A note on the marginal cost approach in maintenance

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Abstract

In this paper we verify the conditions under which the marginal cost approach of Berg [6, 7, 8] holds. As observed by Aven and Dekker [2] the validity of the marginal cost approach gives rise to a useful framework of single component maintenance optimization models which covers almost all models used in practice. For the class of unimodal finite–valued marginal cost functions we show that these optimization models are easy to solve.

Keywords: Maintenance, Marginal Cost Analysis, Lebesgue integral, Unimodality

1 Introduction

In a series of papers Berg [6, 7, 8] introduces the so–called Marginal Cost Analysis (MCA) method for the study of single component replacement models with a block or age replacement structure. This easy approach is applied to a large collection of examples without a proper introduction of the mathematical conditions under which this approach holds. To verify these conditions will be the topic of this paper. At the same time, we will show under which conditions on the so–called marginal cost function this approach can be used to determine the optimal age replacement policy.

2 Analysis

In an infinite horizon single component replacement model with stationary replacement policies characterized by a finite positive parameter T, one can identify for every policy $p(T), T > 0$, a nonnegative increasing stochastic process $\{\boldsymbol{R}(t, p(T)) : t \geq 0\}$ with right continuous sample paths. For each $t > 0$ the random variable $\boldsymbol{R}(t, p(T))$ represents the total maintenance costs up to time t if policy $p(T)$ is used. Well–known examples of such policies are the block and age replacement policies (Barlow and Proschan [4]). In this paper we will only consider the class of age replacement policies. Observe that a completely similar analysis can be given for the class of block replacement policies. Each age replacement policy $p(T)$ is now completely determined by the random variables $T \wedge \boldsymbol{X}_i, i \geq 1$, with $T \wedge \boldsymbol{X}_i := \min\{T, \boldsymbol{X}_i\}$ and $\boldsymbol{X}_i, i \geq 1$, a sequence of nonnegative, independent and identically distributed random variables. Clearly the random variable $T \wedge \boldsymbol{X}_i$ represents the elapsed time between the $(i-1)$th and ith replacement. If $\mathbb{E}(\boldsymbol{Z})$ denotes the expectation of the random variable $\boldsymbol{Z}$ and the function $f : [0, \infty) \to [0, \infty)$ given by $f(T) := \mathbb{E}\left(\boldsymbol{R}(T \wedge \boldsymbol{X}_1, p(T))\right)$ is finite–valued, it is well–known by the regenerative structure of the family of age replacement models (Frenk, Dekker and Kleijn [10]) that the infinite horizon average costs $c(T)$ of policy $p(T)$ equal

$$c(T) = \frac{f(T)}{\mathbb{E}(T \wedge \boldsymbol{X}_1)} \tag{1}$$

In the paper we always assume that the following conditions hold.

(a) The function $f : [0, \infty) \to [0, \infty)$ is continuous on $(0, \infty)$ and right continuous at 0

(b) The distribution function F of the random variable $\boldsymbol{X}_1$ is continuous on $(0, \infty)$, has support $[0, \infty)$ and satisfies $F(0) = 0$ and $F(\infty) = 1$

By condition (b) it is easy to verify that the denominator in (1) is positive and continuously differentiable on $(0, \infty)$. Moreover, by condition (a) this yields that the function $c : (0, \infty) \to [0, \infty)$ is finite–valued and continuous. In the MCA approach it is now assumed by Berg [6, 7, 8] without any proof that the existence of the marginal cost function implies that the numerator in (1) is absolutely continuous and can be represented by the sum of a constant and an integral involving this marginal cost function and the tail of the distribution function F. However, without assuming that the function f is continuous this is not correct. Since this representation yields an easy understandable economic interpretation of the cost function it seems therefore important to specify the proper conditions under which this holds. Observe we will introduce at the same time the terminology used by Berg [6, 7, 8]. To start with our analysis we need the following definition.

Definition 2.1 *Let $g : [0,\infty) \to (-\infty,\infty)$ be a finite–valued function. The upper right Dini derivative $D^+g : [0,\infty) \to [-\infty,\infty]$ at $t \geq 0$ is given by*

$$(D^+g)(t) := \limsup_{h\downarrow 0} \frac{g(t+h)-g(t)}{h}$$

Moreover, the lower right Dini derivative $D_+g : [0,\infty) \to [-\infty,\infty]$ at $t \geq 0$ is given by

$$(D_+g)(t) := \liminf_{h\downarrow 0} \frac{g(t+h)-g(t)}{h}$$

Clearly, the upper and lower right Dini derivative at $t \geq 0$ always exist but may possibly be $-\infty$ or ∞. To analyze the numerator we observe for every $h > 0$ that the policy $p(T)$ is the same as the policy $p(T+h)$ if $\boldsymbol{X}_1 < T$ and so

$$\begin{aligned}
&f(T+h) - f(T) \\
&= I\!\!E\left((\boldsymbol{R}((T+h)\wedge \boldsymbol{X}_1, p(T+h)) - \boldsymbol{R}(T\wedge \boldsymbol{X}_1, p(T)))1_{\{\boldsymbol{X}_1 \geq T\}}\right) \\
&= I\!\!E\left(\boldsymbol{R}((T+h)\wedge \boldsymbol{X}_1, p(T+h)) - \boldsymbol{R}(T\wedge \boldsymbol{X}_1, p(T)) | \boldsymbol{X}_1 \geq T\right)(1-F(T))
\end{aligned}$$

If we define

$$m(T) := \limsup_{h\downarrow 0} \frac{I\!\!E\left(\boldsymbol{R}((T+h)\wedge \boldsymbol{X}_1, p(T+h)) - \boldsymbol{R}(T\wedge \boldsymbol{X}_1, p(T)) | \boldsymbol{X}_1 \geq T\right)}{h}$$

then it follows that $(D^+f)(T) = m(T)(1-F(T))$. As observed by Berg [7] the function $m : [0,\infty) \to [-\infty,\infty]$ measures at T the local increase or decrease in costs between the preventive replacement of a component of age T at time T and the costs of deferring this preventive replacement an infinitesimal time unit later. Therefore it is not surprising that the function $m : [0,\infty) \to [-\infty,\infty]$ is given the following name (Berg [6, 7, 8]).

Definition 2.2 *The function $m : [0,\infty) \to [-\infty,\infty]$ is called the marginal cost function.*

In his papers, Berg does not define the marginal cost function by means of upper right Dini derivatives, but assumes that the right–hand derivative exists. This implies that the lower and upper right Dini derivative must be equal and so his approach, which is only intuitive, is less general. Observe, since we use upper right Dini derivatives, that the marginal cost function is always defined.

It is now possible to prove that the function f is absolutely continuous.

Theorem 2.1 *If the function $f : [0,\infty) \to [0,\infty)$ is continuous on $(0,\infty)$ and right continuous at 0 and the marginal cost function $m : [0,\infty) \to [-\infty,\infty]$ is nonnegative and finite–valued for every $t \geq 0$, then it follows that*

$$f(T) = f(0) + \int_0^T m(y)(1-F(y))dy$$

for every $T > 0$. Moreover, the above integral represents a Lebesgue integral. If additionally the function m is continuous almost everywhere with respect to the Lebesgue measure and bounded on $[0,T]$, then the above integral can be interpreted as a Riemann integral.

Proof: Introduce for each $n \in I\!N$ the function $f_n : [0,\infty) \to [0,\infty)$ given by

$$f_n(t) := n(f(t+\frac{1}{n}) - f(t))$$

By our assumptions f_n is a finite–valued continuous function on $(0,\infty)$ and therefore Riemann integrable on $[0,T]$ for every $0 < T < \infty$. This yields

$$\int_0^T f_n(y)dy = n\int_T^{T+\frac{1}{n}} f(y)dy - n\int_0^{\frac{1}{n}} f(y)dy$$

and by the continuity of the function f we obtain that

$$\lim_{n\to\infty}\int_0^T f_n(y)dy = f(T) - f(0) \tag{2}$$

Since the marginal cost function m is nonnegative it follows for every $t \geq 0$ that $(D^+f)(t) = m(t)(1-F(t)) \geq 0$, and again by the continuity of f and Theorem 1.13 (ii) of Giorgi and

Komlosi [11], the function f is increasing on $(0,\infty)$. Hence, for each $n \in I\!N$ it follows that f_n is nonnegative and so we can apply Fatou's Lemma (Ash [1]) to (2). As a result we obtain

$$f(T) - f(0) = \lim_{n\uparrow\infty} \int_0^T f_n(y)dy \geq \int_0^T \liminf_{n\uparrow\infty} f_n(y)dy \tag{3}$$

Since the function f is increasing it follows by Theorem 2.3.9 of Ash [1] that f is differentiable almost everywhere with respect to the Lebesgue measure, and hence $D^+f = D_+f$ except on a set of Lebesgue measure zero. This finally implies using $(D^+f)(y) \geq \liminf_{n\uparrow\infty} f_n(y) \geq (D_+f)(y)$ for every $y \geq 0$, inequality (3) and Theorem 1.6.5 (b) of Ash [1] that

$$f(T) - f(0) \geq \int_0^T (D^+f)(y)dy = \int_0^T m(y)(1 - F(y))dy$$

To verify the reverse inequality we introduce the function $\mathcal{F} : [0,\infty) \to [0,\infty)$ given by

$$\mathcal{F}(T) := f(0) + \int_0^T m(y)(1 - F(y))dy$$

For every $t \geq 0$ we obtain by the calculus rules for Dini derivatives (Giorgi and Komlosi [11]) that

$$(D^+(\mathcal{F} - f))(t) \geq (D^+\mathcal{F})(t) + (D_+(-f))(t) = (D^+\mathcal{F})(t) - (D^+f)(t)$$

and this yields by Theorem 2.3.10 of Ash [1] that $(D^+(\mathcal{F} - f))(t) \geq 0$ for almost every $t \geq 0$ (with respect to the Lebesgue measure). Moreover, since $\mathcal{F}$ is increasing due to m nonnegative, and hence $(D^+\mathcal{F})(t) \geq 0$ and by assumption $(D^+f)(t) < \infty$ for every $t \geq 0$, we obtain that $(D^+(\mathcal{F} - f))(t) > -\infty$ for every $t \geq 0$. Since by Theorem 2.3.4 of Ash [1] the function $\mathcal{F}$ is absolutely continuous and hence continuous, and by assumption f is continuous, it follows that $\mathcal{F} - f$ is continuous on $(0,\infty)$. The assumptions of Lemma 2 at page 370 of Titchmarsh [13] are now satisfied and so we may conclude that the function $\mathcal{F} - f$ is increasing. This implies for every $T \geq 0$ that

$$\int_0^T m(y)(1 - F(y))dy + f(0) - f(T) = \mathcal{F}(T) - f(T) \geq \mathcal{F}(0) - f(0) = 0$$

and this yields the inequality $f(T) \leq f(0) + \int_0^T m(y)(1 - F(y))dy$. The second part of the above result is an immediate consequence of Theorem 1.7.1 of Ash [1]. $\square$

The condition in Theorem 2.1 that f is continuous on $(0,\infty)$ and right continuous at 0 cannot be omitted. As a counter example we mention the function $f : [0,\infty) \to I\!R$ given by

$$f(t) = \begin{cases} 1 & \text{if } t \geq 1 \\ 0 & \text{if } 0 \leq t < 1 \end{cases}$$

Clearly, for this function $(D^+f)(t) = 0$ for every $t \geq 0$ and so $\int_0^t (D^+f)(y)dy = 0$ for every $t \geq 0$. However, $f(t) = 1$ for every $t \geq 1$ and this yields that $f(t) \neq f(0) + \int_0^t (D^+f)(y)dy$ if $t \geq 1$. Moreover, one also needs in Theorem 2.1 the condition that $m(t)$, or equivalently $(D^+f)(t)$, is *finite* for every $t \geq 0$. As a counter example we mention the Cantor function $f : [0,1] \to I\!R$ discussed in Section 11.72 of Titchmarsh [13]. This function is continuous, increasing and satisfies $(D^+f)(t) = 0$ for almost every $t \geq 0$ and $(D^+f)(t) = \infty$ on a Lebesgue set of measure zero. Moreover, $f(1) = 1$ and $f(0) = 0$ and so $f(1) \neq f(0) + \int_0^1 (D^+f)(y)dy$.

It is easy to adapt the above proof whenever the marginal cost function is finite–valued and uniformly bounded from below on every compact interval. Since in almost all age replacement models the marginal cost function is nonnegative we did not prove this result in detail. Clearly, for $T = 0$ the expected costs in a degenerate cycle of length zero are given by $f(0)$ and this value is assumed to be positive. By the above result the objective function of any age replacement model belonging to the above family has the following representation involving the marginal cost function. This representation gives an intuitive appealing interpretation of the objective function and is mentioned in Dekker [9] and Aven and Dekker [2] without a proper mathematical justification.

Theorem 2.2 *Under the conditions of Theorem 2.1 it follows for every $0 < T < \infty$ that*

$$c(T) = \frac{f(0) + \int_0^T m(y)(1 - F(y))dy}{\int_0^T (1 - F(y))dy}$$

Proof: An easy application of Theorem 2.1 and (1). □

Observe that for the infinite horizon discounted cost criterion with discount factor $\alpha > 0$ under the same conditions on the discounted marginal cost function m_α a similar result can be derived. In this case the discounted cost function c_α has the representation

$$c_\alpha(T) = \frac{f(0) + \int_0^T m_\alpha(y)(1-F(y))dy}{\alpha \int_0^T \exp(-\alpha y)(1-F(y))dy}$$

The optimal policy $0 < T^* < \infty$ (if it exists!) is a solution of the optimization problem (P) given by $\inf\{c(T) : 0 < T < \infty\}$ with c a continuous function on $(0, \infty)$ and $c(0) = \infty$. Denote now by $\mathcal{P}^*$ the possibly empty and closed set of optimal solutions of (P). To verify that under certain conditions the set $\mathcal{P}^*$ is nonempty, we will first compute the upper right Dini derivative of the objective function c. Since the denominator in (1) is positive and continuously differentiable on $(0,\infty)$ and the numerator is a continuous function we obtain by the calculus rules for upper Dini derivatives (see formula (2a) of Giorgi and Komlosi [11]) that for every $T > 0$ the Dini derivative $(D^+c)(T)$ equals

$$(D^+c)(T) = \frac{1-F(T)}{\mathbb{E}(T \wedge \boldsymbol{X}_1)}(m(T) - c(T))$$

Observe this result holds irrespective as to whether $m(T)$ is finite or not, and so we might encounter situations where the above formula holds but the function c does not have the representation of Theorem 2.2. Using the above formula it is relatively easy to derive sufficient conditions for $\mathcal{P}^*$ to be empty or not. These conditions are known under the additional assumption that the marginal cost function m is continuous on $(0, \infty)$ (Berg [6], Dekker [9] and Barros *et al.* [5]).

Lemma 2.1 *If* $c(\infty) := \lim_{t\uparrow\infty} c(t)$ *exists and* $c(\infty) = \infty$ *then* $\mathcal{P}^*$ *is a nonempty compact subset of* $(0,\infty)$*. Moreover, if* $\liminf_{t\uparrow\infty} m(t) > c(\infty)$ *then* $\mathcal{P}^*$ *is also a nonempty compact subset of* $(0,\infty)$*.*

Proof: The first part is trivial to prove. To verify the second part we observe that $\liminf_{t\uparrow\infty} m(t) > c(\infty)$ implies the existence of some finite T_0 such that for every $T \geq T_0$ it follows that $m(T) > c(T)$. This implies that $(D^+c)(T) > 0$ for every $T \geq T_0$ and since c is continuous we may conclude from Theorem 1.14 of Giorgi and Komlosi [11] that c is strictly increasing on (T_0, ∞). By this observation the desired result follows. □

We will now introduce a class of marginal cost functions for which it is easy to compute the optimal age replacement policy.

Definition 2.3 (Avriel *et al.* [3])
A function $g : [0, \infty) \to [-\infty, \infty]$ *is called unimodal with parameter* $0 \leq b < \infty$ *if the function* g *is decreasing on* $[0, b]$ *and increasing on* (b, ∞).

In the sequel it will be assumed that the marginal cost function m is finite–valued and unimodal with parameter $0 \leq b < \infty$. This implies by Theorem 2.2 that the representation for the function c holds. Moreover, by Theorem 4.1.2 of Mikolás [12] the function m is continuous almost everywhere with respect to the Lebesgue measure and so the integral in Theorem 2.1 can be interpreted as a Riemann integral. It is now possible to prove the following result.

Lemma 2.2 *If the optimal solution set* $\mathcal{P}^*$ *is nonempty and* m *is finite–valued and unimodal with parameter* $0 \leq b < \infty$, *then* $\mathcal{P}^*$ *is contained in* $[b, \infty)$.

Proof: Clearly for $b = 0$ the result follows. Assume therefore that $0 < b < \infty$. Since m is decreasing on $[0, b]$ it follows for every $t < b$ that

$$m(t) \int_0^t (1 - F(y))dy - \int_0^t m(y)(1 - F(y))dy - f(0) < 0$$

This yields by Theorem 2.2 that $m(t) - c(t) < 0$ or equivalently $(D^+c)(t) < 0$ for every $t < b$. Hence, by the continuity of c and the remark after Theorem 1.14 of Giorgi and Komlosi [11] the function c is strictly decreasing on $(0, b)$ implying that $\mathcal{P}^* \subseteq [b, \infty)$. $\square$

The same result under the additional assumption that the marginal cost function m is continuous is also shown by Barros *et al.* [5]. Clearly, by the above observation we can restrict our search for an optimal policy to the interval $[b, \infty)$. Finally, we will show under the assumption of unimodality of the marginal cost function that a simple bisection locates an optimal age replacement policy.

Theorem 2.3 *If the optimal solution set $\mathcal{P}^*$ is nonempty and the marginal cost function m is finite–valued and unimodal with parameter $0 \leq b < \infty$, then it follows that $m(T) \geq c(T)$ if and only if the intersection of $\mathcal{P}^*$ and $[b, T]$ is nonempty.*

Proof: If $m(T) \geq c(T)$ and m is unimodal with parameter $0 \leq b < \infty$ it is shown in the proof of Lemma 2.2 that $T \geq b$. Since m is increasing on (b, ∞) and $T \geq b$ we obtain for every $t \geq T$ that

$$\int_T^t (m(t) - m(y))(1 - F(y))dy + (m(t) - m(T)) \int_0^T (1 - F(y))dy \geq 0$$

By Theorem 2.2 it follows for every $t \geq T$ that the identity

$$\begin{aligned}(m(t) - c(t)) \int_0^t (1 - F(y))dy &= (m(T) - c(T)) \int_0^t (1 - F(y))dy \\ &+ \int_T^t (m(t) - m(y))(1 - F(y))dy + (m(t) - m(T)) \int_0^T (1 - F(y))dy\end{aligned}$$

holds, and so $m(t) \geq c(t)$ for every $t \geq T$ if $m(T) \geq c(T)$. This yields $(D^+c)(t) \geq 0$ for every $t \geq T$, and since c is a continuous function we obtain that c is increasing on (T, ∞) implying the desired result. To prove the converse implication we observe that a nonempty intersection of the sets $\mathcal{P}^*$ and $[b, T]$ implies the existence of an optimal solution T^* of (P) within the interval $[b, T]$. This yields $(D^+c)(T^*) \geq 0$ or equivalently $m(T^*) \geq c(T^*)$. Since m is increasing on (b, ∞) and $T^* \leq T$ this implies by the same arguments as in the first part that $m(T) \geq c(T)$ and this shows the reverse implication. □

By Theorem 2.3 it is clear that one has to compare the values $m(t)$ and $c(t)$ in order to determine which part of the positive axis will contain with certainty an optimal solution. Observe by Theorem 2.3 one has to assume that the marginal cost function is finite–valued and unimodal and so continuity of the function m is not required. This result also covers therefore unimodal finite–valued cost functions which are almost everywhere continuous. Under the above conditions it is also not difficult to verify that $m(T) = c(T)$ implies that T is an optimal policy, but the reverse implication might not hold due to the absense of continuity everywhere of the function m.

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