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Weak & Strong Financial Fragility

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WEAK & STRONG FINANCIAL FRAGILITY

J.L. GELUK, L. DE HAAN, AND C. G. DE VRIES

Abstract. The stability of the financial system at higher loss levels is either characterized by asymptotic dependence or asymptotic independence. If asymptotically independent, the dependency, when present, eventually dies out completely at the more extreme quantiles, as in case of the multivariate normal distribution. Given that financial service firms' equity returns depend linearly on the risk drivers, we show that the marginals' distributions maximum domain of attraction determines the type of systemic (in-)stability. A scale for the amount of dependency at high loss levels is designed. This permits a characterization of systemic risk inherent to different financial network structures. The theory also suggests the functional form of the economically relevant limit copulas.

1. Introduction

The financial system is inherently fragile due to its exposure to common and mutual risks and in particular due to the duration mismatch of the assets and liabilities of the banking sector. Financial crises are a recurrent phenomenon with important effects on the real economy. It is therefore of great importance to be able to understand, measure and characterize the systemic stability of the financial service sector. The theoretical literature on systemic stability, using micro information asymmetry and macro risks, provides insightful explanations for the fragility.[1] Some research has tried to measure the amount of fragility potentially present, but a coherent framework within which the fragility of a system can be evaluated is more or less absent from the literature. In this paper we use statistical multivariate extreme value theory to provide such a framework, linking the financial stability theories and the empirical work. This theory implies that the

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[1]Empirically, the macro risk drivers appear to be the dominant source behind financial crises.

interdependency at crisis levels is one of two types. We show that the financial service institutions (FSI) are either asymptotically independent (weakly fragile) albeit possibly being correlated, or the FSI are asymptotically dependent (strongly fragile). If asymptotically independent, the dependency when present, eventually dies out completely at the extreme quantiles. Per contrast, under strong fragility the dependency remains, even in the limit. Thus the systemic risk can be quite different depending on the type of fragility.

Consider the joint loss behavior of correlated bank and other FSI's equity returns, which are dependent due to e.g. loan syndication, interbank loans and macro interest rate exposure. Suppose the weak assumption holds that the asset and liability return distributions are in the domain of attraction of a (univariate) extreme value distribution. Then the banks' equity returns interdependencies display radically different behavior at high loss levels depending on whether the portfolio components's returns are e.g. normally, Student-t or uniformly (credit risk) distributed. We classify the types of different behavior by means of the different domains of attraction of the univariate extreme value distributions. Both continuous and discrete compounding are covered, so that all three limit distributions have relevance.[2] We show that, due to the balance sheet (portfolio) induced linearity, the type of extreme value distribution to which the marginal distributions of the risk factors are attracted determines whether the fragility is weak or strong. For example, normally distributed asset and liability returns imply weak fragility and low systemic risk. But this is quite different if the marginal distributions are Student-t or uniformly distributed. We provide an exhaustive characterization in terms of weak and strong fragility depending on the type of the underlying univariate asset return distributions.

Banks provide an important positive externality to the macro economy through their maintenance of the payment and settlement system and by channeling the monetary policy decisions. The fallout from a bank failure can therefore sort large negative effects.[3] This not withstanding, central banks, ministries of finance, supervisors and regulators are hard pressed when asked to characterize the fragility

[2]Credit risk under discrete compounding has a distribution with bounded support. In contrast, the log returns of equity (market risk) have an unbounded support.

[3]For example, the US S&L crisis is estimated to have cost the US tax payer an amount in the order of 4% of GDP, and the Swedish banking crisis of the 1990's cost as much as 10% of GDP.

of their financial system, even though one readily receives an answer concerning individual bank risk in terms of the Value at Risk (VaR) level. Thus, as is forcefully argued by Borio [6], there is great need for a measure which reflects the amount of systemic risk inherent to a particular financial system and permits comparisons across systems. Motivated by this need, we develop a scale, dubbed the Fragility Index FI, which captures the amount of systemic risk at the system's level. The FI is comparable to the VaR measure for individual bank risk. The scale circumvents the pitfalls of correlation analysis. It reflects different possible intensity levels of systemic dependence, ranging from perfect dependence, to asymptotic dependence, asymptotic independence and just independence and can be applied to any dimension (number of FSI).

In general the dependency structure of a multivariate distribution and the marginal distributions are unrelated concepts, but the linearity of portfolios in the asset returns or risk drivers induces a specific link between the two concepts. We show that popular theoretical economic explanations for systemic breakdowns, such as a macro (interest rate) shock, a sunspot or micro based contagion, fit within our affine setup. The approach is of semi-reduced form, since it takes as given a certain distribution of the assets and liabilities across the system in characterizing the system's stability. It does not analyze how such a network is the outcome of incentives and institutions. The latter question is clearly also of interest, but is outside the scope of a single paper and is well treated elsewhere (e.g. we take as given there is an incentive to diversify); see e.g. Rochet and Tirole [28], Lagunoff and Schreft [22], Freixas, Parigi and Rochet [13] and Allen and Gale [1]. But given a particular FSI network structure, we show how the FI permits a characterization of the systemic risk inherent to such a network. It is shown that rankings based on the correlation structure may give an ordering which differs from the FI, since the FI recognizes better the diversification benefits in the failure regions. The theory also implies the functional form of the economically relevant copulas in the systemically relevant regions.[4]

[4]The variety of copulas is 'large' and the question is which types of copulas are economically relevant for the problem at hand. As it happens, copulas are often chosen for their convenience in estimation, but economic criteria have as of yet scarcely received attention when choosing a particular copula.

There exists some empirical research which has tried to measure the interdependence within and between financial systems by correlating bank stock returns, see e.g. De Nicolo and Kwast [7]. Alternatives for the correlation analysis as measures of systemic risk are copulas and multivariate extreme value analysis. Copulas give the dependency structure embedded in the joint distribution function parametrically; see Longin and Solnik [25] for an early empirical application based on the logistic copula. Extreme value based statistical analysis is a semi-parametric approach which captures the dependency in the tail regions of the joint distribution function without committing to a particular functional form. This literature finds evidence for strong fragility, see Hartmann et al. [18] and Poon et al. [26]. Here we provide the theory behind these empirical results.

The structure of the rest of the paper is as follows: In section 2 we discuss discounting and the linearity of bank portfolios and affine fundamentals based models such as the CAPM. A discussion and comparison of different measures to characterize linkages during periods of market stress is provided in section 3. The analytic claims of the paper on the relationship between the risk drivers' marginal tail properties and the degree of tail dependence are obtained in section 4. The cases of weak and strong fragility are treated in separate subsections. Financial economic analysis is given in section 5. Finally, section 6 provides a summary and conclusions.

2. Affine Portfolios and Compounding

The FSI are linked in a number of ways. An important linkage is through their mutual exposures, yielding similar investments and liabilities. Take e.g. a reinsurance firm which reinsures part of an insurance policy written by an insurance firm, retrocedes part of this reinsurance contract to another reinsurer and invests the premia it receives on the reinsurance policy in a portfolio of stocks and bonds. All this activity is undertaken to diversify the risk of holding an overly specialized portfolio. The diversification activity on the liability side produces direct linkages within the insurance and reinsurance sector. But since these companies invest the premia in well diversified portfolios, there is also an indirect linkage by the risk factors which drive the market risks. Similarly, commercial banks are typically heavily exposed to each other through the interbank money market by which the banks manage their liquidity. Typically commercial banks loan to the same sectors in the economy, which again produces the exposure to the same macro risk drivers (through the movements in the value of the received collateral, e.g. house prices in case of mortgages). Syndicated loans whereby several banks underwrite a large loan directly, expose different banks to the same risk. Investment banks often hold large trading portfolios and hold stakes in commercial companies which belong to the clientele of the bank, yielding direct exposure to market risk. Last but not least, banks in many countries do also hold sizable cross-participations in each other.

In summary, the FSI hold portfolios which are linked and which are directly or indirectly exposed to the same risks or risk factors. Insofar banks hold the same assets or finance the same loans, the linearity is direct. Indirectly, the return on capital of banks is also linearly related through their exposure to macro risk factors. Finance theory, such as the CAPM and APT, often assumes that returns are linearly related to the macro risk factors.[5] The monetary model of exchange rates for example, holds that exchange rate returns Δs_{0j} between the numeraire currency 0 and the j-th currency are linearly related to changes in the relative money supply m, real income y and the interest differential $(R_0 - R_j)$:

$$\Delta s_{0j} = \Delta(m_0 - \phi y_0 + \lambda R_0) - \Delta(m_j - \phi y_j + \lambda R_j) + \varepsilon_{0j}.$$

This is not to say that there are no cases where the relationship between the returns and factors is non-linear. For example, consider a portfolio which both contains options and the underlying stocks.[6] Such portfolios are analyzed in the economics section below.

2.1. **portfolios.** Portfolios are by definition linear in the returns of the assets and liabilities. Indirectly, portfolios are linear in the macro risk factors. Consider two arbitrarily seized portfolios with returns

$$Q_n = \sum_{i=1}^{n} \lambda_i X_i,\ W_n = \sum_{i=1}^{n} \gamma_i X_i,$$

[5]Boyer, Kumagai and Yuan [5] provide evidence that crises spread internationally through asset holdings of investors.

[6]Option theory holds that the returns on both assets are linearly related to the market factor, but this is only a local result.

where the X_i are the individual asset returns or (macro) risk factors and the asset weights satisfy $\sum_{i=1}^{n}\lambda_i = \sum_{i=1}^{n}\gamma_i = 1$. In these portfolios we can allow for short selling and the portfolios can be "unbalanced" in the sense that some assets are not present in both portfolios. In some instances the case of unbalanced portfolios is qualitatively different from the case of balanced portfolios.

We will repeatedly use the following two two-asset portfolios in examples to illustrate the theoretical results. Consider the case of two syndicated loans with independent returns X and Y respectively. Imagine that the loans for the projects are underwritten by two investment banks or sold on to other FSI. Let bank one hold the portfolio with return

$$Q = (1-\gamma)X + \gamma Y, \tag{2.1}$$

while the loan portfolio return to bank two is

$$W = \gamma X + (1-\gamma)Y, \tag{2.2}$$

and where $\gamma\epsilon(1/2, 1)$. This case will be referred to as the zero beta portfolio, given the independence of the two risk drivers.

The second case considers two asset excess returns X and Y related through the CAPM. Both returns have the following single factor structure

$$X = \beta_x R + \varepsilon_x \tag{2.3}$$

and

$$Y = \beta_y R + \varepsilon_y. \tag{2.4}$$

Here R is the excess return on the market portfolio over the risk free rate and ε_x and ε_y are the idiosyncratic risks (independently distributed from the market risk and each other); beta's indicate how much the projects co-vary with the market risk. For simplicity we repeatedly take the beta's equal $\beta_x = \beta_y = \beta$ in the examples.

2.2. **discrete and continuous returns.** Depending on the problem at hand, either continuous or discrete returns are analyzed. Some classes of assets, such as in the case of stocks, have almost continuous price formation in time and hence continuous compounding is typically used for these assets. Other assets only trade or payout at discrete instances in time, for which discrete returns are more appropriate. Portfolio returns can be obtained by summing the weighted discrete returns; using logarithmic returns this is not possible. Per contrast, aggregation over time works well with continuous returns, whereas discrete returns do not add up. For small price movements, the two concepts of a return are close (as a Taylor approximation shows). Let $P(t)$ denote the asset price at time t, and let X be the return. The continuously compounded return is given by $X(t+1) = \log P(t+1)/P(t)$, where $X \in \mathbb{R}$. Discrete returns are computed as $Y(t+1) = P(t+1)/P(t) - 1$, where $(Y+1) \in \mathbb{R}^{+}$. We investigate the implications for the joint loss distribution under continuous compounding and discrete compounding.

Under continuous compounding the loss return can be as large as can be imagined, but the discrete return can not be worse than -1. This lower bound is of particular relevance for credit risk. In the worst case the payoff to a bond is zero and the return on the principal is minus one. Note that the positive returns on such credit instruments at maturity are bounded as well. Such bounds have immediate implications for the possible tail shapes of the multivariate return distribution on the loss side. Though most of the time we deal with continuously distributed returns, we also briefly investigate the systemic stability if there are mass points. The motivation for this is twofold. Many of the theoretical crises models employ the Bernoulli distribution, i.e. there are only a single good and a bad state of the world. In reality, options and other non-linear instruments may have mass points in their return distribution if held to maturity, whereas the distribution of the underlying asset would not.

3. Measures of Dependency

We first argue why there is a need for a measure of dependency to reflect systemic risk and why standard concepts like the correlation measure are less suitable for the question at hand. Then we design a scale which is close in spirit to the popular VaR measure, but which is suitable at the systems level.

Indeed, why would economists be interested in a measure for systemic risk? Take the banking sector's risk regulatory background as laid down in the recently revamped Basel accords for bank capital holding. The surprising fact is that the entire approach has a predominant microprudential orientation, focussing on the risk management practices at individual banks, without much attention for the systemic ramifications, even though the systemic stability is often invoked as the prime motive for the necessity of the Basel rules. The banking sector has important externalities within the banking sector and to the rest of the economy through its maintenance of the clearing and settlement services. These services depend on the reliability of the system as a whole. Moreover, an important part of the risks are endogenous to the sector.[7] Given the specific sensitivity of the banking sector to the macro and endogenous risks, Borio [6] pleads for giving an explicit role to the macropudential aspects of bank regulation and supervision on top of the microprudential framework which is now firmly in place through the Basel II accord. The existing micro prudential operational framework focusses on the VaR of each institution individually. The new macro prudential approach should also care about the tail losses of the banking system as a whole.

The Basel II and Solvency II accords' microprudential oriented regulatory framework are based on the philosophy that the chain is as strong as its weakest link, which would imply that regulators focus on containing

$$1 - P(B_1 \leq s, B_2 \leq s),$$

and where B_i is the i-th bank return exposure. The Basel II accord in practice is even more conservative as it aims to minimize $P(B_1 > s)$ and $P(B_2 > s)$ individually, and where s is the VaR level. In other words

$$1 - P(B_1 \leq s, B_2 \leq s) \leq P(B_1 > s) + P(B_2 > s).$$

Thus the practice is an overly conservative approach, since it safeguards against a systems breakdown twice:

$$1 - P(B_1 \leq s, B_2 \leq s) + P(B_1 > s, B_2 > s) = P(B_1 > s) + P(B_2 > s).$$

Note that in higher dimensions this effect is even stronger, as the joint failure region is taken into account as many times as the number of FSI which are part of the system. From an efficiency point of view overregulation is not desirable, as is too weak supervision. It is therefore of interest to measure the joint failure

[7]To give one stark example from the investment industry, recall the popularity and fall from grace of the portfolio insurance technique for managing risk. While evidently prudential from a micro oriented point of view, the technique faltered when all institutions were trying hedge by selling off at the same time on black Monday in October 1987.

probability $P(B_1 > s, B_2 > s)$ separately and to adjust the individual bank based approach for systemic risk. We discuss a number of alternative ways to evaluate this joint failure probability.

3.1. **the correlation measure.** In case the return distribution is multivariate normal, the joint failure probability can be calculated on basis of the correlation matrix. The coefficient of correlation ρ is perhaps the most commonly used measure of (linear) dependence. One must ask, however, how well ρ captures the dependency if it is unknown whether the data are normally distributed or not. Specifically, the question is whether ρ adequately captures the interdependency at crisis levels. Embrechts, McNeil and Strauman [11] discuss the pitfalls of the normal based correlation analysis as a means to measure systemic risk. The empirical literature moreover, finds little support for normality of the return distribution of many asset classes, which makes correlation analysis less suitable.

A somewhat realistic, note the martingale structure, financial economic example is the following bivariate ARCH inspired volatility model:

$$X_t = N_t H_t, \quad N_t \text{ i.i.d. } N(0,1),$$

$$Y_t = M_t H_t, \quad M_t \text{ i.i.d } N(0,1),$$

with common volatility factor:

$$H_t = w + \beta(X_{t-1}^2 + Y_{t-1}^2), 0 \leq \beta < 1/2.$$

The (stationary) returns X and Y exhibit the characteristic heavy tail property, the clustering of volatility, are interdependent, but nevertheless uncorrelated. The $\rho_{t-k}(X_t, Y_t) = 0, k = 1, 2...$ even though X_t and Y_t are dependent since they are driven by the same conditional variance function H_t.[8] Thus, ρ does not capture the dependency which is in the data.

Another problem is that the correlation concept requires that the first two moments exist. This is at least of some concern for the non-life branch of the insurance industry, where the loss distribution often appears to fail to have a second bounded moment. Lastly, economists evaluating investments within the expected utility theory framework are not so much interested in the correlation

[8]Note that regarding volatility spill-overs the correlations measure gives $\rho_{t-1}(X_t^2, Y_t^2) = 0$, but $\rho_{t-k}(X_t^2, Y_t^2) \neq 0$ for $k > 1$.

measure itself; they rather have an interest in the trade-offs between risk measured as a probability and the associated gains or losses, which are the quantiles of the return distribution. As such, the correlation is therefore only an intermediate step in the calculation of this trade-off between quantile and probability. Therefore we like to turn to measures which are not conditioned on a particular multivariate distribution and which directly reflect the probabilities and associated crash levels.

3.2. **Copulas.** Copulas provide a parametric specification of the dependency structure of multivariate distributions. The popularity of the copula approach partly derives from the fact that copula lend themselves easily to parametric estimation. Given our objective to uncover the dependency in the tail area, however, the parametric copula approach shares the same problem with parametric distribution based approaches, like the Gaussian approach, in that it does not necessarily do justice to the behavior in the tail area. Moreover, as of to date there does not exist an economic motivation for choosing one copula over the other. Often a particular copula specification is chosen for estimation convenience rather than for economic relevance. We shed some light on both issues below, by deriving the relevant copula for the tail area from the underlying economic structure. For supervisors and regulators, using a function rather than an index, the disadvantage is that a function is in general less succinct, and hence may not be acceptable as a summary measure for dependency. It can also be less robust if one lacks part of the necessary information to construct the function. Lastly, copulas are again only an intermediate step in linking loss amounts and their probabilities.

3.3. **The Fragility Index FI.** For all the reasons given above, we turn to develop an index of fragility (FI) which does not hinge on a particular distribution or dependence structure, and which does link losses and probabilities directly. The FI at the system's level is comparable to the VaR measure for individual bank risk. The FI, though, is stated inversely in terms of large loss probabilities, rather than loss levels. This allows for possibly different loss levels for different FSI, while still providing a single index number for the entire system.

3.3.1. *co-crash probabilities.* In the introduction to this section we argued extensively that the macro prudential concern is a heavy loss in one bank going hand in hand with a heavy losses of other banks, creating a breakdown of the financial system. Since we are interested in the probabilities on such joint extreme outcomes, we directly evaluate these probabilities, bypassing the correlation concept. In higher dimensions, though, there are many of such probabilities (bivariate, trivariate, etc.) and somehow one wants to keep the information manageable. To strike a balance between these desiderata we propose to adopt a particular conditional expectation.

Specifically, we ask given that $Y > t$, what is the probability that $X > s$, or vice versa, and where X and Y stand for asset returns and t, s are high loss levels. The probability measure which conditions on any market crash, without indicating the specific institution is the linkage measure[9]

$$\frac{P\{X > s\} + P\{Y > t\}}{1 - P\{X \leq s, Y \leq t\}}$$

first proposed in Huang [19] and evaluated empirically by Hartmann et al. [18]. Here we broaden its sensitivity and extension to higher dimensions.

The linkage measure, even though it is the sum of two conditional probabilities, reflects the expected number of crashes given at least one collapse. To see this, let κ_s denote the number of simultaneously crashing bank stock returns, that is bank returns exceeding s. Write the conditionally expected number of bank crashes given a collapse in at least one bank as $E\{\kappa_s | \kappa_s \geq 1\}$. Then

$$\begin{aligned} E\{\kappa_s|\kappa_s \geq 1\} &= E\{\mathbf{1}_{X>s} + \mathbf{1}_{Y>s}|\kappa_s \geq 1\} \\ &= E\{\mathbf{1}_{X>s}|\kappa_s \geq 1\} + E\{\mathbf{1}_{Y>s}|\kappa_s \geq 1\} \\ &= P\{X > s|\kappa_s \geq 1\} + P\{Y > s|\kappa_s \geq 1\} \\ &= \frac{P\{X > s\} + P\{Y > s\}}{1 - P\{X \leq s, Y \leq s\}} \end{aligned} \tag{3.1}$$

Compare this expression with another quantity which is sometimes used to measure dependence in the tail:

$$\theta_s := \frac{P\{X > s \text{ and } Y > s\}}{P\{X > s \textit{ or } Y > s\}}. \tag{3.2}$$

We have

$$E\{\kappa_s|\kappa_s \geq 1\} = 1 + \theta_s. \tag{3.3}$$

[9]Below we take the all quantiles on which we condition equal to s, but this is by no means necessary. If two banks have quite different levels of capital, one may want to take the loss return thresholds with a systemic impact differently.

The advantage of the expectation measure is that it can be easily extended to higher dimensions, while the θ_s is more cumbersome and less revealing, see below.

Under quite general conditions, including extreme value conditions, the limit

$$\kappa = \lim_{s\to\infty} E\{\kappa_s|\kappa_s \geq 1\} = 1 + \lim_{s\to\infty} \theta_s \tag{3.4}$$

exists. This limit can be used as an indicator for the amount of dependence in the tail between X and Y. One reason to take the limit, rather than using a finite loss level s, is that economics does not say what the critical level is at which systemic failure sets in. Taking the limit thereby removes some arbitrariness. At the same time, the limit is still indicative about what happens at high but finite loss levels, see Balkema and De Haan (1974).

Note that

$$\lim_{s\to\infty} E\{\kappa_s|\kappa_s \geq 1\} = 1$$

is interpreted as asymptotic independence. In the case that

$$\lim_{s\to\infty} E\{\kappa_s|\kappa_s \geq 1\} = 2,$$

one has maximal asymptotic dependence. Hence

$$H := \lim_{s\to\infty} E\{\kappa_s|\kappa_s \geq 1\} - 1 \tag{3.5}$$

is a number between 0 and 1. It can be used as a measure of asymptotic dependence in the tail in a way analogous to the correlation coefficient for the tail distribution. However, there is no direct connection: Even for a normal distribution with coefficient of correlation $r \neq 0$, 1, or -1 one nevertheless has $H = 0$; see below.

In the higher dimensional situation with $d > 2$ random variables, H can be defined in a completely analogous way:

$$H := \frac{\lim_{s\to\infty} E\{\kappa_s|\kappa_s \geq 1\} - 1}{d-1}$$

and the interpretation is the same. Note that the straightforward generalization of (3.2):

$$\lim_{s\to\infty} \theta_s := \lim_{s\to\infty} \frac{P\{X_1 > s, \ldots, X_d > s\}}{P\{X_1 > s \text{ or } \ldots \text{or } X_d > s\}}$$

is not directly linked to H and that $\lim_{s\to\infty} \theta_s$ can be zero even if the random variables are not jointly independent in the tail (even if the random variables are pairwise asymptotically dependent, this limit can still be zero for $d > 2$).

As observed above, for any normal distribution and many other distributions, we find $H = 0$. Even in the case of asymptotic independence one can make a distinction between probability distributions which exhibit more and less dependence by applying a finer scale in the framework of extreme value theory. In the simple case the domain of attraction condition can be written (cf. Resnick, 1987)

$$\lim_{t\to\infty} t\left\{1 - F(tx, ty)\right\} = -\log G(x, y) \tag{3.6}$$

for $x, y > 0$ where F is the initial distribution and G is the limit distribution. Assume a 'second order' or 'speed of convergence' condition: Suppose there exists a positive function A and a limit function H (not identically 0) such that

$$\lim_{t\to\infty} \frac{t(1 - F(tx, ty)) + \log G(x, y)}{A(t)} \to H(x, y) \tag{3.7}$$

for $0 < x, y \leq \infty$ (we include $x = \infty$ and $y = \infty$ since otherwise we would not control the marginal distributions). It can be proved that A is regularly varying with index $\rho \leq 0$. In the case of asymptotic independence

$$G(x, y) = G(x, \infty)G(\infty, y) = e^{-\frac{1}{x}-\frac{1}{y}}. \tag{3.8}$$

Moreover, (3.7) with $y = \infty$ or $x = \infty$ also entails

$$\begin{aligned} \lim_{t\to\infty} \frac{t(1 - F(tx, \infty)) + \log G(x, \infty)}{A(t)} &\to H(x, \infty) \\ \lim_{t\to\infty} \frac{t(1 - F(\infty, ty)) + \log G(x, y)}{A(t)} &\to H(\infty, y) \end{aligned} \tag{3.9}$$

Then, upon combining (3.7), (3.8) and (3.9), it follows that

$$\frac{tP(X > tx, Y > ty)}{A(t)} \to H(x, \infty) + H(\infty, y) - H(x, y). \tag{3.10}$$

Now according to (3.6), $P\{X > tx \text{ or } Y > ty\}$ is asymptotically of order t^{-1}, i.e., regularly varying with index -1, whereas according to (3.10), $P\{X > tx \text{ and } Y > ty\}$ is asymptotically of order $t^{-1}A(t)$, i.e. regularly varying with index $\rho - 1$.

For the bivariate case, Ledford and Tawn [23] introduced the parameter η defined as

$$\eta = \frac{1}{1-\rho} \in [0, 1]$$

as a measure to distinguish between asymptotically independent distributions. Note that if $H > 0$, then $\eta = 1$. Note also that for a bivariate normal distribution

$\eta = (1 + r)/2$, where r is the coefficient of correlation. Moreover, if X and Y are independent, $\eta = 1/2$, but the converse does not hold.

3.3.2. *the FI scale.* If we combine the H scale and the η scale, we can define the Fragility Index FI:

$$FI = \begin{cases} \lim_{s\to\infty} E\left\{\kappa_s | \kappa_s \geq 1\right\} & \text{if } H > 0 \\ \\ \frac{1}{2}\lim_{s\to\infty} \frac{\log P\{X>s\}+\log P\{Y>s\}}{\log P\{X>s, Y>s\}} & \text{if } H = 0. \end{cases}$$

We will say that the financial system is strongly fragile if $FI > 1$, while the system is only weakly fragile if $FI \epsilon [0, 1]$. This index will be used as our scale for the amount of fragility.

We already noted that $E\left\{\kappa_s | \kappa_s \geq 1\right\}$ can be easily extended to higher dimensions. We turn to the extension of Ledford and Tawn's η measure to higher dimensions. We start from the extension of (3.7):

$$\lim_{t\to\infty} \frac{tP\left\{X > tx \text{ } or \text{ } Y > ty \text{ } or \text{ } Z > tz\right\} - \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)}{A(t)} = H(x, y, z).$$

As in (3.9) and the reasoning thereafter, we use

$$\begin{aligned} &P\{X > tx \text{ or } Y > ty \text{ or } Z > tz\} - P\{X > tx\} - P\{Y > ty\} - P\{Z > tz\} \\ &= -P\left\{X > tx, Y > ty\right\} - P\left\{X > tx, Z > tz\right\} - P\left\{Y > ty, Z > tz\right\} \\ &\quad +P\left\{X > tx, Y > ty, Z > tz\right\}. \end{aligned} \tag{3.11}$$

Suppose that all two-dimensional marginal distributions satisfy (3.10), i.e.

$$\lim_{t\to\infty} \frac{tP(X > tx, Y > ty)}{A(t)} \to H(x, \infty, \infty) + H(\infty, y, \infty) - H(x, y, \infty)$$

$$\lim_{t\to\infty} \frac{tP(Y > ty, Z > tz)}{A(t)} \to H(\infty, y, \infty) + H(\infty, \infty, z) - H(\infty, y, z)$$

$$\lim_{t\to\infty} \frac{tP(X > tx, Z > tz)}{A(t)} \to H(\infty, x, \infty) + H(\infty, \infty, z) - H(x, \infty, z)$$

Then (cf. (3.11)) it remains to deal with $P\left\{X > tx, Y > ty, Z > tz\right\}$. It is possible that this probability is of the same order as the two-dimensional distributions, i.e. of order $t^{-1}A(t)$, or this probability is of lower order and then we are dealing with a new parameter, smaller than the η that comes from A, determining the joint excess of all three variables. Similar arguments can be made for any $d > 2$. The d-dimensional extension of FI thus reads

$$FI = \begin{cases} \lim_{s\to\infty} \frac{P\{X_1>s\}+\ldots+P\{X_d>s\}}{1-P\{X_1\leq s,\ldots,X_d\leq s\}} & \text{if } H>0 \\ \\ \frac{1}{d}\lim_{s\to\infty} \frac{\log P\{X_1>s\}+\ldots+\log P\{X_d>s\}}{\log P\{X_1>s,\ldots,X_d>s\}} & \text{if } H=0. \end{cases} \tag{3.12}$$

This is the scale we will employ throughout the rest of the paper to judge the amount of systemic fragility. Note that if $H > 0$ the fragility is strong, whereas $H = 0$ refers to weak fragility.

4. Weak and Strong Financial Fragility

To determine the amount of fragility of a network in the affine FSI portfolio framework, we need a theory about the dependence between weighted sums of random variables (asset returns, risk factors) in the tail areas. This section first deals with discrete returns, then turns to continuous returns. The part on continuous returns is divided into a part with light and a part with heavy tails. The main results are summarized at the end of the section in terms of their economic implications.

The fragility of a bivariate system is linked to the joint tail behavior of linear portfolio combinations $Q = \sum \lambda_i X_i$ and $W = \sum \mu_i X_i$. It is assumed that the loss returns X_i $(i = 1, \ldots, n)$ are (cross sectionally) i.i.d.[10] We are interested in high values of the vector (Q, W) and in particular in the dependence between Q and W in the tail area. Assume that the distribution of X_i is in the domain of attraction of an extreme value distribution G_τ.[11] The marginal distributions which are in this class do have a proper limit distribution for the linearly scaled maximum (loss).[12] In order to be able to calculate the fragility we need to find the joint tail behavior of Q and W for different values of τ. For the cases $\tau > 0$ and $\tau < 0$ we give a complete characterization; in case $\tau = 0$ we cover the cases of subexponential and

[10]In case the factors or returns in our linear portfolios are time dependent but stationary, this does not affect the results in the paper.

[11]Where $\tau < 0$ refers to the Weibull limit law, $\tau > 0$ is the Frechet case, and $\tau = 0$ represents the Gumbel limit law. The extreme value theorem holds that the limit law for the maximum is either one of these distributions.

[12]The tail behavior is closely related to the type of relevant G_τ.

superexponential distributions (to be defined below)[13]. Throughout we use the notation

$$\alpha =: 1/|\tau| \text{ if } \tau \neq 0. \tag{4.1}$$

4.1. **Discrete returns, Case $\tau < 0$.** We start by showing that the class of distributions in the domain of attraction of the Weibull limit law is closed under addition.

4.1.1. *Closure under addition.* Let $F(x)$ be a loss distribution with bounded support $[0, a]$, hence $F(a) = 1$, $0 < a < \infty$. Suppose $F(x)$ is in the domain of attraction of a Weibull extreme value distribution. It is first shown that convolutions remain in this domain of attraction.

Define the "upper tail distribution" $H(x)$ as

$$H(x) \equiv 1 - F(-x + a)$$

Introduce its Laplace transform $\widetilde{H}(y)$

$$\widetilde{H}(y) \equiv \int_0^\infty y e^{-yx} H(x) dx. \tag{4.2}$$

Hence for $t > 0$ by a transformation of variable

$$\frac{\widetilde{H}(y/t)}{H(t)} = y \int_0^\infty e^{-yx} \frac{H(tx)}{H(t)} dx. \tag{4.3}$$

Recall that if $F(x) = P\{X_i \leq x\}$, for $i = 1, 2$ and if X_1 and X_2 are independent, their convolution is $F^{2*}(x) := P\{X_1 + X_2 \leq x\}$. We have the following result:

Lemma 1. *Suppose $H(x)$ is in the domain of attraction of the Weibull extreme value distribution, i.e.*

$$\lim_{t \downarrow 0} H(tx)/H(t) = x^\alpha, \alpha > 0. \tag{4.4}$$

Then the convolution H^{2*} of $H(x)$ is again in the domain of attraction of the Weibull extreme value distribution and satisfies

$$\lim_{t \downarrow 0} \frac{H^{2*}(tx)}{H^{2*}(t)} = x^{2\alpha}.$$

[13]Only subsets of the Gumbel class are covered as it is unknown whether this class is closed under addition.

Proof. By (4.3), (4.4) and a transformation of variable

$$\begin{aligned}
\lim_{t\downarrow 0}\frac{\widetilde{H}(y/t)}{H(t)} &= y\int_0^\infty e^{-yx}\lim_{t\downarrow 0}\frac{H(tx)}{H(t)}dx \\
&= y\int_0^\infty e^{-yx}x^\alpha dx \\
&= y^{-\alpha}\Gamma(1+\alpha) \qquad (4.5)
\end{aligned}$$

The justification for interchanging the limit and the integral is as in Feller (1971, XIII.5). Hence the Laplace transform $\widetilde{H}$ varies regularly at infinity with tail index $-\alpha$ whenever H varies regularly at zero with index α. By the convolution theorem for Laplace transforms

$$\widetilde{H}^{2*}(y) = \left(\widetilde{H}(y)\right)^2.$$

Hence, $\widetilde{H}^{2*}(y)$ varies regularly at infinity with index -2α. By (4.5) this implies that the convolution $H^{2*}(x)$ varies regularly at zero with index 2α. □

Thus the class is closed under addition, but the index of regular variation changes!

Remark 1. *This convolution result implies that if portfolios contain an equal number of assets (with returns in that class), the portfolios have the same index of regular variation, while if two portfolios differ with respect to the number of assets, their indices differ.*

4.1.2. *Domain of attraction.* We now set out to prove that the portfolio return distribution is in the domain of attraction of a multivariate extreme value distribution, if the univariate distributions of the composite parts are in the domain of attraction of the Weibull extreme value distribution. For simplicity of writing we assume that the right end point of the distribution function (which must be finite) is zero. Recall that $\alpha = -1/\tau$ when $\tau < 0$. Then for $x > 0$, $i = 1, 2$

$$\lim_{n\to\infty} nP\{-X_i \le xa(n)\} = x^\alpha$$

where $a(x)$ is the inverse function of the distribution function of $-X_i$ at the point $1-1/x$. Hence as $n \to \infty$ for $x, y > 0$

$$\begin{aligned} n^2 P\{-X_1 &\leq xa(n) \text{ and } -X_2 \leq ya(n)\} \\ &= nP\{-X_1 \leq xa(n)\} nP\{-X_2 \leq ya(n)\} \\ &\to \alpha^2 \int_0^y \int_0^x s^{\alpha-1} t^{\alpha-1} ds dt = (xy)^\alpha. \end{aligned} \tag{4.6}$$

We claim that the distribution of the random vector

$$(Q, W) := (\lambda_1 X_1 + \lambda_2 X_2, \mu_1 X_1 + \mu_2 X_2) \ \ (\lambda_i, \mu_i > 0)$$

is in the domain of attraction of an extreme value distribution. In particular we claim that for $x, y > 0$ as $n \to \infty$

$$P^{n^2}\{Q \leq -a(n)x, W \leq -a(n)y\} \to \exp\left(-\alpha^2 \int_S (st)^{\alpha-1}\, ds dt\right),$$

where $S = \{(s,t) : \lambda_1 s + \lambda_2 t \leq x \text{ or } \mu_1 s + \mu_2 t \leq y, s > 0, t > 0\}$. Or equivalently

$$\begin{aligned} \lim_{n\to\infty} n^2 P\{-(\lambda_1 X_1 + \lambda_2 X_2) &\leq a(n)x \text{ or } -(\mu_1 X_1 + \mu_2 X_2) \leq a(n)y\} \\ &= \alpha^2 \int_S (st)^{\alpha-1} ds dt \end{aligned} \tag{4.7}$$

The limit (4.6) entails

$$\begin{aligned} \lim_{n\to\infty} n^2 P\{x_1 a(n) &\leq -X_1 < x_2 a(n), y_1 a(n) \leq -X_2 < y_2 a(n)\} \\ &= \alpha^2 \int_{y_1}^{y_2} \int_{x_1}^{x_2} s^{\alpha-1} t^{\alpha-1} ds dt, \end{aligned} \tag{4.8}$$

for $0 \leq x_1 < x_2 < \infty, 0 \leq y_1 < y_2 < \infty$, i.e. we have convergence for rectangles and for finite unions of rectangles.

It clearly suffices for the proof of (4.7) to give a proof for sets A_m and B_m such that $A_m \subset S \subset B_m$ and $B_m \backslash A_m \downarrow \emptyset, m \to \infty$. Note that in case $\lambda_1\mu_2 \neq \lambda_2\mu_1$ and $x/y \in (\lambda_1/\mu_1, \lambda_2/\mu_2)$, the boundary of S consists of 4 line segments. The vertices are $(0,0)$, $(a, 0) := (\max(\frac{x}{\lambda_1}, \frac{y}{\mu_1}), 0)$, $(0, b) := (0, \max(\frac{x}{\lambda_2}, \frac{y}{\mu_2}))$ and

$$(s_0, t_0) := \frac{1}{\begin{vmatrix} \lambda_1 & \lambda_2 \\ \mu_1 & \mu_2 \end{vmatrix}} \left(\begin{vmatrix} x & \lambda_2 \\ y & \mu_2 \end{vmatrix}, - \begin{vmatrix} x & \lambda_1 \\ y & \mu_1 \end{vmatrix} \right). \tag{4.9}$$

We concentrate on the subarea S_1 with vertices $(0,0),(0,b),(s_0,t_0)$ and $(s_0,0)$. Define for $i=1,\dots,m$

$$s_i := \frac{(m-i)s_0}{m} \text{ and } t_i := \frac{t_0-b}{s_0}s_i+b$$

and the sets

$$L_i := (s,t) : s_i \le s < s_{i-1}, 0 \le t \le t_i$$

and

$$U_i := (s,t) : s_i \le s < s_{i-1}, 0 \le t \le t_{i+1}.$$

Then $\cup_{i=1}^m L_i \subset S_1 \subset \cup_{i=1}^m U_i$, and by (4.8)

$$\lim_{n\to\infty} n^2 P\{a(n)^{-1}(X_1,X_2) \in \cup_{i=1}^m L_i\} = \alpha^2 \int_{(s,t)\in\cup_{i=1}^m L_i} (st)^{\alpha-1}\, dsdt$$

and

$$\lim_{n\to\infty} n^2 P\{a(n)^{-1}(X_1,X_2) \in \cup_{i=1}^m U_i\} = \alpha^2 \int_{(s,t)\in\cup_{i=1}^m U_i} (st)^{\alpha-1}\, dsdt.$$

Since clearly $\cup_{i=1}^m U_i \setminus \cup_{i=1}^m L_i \to \emptyset$ as $m\to\infty$, we have proved (4.7).

Let us now simplify the integral at the right-hand side of (4.7). Write D_1 for the triangle with vertices $(0,b)$, (s_0,t_0), $(0,t_0)$; D_2 for the rectangle with vertices $(0,0)$, $(s_0,0)$, (s_0,t_0), $(0,t_0)$; D_3 for the triangle with vertices $(s_0,0)$, (s_0,t_0), $(c,0)$. Then

$$\alpha^2 \int\int_{(s,t)\in S_1} s^{\alpha-1}t^{\alpha-1} dsdt =: I_1 + I_2 - I_3,$$

where I_1, I_2, and I_3 are the integrals over $D_1\cup D_2$, $D_3\cup D_2$, and D_2 respectively. The first integral can be written as

$$\begin{aligned} I_1 &= \alpha^2 \int_0^{s_0} s^{\alpha-1} \int_0^{b-s(b-t_0)/s_0} t^{\alpha-1} dtds \\ &= \alpha \int_0^{s_0} s^{\alpha-1} \left(b - \frac{s}{s_0}(b-t_0)\right)^\alpha ds \\ &= \alpha \left(\frac{b^2 s_0}{b-t_0}\right)^\alpha \int_0^{1-t_0/b} s^{\alpha-1}(1-s)^\alpha ds. \end{aligned}$$

Define the function φ:

$$\varphi(s,\alpha) := \int_0^s \theta^{\alpha-1}(1-\theta)^\alpha\, d\theta.$$

Note that this is a special case of the incomplete beta function. Then

$$I_1 = \alpha \left(\frac{b^2 s_0}{b-t_0}\right)^\alpha \varphi(1-t_0/b,\alpha),$$

and, similarly

$$I_2 = \alpha \left(\frac{c^2 t_0}{c - s_0} \right)^{\alpha} \varphi(1 - s_0/c, \alpha).$$

It follows that the right-hand side of (4.7) is

$$\alpha \left(\frac{b^2 s_0}{b - t_0} \right)^{\alpha} \varphi(1 - t_0/b, \alpha) + \alpha \left(\frac{c^2 t_0}{c - s_0} \right)^{\alpha} \varphi(1 - s_0/c, \alpha) - (s_0 t_0)^{\alpha}. \tag{4.10}$$

Reformulation of both sides in (4.7) shows that we have proved the following theorem.

Theorem 1. *With the notation given in the introduction of the section, in case* $\tau < 0$*, it follows that the vector* (Q, W) *is in the bivariate domain of attraction of an extreme value distribution, i.e. for* $x, y > 0$

$$\begin{aligned} P^n\{\lambda_1 X_1 + \lambda_2 X_2 \; &\leq \; -a(\sqrt{n})x, \mu_1 X_1 + \mu_2 X_2 \leq -a(\sqrt{n})y\} \\ &\to \; \exp\left\{ -\alpha^2 \int_S (st)^{\alpha-1} ds dt \right\}, \; \text{as } n \to \infty, \end{aligned}$$

where $S = \{(s,t) : \lambda_1 s + \lambda_2 t \leq x \text{ or } \mu_1 s + \mu_2 t \leq y, s > 0, t > 0\}$ *so that the logarithm of the right-hand side equals minus (4.10) with* (s_0, t_0) *as in (4.9).*

So we conclude that the vector $(\lambda_1 X_1 + \lambda_2 X_2, \; \mu_1 X_1 + \mu_2 X_2)$ is in the domain of attraction of a multivariate extreme value distribution, with extreme value index $\tau/2$ and non-discrete spectral measure.

To provide some intuition and interpretation, we conclude this section by calculating the fragility for the two example portfolios.

Example 1. *Consider the zero beta portfolios (2.1) and (2.2), where* $\gamma \in (\frac{1}{2}, 1)$*, and* X, Y *are as* X_1, X_2 *in Theorem 1. Note that in this case we take limits as* $s \uparrow 0$*. Similar to (3.1) and (3.4) we have*

$$\kappa = \lim_{s \uparrow 0} E\{\kappa_s | \kappa_s \geq 1\} = \lim_{s \uparrow 0} \frac{P\{Q > s\} + P\{W > s\}}{P\{Q > s \text{ or } W > s\}} \tag{4.11}$$

First suppose $\frac{1}{2} < \gamma < 1$*. In order to evaluate* κ *we use (4.7) with* $\lambda_1 = \mu_2 = 1 - \gamma, \lambda_2 = \mu_1 = \gamma$ *and* $x = y = 1$ *to find that*

$$\begin{aligned} n^2 P\{-\left(\left(1-\gamma\right) X_1 + \gamma X_2\right) \leq a(n) \text{ or } -\left(\gamma X_1 + \left(1-\gamma\right) X_2\right) \leq a(n)\} \\ \to 2\alpha \left[\gamma\left(1-\gamma\right)\right]^{-\alpha} \int_0^{\gamma} s^{\alpha-1} \left(1-s\right)^{\alpha} ds - 1. \end{aligned}$$

For the marginal distribution we find similarly

$$n^2 P\{-\left(\left(1-\gamma\right) X_1 + \gamma X_2\right) \leq a(n)\}$$

$$\to \alpha^2 \int_0^{1/\gamma} \int_0^{\frac{1-\gamma t}{1-\gamma}} (st)^{\alpha-1} ds dt = \frac{\Gamma(\alpha+1)^2}{[\gamma(1-\gamma)]^\alpha \Gamma(2\alpha+1)}.$$

By combining these results, the FI index is found as

$$\kappa = \frac{2\Gamma(\alpha+1)^2}{\Gamma(2\alpha+1)\{2\alpha\int_0^\gamma s^{\alpha-1}(1-s)^\alpha ds - [\gamma(1-\gamma)]^\alpha\}}. \tag{4.12}$$

Example 2. *Consider again the zero beta portfolios Q and W, but now suppose more specifically that the marginal loss distributions of X and Y are uniformly distributed on $[0,1]$. By computing the areas above the portfolio lines in the upper right hand corner of the unit square by direct integration, one readily finds*

$$P\{Q > t\} = P\{W > t\} = \frac{1}{2}\frac{1}{\gamma(1-\gamma)}(1-t)^2$$

and

$$P\{Q > t, W > t\} = \frac{1}{\gamma}(1-t)^2.$$

Thus

$$FI = \lim_{t\uparrow 1} E\{\kappa_t | \kappa_t \geq 1\} = \frac{1}{1 - \frac{\frac{1}{\gamma}(1-t)^2}{\frac{1}{\gamma(1-\gamma)}(1-t)^2}} = \frac{1}{\gamma} > 1$$

as $\gamma \epsilon (1/2, 1)$ and the portfolios are asymptotically dependent. Note that if one takes $\alpha = 1$ in (4.12), which is the extreme value index for the uniform distribution, one obtains $\kappa = 1/\gamma$, confirming this specific case for the uniform distribution.

Remark 2. *Consider the case of unbalanced portfolios. Suppose the portfolio W is changed into $W = X$, but the portfolio Q remains as it is. For the uniform distribution based example, the unbalanced portfolio implies $P\{W > t\} = 1 - t$, while $P\{Q > t\}$ remains as before. The joint loss probability becomes*

$$P\{Q > t, W > t\} = \left(\frac{1}{2\gamma} + \frac{1}{2}\right)(1-t)^2.$$

In this case $\kappa = \lim_{t\uparrow 1} E\{\kappa_t | \kappa_t \geq 1\} = 1$, as $P\{W > t\}$ is of larger order than the other two probabilities; recall the Remark 1 at the beginning of the subsection. Thus the portfolios are asymptotically independent. One also calculates that the FI in this case is $3/4$, reflecting the weak fragility.

Example 3. *Consider the equal beta asset returns X and Y from (2.3) and (2.4) respectively, with $\beta_x = \beta_y = \beta$. Suppose the market factor R, idiosyncratic risk ε_x*

and ε_y are independently uniformly distributed on $[0,1]$. *In this case, the area of the upper triangle in the* (R,ε_x) *space gives for* $1+\beta > s > \max[1,\beta]$

$$P\{\beta R+\varepsilon_x > s\} = P\{\beta R+\varepsilon_y > s\} = \frac{1}{2\beta}(1+\beta-s)^2 .$$

The joint failure probability is

$$P\{\beta R+\varepsilon_x > s, \beta R+\varepsilon_y > s\} = \frac{1}{3\beta}(1+\beta-s)^3 .$$

Note that the joint failure probability is of smaller order than the individual failure probability due to the fact that the idiosyncratic risks ε_x *and* ε_y*, per definition do not drive both asset returns. This implies that the FI will be below one and that the system's fragility is only weak:*

$$FI = \lim_{s\to 1+\beta} \frac{1}{2}\frac{2\ln\frac{1}{2\beta}(1+\beta-s)^2}{\ln\frac{1}{3\beta}(1+\beta-s)^3} = \frac{2}{3}.$$

4.2. **Continuous returns with light tails, Case $\tau = 0$.** The class of continuous distributions which are in the domain of attraction of the Gumbel extreme value distribution can be divided into two subclasses. One subclass exists of subexponential distributions, and the complement constitutes the other subclass. The subexponential distributions comprise a subset of the distributions which are partly in the domain of attraction of the Gumbel limit law, i.e. for which $\tau = 0$, and contain the entire class of distributions in the domain of the Frechet limit law, i.e. for which $\tau > 0$. The subexponential distributions are exhaustively treated in the next section. If a distribution is subexponential, then for all $\varepsilon > 0$

$$e^{\varepsilon x}(1-F(x)) \to \infty \text{ as } x\to\infty.$$

This justifies the name subexponential, since the tail of $F(x)$ decays at rate slower than any exponential, see Embrechts et al. [10]. In particular for Weibull type distributions we have $1-F(x)\sim \exp\{-x^{\beta}\}$, so that the above condition holds if $0<\beta<1$.

In case $\beta = 1$, we are in a class similar to the exponential distribution, see De Vries [33]. The case $\beta > 1$ is treated as an example of the superexponential class defined below. This class of distributions also comprises the normal distribution as a special case. The case of the normal distribution was first treated by Sibuya [34]. The complement of the subexponential class within the domain of attraction of the Gumbel law is, however, larger than the superexponential class. But its closure properties under addition are not known. The treatment below is motivated by the results in Rootzen [30], for which the closure property holds.

4.2.1. *The superexponential distributions.* The definition of the superexponential class is based on Rootzen [30]. Consider the following class of distributions.

Definition 1. *A distribution function F is superexponential if it satisfies the following conditions:*

(1) *F has a density f which satisfies*

$$f(x) \sim Kx^{\alpha}e^{-x^p}, \text{ as } x \to \infty, \text{ with } p > 1, \tag{4.13}$$

for some constants $K > 0$, α and p.

(2) *The function D defined by $D(x) = f(x)e^{x^p}$ satisfies*

$$\limsup_{x\to\infty} \left| \frac{xD'(x)}{D(x)} \right| < \infty. \tag{4.14}$$

For $x < 0$, it is assumed that (4.13) and (4.14) hold with the same p, but possibly with different $D(x)$, α and K.

For further reference let q be the conjugate exponent of p, defined by $1/p+1/q = 1$. Note that $p > 1$ in the above definition, which implies $q > 1$.

For the so defined class of superexponential distributions, Rootzen [30] proves that the portfolios with n assets and positive portfolio weights summing to one $\sum_{i=1}^n \lambda_i = \sum_{i=1}^n \mu_i = 1$, are in the domain of attraction of the Gumbel limit law. Moreover, Rootzen showed that

$$P\{\sum_{i=1}^{n} \lambda_i X_i > x\} \sim A(\lambda) \left(\frac{x}{(\sum_{i=1}^n \lambda_i^q)^{1/q}} \right)^{\theta} \exp\left(-\frac{x^p}{(\sum_{i=1}^n \lambda_i^q)^{p/q}} \right), \tag{4.15}$$

as $x \to \infty$, where

$$\theta = n(\frac{1}{2} + \alpha - \frac{p}{2q}) - \frac{p}{2}$$

and $A(\lambda)$ equals

$$K^n \left(\frac{2\pi}{p(p-1)p^{-q}} \right)^{(n-1)/2} p^{-nq/2+(q/p-1)/2} \left(\frac{\prod_{i=1}^{n} \lambda_i}{(\sum_{i=1}^n \lambda_i^q)^{2/q}} \right)^{(\alpha+1/2)p/q-1/2}.$$

We can now turn to the question of asymptotic dependence. The derivation follows the same strategy as was used by Sibuya [34] in the proof for the normal distribution. Suppose that

$$(\sum_{i=1}^{n}\lambda_i^q)^{1/q} \geq (\sum_{i=1}^{n}\mu_i^q)^{1/q}. \tag{4.16}$$

If the opposite inequality holds, it is treated similarly. First note that the conditional expectation measure (3.1) can be bounded from above as follows:

$$\begin{aligned} E\left\{\kappa_s|\kappa_s \geq 1\right\} &= \frac{1}{1-\frac{P\left\{\sum_{i=1}^{n}\lambda_iX_i>s,\sum_{i=1}^{n}\mu_iX_i>s\right\}}{P\left\{\sum_{i=1}^{n}\lambda_iX_i>s\right\}+P\left\{\sum_{i=1}^{n}\mu_iX_i>s\right\}}} \\ &\leq \frac{1}{1-\frac{P\left\{\sum_{i=1}^{n}(\lambda_i+\mu_i)X_i>2s\right\}}{P\left\{\sum_{i=1}^{n}\lambda_iX_i>s\right\}+P\left\{\sum_{i=1}^{n}\mu_iX_i>s\right\}}} \\ &\leq \frac{1}{1-\frac{P\left\{\sum_{i=1}^{n}(\lambda_i+\mu_i)X_i/2>s\right\}}{P\left\{\sum_{i=1}^{n}\lambda_iX_i>s\right\}}}. \end{aligned} \tag{4.17}$$

Consider first the case

$$\lambda_i = c\mu_i \text{ for } i = 1, ..., n \text{ and some } c > 0. \tag{4.18}$$

Since the wealth constraints imply $\sum_{i=1}^{n}\lambda_i = \sum_{i=1}^{n}\mu_i = 1$, it follows that in this case $c = 1$ and $\lambda_i = \mu_i$ for all i. But the case of complete dependence is obviously not of interest. Hence, assume that (4.18) does not hold. In that case Minkowski's inequality gives

$$(\sum_{i=1}^{n}\lambda_i^q)^{1/q} + (\sum_{i=1}^{n}\mu_i^q)^{1/q} > (\sum_{i=1}^{n}(\lambda_i+\mu_i)^q)^{1/q}. \tag{4.19}$$

Note that (4.16) and (4.19) imply

$$(\sum_{i=1}^{n}\lambda_i^q)^{1/q} > \frac{1}{2}(\sum_{i=1}^{n}(\lambda_i+\mu_i)^q)^{1/q}.$$

From Rootzen [30] we have as $s \to \infty$

$$P\{\sum_{i=1}^{n}(\lambda_i+\mu_i)X_i/2 > s\} \sim$$

$$A(\lambda+\mu)\left(\frac{2s}{(\sum_{i=1}^{n}(\lambda_i+\mu_i)^q)^{1/q}}\right)^{\theta}\exp\left(-\frac{2^ps^p}{(\sum_{i=1}^{n}(\lambda_i+\mu_i)^q)^{p/q}}\right).$$

Note that, if

$$P\{Z_j > s\} \sim K_jx^{\theta}\exp\left(-\left(\frac{s}{a_j}\right)^p\right),\ s \to \infty,\ \text{for } j = 1, 2$$

and if $a_1 > a_2$, then

$$\lim_{s\to\infty}\frac{P\{Z_2 > s\}}{P\{Z_1 > s\}} = 0.$$

A combination of the above gives

$$\lim_{s\to\infty} \frac{P\{\sum_{i=1}^{n}(\lambda_i+\mu_i)X_i/2 > s\}}{P\{\sum_{i=1}^{n}\lambda_i X_i > s\}} = 0. \tag{4.20}$$

In view of (4.17), we have proved the following theorem.

Theorem 2. *Suppose the i.i.d. random asset returns X_i have a distribution in the superexponential class as in Definition 1, then for portfolios $\sum_{i=1}^{n}\lambda_i X_i$ and $\sum_{i=1}^{n}\mu_i X$ it holds that $\kappa = \lim_{s\to\infty} E\{\kappa_s|\kappa_s \geq 1\} = 1$.*

Example 4. *Consider again the zero beta portfolios (2.1) and (2.2). Suppose that the marginal loss distributions of X and Y are standard normally distributed. One readily finds*

$$P\{Q>s\} = P\{W>s\} \sim \frac{1}{\sqrt{2\pi}} \frac{\sqrt{(1-\gamma)^2+\gamma^2}}{s} \exp\left(-\frac{s^2/2}{(1-\gamma)^2+\gamma^2}\right)$$

as $t\to\infty$. Similarly

$$P\{Q+W>2s\} = P\{X+Y>2s\} \sim \frac{1}{2\sqrt{\pi}s}\exp(-s^2)$$

and hence (4.20) follows. Thus the system is only weakly fragile. In order to compute the joint failure probability for the FI we need more precision in computing the joint failure probability than is provided through the above bound. Ruben [31] *gives accurate multivariate first order expressions for the joint probabilities; see also (5.2) below. Using these approximations, we have*

$$P\{Q>s, W>s\} \sim \frac{1}{2\pi s^2}\frac{1}{\sqrt{(1-\gamma)^2+\gamma^2}}\exp(-s^2)$$

as $s\to\infty$. Thus, recalling $\gamma \neq 1/2$,

$$\begin{aligned} FI &= \lim_{s\to\infty}\frac{1}{2}\frac{\log P\{Q>s\}+\log P\{W>s\}}{\log P\{Q>s, W>s\}} \\ &= \frac{1}{2}\frac{1}{(1-\gamma)^2+\gamma^2} < 1. \end{aligned}$$

Example 5. *Consider the equal beta asset returns X and Y from (2.3) and (2.4) respectively, with $\beta_x = \beta_y = \beta$. Suppose the market factor R, idiosyncratic risk ε_x and ε_y are independently standard normally distributed. Using similar arguments as in Example 4, we find that the FI again indicates weak fragility:*

$$FI = \lim_{s\to\infty}\frac{\ln P\{\beta R+\varepsilon_x>s\}}{\ln P\{\beta R+\varepsilon_x>s, \beta R+\varepsilon_y>s\}} = \frac{1+2\beta^2}{2+2\beta^2}$$

as $t\to\infty$.

4.3. **Continuous returns with heavy tails.** Since there is no uniform terminology in the literature, we will focus on the well known class of subexponential distributions S and formulate the case $\tau > 0$ as a special case. The class comprises all the distributions in the domain of attraction of the extreme value distribution with $\tau > 0$ (Frechet limit law). The rest of the class S are distributions which are in the maximum domain of attraction with $\tau = 0$ (Gumbel limit law); but it excludes distributions like the superexponentials. The case of subexponential distributions in the domain of attraction of the Gumbel limit law when $\tau = 0$, was studied before by Willekens and Resnick [35].

The class S is defined by the property

$$P(X_1 + X_2 > x) \sim 2P(X_1 > x) \; as \; x \to \infty, \tag{4.21}$$

where X_1, X_2 are i.i.d. random variables.[14] Since S is closed under asymptotic equivalence of the tail of the d.f., the class is closed under addition of i.i.d. random variables. The result (4.21) was obtained by Feller [12] for distributions with regularly varying tails at infinity (when $\tau > 0$).

The subclass of S for which $\tau > 0$, are the distribution functions F which have a first order term similar to the Pareto distribution, i.e.

$$F(s) = 1 - s^{-\alpha} L(s) \quad \text{as } s \to \infty, \tag{4.22}$$

where $L(s)$ is a slowly varying function such that

$$\lim_{t\to\infty} \frac{L(ts)}{L(t)} = 1, \quad s > 0. \tag{4.23}$$

It is easy to see that conditions (4.22)-(4.23) are equivalent to

$$\lim_{t\to\infty} \frac{\overline{F}(ts)}{\overline{F}(t)} = s^{-\alpha}, \quad \alpha > 0, \; s > 0, \tag{4.24}$$

i.e., the tail of the distribution function $\overline{F} := 1 - F$ varies regularly at infinity.[15]

[14] Alternatively subexponential distributions are characterized by the fact that the ratio of probability of the sum and probability of the maximum exceeding a threshold from an i.i.d. sample converges to one for large thresholds, implying that samples of subexponentials are dominated by the largest observations.

[15] The tail index α can be interpreted as the number of bounded distributional moments. And as not all moments are bounded, we speak of heavy tails. Distributions like the Student-t, F-distribution, Burr distribution, sum-stable distributions with unbounded variance all fall into

Apart from the distributions functions with a regularly varying tail, the class S contains distributions like the lognormal and the Weibull distribution $1-\exp\{-x^\beta\}$ with parameter $\beta < 1$. The class S is of importance in ruin theory in insurance, queueing theory and other areas of applied probability. For applications the reader is referred to Asmussen [2], Embrechts et al. [10] or Rolski et al. [29].

For a portfolio $Q_n = \sum_{i=1}^n \lambda_i X_i$ with positive portfolio weights λ_i and i.i.d. subexponential X_i's, it is well known that as $s \to \infty$

$$P\{\sum_{i=1}^n \lambda_i X_i > s\} \sim \sum_{i=1}^n P\{\lambda_i X_i > s\}. \tag{4.25}$$

See e.g. Tang [32], or Geluk and de Vries [15]. We have the following result for the fragility in this case.

Theorem 3. *Suppose the random variables X, X_i $(i = 1, \ldots, n)$ are i.i.d. with a subexponential distribution. Then for portfolios $Q_n = \sum_{i=1}^n \lambda_i X_i$ and $W_n = \sum_{i=1}^n \mu_i X$ we have*

$$\kappa = \lim_{s\to\infty} E\{\kappa_s | \kappa_s \geq 1\} = 1 + \lim_{s\to\infty} \frac{\sum_{i=1}^n P\{X > \frac{s}{\lambda_i} \vee \frac{s}{\mu_i}\}}{\sum_{i=1}^n P\{X > \frac{s}{\lambda_i} \wedge \frac{s}{\mu_i}\}}. \tag{4.26}$$

Proof of Theorem 3. In order to prove (4.26), we need to prove the relations (4.27) and (4.28) below in case $x = y = 1$.

It is well known (see Theorem 5 in [15]) that under the given conditions as $s \to \infty$, for $n = 2$ and $x, y > 0$ fixed

$$P\{\sum_{i=1}^n \lambda_i X_i > sx \cap \sum_{i=1}^n \mu_i X_i > sy\} \sim \sum_{i=1}^n P\{X > \frac{sx}{\lambda_i} \vee \frac{sy}{\mu_i}\} \tag{4.27}$$

and

$$P\{\sum_{i=1}^n \lambda_i X_i > sx \cup \sum_{i=1}^n \mu_i X_i > sy\} \sim \sum_{i=1}^n P\{X > \frac{sx}{\lambda_i} \wedge \frac{sy}{\mu_i}\}. \tag{4.28}$$

Since the proof of the second relation is similar, we only prove (4.27). The proof is by induction. For $n = 2$, (4.27) follows from Theorem 5 in Geluk and de Vries [15]. Note that linear combinations of i.i.d. subexponential random variables with positive coefficients have a subexponential distribution function. This follows from a combination of Theorems 1 and 2 in Embrechts and Goldie [9]. See also Corollary 1 in [15].

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this class. It can be shown that the distribution of the stationary solution to ARCH and GARCH processes also belongs to this class, see De Haan et al. [17] for a proof.

Next we show that the result holds with $n+1$ instead of n. Since $x, y > 0$ are arbitrary, we may assume that the additional term satisfies $\lambda_{n+1} = \mu_{n+1} = 1$. It follows from (4.27) that $Q_n \wedge W_n$ has a subexponential distribution function. Application of Theorem 1 in [9] then gives

$$\begin{aligned} P\{Q_n + X &> s, W_n + X > s\} = P\{(Q_n \wedge W_n) + X > s\} \sim \\ P\{(Q_n \wedge W_n) &> s\} + P\{X > s\} \sim \sum_{i=1}^{n+1} P\{X > \frac{s}{\lambda_i} \vee \frac{s}{\mu_i}\}, \end{aligned}$$

where the last equivalence follows from the induction hypothesis. This completes the induction step. □

We present several examples to illustrate the use of Theorem 3 in the (heavy tailed) cases $\tau > 0$ and $\tau = 0$.

4.3.1. *Frechet domain of attraction $\tau > 0$.* The subexponential distributions in the Frechet domain of attraction are regularly varying as defined in (4.24). For this class and two risk factors (4.26) specializes to (4.29) below.

Example 6. *In case two portfolios have the structure $Q_2 = \lambda_1 X_1 + \lambda_2 X_2$ and $W_2 = \mu_1 X_1 + \mu_2 X_2$ where X_1 and X_2 are i.i.d. with a regularly varying distribution function tail with tail index α, using (4.26) and the regular variation of the d.f. tail it follows that*

$$FI = 1 + \frac{(\lambda_1 \wedge \mu_1)^\alpha + (\lambda_2 \wedge \mu_2)^\alpha}{(\lambda_1 \vee \mu_1)^\alpha + (\lambda_2 \vee \mu_2)^\alpha} \tag{4.29}$$

The zero beta example for the regularly varying distributed returns as given in de Vries [33] *is a special case. Thus for the portfolios (2.1) and (2.2), (4.29) implies that*

$$\kappa = 1 + (\frac{1}{\gamma} - 1)^{\pm\alpha}, \;\; as\ \gamma > (<)\frac{1}{2}. \tag{4.30}$$

So the system is strongly fragile.

Remark 3. *It pays for intuition to demonstrate this result in a more direct way. The Feller* [12] *additivity result (4.21) implies that $P\{Q > s\} \sim [\gamma^\alpha + (1-\gamma)^\alpha] s^{-\alpha} L(s)$ for large s. What transpires is that the probability to be above the portfolio line is, to a first order, dictated by the marginal probabilities $P\{\gamma Y > s\}$ and $P\{(1-\gamma)X > s\}$. The rest of the area does not contribute significantly (has* *mass of lower order). Similarly, the joint probability* $1 - P\{Q \leq s, W \leq s\}$ *is determined by the amount of the probability mass along the* X *and* Y *axes, starting from where the failures area cuts the* X *and* Y *axes closest to the origin, i.e. at* $(s/\gamma, 0)$ *and* $(0, s/\gamma)$. *Thus* $1 - P\{Q \leq s, W \leq s\} \sim 2\gamma^{\alpha} s^{-\alpha} L(s)$ *for large* s. *A direct proof of this result based on regular variation rather than subexponentiality, is provided in the Appendix for the benefit of the reader. Taking ratios then gives (4.30).*

Example 7. *Consider the equal beta asset returns* X *and* Y *from (2.3) and (2.4) respectively, with* $\beta_x = \beta_y = \beta$. *Suppose the market factor* R, *idiosyncratic risk* ε_x *and* ε_y *are i.i.d. with regularly varying tail and tail index* α. *Application of Theorem 3 shows that the system is strongly fragile and the fragility measure reads*

$$FI = \kappa = 1 + \frac{\beta^{\alpha}}{\beta^{\alpha} + 2}. \tag{4.31}$$

For the subexponential distributions in the Frechet domain of attraction, i.e. which are regularly varying as defined in (4.24), (4.26) specializes to (4.29), and hence the asymptotic dependency is generic. But for the subexponential distributions with a rapidly varying distribution tail, i.e. those which are in the Gumbel domain of attraction, the results are more subtle as the examples in the next subsection show.

4.3.2. *Gumbel domain of attraction* $\tau = 0$. For the subexponential distributions in the domain of the Gumbel distribution it depends on the specifics of the case whether the system exhibits asymptotic dependence or independence.

Example 8. *Consider the zero beta portfolios (2.1) and (2.2). Suppose that the returns* X *and* Y *are subexponentially distributed and are in the maximum domain of attraction of the Gumbel distribution. From Theorem 3 and the rapid variation of* $1-F$, *it follows that* $\kappa = 1$ *(assume* $\gamma \neq 1/2$*). To compute the* FI *index, assume in particular that the returns follow a subexponential Weibull type distribution*

$$F(x) = 1 - \exp\{-x^{\xi}\}, \ x > 0 \text{ with } 0 < \xi < 1.$$

Using (4.27), we find weak fragility

$$FI = \lim_{s \to \infty} \frac{1}{2} \frac{\ln P\{Q > s\} + \ln P\{W > s\}}{\ln P\{Q > s, W > s\}} = \left(\frac{1-\gamma}{\gamma} \right)^{\xi}.$$

Example 9. *Consider again the equal beta asset returns X and Y, with $\beta_x = \beta_y = \beta$. Suppose the market factor R, idiosyncratic risks ε_x and ε_y are i.i.d. with a subexponential distribution in the Gumbel domain of attraction and support on $\mathbb{R}^+$; for example a lognormal distribution. The example is used to build some intuition for the Theorem 3 result in the Gumbel domain of attraction case. To do this, we make use of the rule (4.25) and the following result*

$$\lim_{s\to\infty} \frac{P\{R > ts\}}{P\{R > s\}} = \begin{cases} 0 & \text{if} \quad t > 1 \\ \infty & \text{if} \quad 0 < t < 1 \end{cases}, \tag{4.32}$$

which stems from the rapid variation of the upper tail, see Embrechts et al. [10]*. The first result is ($i = x, y$)*

$$P\{\beta R + \varepsilon_i > s\} \sim \begin{cases} P\{\beta R > s\} \text{ if } \beta > 1 \\ 2P\{R > s\} \text{ if } \beta = 1 \\ P\{R > s\} \text{ if } \beta < 1 \end{cases}. \tag{4.33}$$

To show this, note that by (4.25)

$$P\{\beta R + \varepsilon_x > s\} = P\{\beta R + \varepsilon_y > s\} \sim P\{\beta R > s\} + P\{R > s\}.$$

Dividing by $P\{\beta R > s\}$ in case $\beta > 1$, and by $P\{R > s\}$ in case $\beta < 1$, taking limits and using (4.32) yields the asymptotic equivalences (4.33).

To evaluate the joint failure probability, we resort to using bounds. Note that an upper bound for the joint failure probability is

$$P\{\beta R + \varepsilon_x > s, \beta R + \varepsilon_y > s\} \le P\{2\beta R + \varepsilon_x + \varepsilon_y > 2s\}.$$

By (4.25) this bound is asymptotic to

$$P\{\beta R + \frac{1}{2}\varepsilon_x + \frac{1}{2}\varepsilon_y > s\} \sim P\{\beta R > s\} + 2P\{\frac{1}{2}R > s\} \tag{4.34}$$

for large s. For $\beta < 1$, by the rule (4.32)

$$\lim_{s\to\infty} \frac{P\{\beta R > s\}/P\{R > s\} + 2P\{\frac{1}{2}R > s\}/P\{R > s\}}{2P\{\beta R > s\}/P\{R > s\} + 2} = 0.$$

Thus for $\beta < 1$

$$\kappa \le \lim_{s\to\infty} \frac{1}{1 - \frac{P\{2\beta R + \varepsilon_x + \varepsilon_y > 2s\}}{P\{\beta R + \varepsilon_x > s\} + P\{\beta R + \varepsilon_y > s\}}} = 1.$$

The two assets are therefore asymptotically independent.

For $\beta = 1$, combine the upper bound (4.34) with the lower bound (recall the random variables are non-negative)

$$P\{\beta R + \varepsilon_x > s, \beta R + \varepsilon_y > s\} \geq P\{\beta R > s\}, \tag{4.35}$$

to sandwich κ

$$\frac{4}{3} = \lim_{s\to\infty} \frac{1}{1 - \frac{P\{R>s\}}{P\{R+\varepsilon_x>s\}+P\{R+\varepsilon_y>s\}}} \leq \kappa \leq \lim_{s\to\infty} \frac{1}{1 - \frac{1+2P\{\frac{1}{2}R>s\}/P\{R>s\}}{2+2}} = \frac{4}{3}.$$

Hence $FI = 4/3$, meaning that the two assets are asymptotically dependent.

Lastly, we investigate the case $\beta > 1$. Using the upper bound again and dividing by $P\{\beta R > s\}$ gives

$$\kappa \geq \lim_{s\to\infty} \frac{1}{1 - \frac{1}{2+2P\{R>s\}/P\{\beta R>s\}}} = 2.$$

Thus if $\beta > 1$ the dependency is maximal (in the tail area).

One can also verify these results directly by application of Theorem 3 with $\lambda_1 = \mu_1 = \beta$, $\lambda_2 = 1, \mu_2 = 0, \lambda_3 = 0$, and $\mu_3 = 1$. This shows that

$$\kappa = 1 + \lim_{s\to\infty} \frac{P\{R > s/\beta\}}{P\{R > s/\beta\} + 2P\{R > s\}}.$$

Then use (4.32) to find the κ value.

Remark 4. *Note the stark contrast between this last example and the example 7 for the subclass of regular varying distributions. In the example 7 the linkage measure changes continuously with the value of β, cf. (4.31). At $\beta = \{0, 1, \infty\}$ the κ values for the two cases coincide, but at intermediate values the subexponential distributions in the maximum domain of attraction of the Gumbel either imply maximal asymptotic dependence or asymptotic independence. Also note that for both cases the correlation coefficient is $\beta^2/(1+\beta^2)$ and hence changes continuously with β as well.*

4.4. **Summary.** We summarize these results by returning to the question of systemic risk. As noted before, in the banking industry the risks of an individual bank are captured by the VaR measure. The systemic risk is that there may occur multiple failures. Let the critical VaR failure level be denoted by s. For the zero beta portfolio example, the probability of an individual failure reads

$$P\{Q > s\} = P\{W > s\} = q,$$

Table 1. Individual and Systemic Failure Probabilities

$q \sim$		$P\{Q > s, W > s\} \sim$	limit
Uniform			
$\frac{1}{2}\frac{1}{\gamma(1-\gamma)}(1-s)^2$	$\asymp$	$\frac{1}{\gamma}(1-s)^2$	$s \to 1-$
Pareto, $\alpha > 1$			
$[(1-\gamma)^\alpha + \gamma^\alpha]\, s^{-\alpha}$	$\asymp$	$2(1-\gamma)^\alpha s^{-\alpha}$	$s \to \infty$
Unit exponential			
$\frac{\gamma}{2\gamma-1}\exp\left(-\frac{s}{\gamma}\right)$	$\gg$	$\frac{1}{2\gamma-1}\exp(-2s)$	$s \to \infty$

say. Thus q is the inverse of the VaR measure s. Focussing on the probability q rather than the VaR level is useful for comparison with the systemic risk. The probability of a systemic breakdown is $P\{Q > s, W > s\}$. On the basis of the above, we have collected the following results in the Table 1.[16]

For $\gamma \in (1/2, 1)$ the systemic risk in case of the exponential distribution is of lower order than the univariate VaR risk. But for the Pareto and uniform distribution the systemic risk is of the same order as the univariate VaR risk. In fact, we have the following summary result.

Proposition 1. *Suppose that the FSI' asset and liability risks are affine combinations of independent risk drivers. The Basel II and Solvency II per individual FSI based VaR criterion is to a first order also the appropriate criterion for safeguarding against systemic risk in case the marginal distributions of the risk drivers are from the class of the superexponential distributions since the systemic risk is relatively unimportant.*[17] *If the risk drivers are in the domain of attraction of the Frechet or Weibull limit distribution, however, the systemic risk is of the same order as the marginal VaR risk.*[18] *From a system's perspective the per individual FSI based VaR criterion is then overly conservative.*

[16]Only the dominant first order terms are represented, $f \asymp g$, means that f/g is bounded away from 0 and ∞, $f \gg g$ means $f/g \to \infty$.

[17]The subexponential distributions in the maximum domain of the Gumbel require a case by case treatment as the examples showed.

[18]For the Weibull case, all bank portfolios must have an exposure to the same risk drivers for this result to hold. Otherwise the orders of magnitude differ and systemic risk is relatively unimportant.

For two reasons it is important to have a scale like the FI as a measure of the potential for simultaneous failures. First, like the individual FSI based VaR measure, however imperfect, it is necessary to have a measure to be able to discuss and compare the stability of different financial networks. Measurement predates eventual regulatory and supervisory action. Second, we argued before that the individual FSI based approach can be overly conservative as it counts the joint failure as often as the number of FSI in the system. In a non-normal world, the systemic risk can be of the same order as the univariate VaR risk.

5. ECONOMICS

We return to the economic issues outlined before and discuss implications of the probability results. We derive economically relevant limit copulas, we investigate different banking networks and their FI scale, we briefly deal with non-linear instruments and discuss sunspot equilibria.

5.1. **Copulas.** Copulas have gained in popularity as a measure of dependency in economics and finance due to the dismay over the standard use of correlation. For this reason it appears important to connect the concept of a copula to our measure. Recall that the FI only has something to say about the tail region, while a copula is a global dependency measure. By taking limits, the two concepts can be connected

$$\kappa = \lim_{s\to\infty} E\left\{\kappa_s | \kappa_s \geq 1\right\} = \lim_{p\uparrow 1} \frac{2(1-p)}{1-C(p,p)}, \tag{5.1}$$

where $C(.,.)$ is the bivariate copula. The Morgenstern copula for example,

$$C(x,y;\delta) = xy[1+\delta(1-x)(1-y)], \ \ -1 \leq \delta \leq 1,$$

implies asymptotic independence as $\kappa = 1$. whereas The logistic copula,

$$C(x,y;\beta) = \exp[-\{(-\ln x)^{1/\beta} + (-\ln y)^{1/\beta}\}^{\beta}], \ 0 < \beta \leq 1,$$

induces asymptotic dependence with $\kappa = 2^{1-\beta} \geq 1$. Longin and Solnik [25] used the logistic copula to estimate the dependency between equity markets. There exist, however, many other copulas and the question is which copula should be chosen for the analysis of systemic stability.

Let us come back to the example portfolios and derive an economic motivated property of the copula. Recall the zero beta portfolios (2.1) and (2.2). Let X and

Y be i.i.d. random variables with (identical) regularly varying tails as in (4.22). Suppose $\gamma \in (1/2, 1)$ and let $\lim_{t,s\to\infty} t/s = c > 0$, some positive constant. For the joint distribution function $P\{Q \leq s, W \leq t\}$, one has

$$P\{Q \leq s, W \leq t\} = \begin{cases} 1 - [\gamma^\alpha + (1-\gamma)^\alpha] c^{-\alpha} s^{-\alpha} L(s) & \text{as} \quad \frac{1}{c} > \frac{\gamma}{1-\gamma} \\ 1 - \gamma^\alpha [1 + c^{-\alpha}] s^{-\alpha} L(s) & \text{as} \quad \frac{1-\gamma}{\gamma} \leq \frac{1}{c} \leq \frac{\gamma}{1-\gamma} \\ 1 - [\gamma^\alpha + (1-\gamma)^\alpha] s^{-\alpha} L(s) & \text{as} \quad \frac{1}{c} < \frac{1-\gamma}{\gamma} \end{cases}$$

as $s, t \to \infty$. Recall the Remark 3, where we argued that for $c = 1$, $P\{Q \leq s, W \leq s\} \sim 1 - 2\gamma^\alpha s^{-\alpha} L(s)$.

Alternatively, we verify the limit on the right hand side in (5.1). The limit copula associated with $P\{Q \leq s, W \leq t\}$ reads

$$C_{Q,W}(x,y) = \begin{cases} y & \text{as } \frac{1-y}{1-x} > (\frac{\gamma}{1-\gamma})^\alpha \\ 1 - \frac{\gamma^\alpha}{\gamma^\alpha + (1-\gamma)^\alpha}[2 - x - y] & \text{as } (\frac{1-\gamma}{\gamma})^\alpha \leq \frac{1-y}{1-x} \leq (\frac{\gamma}{1-\gamma})^\alpha \\ x & \text{as } \frac{1-y}{1-x} < (\frac{1-\gamma}{\gamma})^\alpha \end{cases}$$

for x and y in a neighborhood of 1.[19] Note that if α is close to zero, $C_{Q,W}(x,y)$ is close to the maximal dependent copula $\min(x,y)$. While if α is large, $C_{Q,W}(x,y)$ is close to $x + y - 1$, which for x, y in a neighborhood of 1 is the first order Taylor approximation to the independent copula.[20] It readily follows that[21]

$$\lim_{p\uparrow 1} \frac{2(1-p)}{1 - C(p,p)} = \lim_{p\uparrow 1} \frac{2(1-p)}{1 - [1 - \frac{\gamma^\alpha}{\gamma^\alpha + (1-\gamma)^\alpha} 2(1-p)]} = 1 + (\frac{1}{\gamma} - 1)^\alpha.$$

Note how the limit copula differs from the standard type of copulas. The upshot of all this is that economically relevant copulas may be quite different from the popular functional forms from the literature. The choice of copulas in applied work is partly driven by the ease to which these lend themselves to estimation. But this may not always yield the economically relevant specification. For the economic problem of systemic risk in the financial sector, we now at least have a

[19]A similar result holds for the case of the distributions in the domain of attraction of the Weibull, i.e. when $\tau < 0$. This is particularly easy to see for the Example 2 of uniform distributions.

[20]These are the Frechet Hoeffding bounds.

[21]Note that for the case of the uniform distributions discussed in the example 2, one finds $C(p,p) = 1 - 2\gamma(1-p)$.

theory about the relevant economic (limiting) functional form of the copula in the failure region.[22]

5.2. **options.** Consider how the analysis has to be adapted when the portfolio includes non-linear instruments like an option. Note this does not necessarily destroy the linearity of a portfolio, as this is still linear in the option's return. Suppose, however, that one portfolio consists in a call option which is held till maturity and that another portfolio comprises the underlying. Then one has to take care of the nonlinearities due to the fact that the option may be out of the money. Let S_t denote the share price at time t and let C_t denote the price of the (European) call option. Suppose the call is at the money at the time of purchase t. The option expires at time $T > t$. If held till maturity, the gross returns on the underlying and the call are respectively

$$\frac{S_T}{S_t} \text{ and } \max[0, (\frac{S_T}{S_t} - 1)\frac{S_t}{C_t}],$$

where $S_t/C_t > 1$. Suppose that S_T/S_t follows a (continuous) distribution $F(S_T/S_t)$ say, for which the left tail is in the domain of attraction of the Weibull distribution, and

$$\lim_{s \downarrow 0} P\{\frac{S_T}{S_t} \leq s\} = 0,$$

while $F(1) > 0$. So there is a non-zero probability that the option ends out of the money. It follows that

$$\lim_{s \downarrow 0} P\{\max[0, (\frac{S_T}{S_t} - 1)\frac{S_t}{C_t}] \leq s\} = F(1).$$

Note moreover that the joint probability for $s < 1$ collapses to

$$\lim_{s \downarrow 0} P\{\frac{S_T}{S_t} \leq s, \max[0, (\frac{S_T}{S_t} - 1)\frac{S_t}{C_t}] \leq s\} = P\{\frac{S_T}{S_t} \leq s\}.$$

Since if $S_T < S_t$, the option return is zero and the hence

$$\max[0, (\frac{S_T}{S_t} - 1)\frac{S_t}{C_t}] \leq s < 1$$

is automatically satisfied. It follows that $FI = 1/2$, i.e. the two portfolios are asymptotically independent.

[22] In case there are more than two assets, the middle line segment in the limit copula is cut up into multiple straight line segments with different slopes, where the kinks are driven by where the portfolio hyperplanes cut the portfolio axes.

To give one other example, consider two portfolios each consisting of an at the money call option, where each option is written on a different stock. Suppose the two stock returns are independently distributed, with distribution functions $F(.)$ and $G(.)$. Thus the loss returns on the two stocks are evidently asymptotically independent. Assume again that the two stock returns have continuous distributions and that the there is a non-zero probability that the stock prices fall below their level at the date of purchase of the option, i.e. $F(1) > 0$, and $G(1) > 0$.[23] It follows that for the two option portfolios

$$\lim_{s \downarrow 0} E\left\{\kappa_s | \kappa_s \geq 1\right\} = \frac{1}{1 - \frac{F(1)G(1)}{F(1)+G(1)}} > 1,$$

since $F(1)G(1) > 0$. Thus the option portfolios are asymptotically dependent even though the stock portfolios are asymptotically independent! While these cases may appear contrived, they are actually quite relevant given the huge exposures of the banking book to derivatives.

5.3. **sequence of FSI networks and large portfolios.** In this section we investigate how different dependence measures and different tail shapes affect the ranking of different FSI networks regarding their systemic risk. The network configurations we discuss are motivated by the cases discussed in the banking literature on systemic risk, such as Rochet and Tirole [28], Allen and Gale [1] and Freixas, Parigi and Rochet [13].

Suppose there are four projects with returns: $4U$, $4X$, $4Y$, and $4T$. The returns to the projects U, X, Y, and T are random and are independently distributed. We investigate and compare the cases where these random variables either follow a standard normal distribution or have a Student-t distribution with α degrees of freedom. The projects can be broken down into four equally sized participations. There are also four distinct banks with returns: B_1, B_2, B_3, and B_4. Consider the following cases of syndicated loans.

Case 1. *Each bank finances one entire project. In particular the returns are*

$$B_1 = 4U, \; B_2 = 4X, \; B_3 = 4Y, \; B_4 = 4T,$$

where we identify the portfolio return of each bank with its name tag B_i.

[23]The options have the same expiration date and are bought on the same date.

Case 2. *Each bank participates in two projects. The specific portfolios are*

$$B_1 = 2U + 2X, B_2 = 2X + 2Y,\ B_3 = 2Y + 2T,\ B_4 = 2T + 2U.$$

Case 3. *There is further diversification. In particular, the portfolios are*

$$B_1 = 2U + X + Y, B_2 = 2X + Y + T,\ B_3 = 2Y + T + U,\ B_4 = 2T + U + X.$$

Case 4. *All bank portfolios are fully diversified:*

$$B_i = U + X + Y + T,\ \textit{for } i = 1, ..., 4.$$

Note that these syndicates represent different network configurations as discussed in Allen and Gale [1] and Freixas, Parigi and Rochet [13], but in this case for loan syndication rather than for the interbank market. In particular the last portfolio is reminiscent to the diversified lending case and the second portfolio resembles the credit chain funding of Freixas et al. [13].

5.3.1. *normally distributed returns.* Suppose the project returns U, X, Y, and T are independent and follow a standard normal distribution. For these cases we evaluate the FI (3.12). With normally distributed returns, the returns are asymptotically independent except for the fully diversified syndicates and we have to use the second part of the FI scale. To evaluate the numerator we use Laplace's asymptotic expansion, for large s

$$P\{B_i > s\} \sim \frac{1}{\sqrt{2\pi}} \frac{\sigma}{s} \exp(-\frac{s^2}{2\sigma^2}),$$

where the variance σ^2 is 16, 8, 6 and 4 for the cases 1 to 4 respectively.

For the denominator we need the multivariate analogue of this approximation which is available from Ruben [31]. Suppose (B_1, B_2, B_3, B_4) has a (non-degenerate) multivariate normal distribution with mean zero and covariance matrix V. Let $M = V^{-1}$ be the inverse of the covariance matrix and write its determinant as $|M|$. Denote the vector $(1, ..., 1)$ by ι and write

$$(\delta_1, \delta_2, \delta_3, \delta_4) = \iota M.$$

Then it follows from Ruben [31] that for s large

$$P\{B_1 > s, B_2 > s, B_3 > s, B_4 > s\} \simeq (2\pi)^{-4/2} \frac{\sqrt{|M|}}{s^4 \delta_1 \delta_2 \delta_3 \delta_4} \exp\left(-\frac{1}{2} s^2 \iota M \iota^T\right). \tag{5.2}$$

Since the terms in the exponent dominate, it follows that

$$\frac{1}{4}\lim_{s\to\infty}\frac{\sum_{i=1}^{4}\log P\{B_i>s\}}{\log P\{B_1>s,B_2>s,B_3>s,B_4>s\}}=\frac{1}{\sigma^2}\frac{1}{\iota M\iota^T}. \tag{5.3}$$

Thus for normal distribution in combination with the FI the covariance matrix V is a natural representation of the network dependencies, even far into the tails.

Turning now to the specific cases, in case one the $V = 16I$, where I is the identity matrix. Hence $\iota M\iota^T = 4/16$, and from (5.3) it follows that $FI = 1/4$.

The second case is somewhat non-standard and we cannot immediately apply (5.3). Note that banks are only partially connected, which is reflected through the zero's in the covariance matrix

$$V=\begin{pmatrix} 8 & 4 & 0 & 4\\ 4 & 8 & 4 & 0\\ 0 & 4 & 8 & 4\\ 4 & 0 & 4 & 8\end{pmatrix}.$$

But knowing the returns for the first three bank portfolios implies the fourth

$$B_4 = B_1 + B_3 - B_2.$$

Thus it follows that $|M| = 0$ and the correlation matrix is singular. Thus we cannot directly apply Ruben's result. One can write, however,

$$\begin{aligned} P\{B_1 &> s, B_2 > s, B_3 > s, B_4 > s\} = \\ P\{B_1 &> s, B_2 > s, B_3 > s, B_1 + B_3 - B_2 > s\} \end{aligned}$$

and use Ruben's [31] result in the three dimensional space with the restriction $B_1 + B_3 - B_2 > s$. We have from Ruben that

$$P\{B_1>s,B_2>s,B_3>s\}\sim(2\pi)^{-3/2}\frac{\sqrt{|M|}}{s^3\delta_1\delta_2\delta_3}\exp\left(-\frac{1}{2}s^2\iota M\iota^T\right) \tag{5.4}$$

and where M is the covariance matrix of $\{B_1, B_2, B_3\}$. The linear restriction $B_1 + B_3 - B_2 > s$ only affects the constants $\delta_1\delta_2\delta_3$ in (5.4). Hence, the FI measure is

$$\frac{1}{4}\lim_{s\to\infty}\frac{\sum_{i=1}^{4}\log P\{B_i>s\}}{\log P\{B_1>s,B_2>s,B_3>s\}}=\frac{1}{8}\frac{1}{1/4}=\frac{1}{2}$$

For case three the covariance matrix has full rank and reads

$$V = \begin{pmatrix} 6 & 3 & 4 & 3 \\ 3 & 6 & 3 & 4 \\ 4 & 3 & 6 & 3 \\ 3 & 4 & 3 & 6 \end{pmatrix}.$$

One computes that $\iota M \iota^T = 1/4$, so that $FI = 4/6$.

The fully diversified portfolios imply that all banks become perfectly correlated, so that we are in a case of maximal asymptotic dependence and $FI = \lim_{s\to\infty} E\{\kappa_s | \kappa_s \geq 1\} = 4$. The measure nicely reflects the increases in network connectedness as we move from first case to the last case.

5.3.2. *fat tails.* Now suppose the project returns are rescaled Student-t distributed, so that the projects still have the same first two moments as under normality but exhibit heavy tails. This implies that for large s

$$\begin{aligned} \Pr\{U > s\} &= \Pr\{X > s\} = \Pr\{Y > s\} = \Pr\{T > s\} \\ &\sim as^{-\alpha} \ (a, \alpha > 0, s \to \infty), \end{aligned}$$

where α is the number of degrees of freedom. In this case (3.4) is a natural measure for the extreme network dependencies. It is immediate that for case 1 we have $FI = \kappa = 1$.

For case 2, using Feller's convolution theorem [12] as $s \to \infty$

$$\Pr\{2U + 2X > s\} \sim 2a2^{\alpha} s^{-\alpha}.$$

Moreover by the arguments above, the probability of no failures

$$\Pr\{2U + 2X \leq s, 2X + 2Y \leq s, 2Y + 2T \leq s, 2T + 2U \leq s, \}$$

can be found by noting that the set

$$(2U + 2X = s, 2X + 2Y = s, 2Y + 2T = s, 2T + 2U = s)$$

cuts the four axes at the points

$$(\frac{s}{2}, 0, 0, 0); (0, \frac{s}{2}, 0, 0); (0, 0, \frac{s}{2}, 0); (0, 0, 0, \frac{s}{2}).$$

Above each of the these points there is (approximate) mass $a2^{\alpha}s^{-\alpha}$ along the axes. The mass above the four points together gives the probability

$$\begin{aligned} 1-\Pr\{2U+2X &\leq s, 2X+2Y \leq s, 2Y+2T \leq s, 2T+2U \leq s,\} \\ &\sim 4 \times a2^{\alpha}s^{-\alpha}. \end{aligned}$$

Combining gives

$$FI = \kappa = \frac{4 \times 2a2^{\alpha}s^{-\alpha}}{4 \times a2^{\alpha}s^{-\alpha}} = 2.$$

The third case is interesting since the denominator is as in the second case. Note that the set

$$(2U + X + Y = s, 2X + Y + T = s, 2Y + T + U = s, 2T + U + X = s)$$

cuts the axes at the points

$$(\frac{s}{2}, 0, 0, 0); (0, \frac{s}{2}, 0, 0); (0, 0, \frac{s}{2}, 0); (0, 0, 0, \frac{s}{2})$$

as in the previous case. Since we need the probability to be below planes like

$$2U + X + Y = s,$$

the binding point is where this plane cuts the axes closest to the origin, i.e. along the $U-$axis where the triangle cuts at $s/2$. Thus

$$\begin{aligned} 1-\Pr\{2U+X+Y &\leq s, 2X+Y+T \leq s, \\ 2Y+T+U &\leq s, 2T+U+X \leq s,\} \\ &= 4 \times a2^{\alpha}s^{-\alpha}. \end{aligned}$$

The numerator is straightforward by Feller's convolution result and equals

$$4 \times \Pr\{2U + X + Y > s\} \sim 4(2^{\alpha} + 2)as^{-\alpha}.$$

Hence, the third network implies

$$FI = \kappa = \frac{4(2^{\alpha}+2)as^{-\alpha}}{4a2^{\alpha}s^{-\alpha}} = 1 + \frac{1}{2^{\alpha-1}}.$$

Note that for $\alpha > 1$, this network is less fragile than the previous one.

Lastly, with full diversification, the four portfolios become totally dependent so that $FI = \kappa = 4$.

We compare the sequence of networks by their ranking of systemic dependencies, see the Table 2. If we were to use the correlation matrices, we would conclude that

Table 2. Fragility Index and Bank Portfolios

Portfolio B_1	FI Index	
	Normal	Student-t
4U	1/4	1
2U+2X	1/2	2
2U+X+Y	2/3	$1+\frac{1}{2^{\alpha-1}}$
U+X+Y+T	4	4

the networks become increasingly more interdependent and exposed to systemic risk. A different picture emerges if we use the FI scale. Indeed, for networks 1, 2 and 4, the measure κ is also increasing under the Student-t assumption. But the third network has a lower asymptotic dependency measure than the second network as long as the first moment is bounded, i.e. $\alpha > 1$. Thus the monotonicity is upset under fat tails. The intuition for this non-monotonicity is as follows. Both under normality and in case of the Student-t assumption, the univariate failure probabilities decrease as we move from case 2 to case 3. This is the benefit of diversification. In case of the normal, this reduction does not lower the joint failure probability[24], so that the mass is moved in the direction where there are no failures and the multidimensional disc representing normal iso probability sets becomes more pointed, i.e. the dependence increases in the center. In case of the fat tailed distributions though, when moving from case 2 to 3, the joint failure probability is reduced even though the probability on at least one failure remains constant. This causes the drop in the FI index.

5.4. **sunspots.** The theoretical economics literature also devotes a considerable attention to the issue of multiple equilibria and how agents coordinate on these equilibria. With multiple equilibria fundamentals do not fully determine outcomes, somehow one of the equilibria is being played by coordination on a sunspot. We now show that this approach can also be subsumed under our reduced form approach.

Consider the Diamond and Dybvig liquidity preference model of banking. In this model there are two Nash equilibria, one with and the other without a bank

[24] In fact $P\{B_1 > s, B_2 > s, B_3 > s\}$ is the same for all four cases under normality.

run. Assume that agents can coordinate on one of the equilibria via the exogenous device of a sunspot. The sunspot is a random variable that has no direct effect on the economy. Suppose there is a Bernoulli random variable representing the sunspot that indicates which of the two equilibria is relevant. More specifically, take three independently distributed asset returns X, Y and Z. Let the portfolio returns be $Q = X + Y$ with probability π, and $Q = Y$ with probability $1 - \pi$. Similarly, let $W = X + Z$ with probability π, and $W = Z$ with probability $1 - \pi$. Note that with probability $1 - \pi$ the portfolio returns are independent. All three assets X, Y and Z are i.i.d. distributed with Pareto type tails

$$P\{X > s\} = P\{Y > s\} = P\{Z > s\} = cs^{-\alpha}, c > 0, s\epsilon[c^{1/\alpha}, \infty).$$

Application of Theorem 3 then gives

$$\begin{aligned} \kappa &= \lim_{s\to\infty} \frac{P\{Q > s\} + P\{W > s\}}{1 - P\{Q \leq s, W \leq s\}} \\ &= 1 + \frac{\pi}{2 + \pi}. \end{aligned}$$

In case the random variables are normally distributed one also checks that the fragility index reduces to $FI = 3/4$, since the correlation is $1/2$ in case of the π state, while the other state does not count.

6. Conclusion

FSI systems are well known to be inherently unstable. Thus there is need for a measure of the potential for systemic breakdown. It is well understood that the correlation measure may not accurately reflect the risk of joint breakdowns. To remedy this deficiency we have constructed a scale FI which accurately measures the severity of joint tail risk in higher dimensions. The FI scale reveals the type of tail dependence (asymptotic dependence or independence), which either implies strong or weak fragility of the system, and also indicates the amount of such dependence. FSI portfolios are essentially linear in their exposures, either directly through the portfolio asset and liability returns or indirectly through the relation with macro risk factors. This permits an evaluation of the FI under a very wide range of asset return distributions. Assuming that the marginal distributions of the risk drivers are in the domain of attraction of an (univariate) extreme value limit law, we showed when the system would be weakly and when it is strongly fragile. For example, discrete returns and with all types of assets present in all bank portfolios imply strong fragility. Under continuous discounting, normal distributed returns induce weak fragility, but Student-t type returns render the system strongly fragile. Subsequently we studied different kind of banking networks and their fragility in terms of FI and used the arguments to construct the characteristics of economically relevant copula. It is hoped that this characterization of the financial fragility will help in bridging the gap between theory and practice and be especially helpful in evaluating the systemic aspects of FSI supervision.

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7. Appendix

Since the proof for the general case of subexponentials comprising the Gumbel part is quite involved, we provide a simple proof for the Frechet subcase $\tau > 0$ to enhance the reader's intuition. By the domain of attraction assumption

$$\lim_{n\to\infty} nP\left\{X_1 > xa(n)\right\} = x^{-\alpha} \text{ for } x > 0,\ i = 1, 2, \tag{7.1}$$

where a is again the inverse function of the distribution function of X_i at the point $1 - 1/x$. Hence, for x, $y > 0$ and independence

$$\lim_{n\to\infty} nP\left\{X_1 > xa(n) \text{ or } X_2 > ya(n)\right\} =$$

$$\begin{aligned} \lim_{n\to\infty} nP\{X_1 &> xa(n)\} + \lim_{n\to\infty} nP\{X_2 > ya(n)\} \\ -\lim_{n\to\infty} nP\{X_1 &> xa(n)\}P\{X_2 > ya(n)\} = \\ & x^{-\alpha} + y^{-\alpha} + 0. \end{aligned} \tag{7.2}$$

From (7.1) and (7.2) we get for x, $y > 0$ as $n \to \infty$

$$\begin{aligned} nP\{X_1 &\le xa(n) \text{ and } X_2 > ya(n)\} = \\ nP\{X_1 &> xa(n) \text{ or } X_2 > ya(n)\} - nP\{X_1 > xa(n)\} \to y^{-\alpha}. \end{aligned}$$

By subtracting two similar expressions this gives for $0 < x_1 < x_2$ and $y > 0$

$$nP\{x_1 a(n) \le X_1 < x_2 a(n) \text{ and } X_2 > ya(n)\} \to 0. \tag{7.3}$$

Also by (7.1) and (7.2)

$$\lim_{n\to\infty} nP\{X_1 > xa(n) \text{ and } X_2 > ya(n)\} = 0. \tag{7.4}$$

Next, consider the joint tails of the portfolios. We need to evaluate asymptotically

$$nP\{\lambda_1 Y_1 + \lambda_2 Y_1 > xa(n) \text{ or } \mu_1 Y_1 + \mu_2 Y_2 > ya_{(n)}\}. \tag{7.5}$$

Look at the problem from a geometric point of view. Consider the area

$$S := \{(s,t) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid \lambda_1 s + \lambda_2 t > x \text{ or } \mu_1 s + \mu_2 t > y\}.$$

For simplicity let $\mu_2 x < \lambda_2 y$ and $\mu_1 x > \lambda_1 y$. Now

$$S \supset \{(s,t) \mid s > \frac{y}{\mu_1} \text{ or } t > \frac{x}{\lambda_2}\}.$$

Hence by (7.2) the limit (7.5) is at least

$$\left(\frac{x}{\lambda_2}\right)^{-\alpha} + \left(\frac{y}{\mu_1}\right)^{-\alpha}.$$

Also

$$\begin{aligned} S \;\subset\; & \{(s,t) \mid s > \varepsilon\frac{x}{\lambda_1} \text{and } t > \varepsilon\frac{y}{\mu_1}\} \cup \left[0, \varepsilon\frac{x}{\lambda_1}\right] \times \left[(1-\varepsilon)\frac{x}{\lambda_2}, \infty\right) \cup \\ & \cup \left[0, \varepsilon\frac{y}{\mu_1}\right] \times \left[(1-\varepsilon)\frac{x}{\mu_2}, \infty\right), \end{aligned}$$

and thus by (7.3) and (7.4) the limit (7.5) is at most

$$\left((1-\varepsilon)\frac{x}{\lambda_2}\right)^{-\alpha} + \left((1-\varepsilon)\frac{y}{\mu_1}\right)^{-\alpha}.$$

It follows that in the limit (7.5) is

$$\left(\frac{x}{\lambda_2}\right)^{-\alpha} + \left(\frac{y}{\mu_1}\right)^{-\alpha}.$$

So far we have considered the case $\mu_2 x < \lambda_2 y$ and $\mu_1 x > \lambda_1 y$. Checking the other cases yields that

$$\lim_{n\to\infty} nP\left\{\lambda_1 Y_1 + \lambda_2 Y_2 > xa(n) \text{ or } \mu_1 Y_1 + \mu_2 Y_2 > ya(n)\right\}$$

$$= \left(\frac{x}{\lambda_1} \wedge \frac{y}{\mu_1}\right)^{-\alpha} + \left(\frac{x}{\lambda_2} \wedge \frac{y}{\mu_2}\right)^{-\alpha}.$$

It follows that the vector (X_1, X_2) is in de domain of attraction of an extreme value distribution with marginal extreme value index $\tau = 1/\alpha$ and discrete spectral measure concentrated on two points in the interior of its range. In particular, we have the denominator for (4.29).

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