A unified treatment of single component replacement models

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Abstract

In this paper we discuss a general framework for single component replacement models. This framework is based on the regenerative structure of these models and by using results from renewal theory a unified presentation of the discounted and average finite and infinite horizon cost models is given. Moreover, we present sufficient conditions for the numerator of the cost function to have a vanishing singular part. Finally, some well–known replacement models are discussed, and making use of the previous results an easy derivation of their cost functions is presented.

Keywords: Maintenance, renewal theory, Lebesgue integration, intensity cost function.

1 Introduction

In the extensive literature on infinite horizon single component replacement models (for surveys, see McCall [18], Pierskalla and Voelker [22], Sherif and Smith [26] and Valdez–Flores and Feldman [29]) a lot of different models are analyzed. However, as observed by Zijlstra [32, 33], in most of these models the replacement action, prescribed by a given stationary policy, regenerates the system, i.e. the system is "as good as new" immediately after the completion of such an action. Although this observation is always used in the literature, no paper discusses in detail the consequences of the regenerative structure of infinite horizon single component replacement models. In this paper we will start in Section 2 with such a regenerative structure. Using well–known techniques and results from the theory of regenerative processes, we will unify the derivation of the infinite horizon average and expected discounted costs associated with a given stationary policy. Moreover, we show that under some weak nonlattice assumption the difference between the expected discounted costs (with discount factor $\alpha > 0$) up to time t and the infinite horizon expected discounted costs, converges exponentially fast with factor α to an easy function of the corresponding infinite horizon average costs. Also in this section we derive easy upper and lower bounds on the cost functions of the corresponding finite horizon model. Since the infinite horizon average and expected discounted costs are given by a ratio of functions we discuss in Section 3 under which conditions the singular part of the numerator vanishes. Moreover, these conditions are given a "probabilistic" interpretation using a so–called cost intensity function. By this interpretation it is relatively easy to derive the cost function of a large class of replacement models with or without minimal repairs. This will be discussed in Section 4 and due to the absense of differentiability requirements for the preventive repair cost function and the regular maintenance cost function, these models are more general than those considered by Berg [4, 5, 6]. In fact, one cannot apply to these models the intuitively appealing Marginal Cost Analysis (MCA) method of Berg. Finally, we like to observe that under some conditions the MCA method can be validated using the results of Section 3. This will be worked out in another paper (Frenk, Dekker and Kleijn [13]).

2 General setup

In most single–component maintenance models we encounter the following framework. Let $\boldsymbol{c}_i$, $i \geq 1$, be a sequence of independent and identically distributed random variables defined on a probability space $(\Omega, \mathcal{F}, \mu)$ with finite mean and (right continuous) distribution function F satisfying $F(0) = 0$ (Feller [11]). Moreover, set $\boldsymbol{c}_0 := 0$. The nth partial sum $\boldsymbol{S}_n, n \geq 1$, of the random variables $\boldsymbol{c}_i, i \geq 1$, denotes now the occurrence of the nth regeneration point associated with a given stationary replacement policy $p \in \mathcal{P}$. For notational convenience set also $\boldsymbol{S}_0 := 0$. Associated with this policy $p \in \mathcal{P}$ is an increasing nonnegative stochastic process $\{\boldsymbol{R}_\alpha(t), t \geq 0\}$, defined on the same probability space $(\Omega, \mathcal{F}, \mu)$ with right continuous sample paths, $\boldsymbol{R}_\alpha(0) = 0$ and $\boldsymbol{R}_\alpha(t, \cdot)$ for each $t > 0$ a Borel measurable function. This process denotes the total discounted costs, with discount factor $\alpha \geq 0$, up to time t due to policy $p \in \mathcal{P}$, and satisfies the following important regenerative property.

Regeneration property.
For each $n \geq 0$ the stochastic process

$$\{\boldsymbol{c}_{n+1}, \boldsymbol{c}_{n+2}, \ldots, \{\exp(\alpha \boldsymbol{S}_n)(\boldsymbol{R}_\alpha(t + \boldsymbol{S}_n) - \boldsymbol{R}_\alpha(\boldsymbol{S}_n)) : t \geq 0\}\}$$

is independent of $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n$ and its distribution does not depend on n.

To relate the above setup with replacement models, we consider the following examples. For the age replacement family of single component maintenance models it follows that $\boldsymbol{c}_i = T \wedge \boldsymbol{X}_i$, with T some positive parameter and $\boldsymbol{X}_i, i \geq 1$, a sequence of independent and identically distributed random variables on $(0, \infty)$. Moreover, $T \wedge \boldsymbol{X}_i := \min\{T, \boldsymbol{X}_i\}$. Also, for the block replacement family we obtain that $\boldsymbol{c}_i = T$, with T some positive parameter. Examples of generalizations of the latter policy are given by the opportunity–based block replacement policies (Dekker and Smeitink [10]) and the so–called T–policy which replaces the component at the first failure after reaching the age $T > 0$ (Muth [20], Makis and Jardine [17]). If a component fails before reaching age T it is assumed in most of the above models that it is minimally repaired. For both extensions of the block replacement rule it is clear that $\boldsymbol{c}_i = T + \boldsymbol{Y}_i(T)$ with $\boldsymbol{Y}_i(T), i \geq 1$, a sequence of independent and identically distributed nonnegative random variables depending on T. Finally, if one takes the random down time of the system into account in a block (Sivazlian [27]) or age replacement model, it is also easy to write down an explicit formula for $\boldsymbol{c}_i$. This shows some examples of replacement models fitting into the above framework. We will now continue with the analysis of this framework. In the remainder it is always assumed that the random variable

$\boldsymbol{R}_\alpha(\boldsymbol{c}_1)$ is μ–integrable (page 37 of Ash [1]) or equivalently $\mathbb{E}(\boldsymbol{R}_\alpha(\boldsymbol{c}_1))$ is finite. Since $\boldsymbol{R}_\alpha(\boldsymbol{c}_1) \leq \boldsymbol{R}_0(\boldsymbol{c}_1)$ for every $\alpha \geq 0$ it is sufficient to demand that $\mathbb{E}(\boldsymbol{R}_0(\boldsymbol{c}_1))$ is finite. Since we need in Lemma 2.2 that $\mathbb{E}(\boldsymbol{R}_\alpha(t)) < \infty$ for every $t \geq 0$, we make the following observations. Although this line of reasoning is well–known within renewal theory it is listed here for completeness. Let $\boldsymbol{N}(t) := \inf\{n \geq 1 : \boldsymbol{S}_n > t\}$ denote the renewal counting process associated with the random variables $\boldsymbol{c}_i, i \geq 1$. Applying now the monotonicity of the stochastic process $\{\boldsymbol{R}_\alpha(t) : t \geq 0\}$ it follows for every $t \geq 0$ that

$$0 \leq \boldsymbol{R}_\alpha(t) \leq \sum_{n=1}^{\boldsymbol{N}(t)} (\boldsymbol{R}_\alpha(\boldsymbol{S}_n) - \boldsymbol{R}_\alpha(\boldsymbol{S}_{n-1})) \leq \sum_{n=1}^{\boldsymbol{N}(t)} \exp(\alpha \boldsymbol{S}_{n-1})(\boldsymbol{R}_\alpha(\boldsymbol{S}_n) - \boldsymbol{R}_\alpha(\boldsymbol{S}_{n-1}))$$

By the regeneration property we obtain that the distribution of the random variable $\exp(\alpha \boldsymbol{S}_{n-1})(\boldsymbol{R}_\alpha(\boldsymbol{S}_n) - \boldsymbol{R}_\alpha(\boldsymbol{S}_{n-1}))$ is independent of n and this yields that the expectation $\mathbb{E}(\exp(\alpha \boldsymbol{S}_{n-1})(\boldsymbol{R}_\alpha(\boldsymbol{S}_n) - \boldsymbol{R}_\alpha(\boldsymbol{S}_{n-1}))) = \mathbb{E}(\boldsymbol{R}_\alpha(\boldsymbol{c}_1)) < \infty$. Moreover, by the same regeneration property the random variable $\exp(\alpha \boldsymbol{S}_{n-1})(\boldsymbol{R}_\alpha(\boldsymbol{S}_n) - \boldsymbol{R}_\alpha(\boldsymbol{S}_{n-1}))$ is independent of $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_{n-1}$ and so it follows by Wald's identity (Wolff [31]) and $\boldsymbol{N}(t)$ is a stopping time with respect to $\boldsymbol{c}_i, i \geq 1$ satisfying $\mathbb{E}(\boldsymbol{N}(t)) < \infty$ for every $t > 0$ that

$$0 \leq \mathbb{E}(\boldsymbol{R}_\alpha(t)) \leq \mathbb{E}(\sum_{n=1}^{\boldsymbol{N}(t)} \exp(\alpha \boldsymbol{S}_{n-1})(\boldsymbol{R}_\alpha(\boldsymbol{S}_n) - \boldsymbol{R}_\alpha(\boldsymbol{S}_{n-1}))) = \mathbb{E}(\boldsymbol{N}(t))\mathbb{E}(\boldsymbol{R}_\alpha(\boldsymbol{c}_1)) < \infty$$

To analyze the behaviour of this expectation as t approaches infinity we introduce the related stochastic process $\{\boldsymbol{R}_\alpha(t \wedge \boldsymbol{c}_1) : t \geq 0\}$ and the function $f_\alpha : [0, \infty) \to [0, \infty]$ given by $f_\alpha(t) := \mathbb{E}(\boldsymbol{R}_\alpha(t \wedge \boldsymbol{c}_1))$. Clearly this process represents the total discounted costs with discount factor $\alpha \geq 0$ made in the first cycle up to time t. It is now easy to establish the following result.

Lemma 2.1 *The function $f_\alpha : [0, \infty) \to [0, \infty]$ is uniformly bounded, nonnegative, increasing and right continuous with at most a countable number of discontinuities. Also it holds that $f_\alpha(\infty) := \lim_{t\to\infty} f_\alpha(t) = \mathbb{E}(\boldsymbol{R}_\alpha(\boldsymbol{c}_1)) < \infty$.*

Proof: Since the stochastic process $\{\boldsymbol{R}_\alpha(t) : t \geq 0\}$ is increasing and nonnegative it follows that $0 \leq \boldsymbol{R}_\alpha(t \wedge \boldsymbol{c}_1) \leq \boldsymbol{R}_\alpha(\boldsymbol{c}_1)$ for every $t \geq 0$ and hence $0 \leq f_\alpha(t) \leq \mathbb{E}(\boldsymbol{R}_\alpha(\boldsymbol{c}_1)) < \infty$ for every $t \geq 0$. By the same monotonicity of $\boldsymbol{R}_\alpha(\cdot)$ we obtain that $\boldsymbol{R}_\alpha((t+h)\wedge\boldsymbol{c}_1) \geq \boldsymbol{R}_\alpha(t\wedge\boldsymbol{c}_1)$ for every $h \geq 0$ implying f_α is increasing. To show that the function f_α is right continuous we observe since $\{\boldsymbol{R}_\alpha(t) : t \geq 0\}$ has right continuous sample paths that $\lim_{h\downarrow 0} \boldsymbol{R}_\alpha((t+h)\wedge \boldsymbol{c}_1) = \boldsymbol{R}_\alpha(t\wedge\boldsymbol{c}_1)$ and this proves by the dominated convergence theorem (Ash [1]) that the function f_α is right continuous. Also, since $\boldsymbol{c}_1 < \infty$ μ–almost everywhere we obtain that $\lim_{t\to\infty} \boldsymbol{R}_\alpha(t\wedge\boldsymbol{c}_1) = \boldsymbol{R}_\alpha(\boldsymbol{c}_1)$ μ–almost everywhere and again by the dominated convergence theorem and $0 \leq \boldsymbol{R}_\alpha(t\wedge\boldsymbol{c}_1) \leq \boldsymbol{R}_\alpha(\boldsymbol{c}_1)$ it follows that $\lim_{t\to\infty} f_\alpha(t) = \mathbb{E}(\boldsymbol{R}_\alpha(\boldsymbol{c}_1))$. Finally, due to f_α increasing and finite we obtain by Theorem 4.1.2 of Mikolás [19] that on every finite interval (a, b) the function f_α has a set of discontinuities which is at most countable and this shows the last part. □

To relate the stochastic process $\boldsymbol{R}_\alpha(\cdot)$ to the stochastic process $\boldsymbol{R}_0(\cdot)$ we observe, since the increasing stochastic process $\{\boldsymbol{R}_0(t\wedge\boldsymbol{c}_1) : t \geq 0\}$ has right continuous sample paths and $\mathbb{E}(\boldsymbol{R}_0(t\wedge\boldsymbol{c}_1))$ is uniformly bounded implying $\boldsymbol{R}_0(t\wedge\boldsymbol{c}_1) < \infty$ μ–almost everywhere, that by Theorem 5.7.3 of Mikolás [19] for almost every realization $\omega \in \Omega$ the value $\boldsymbol{R}_\alpha((t\wedge\boldsymbol{c}_1)(\omega), \omega)$ can be written as the Riemann–Stieltjes integral

$$\int_0^{(t\wedge\boldsymbol{c}_1)(\omega)} \exp(-\alpha y)\boldsymbol{R}_0(dy, \omega)$$

This implies that

$$f_\alpha(t) := \mathbb{E}(\boldsymbol{R}_\alpha(t\wedge\boldsymbol{c}_1)) = \mathbb{E}(\int_0^{t\wedge\boldsymbol{c}_1} \exp(-\alpha y)\boldsymbol{R}_0(dy))$$

and by the dominated convergence theorem (Ash [1]) and Lemma 2.1 it follows that

$$\mathbb{E}(\boldsymbol{R}_\alpha(\boldsymbol{c}_1)) = \mathbb{E}(\int_0^{\boldsymbol{c}_1} \exp(-\alpha y)\boldsymbol{R}_0(dy))$$

To unify the derivation of the finite–valued function $v_\alpha : [0, \infty) \to [0, \infty)$ defined by $v_\alpha(t) := \mathbb{E}(\boldsymbol{R}_\alpha(t))$ we show in the next lemma that for every discount factor $\alpha \geq 0$ it is the unique solution of a renewal type equation. However, before presenting this lemma we introduce for $\alpha \geq 0$ the (possibly) defective distribution function F_α given by

$$F_\alpha(x) := \int_0^x \exp(-\alpha y)F(dy)$$

Moreover, we introduce the renewal function $U_\alpha : [0,\infty) \to [0,\infty)$ associated with F_α and this function has the well–known representation

$$U_\alpha(x) := \sum_{n=0}^{\infty} F_\alpha^{n*}(x) \tag{1}$$

where F_α^{n*} denotes the n–fold convolution of F_α.
Observe now by Lemma 2.1 and U_α is right continuous due to F_α is right continuous that by Exercise 2 at page 56 of Ash [1] the integral in the next lemma can be interpreted as a Riemann–Stieltjes integral.

Lemma 2.2 *It follows for every discount factor $\alpha \geq 0$ that*

$$v_\alpha(t) = \int_{0-}^{t} f_\alpha(t-y)U_\alpha(dy)$$

for every $t \geq 0$.

Proof: Clearly for every $t \geq 0$ we obtain that

$$v_\alpha(t) = \mathbb{E}(\boldsymbol{R}_\alpha(t \wedge \boldsymbol{c}_1)) + \mathbb{E}(\boldsymbol{R}_\alpha(t) - \boldsymbol{R}_\alpha(t \wedge \boldsymbol{c}_1)) = f_\alpha(t) + \int_0^t \mathbb{E}(\boldsymbol{R}_\alpha(t) - \boldsymbol{R}_\alpha(t \wedge \boldsymbol{c}_1) | \boldsymbol{c}_1 = y)F(dy)$$

By the regeneration property of the stochastic process $\{\boldsymbol{R}_\alpha(t) : t \geq 0\}$ it follows for every $y \leq t$ that

$$\begin{aligned}
\mathbb{E}(\boldsymbol{R}_\alpha(t) - \boldsymbol{R}_\alpha(t \wedge \boldsymbol{c}_1) | \boldsymbol{c}_1 = y) &= \mathbb{E}(\boldsymbol{R}_\alpha(t - y + \boldsymbol{c}_1) - \boldsymbol{R}_\alpha(\boldsymbol{c}_1) | \boldsymbol{c}_1 = y) \\
&= \exp(-\alpha y)\mathbb{E}(\exp(\alpha \boldsymbol{c}_1)(\boldsymbol{R}_\alpha(t - y + \boldsymbol{c}_1) - \boldsymbol{R}_\alpha(\boldsymbol{c}_1)) | \boldsymbol{c}_1 = y) \\
&= \exp(-\alpha y)\mathbb{E}(\exp(\alpha \boldsymbol{c}_1)(\boldsymbol{R}_\alpha(t - y + \boldsymbol{c}_1) - \boldsymbol{R}_\alpha(\boldsymbol{c}_1))) \\
&= \exp(-\alpha y)\mathbb{E}(\boldsymbol{R}_\alpha(t - y))
\end{aligned}$$

and this implies

$$v_\alpha(t) = f_\alpha(t) + \int_0^t \exp(-\alpha y) v_\alpha(t-y)F(dy) = f_\alpha(t) + \int_0^t v_\alpha(t-y)F_\alpha(dy)$$

By Lemma 2.1 and $\mathbb{E}(\boldsymbol{R}_\alpha(t))$ is bounded on every compact interval the desired result follows by applying Theorem 2.4 in Chapter 4 of Asmussen [2]. $\square$

By Lemma 2.1 it follows immediately that the function $G_\alpha : [0, \infty) \to [0, 1]$ given by $G_\alpha(t) := f_\alpha(t)/f_\alpha(\infty)$ is a right continuous nondefective distribution function satisfying $G_\alpha(0) = 0$. Hence, by Lemma 2.1 and Lemma 2.2 we obtain that

$$v_\alpha(t) = f_\alpha(\infty)U_\alpha^d(t) = \mathbb{E}(\boldsymbol{R}_\alpha(\boldsymbol{c}_1))U_\alpha^d(t) \tag{2}$$

with $U_\alpha^d(t) := \int_{0-}^t G_\alpha(t-y)U_\alpha(dy)$ the renewal function associated with the delayed renewal process $\{\boldsymbol{S}_n^d : n \geq 0\}$ where $\boldsymbol{S}_n^d = \boldsymbol{S}_n + \boldsymbol{d}_\alpha$ and $\boldsymbol{d}_\alpha$ a random variable with distribution function G_α. By this observation one can easily verify for $\alpha = 0$ the following result.

Theorem 2.1 *It follows that the infinite horizon average costs associated with a given stationary policy $p \in \mathcal{P}$ equals*

$$c_0(p) := \lim_{t\to\infty} \frac{v_0(t)}{t} = \frac{\mathbb{E}(\boldsymbol{R}_0(\boldsymbol{c}_1))}{\mathbb{E}(\boldsymbol{c}_1)}$$

Also, if the distribution function F is nonlattice and $\mathbb{E}(\boldsymbol{c}_1^2)$ is finite, we obtain that

$$\lim_{t\to\infty} v_0(t) - \frac{1}{\mathbb{E}(\boldsymbol{c}_1)}\int_0^t \mathbb{E}(\boldsymbol{R}_0(y \wedge \boldsymbol{c}_1))dy = \frac{\mathbb{E}(\boldsymbol{c}_1^2)}{2\mathbb{E}(\boldsymbol{c}_1)}c_0(p)$$

Moreover, the above integral can be interpreted as a Riemann integral.

Proof: By the elementary renewal theorem (see Proposition 1.4 of Chapter 4 in Asmussen [2]) we obtain that $\lim_{t\to\infty} U_0^d(t)/t = 1/\mathbb{E}(\boldsymbol{c}_1)$ and this shows by (2) the first part. To verify the second part we first observe by Lemma 2.1 and Theorem 1.7.1 of Ash [1] that the integral $\int_0^t \mathbb{E}(\boldsymbol{R}_0(y \wedge \boldsymbol{c}_1))dy$ can be interpreted as a Riemann integral for every $0 < t < \infty$. Moreover, if F is nonlattice and $\mathbb{E}(\boldsymbol{c}_1^2)$ is finite we obtain by Proposition 4.1 of Chapter 6 in Asmussen [2] that

$$\lim_{t\to\infty} U_0(t) - \frac{t}{\mathbb{E}(\boldsymbol{c}_1)} = \frac{\mathbb{E}(\boldsymbol{c}_1^2)}{2(\mathbb{E}(\boldsymbol{c}_1))^2}$$

Hence, it follows that

$$\begin{aligned}\lim_{t\to\infty} U_0^d(t) - \frac{1}{\mathbb{E}(\boldsymbol{c}_1)}\int_0^t G_0(y)dy &= \lim_{t\to\infty}\int_0^t \left(U_0(t-y) - \frac{(t-y)}{\mathbb{E}(\boldsymbol{c}_1)}\right)G_0(dy) \\ &= \frac{\mathbb{E}(\boldsymbol{c}_1^2)}{2(\mathbb{E}(\boldsymbol{c}_1))^2}\end{aligned}$$

and this shows by the first part, relation (2) and the definition of G_0 the desired result. □

As already noticed the random variable $\boldsymbol{c}_i$ associated with the age replacement family of single component maintenance models has the representation $\boldsymbol{c}_i = T \wedge \boldsymbol{X}_i$ and so we immediately obtain that $\mathbb{E}(\boldsymbol{c}_i^2)$ is finite. If additionally the distribution function F of the random variable $\boldsymbol{c}_i = T \wedge \boldsymbol{X}_i$ is nonlattice, it follows that the conditions for the second part of Theorem 2.1 are satisfied. Observe in Section 4 we will write out the formulas of Theorem 2.1 for a large class of age replacement cost models. Similar remarks can be made for the class of random variables $\boldsymbol{c}_i = T + \boldsymbol{Y}_i(T)$. For this class we need to assume that $\mathbb{E}\left((\boldsymbol{Y}_i(T))^2\right)$ is finite and the distribution function of the random variable $\boldsymbol{Y}_i(T)$ is nonlattice. Finally, for the block replacement family of single component maintenance models it follows due to $\boldsymbol{c}_i = T$ that the corresponding degenerate distribution function is cleary lattice. This means that one cannot apply the second part of Theorem 2.1. However, it is easy to derive a similar type of result by using Lemma 2.2. This is possible since the renewal function is an easy stepfunction with unit steps at multiples of T if the underlying distribution function is degenerate at T.

Without any conditions on the distribution function F except finiteness of the second moment $\mathbb{E}(\boldsymbol{c}_1^2)$ it is relatively easy to derive upper and lower bounds on the function v_0. This is a consequence of the existence of upper and lower bounds for the renewal function U_0 and therefore we will first discuss these bounds. If $\boldsymbol{c}_0$ is a random variable independent of the nth partial sum $\boldsymbol{S}_n, n \geq 1$, with distribution function F_e given by

$$F_e(x) = \frac{1}{\mathbb{E}(\boldsymbol{c}_1)} \int_0^x (1 - F(y))dy$$

then we consider the delayed renewal process $\{\boldsymbol{S}_n^d : n \geq 0\}$ with $\boldsymbol{S}_n^d = \boldsymbol{S}_n + \boldsymbol{c}_0, n \geq 0$. The distribution function F_e is called the equilibrium distribution associated with F and it is well–known (Asmussen [2]) that the renewal function U_e of this delayed renewal process satisfies

$$U_e(x) = \mathbb{E}(\sum_{n=0}^{\infty} 1_{\{\boldsymbol{S}_n^d \leq x\}}) = E\left(U_0(x - \boldsymbol{c}_0) 1_{\{\boldsymbol{c}_0 \leq x\}}\right) = \frac{x}{\mathbb{E}(\boldsymbol{c}_1)} \tag{3}$$

By (3) and U_0 nonnegative and increasing we immediately obtain the well–known linear lower bound $x/\mathbb{E}(\boldsymbol{c}_1) = E\left(U_0(x - \boldsymbol{c}_0) 1_{\{\boldsymbol{c}_0 \leq x\}}\right) \leq U_0(x)$. Although it is relatively easy to improve this lower bound around zero (Van Weeren [30]) we will not pursue this. To construct an upper bound we observe the following. If $\boldsymbol{c}^*$ is a random variable on $(0,\infty)$ independent of the random variables $\boldsymbol{c}_0$ and $\boldsymbol{S}_n, n \geq 0$ it follows by (3) that

$$\begin{aligned} \mathbb{E}(U_0(x+\boldsymbol{c}^*-\boldsymbol{c}_0)) &= \mathbb{E}(\mathbb{E}(U_0(x+\boldsymbol{c}^*-\boldsymbol{c}_0)1_{\{\boldsymbol{c}_0\leq x+\boldsymbol{c}^*\}}|\boldsymbol{c}^*)) \\ &= \frac{1}{\mathbb{E}(\boldsymbol{c}_1)}\mathbb{E}(x+\boldsymbol{c}^*) \end{aligned} \tag{4}$$

It is now possible to prove the following upper bound. This upper bound is known as Lorden's inequality and for completeness we list an easier proof as the one given in Proposition 4.2 in Chapter 6 of Asmussen [2].

Lemma 2.3 *If $\mathbb{E}(\boldsymbol{c}_1^2)$ is finite then for every $x \geq 0$ it follows that*

$$U_0(x) \leq \frac{x}{\mathbb{E}(\boldsymbol{c}_1)} + \frac{\mathbb{E}(\boldsymbol{c}_1^2)}{(\mathbb{E}(\boldsymbol{c}_1))^2}$$

Proof: Let $\boldsymbol{c}^*$ and $\boldsymbol{c}_0$ be independent and identically distributed nonnegative random variables with distribution function F_e and assume that these random variables are independent of the random variables $\boldsymbol{S}_n, n \geq 0$. Using the subadditivity of the renewal function U_0, i.e. $U_0(x+y) \leq U_0(x) + U_0(y)$ for every $x, y \geq 0$ we obtain that

$$U_0(x) \leq \mathbb{E}(U_0(x+\boldsymbol{c}^*-\boldsymbol{c}_0)) + \mathbb{E}(U_0(\boldsymbol{c}_0-\boldsymbol{c}^*))$$

and this implies by (4) that $U_0(x) \leq \frac{\mathbb{E}(x+\boldsymbol{c}^*)}{\mathbb{E}(\boldsymbol{c}_1)} + \frac{\mathbb{E}(\boldsymbol{c}_0)}{\mathbb{E}(\boldsymbol{c}_1)}$. Since $\mathbb{E}(\boldsymbol{c}^*) = \mathbb{E}(\boldsymbol{c}_0) = \frac{\mathbb{E}(\boldsymbol{c}_1^2)}{2\mathbb{E}(\boldsymbol{c}_1)}$ the desired result follows. □

It is now possible to derive the following upper and lower bound for the average cost up to time t if the second moment $\mathbb{E}(\boldsymbol{c}_1^2)$ is finite. This will be discussed in the next theorem, and the listed inequality should be compared with the second part of Theorem 2.1.

Theorem 2.2 *If $\mathbb{E}(\boldsymbol{c}_1^2)$ is finite then it follows for every $t > 0$ that*

$$0 \leq v_0(t) - \frac{1}{\mathbb{E}(\boldsymbol{c}_1)}\int_0^t \mathbb{E}(\boldsymbol{R}_0(y \wedge \boldsymbol{c}_1))dy \leq \frac{\mathbb{E}(\boldsymbol{c}_1^2)}{\mathbb{E}(\boldsymbol{c}_1)}c_0(p)$$

Proof: By Lemma 2.2 we obtain since $f_0(0) = 0$ that

$$v_0(t) = \int_{0-}^{t} f_0(t-y)U_0(dy) = \int_{0+}^{t} U_0(t-y) f_0(dy)$$

Due to f_0 increasing this yields the desired inequality by substituting the lower and upper bound for the renewal function, given by Lemma 2.3, into the above expression. $\square$

By Theorem 2.2 it is clear for t large enough and $\mathbb{E}(\boldsymbol{c}_1^2)$ finite that a reasonable approximation for the average costs up to time t is given by $\frac{1}{\mathbb{E}(\boldsymbol{c}_1)} \int_0^t \mathbb{E}(\boldsymbol{R}_0(y \wedge \boldsymbol{c}_1))dy$. This observation concludes our discussion of the case $\alpha = 0$. We will now analyze the behaviour of the objective function for the discounted case. To start with this analysis we will first analyze the behaviour of the renewal function U_α, introduced in (1).

Lemma 2.4 *If $\alpha > 0$ it follows for every $t > 0$ that*

$$\frac{-1}{1 - F_\alpha(\infty)} \leq \exp(\alpha t) \left(U_\alpha(t) - \frac{1}{1 - F_\alpha(\infty)} \right) \leq 0$$

Moreover, if F is nonlattice we obtain that

$$\lim_{t \to \infty} \exp(\alpha t) \left(U_\alpha(t) - \frac{1}{1 - F_\alpha(\infty)} \right) = \frac{-1}{\alpha \mathbb{E}(\boldsymbol{c}_1)}$$

Proof: To prove the second part we observe with $z_1 : [0, \infty) \to (-\infty, 0]$ given by

$$z_1(t) := \frac{1}{1 - F_\alpha(\infty)} \left(F_\alpha(t) - F_\alpha(\infty) \right)$$

that

$$\begin{aligned}
\int_{0-}^{t} U_\alpha(t-y) z_1(dy) &= \frac{1}{1 - F_\alpha(\infty)} \left(\int_{0+}^{t} U_\alpha(t-y) F_\alpha(dy) - F_\alpha(\infty) U_\alpha(t) \right) \\
&= \frac{1}{1 - F_\alpha(\infty)} \left(U_\alpha(t) - 1 - F_\alpha(\infty) U_\alpha(t) \right) \\
&= U_\alpha(t) - \frac{1}{1 - F_\alpha(\infty)} \quad =: \quad w(t)
\end{aligned}$$

By this equation it follows that the function $w_1(t) := \exp(\alpha t) w(t)$ satisfies the renewal type equation

$$w_1(t) = \exp(\alpha t) z_1(t) + \int_0^t w_1(t-y) F(dy)$$

and this implies by Theorem 2.4 in Chapter 4 of Asmussen [2] that

$$w_1(t) = \int_{0-}^{t} \exp(\alpha(t-y))z_1(t-y)U_0(dy)$$

Observe now for every $t \geq 0$ that

$$\begin{aligned} 0 \quad \leq \quad -\exp(\alpha t)z_1(t) &= \frac{\exp(\alpha t)}{1-F_\alpha(\infty)}\int_t^\infty \exp(-\alpha y)F(dy) \\ &\leq \frac{1-F(t)}{1-F_\alpha(\infty)} \end{aligned}$$

and this yields by (iv) of Proposition 4.1 in Chapter 4 of Asmussen [2] that the function $t \to \exp(\alpha t)z_1(t)$ is directly Riemann integrable. Applying now the key renewal theorem (see Theorem 4.3 in Chapter 4 of Asmussen [2]) to the renewal type equation satisfied by $w_1(t)$ we obtain that

$$\begin{aligned} \lim_{t\to\infty} w_1(t) &= \frac{1}{\mathbb{E}(\boldsymbol{c}_1)}\int_0^\infty \exp(\alpha y)z_1(y)dy \\ &= \frac{-1}{\mathbb{E}(\boldsymbol{c}_1)(1-F_\alpha(\infty))}\int_0^\infty \exp(\alpha y)\left(F_\alpha(\infty)-F_\alpha(y)\right)dy \\ &= \frac{-1}{\alpha\mathbb{E}(\boldsymbol{c}_1)(1-F_\alpha(\infty))}\int_0^\infty \left(\exp(\alpha z)-1\right)F_\alpha(dz) \\ &= \frac{-(1-F_\alpha(\infty))}{\alpha\mathbb{E}(\boldsymbol{c}_1)(1-F_\alpha(\infty))} \\ &= \frac{-1}{\alpha\mathbb{E}(\boldsymbol{c}_1)} \end{aligned}$$

and this shows the second part. To prove the first part we observe by the inequality $-\exp(\alpha t)z_1(t) \leq (1-F(t))/(1-F_\alpha(\infty))$ and U_0 increasing that

$$\begin{aligned} 0 \quad \leq \quad -w_1(t) &= -\int_{0-}^{t} \exp(\alpha(t-y))z_1(t-y)U_0(dy) \\ &\leq \frac{1}{1-F_\alpha(\infty)}\int_{0-}^{t}(1-F(t-y))U_0(dy) \\ &= \frac{1}{1-F_\alpha(\infty)} \end{aligned}$$

and this shows together with $U_\alpha(t) \leq U_\alpha(\infty) = 1/(1-F_\alpha(\infty))$ the first part. □

An easy application of Lemma 2.4 is now given by the following result.

Theorem 2.3 *It follows for $\alpha > 0$ that*

$$\begin{aligned}\lim_{t\to\infty} v_\alpha(t) &= \frac{\mathbb{E}\left(\int_0^{\boldsymbol{c}_1} \exp(-\alpha y)\boldsymbol{R}_0(dy)\right)}{1 - F_\alpha(\infty)} \\ &= \frac{\mathbb{E}\left(\int_0^{\boldsymbol{c}_1} \exp(-\alpha y)\boldsymbol{R}_0(dy)\right)}{\alpha\int_0^\infty (1 - F(y))\exp(-\alpha y)dy}\end{aligned}$$

Moreover, for every $t > 0$ the inequality

$$-c_0(p)\frac{\mathbb{E}(\boldsymbol{c}_1)}{1 - F_\alpha(\infty)} \leq \exp(\alpha t)(v_\alpha(t) - v_\alpha(\infty)) \leq 0$$

holds with $c_0(p)$ the value of the corresponding infinite horizon average costs. Finally, if additionaly the distribution function F is nonlattice we obtain that

$$\lim_{t\to\infty} \exp(\alpha t)(v_\alpha(t) - v_\alpha(\infty)) = \frac{-c_0(p)}{\alpha}$$

Proof: By (2) it follows that

$$\begin{aligned}\lim_{t\to\infty} v_\alpha(t) &= f_\alpha(\infty)\lim_{t\to\infty} U_\alpha^d(t) \\ &= f_\alpha(\infty)\lim_{t\to\infty}\int_0^t U_\alpha(t-y)G_\alpha(dy) \\ &= f_\alpha(\infty)U_\alpha(\infty) \\ &= \frac{f_\alpha(\infty)}{1 - F_\alpha(\infty)}\end{aligned}$$

As observed before Lemma 2.2 we know that $f_\alpha(\infty) = \mathbb{E}(\boldsymbol{R}_\alpha(\boldsymbol{c}_1)) = \mathbb{E}\left(\int_0^{\boldsymbol{c}_1}\exp(-\alpha y)\boldsymbol{R}_0(dy)\right)$ and this shows the first part. To prove the remaining results we first observe by Lemma 2.1 and Theorem 5.7.3 of Mikolás [19] that $f_\alpha(t) = \int_0^t \exp(-\alpha t) f_0(dy)$ for every $0 \leq t < \infty$. This implies using $v_\alpha(\infty) = f_\alpha(\infty)U_\alpha(\infty)$ and Lemma 2.2 that

$$\begin{aligned}\exp(\alpha t)(v_\alpha(t) - v_\alpha(\infty)) &= \int_{0-}^t \exp(\alpha(t-y))(U_\alpha(t-y) - U_\alpha(\infty))f_0(dy) \\ &\quad + U_\alpha(\infty)\exp(\alpha t)(f_\alpha(t) - f_\alpha(\infty))\end{aligned}$$

Since $0 \leq \exp(\alpha t)(f_\alpha(\infty) - f_\alpha(t)) \leq f_0(\infty) - f_0(t)$ the third part follows by the second part of Lemma 2.4. Moreover, by the above inequality, Theorem 2.1 and the first part of

Lemma 2.4 we obtain for every $t \geq 0$ that

$$\begin{aligned} 0 &\geq \exp(\alpha t)(v_\alpha(t) - v_\alpha(\infty)) \\ &\geq \frac{-f_0(t)}{1 - F_\alpha(\infty)} + U_\alpha(\infty)(f_0(t) - f_0(\infty)) \\ &= \frac{-f_0(\infty)}{1 - F_\alpha(\infty)} \quad = \quad -c_0(p)\frac{\mathbb{E}(\boldsymbol{c}_1)}{1 - F_\alpha(\infty)} \end{aligned}$$

and this shows the second part. □

Clearly the above lemma can be used to justify the approximation $v_\alpha(\infty)$ of the discounted cost $v_\alpha(t)$ up to time t for t large enough. To be more precise, it follows by Theorem 2.3 that

$$0 \leq v_\alpha(\infty) - v_\alpha(t) \leq \exp(-\alpha t) c_0(p) \frac{\mathbb{E}(\boldsymbol{c}_1)}{1 - F_\alpha(\infty)}$$

Since in the previous theorems the function $t \to \mathbb{E}(\boldsymbol{R}_\alpha(t \wedge \boldsymbol{c}_1))$ shows up, we will focus in the next section our attention on the computation of this function. In particular we will discuss under which conditions this function does not have a singular component. These results enable us to compute in an easy and intuitive way the objective function of most of the replacement models discussed in the literature.

3 On the absolute continuity of the numerator

By Lemma 2.1 it follows that $f_\alpha : [0, \infty) \to [0, \infty)$ given by $f_\alpha(t) = \mathbb{E}(\boldsymbol{R}_\alpha(t \wedge \boldsymbol{c}_1))$ can be identified (up to a normalization constant) with a right continuous distribution function and this yields by the Lebesgue decomposition Theorem (Ash [1]) that f_α can be written uniquely as the sum of an increasing atomic component $f_\alpha^{(a)}$, an increasing absolutely continuous component $f_\alpha^{(ac)}$ and an increasing continuous singular component $f_\alpha^{(s)}$ (Feller [11], Loeve [15]). Applying now the Radon–Nikodym Theorem (Ash [1]) to the increasing absolutely continuous component $f_\alpha^{(ac)}$ we obtain that there exists a nonnegative Borel measurable function g satisfying

$$f_\alpha^{(ac)}(t) = \int_0^t g(y)dy$$

for every $t \geq 0$ with the integral representing a Lebesgue integral. Since the increasing singular component $f_\alpha^{(s)}$ is a continuous function which increases only strictly on a set of Lebesgue measure zero, it is clear that the setup in Section 2 is too general to give nice analytical expressions for the function f_α. Fortunately, in most single component maintenance models the stochastic process $\{\boldsymbol{R}_\alpha(t \wedge \boldsymbol{c}_1) : t \geq 0\}$ satisfies the following decomposition property.

Decomposition property.

There exists an increasing stochastic process $\{\boldsymbol{Z}_\alpha(t) : 0 \leq t < \infty\}$ with right continuous sample paths, $\boldsymbol{Z}_\alpha(0) = 0$ and $t \to \mathbb{E}(\boldsymbol{Z}_\alpha(t))$ a continuous function and a stochastic process $\{g_\alpha(t \wedge \boldsymbol{c}_1) : 0 \leq t < \infty\}$ with g_α an increasing right continuous finite–valued function satisfying $g_\alpha(0) = 0$ such that

$$\boldsymbol{R}_\alpha(t \wedge \boldsymbol{c}_1) = \boldsymbol{Z}_\alpha(t) + g_\alpha(t \wedge \boldsymbol{c}_1)$$

for every $t \geq 0$.

Clearly, if the above decomposition property holds we obtain that

$$\mathbb{E}\left(\boldsymbol{R}_\alpha(t \wedge \boldsymbol{c}_1)\right) = \mathbb{E}(\boldsymbol{Z}_\alpha(t)) + \mathbb{E}(g_\alpha(t \wedge \boldsymbol{c}_1))$$

for every $t \geq 0$. The next lemma discusses under which conditions on the continuous increasing function $p_\alpha : [0, \infty) \to [0, \infty)$ given by $p_\alpha(t) := \mathbb{E}(\boldsymbol{Z}_\alpha(t))$ this function is absolutely continuous and can therefore be written as the Lebesgue integral of a Lebesgue integrable function. Before discussing this lemma we introduce the so–called upper and lower right Dini derivative D^+g, resp. D_+g of a function $g : [0, \infty) \to (-\infty, +\infty]$ given by

$$(D^+g)(t) := \limsup_{h \downarrow 0} \frac{g(t+h) - g(t)}{h}$$

and

$$(D_+g)(t) := \liminf_{h \downarrow 0} \frac{g(t+h) - g(t)}{h}$$

(For a survey on the properties of Dini derivatives and their uses in optimization, see Giorgi and Komlosi [14]). Observe it follows that $(D^+g)(t) \geq (D_+g)(t)$ and this upper and lower right Dini derivative always exists within the interval $[-\infty, \infty]$. Moreover, since the function $p_\alpha(t)$ is an increasing function we obtain that $(D^+p_\alpha)(t) \geq (D_+p_\alpha)(t) \geq 0$ for every $t \geq 0$.

Lemma 3.1 *If the function $p_\alpha : [0,\infty) \to [0,\infty)$ given by $p_\alpha(t) := I\!E(\boldsymbol{Z}_\alpha(t))$ is continuous and the function $t \to (D^+p_\alpha)(t)$ is finite–valued then*

$$p_\alpha(t) = \int_0^t (D^+p_\alpha)(y)dy$$

for every $t \geq 0$ with the integral representing a Lebesgue integral. Moreover, if the finite–valued function $y \to (D^+p_\alpha)(y)$ is continuous almost everywhere on $[0,t]$ with respect to the Lebesgue measure this integral can be interpreted as a Riemann integral.

Proof: We first give a proof of the inequality $p_\alpha(t) \geq \int_0^t (D^+p_\alpha)(y)dy$. Introduce now for each $n \in I\!N$ the function $p_\alpha^{(n)} : [0,\infty) \to [0,\infty)$ given by

$$p_\alpha^{(n)}(t) := n(p_\alpha(t + \frac{1}{n}) - p_\alpha(t))$$

Clearly for each $n \in I\!N$ this function is nonnegative, uniformly bounded and continuous and

$$\int_0^t p_\alpha^{(n)}(y)dy = n\int_t^{t+1/n} p_\alpha(y)dy - n\int_0^{1/n} p_\alpha(y)dy$$

By the above remarks it follows that $\lim_{n\uparrow\infty} \int_0^t p_\alpha^{(n)}(y)dy = p_\alpha(t) - p_\alpha(0) = p_\alpha(t)$ and by Fatou's Lemma (Ash [1]) we obtain that

$$p_\alpha(t) = \lim_{n\uparrow\infty} \int_0^t p_\alpha^{(n)}(y)dy \geq \int_0^t \liminf_{n\uparrow\infty} p_\alpha^{(n)}(y)dy$$

Since it follows for every $t \geq 0$ that

$$(D^+p_\alpha)(t) \geq \liminf_{n\uparrow\infty} p_\alpha^{(n)}(t) \geq (D_+p_\alpha)(t)$$

and $(D_+p_\alpha)(t) = (D^+p_\alpha)(t)$ almost everywhere (with respect to the Lebesgue measure) due to p_α increasing and Theorem 2.39 of Ash [1], we obtain by Theorem 1.6.5 of Ash [1] that

$$p_\alpha(t) = \lim_{n\uparrow\infty} \int_0^t p_\alpha^{(n)}(y)dy \geq \int_0^t (D^+p_\alpha)(y)dy$$

To verify the reverse inequality we introduce the function $P_\alpha : [0,\infty) \to [0,\infty)$ given by

$$P_\alpha(t) := \int_0^t (D^+p_\alpha)(y)dy$$

Clearly for every $t \geq 0$ we obtain by the calculus rules for Dini derivatives (Giorgi and Komlosi [14]) that

$$(D^+(P_\alpha - p_\alpha))(t) \geq (D^+P_\alpha)(t) + (D_+(-p_\alpha))(t) = (D^+P_\alpha)(t) - (D^+p_\alpha)(t)$$

and this yields by Theorem 2.3.10 of Ash [1] that $(D^+(P_\alpha - p_\alpha))(t) \geq 0$ for almost every t (with respect to the Lebesgue measure). Moreover, since P_α is increasing we obtain that $(D^+P_\alpha)(t) \geq 0$ and due to $t \to (D^+p_\alpha)(t)$ is a finite–valued function this yields for every $t \geq 0$ that $(D^+(P_\alpha - p_\alpha))(t) \geq -(D^+p_\alpha)(t) > -\infty$. Since by Theorem 2.3.4 of Ash [1] the function P_α is absolutely continuous, and hence continuous, and p_α is continuous by assumption it follows by Lemma 2 at page 370 of Titchmarsh [28] that the function $P_\alpha - p_\alpha$ is increasing. This implies for every $t \geq 0$ that

$$\int_0^t (D^+p_\alpha)(y)dy - p_\alpha(t) = P_\alpha(t) - p_\alpha(t) \geq P_\alpha(0) - p_\alpha(0) = 0$$

and this proves the desired result. □

The condition that the function in Lemma 3.1 is continuous cannot be omitted. As an example we mention the function $p : [0,\infty) \to [0,\infty)$ given by

$$p(t) = \begin{cases} 0 & \text{if } t < 1 \\ 1 & \text{if } t \geq 1 \end{cases}$$

For this function it follows that $(D^+p)(t) = 0$ for every $t \geq 0$ and since p is not continuous it can obviously not be written as a Lebesgue integral. Moreover, the condition that D^+p_α

is a finite–valued function can also not be omitted. As a counter example we refer to the well–known Cantor function f discussed on p. 366 of Titchmarsh [28]. The derivative of this function exists almost everywhere and vanishes, and it does not hold that $1 = f(1) - f(0) = \int_0^1 (D^+ f)(y)dy = 0$.

We will now relate $(D^+ p_\alpha)(t)$ to another limit which has a clear probabilistic interpretation. Observe for every realization $\omega \in \Omega$ satisfying $\boldsymbol{c}_1(\omega) \leq t$ that $\boldsymbol{R}_\alpha((t \wedge \boldsymbol{c}_1)(\omega), \omega) = \boldsymbol{R}_\alpha(\boldsymbol{c}_1(\omega), \omega)$ and $\boldsymbol{R}_\alpha((t \wedge \boldsymbol{c}_1)(\omega), \omega) = \boldsymbol{Z}_\alpha(t, \omega) + g_\alpha(\boldsymbol{c}_1(\omega))$. Hence we obtain that $\boldsymbol{Z}_\alpha(t, \omega) = \boldsymbol{Z}_\alpha(\boldsymbol{c}_1(\omega), \omega)$ and this implies for every $h > 0$ that

$$\begin{aligned} \mathbb{E}(\boldsymbol{Z}_\alpha(t+h) - \boldsymbol{Z}_\alpha(t)) &= \mathbb{E}((\boldsymbol{Z}_\alpha(t+h) - \boldsymbol{Z}_\alpha(t)) 1_{\{\boldsymbol{c}_1 > t\}}) \\ &= \mathbb{E}(\boldsymbol{Z}_\alpha(t+h) - \boldsymbol{Z}_\alpha(t) | \boldsymbol{c}_1 > t)(1 - F(t)) \end{aligned}$$

If we define the cost intensity function of the stochastic process $\{\boldsymbol{Z}_\alpha(t) : t \geq 0\}$ by

$$m_\alpha(t) := \limsup_{h \downarrow 0} \frac{\mathbb{E}(\boldsymbol{Z}_\alpha(t+h) - \boldsymbol{Z}_\alpha(t) | \boldsymbol{c}_1 > t)}{h}$$

then it follows by the above relation that $(D^+ p_\alpha)(t) = m_\alpha(t)(1 - F(t))$. Moreover, if the stochastic process $\{\boldsymbol{Z}_0(t) : t \geq 0\}$ denotes the related undiscounted version of the stochastic process $\{\boldsymbol{Z}_\alpha(t) : t \geq 0\}$ we clearly obtain that

$$\exp(-\alpha(t+h)) \mathbb{E}(\boldsymbol{Z}_0(t+h) - \boldsymbol{Z}_0(t) | \boldsymbol{c}_1 > t) \leq \mathbb{E}(\boldsymbol{Z}_\alpha(t+h) - \boldsymbol{Z}_\alpha(t) | \boldsymbol{c}_1 > t)$$

and

$$\mathbb{E}(\boldsymbol{Z}_\alpha(t+h) - \boldsymbol{Z}_\alpha(t) | \boldsymbol{c}_1 > t) \leq \exp(-\alpha t) \mathbb{E}(\boldsymbol{Z}_0(t+h) - \boldsymbol{Z}_0(t) | \boldsymbol{c}_1 > t)$$

Hence, if $m_0(t)$ is the cost intensity function of the undiscounted version $\{\boldsymbol{Z}_0(t) : t \geq 0\}$ it follows by the above inequalities that $m_\alpha(t) = \exp(-\alpha t) m_0(t)$. By these observations the result in Lemma 3.1 can be rewritten as follows.

Lemma 3.2 *If the function $p_\alpha : [0, \infty) \to [0, \infty)$ given by $p_\alpha(t) := \mathbb{E}(\boldsymbol{Z}_\alpha(t))$ is continuous and the cost intensity function $m_0(\cdot)$ of the undiscounted version $\boldsymbol{Z}_0(\cdot)$ of the stochastic process $\boldsymbol{Z}_\alpha(\cdot)$ is finite–valued, i.e.*

$$m_0(t) := \limsup_{h \downarrow 0} \frac{\mathbb{E}(\boldsymbol{Z}_0(t+h) - \boldsymbol{Z}_0(t) | \boldsymbol{c}_1 > t)}{h} < \infty$$

for every $t > 0$ then it follows that $E(\boldsymbol{Z}_\alpha(t)) = \int_0^t \exp(-\alpha y) m_0(y)(1 - F(y))dy$ for every $t > 0$.

Proof: Apply Lemma 3.1 and the observations before Lemma 3.2. □

By the above relation between $(D^+ p_\alpha)(t)$ and $m_0(t)$ it follows clearly that $t \to (D^+ p_\alpha)(t)$ is finite–valued if and only if $t \to m_0(t)$ is finite–valued. To relate the stochastic process $\{g_\alpha(t \wedge \boldsymbol{c}_1) : t \geq 0\}$ to its undiscounted version $\{g_0(t \wedge \boldsymbol{c}_1) : t \geq 0\}$ with $g_0 : [0, \infty) \to [0, \infty)$ a finite–valued right continuous increasing function and $g_0(0) = 0$ we observe by Theorem 5.7.3 of Mikolás [19] that

$$g_\alpha(t) = \int_0^t \exp(-\alpha y) g_0(dy)$$

for every $t > 0$. This implies that $E(g_\alpha(t \wedge \boldsymbol{c}_1)) = E(\int_0^{t \wedge \boldsymbol{c}_1} \exp(-\alpha y) g_0(dy))$ for every $t > 0$ and applying Fubini's theorem (Mikolás [19]) it is easy to verify that

$$\begin{aligned} E\left(\int_0^{t\wedge \boldsymbol{c}_1} \exp(-\alpha y) g_0(dy)\right) &= \int_0^t \int_0^z \exp(-\alpha y) g_0(dy) F(dz) + \int_0^t \exp(-\alpha y) g_0(dy)(1 - F(t)) \\ &= \int_0^t \int_y^t F(dz) \exp(-\alpha y) g_0(dy) + \int_0^t \exp(-\alpha y) g_0(dy)(1 - F(t)) \\ &= \int_0^t \exp(-\alpha y)(1 - F(y)) g_0(dy) \end{aligned}$$

By the above observations the next result follows immediately.

Theorem 3.1 *If the decomposition property and the conditions of Lemma 3.2 hold then for every $t \geq 0$ we obtain that*

$$E(\boldsymbol{R}_\alpha(t \wedge \boldsymbol{c}_1)) = \int_0^t \exp(-\alpha y) m_0(y)(1 - F(y))dy + \int_0^t \exp(-\alpha y)(1 - F(y)) g_0(dy)$$

This concludes our discussion on the absolute continuity of the numerator and in the next section we will consider some examples of maintenance models fitting into the above framework.

4 Examples of maintenance models

To show the implications of the previous results we will consider in some detail a few maintenance models. Observe this section considers similar models, with a more general cost structure, as considered in an intuitive way by Berg [4, 5, 6]. The first model we will consider is the age replacement model with running costs and age–dependent replacement costs. In this model a component is replaced by a new component either at failure (failure replacement) or preventively (preventive replacement) if it reaches a critical age $0 < T < \infty$. In this model the regeneration points are given by the replacement instances and so $\boldsymbol{c}_i = T \wedge \boldsymbol{X}_i$ with $\boldsymbol{X}_i, i \geq 1$, a sequence of independent and identically distributed random variables with distribution function G representing the random lifetime of the ith component. We assume for convenience that the distribution function G is continuous satisfying $(D^+G)(t) < \infty$ for every $t \geq 0$ and this implies by Lemma 3.1 that the upper right Dini derivative can be interpreted as a possibly discontinuous density. Moreover, the replacement costs both at a preventive and a failure replacement depend on the age of the component and are given by the increasing, finite–valued and right continuous functions $c_p(\cdot)$, resp. $c_f(\cdot)$. This means if a component of age $x < T$ breaks down, the replacement costs are given by $c_f(x)$, while at a preventive replacement of a component of age T the corresponding costs are $c_p(T)$. Finally, it is assumed that the running costs of a component are given by the increasing, finite–valued and right continuous function $K : [0, \infty) \to [0, \infty)$ with $K(0) = 0$ and these costs represent for example regular maintenance costs, depreciation costs, costs of reduced output and so on. For the above model the function g_0 is given by

$$g_0(x) = \begin{cases} K(x) & \text{if } x < T \\ c_p(T) + K(T) & \text{if } x \geq T > 0 \end{cases}$$

while the (right continuous) stochastic process $\boldsymbol{Z}_0(t)$ is defined by

$$\boldsymbol{Z}_0(t) = \text{cost in the first cycle up to time } t \text{ due to a replacement at failure}$$

Observe now for $t \geq T > 0$ that $\boldsymbol{Z}_0(t-) = \boldsymbol{Z}_0(t)$ and for $t < T$ that $\boldsymbol{Z}_0(t-) < \boldsymbol{Z}_0(t)$ if and only if the first failure occurs at time t. By the continuity of the distribution function G this implies that $\boldsymbol{Z}_0(t-) = \boldsymbol{Z}_0(t)$ μ–almost everywhere and hence by a standard application of the dominated convergence theorem (Ash [1]) we obtain that $t \to I\!\!E(\boldsymbol{Z}_0(t))$ is a continuous function. To determine the cost intensity function notice for $t+h<T$ and $h>0$ that $\boldsymbol{Z}_0(t+h)=\boldsymbol{Z}_0(t)=0$ conditional on the event $\{\boldsymbol{c}_1 \geq t+h\}$ and so

$$\mathbb{E}(\boldsymbol{Z}_0(t+h)-\boldsymbol{Z}_0(t)|\boldsymbol{c}_1>t)=\mathbb{E}((\boldsymbol{Z}_0(t+h)-\boldsymbol{Z}_0(t))1_{\{\boldsymbol{c}_1\leq t+h\}}|\boldsymbol{c}_1>t)$$

This implies by the monotonicity of the function c_f and the representation of $\boldsymbol{c}_1$ that

$$c_f(t)\frac{G(t+h)-G(t)}{1-G(t)}\leq \mathbb{E}(\boldsymbol{Z}_0(t+h)-\boldsymbol{Z}_0(t)|\boldsymbol{c}_1>t)\leq c_f(t+h)\frac{G(t+h)-G(t)}{1-G(t)}$$

for every $t+h<T$ and $h>0$. Since $c_f(\cdot)$ is a right continuous function and G has a failure rate $r(\cdot)$ given by $r(t)=\frac{(D^+G)(t)}{1-G(t)}$ for every $t\geq 0$ we finally obtain for $t<T$ that

$$m_0(t):=\limsup_{h\downarrow 0}\frac{\mathbb{E}(\boldsymbol{Z}_0(t+h)-\boldsymbol{Z}_0(t)|\boldsymbol{c}_1>t)}{h}=c_f(t)r(t)$$

Moreover, for $t\geq T$ and $h>0$ it follows immediately that $\boldsymbol{Z}_0(t+h)=\boldsymbol{Z}_0(t)$ and so $\mathbb{E}(\boldsymbol{Z}_0(t+h)-\boldsymbol{Z}_0(t))=0$ or equivalently $m_0(t)=0$. Using Theorem 3.1 we obtain

$$\mathbb{E}(\boldsymbol{R}_\alpha(t\wedge\boldsymbol{c}_1))=\begin{cases} z(t) & \text{if } t<T\\ z(T)+\exp(\alpha T)c_p(T)(1-G(T)) & \text{if } t\geq T\end{cases}$$

with

$$z(t):=\int_0^t \exp(-\alpha y)c_f(y)r(y)(1-G(y))dy+\int_0^t\exp(-\alpha y)(1-G(y))K(dy)$$

and so we can determine by Theorem 2.1 and Lemma 2 the behaviour of the function $v_\alpha(\cdot)$ (the details are left to the reader). Observe, if $K(\cdot)$ and/or $c_p(\cdot)$ are not differentiable, the MCA approach of Berg [4, 5, 6] cannot be used. Special cases of the above model are discussed by Barlow and Proschan [3], Schaeffer [25], Cleroux and Hanscom [9], Fox [12], Berg and Epstein [7] and Ran and Rosenlund [23]. If we also like to incorporate down times due to repairs we obtain that $\boldsymbol{c}_i, i\geq 1$, equals $\boldsymbol{c}_i=(T\wedge\boldsymbol{X}_i)+\boldsymbol{L}_i$ with $\boldsymbol{L}_i$ the length of the ith down time and the random variables $\boldsymbol{L}_i, i\geq 1$ independent and identically distributed. In this case the stochastic process $\{\boldsymbol{Z}_0(t):t\geq 0\}$ not only counts the costs due to a failure replacement but also incorporates the costs due to down time. In the above model we only allow replacements and we do not allow minimal repairs (Barlow and Proschan [3]) if a component has failed before reaching its critical age $0 < T < \infty$. A minimal repair means that the component is restored to its state just before failing or equivalently the failure rate function following a minimal repair is undisturbed (for a mathematical definition see Nakagawa and Kowada [21]). However, if we may choose either to replace a failed component or to repair it minimally we first need to introduce the cost of a minimal repair at age t. To model this we introduce a stochastic process $\{\boldsymbol{M}(t) : t \geq 0\}$ with right continuous sample paths and define

$$\boldsymbol{M}(t) := \text{costs of a minimal repair if a component fails at age } t.$$

It is realistic to assume that the stochastic process $\{\boldsymbol{M}(t) : t \geq 0\}$ is increasing and to determine whether a failed component is minimally repaired we also introduce a so–called repair–cost limit function $l : [0, \infty) \to [0, \infty]$. This function is increasing and right continuous and represents the maximum costs one is willing to pay for a minimal repair of a component of age t. This implies that a minimal repair on a failed component of age t is performed if and only if $\boldsymbol{M}(t) \leq l(t)$ and so the probability $p(t)$ that a failed component of age t is minimally repaired is given by $Pr\{\boldsymbol{M}(t) \leq l(t)\}$. Since the function l is increasing and right continuous and the increasing stochastic process $\{\boldsymbol{M}(t) : t \geq 0\}$ has right continuous sample paths we obtain that the function $p : [0, \infty) \to [0, 1]$ is right continuous. To determine the random time until the first replacement we consider the stochastic process $\{(\boldsymbol{Y}_n, \boldsymbol{I}_n) : n \geq 1\}$ with $\boldsymbol{I}_n = 0$, resp. 1, if a minimal repair, resp. a replacement, is carried out at the instance $\boldsymbol{Y}_n$, with $\boldsymbol{Y}_n, n \geq 1$, the instance of the nth failure of the first component. Clearly, if $\tau := \inf\{n \geq 1 : \boldsymbol{I}_n = 1\}$ then by definition $\boldsymbol{Y}_n = \infty$ for every $n \geq \tau + 1$ and the random variable $\boldsymbol{c}_1$ is represented by $\boldsymbol{c}_1 = T \wedge \boldsymbol{Y}_\tau$. By the definition of minimal repair the point process $\{\boldsymbol{N}(t) : 0 \leq t < \infty\}$ associated with $\{\boldsymbol{Y}_n : n \geq 0\}$ (define $\boldsymbol{Y}_0 := 0$!) and given by $\boldsymbol{N}(t) := \sup\{n \geq 0 : \boldsymbol{Y}_n \leq t\}$ is a nonhomogeneous Poisson process with random intensity function $r(t, \omega) = r(t)$ for every $t \leq \boldsymbol{Y}_\tau(\omega)$ and $r(t, \omega) = 0$ otherwise. This implies that the failure rate at time t of the distribution of $\boldsymbol{Y}_\tau$ equals $(1 - p(t))r(t)$ and hence its distribution function G is given by

$$1 - G(x) := Pr\{\boldsymbol{Y}_\tau > x\} = \exp(-\int_0^x (1 - p(y))r(y)dy)$$

This yields as already observed by Berg [5] a much easier way to derive the above distribution function than the approach of Block, Borges and Savits [8]. For this model the function g_0 is the same as for the first model and the right continuous stochastic process $\{\boldsymbol{Z}_0(t) : t \geq 0\}$ is given by

$$\boldsymbol{Z}_0(t) = \text{cost in the first cycle up to time } t \text{ due to minimal repairs and a replacement at failure}$$

As before it is not difficult to verify that $t \to \mathbb{E}(\boldsymbol{Z}_0(t))$ is a continuous function. Moreover, by the definition of $\boldsymbol{c}_1$ it follows for every $t < T$ that

$$\begin{aligned} \mathbb{E}(\boldsymbol{Z}_0(t+h) - \boldsymbol{Z}_0(t) | \boldsymbol{c}_1 > t) &= \mathbb{E}((\boldsymbol{Z}_0(t+h) - \boldsymbol{Z}_0(t)) 1_{\{\boldsymbol{Y}_{\boldsymbol{N}(t)+1} \leq t+h, \nu = \boldsymbol{N}(t)+1\}} | \boldsymbol{Y}_\tau > t) \\ &+ \mathbb{E}((\boldsymbol{Z}_0(t+h) - \boldsymbol{Z}_0(t)) 1_{\{\boldsymbol{Y}_{\boldsymbol{N}(t)+1} \leq t+h, \nu > \boldsymbol{N}(t)+1\}} | \boldsymbol{Y}_\tau > t) \end{aligned}$$

and this yields by the definition of a nonhomogeneous Poisson point process (Ross [24]) that

$$\begin{aligned} \lim_{h \downarrow 0} \frac{\mathbb{E}(\boldsymbol{Z}_0(t+h) - \boldsymbol{Z}_0(t) | \boldsymbol{c}_1 > t)}{h} &= (1 - p(t))r(t)c_f(t) + r(t)E\left(\boldsymbol{M}(t) 1_{\{\boldsymbol{M}(t) \leq l(t)\}}\right) \\ &= (1 - p(t))r(t)c_f(t) + p(t)r(t)E\left(\boldsymbol{M}(t) | \boldsymbol{M}(t) \leq l(t)\right) \end{aligned}$$

Moreover, it is also easy to see that $\mathbb{E}(\boldsymbol{Z}_0(t+h) - \boldsymbol{Z}_0(t)) = 0$ for every $t \geq T$ and $h > 0$ and so as before one can work out the expressions for the discounted and average cost criterium (the details are left for the reader). Finally, we like to observe that a special case of the above model is considered by Berg [5] and Block, Borges and Savits [8]. To conclude this section we observe that one can also very easily include in the above analysis a cost function for minimal repairs which not only depends on the age of the component but also on the number of minimal repairs already performed on this component. If one additionally introduces a third option that a minimal repair is unsuccesfull with a probability depending on the age and the number of previous minimal repairs and if this happens that one has to replace it by a new component this can still be analyzed using the above approach. This generalization is considered by Makis and Jardine [17]. Also other classes of maintenance models can be analyzed using the above approach. As an example we mention the so–called T–policy of Muth [20] already considered in Section 2 (see also Makis and Jardine [16]). To conclude this paper we finally observe that an overview on these other classes of maintenance models is given by Sherif and Smith [26] and Valdez–Flores and Feldman [29].

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