren over de tijd. In het bijzonder richten we ons op het model van Bhattacharjee and Ramesh (2000), die twee heuristieken ontwikkelden voor een probleem met constante vraagfuncties. In dit hoofdstuk laten we zien dat het probleem exact kan worden opgelost door een (praktisch) efficiënt algoritme door de methode van Thomas (1970) toe te passen. Vervolgens laten we zien dat het algoritme verder kan worden versneld door de resultaten van Orlin (1985) toe te passen.

Omdat het in sommige gevallen niet gewenst is om verschillende verkoopprijzen over de tijd te hebben, nemen we in Hoofdstuk 5 aan dat er één prijs gezet moet worden in alle tijdsperiodes. Kunreuther and Schrage (1973) ontwikkelden een heuristiek voor dit probleem. Gilbert (1999) laat zien dat een speciaal geval van het probleem (exact) opgelost kan worden in $\mathcal{O}(T^2)$ tijd (waarbij T de planningshorizon is). In dit hoofdstuk verbeteren beide resultaten. Ten eerste ontwikkelen een exact algoritme voor het algemene probleem met een looptijd van $\mathcal{O}(T^3 \log T)$. Ten tweede laten we zien dat de looptijd van ons algoritme reduceert tot $\mathcal{O}(T^2)$ voor het model van Gilbert (1999).

In Deel III breiden we het ELS model met een mogelijkheid tot herproductie. Door wetgeving van de overheid, het milieubewust zijn van de klant en economische redenen nemen bedrijven steeds meer gebruikte producten (returns) terug van de klant. In ons model nemen we aan dat deze returns kunnen worden geherproduceerd en daarna zo goed als nieuw zijn. De vraag kan nu dus worden voldaan door zowel reguliere productie als herproductie van returns.

In Hoofdstuk 6 nemen we aan dat er gezamenlijke setupkosten zijn voor productie en herproductie. Dit is bijvoorbeeld het geval wanneer er een enkele productielijn voor productie en herproductie is. We laten zien dat dit probleem efficiënt opgelost kan worden in het geval van constante kostenparameters. We hebben echter niet kunnen aantonen dat het probleem met algemene kosten $\mathcal{NP}$-moeilijk of polynomiaal oplosbaar is. Dit is dus een probleem voor verder onderzoek.

Vervolgens bekijken we een speciaal geval van het model met voldoende returns om aan de vraag te voldoen en geen reguliere productie. We laten zien dat dit probleem kan worden opgelost in $\mathcal{O}(T^2 \log T)$ tijd. Bovendien tonen we aan dat dit probleem equivalent is aan het ELS model met capaciteiten op de hoeveelheid voorraad in elke periode. Voor zover wij weten heeft het snelste algoritme voor dit probleem een looptijd van $\mathcal{O}(T^3)$ (Love, 1973). Toepassing van ons algoritme levert dus een verbetering op.

In plaats van gezamenlijke setupkosten voor productie en herproductie nemen we in Hoofdstuk 7 aan dat er aparte setupkosten zijn. Dit is bijvoorbeeld het geval wanneer er aparte productielijnen voor productie en herproductie zijn. In dit hoofdstuk laten we zien dat dit probleem $\mathcal{NP}$-moeilijk is, zelfs in het geval van constante kostenparameters. Verder leiden we in dit hoofdstuk complexiteitsresultaten af voor gerelateerde problemen. Ten slotte ontwikkelen een genetisch algoritme voor het probleem. Numerieke experimenten tonen aan dat de gemiddelde performance goed is (op onze verzameling van testproblemen): de afwijking ten opzichte van het optimum ligt gemiddeld binnen 1%. De maximale afwijking ligt echter boven 20%. Hier ligt dus nog een mogelijkheid tot verbetering. Verder is de ontwikkeling van een heuristiek met een performance garantie ook een issue voor verder onderzoek.

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The Economic Lot-Sizing Problem
New Results and Extensions

One way for firms to reduce cost is efficient production planning. The main theme in this thesis is a classical production planning problem: the economic lot-sizing (ELS) problem. The objective of this problem is to find a production plan that satisfies the given demand for a finite, discrete planning horizon, and minimizes the total setup, production and holding costs. We study aspects of the classical problem as well as extensions of this problem.

In the first part of the thesis we consider the ELS model with time-invariant cost parameters. We analyze properties of an optimal solution and, in particular, we are interested in the proportion of holding cost and setup cost in an optimal solution. Furthermore, we perform a worst case analysis on a broad class of on-line heuristics for the problem.

Because the classical model is relatively simple, we also consider extensions of the model. We are interested whether there exist algorithms to solve the extensions efficiently. In the first extension we incorporate pricing decisions in the ELS model. The problem is now to find optimal price(s) and an optimal production plan simultaneously. We consider models with variable prices and a constant price over time.

Furthermore, we extend the ELS model with a remanufacturing option. It is assumed that a known quantity of products returns from the customer in each period and those returned products can be remanufactured to satisfy demand (besides regular manufacturing). We derive algorithms and complexity results for models with a joint setup cost for manufacturing and remanufacturing (in case of a single production line) and a separate setup cost (in case of separate production lines).

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