SENSITIVITY ANALYSIS IN (DEGENERATE) QUADRATIC PROGRAMMING

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Abstract

In this paper we deal with sensitivity analysis in convex quadratic programming, without making assumptions on nondegeneracy, strict convexity of the objective function, and the existence of a strictly complementary solution. We show that the optimal value as a function of a right–hand side element (or an element of the linear part of the objective) is piecewise quadratic, where the pieces can be characterized by maximal complementary solutions and tripartitions. Further, we investigate differentiability of this function. A new algorithm to compute the optimal value function is proposed. Finally, we discuss the advantages of this approach when applied to mean–variance portfolio models.

Key words: Quadratic Programming, Parametric Programming, Sensitivity Analysis, Maximal Complementary Solutions, Tripartitions, Mean–Variance Portfolio Theory.

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1 Introduction

In this paper we are concerned with parametric versions of the convex quadratic programming (CQP) problem, where elements in the right–hand side or linear term of the objective are varied. Markowitz [25, 26] was one of the first to consider this problem, being particularly interested in applications to mean–variance analysis of investment portfolios. Other early references include Houthakker [17] and Wolfe [37] who use a parametric algorithm to solve the quadratic programming problem (QP) itself (for a survey see Boot [3]). Eaves [8] also considers the parametric CQP problem. The basic result already obtained by Markowitz is that the efficient frontier (or optimal value function) is piecewise quadratic and can be obtained by computing successive corner portfolios, while in between these corner portfolios optimal solutions 'vary linearly'. The analyses in the cited references use some variant of the simplex method [5], where the corner portfolios correspond to a kind of basis change.

In the area of linear programming (LP) there is a lot of literature on the difficulties that may occur in parametric analysis, when the problem under consideration is degenerate. This involves that the optimal basis need not be unique, that multiple optimal solutions may exist, and that the notion of shadow price may not be well defined. We refer to Gal [11] for a literature survey and Rubin and Wagner [32] for an overview of practical implications. Recently, research on interior point methods (Gonzaga [14]) has triggered interest in investigating parametric analysis without using bases, instead making use of strictly complementary solutions (Adler and Monteiro [1] and Jansen et al. [20]). In LP such a solution was already shown to exist by Goldman and Tucker [13] in 1956.

In the literature often assumptions are made to prevent situations concerning degeneracy. For nonlinear programming the research of sensitivity and stability analysis may be divided into two fields: directional differentiability of the optimal value function (e.g. [19, 31]) and directional differentiability of the optimal solutions (e.g. [6, 21, 23, 24]). Lipschitz continuity plays an important role in such analyses. Standard assumptions include strict convexity [22, 33, 36], strict complementarity Fiacco and McCormick [10], Fiacco [9], and others [6, 22, 33, 36], second-order necessary or sufficient conditions for optimality [21, 30] and some regularity conditions [12, 19]. Dubeau and Gauvin [7] give some simple examples and counterexamples to illustrate the relative importance of some of these assumptions. Jittorntrum [21] showed that the strict complementarity condition is often too restrictive and does not result in any significant advantage, while dropping the assumption does not result in losses of sensitivity information, stability of the solution and computational efficiency (He states: 'strict complementarity should no longer be a standard requirement imposed on the problem').

In this paper we shall not make any assumption. The main tool we use is the existence of a *maximal complementary solution* in any CQP problem (Güler and Ye [16]). Such a solution uniquely defines the *tripartition* of the problem, which gives a partition of the variables in 'important' and 'not important' variables.

We give a new proof of the fact that the optimal value function as a function of a right–hand side element (or an element of the linear term in the objective function) is piecewise quadratic. We show that the pieces of this function correspond to intervals for the parameter on which the tripartition is constant, while it changes exactly in those points where either the first or the second order derivative does not exist (instead directional derivatives need to be considered). We show how these intervals and (directional) derivatives can be computed with auxiliary linear or quadratic problems. Using these, we propose an algorithm to compute the optimal value function.

To show important implications of our results we consider the same problem as Markowitz [26]: the mean–variance model for portfolio selection. We argue that our approach gives more practically relevant information than a simplex solution, and that some disadvantages of the model (Michaud [28]) can be alleviated.

This paper is built up as follows. In Section 2 we introduce our problem, maximal complementary solutions and the tripartition. In Section 3 we derive the main theoretical results, while Section 4 contains the application to mean–variance models.

2 Problem description

The general CQP problem is given by

$$(\mathrm{QP}) \quad \min_{x} \left\{ c^T x + \frac{1}{2} x^T Q x \ : \ Ax = b, \ x \geq 0 \right\},$$

where $c, x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, A an $m \times n$ matrix with full row rank and Q a symmetric positive semidefinite $n \times n$ matrix. The Wolfe–dual of (QP) is given by

$$(\mathrm{QD}) \quad \max_{u,y,s} \left\{ b^T y - \frac{1}{2} u^T Q u \ : \ A^T y + s - Qu = c, \ s \geq 0 \right\}.$$

We assume that both (QP) and (QD) have optimal solutions. It is well–known that if x is feasible for (QP) and (u, y, s) for (QD) then these solutions are optimal if and only if $Qx = Qu$ and $x^T s = 0$. This last is the so–called complementary slackness property. Furthermore, there exist optimal solutions for which $x = u$; we will only be interested in those and henceforth denote an optimal solution just by (x, y, s). We will later use the following well–known result.

Lemma 2.1 *Let (x^*, y^*, s^*) and $(\tilde{x}, \tilde{y}, \tilde{s})$ both be optimal solutions of (QP) and (QD). Then $Qx^* = Q\tilde{x}$, $c^T x^* = c^T \tilde{x}$ and $b^T y^* = b^T \tilde{y}$.*

Proof: Since (x^*, y^*, s^*) and $(\tilde{x}, \tilde{y}, \tilde{s})$ are both optimal, we conclude that

$$c^T x^* + \frac{1}{2} x^* Q x^* = b^T \tilde{y} - \frac{1}{2} \tilde{x}^T Q \tilde{x}.$$

Using that $A^T \tilde{y} + \tilde{s} - Q\tilde{x} = c$ and multiplying by $(x^*)^T$ it follows,

$$\frac{1}{2}(x^* Q x^* + \tilde{x} Q \tilde{x} - 2\tilde{x} Q x^*) = b^T \tilde{y} - c^T x^* - \tilde{x} Q x^* = 0.$$

Thus, $(x^* - \tilde{x})^T Q(x^* - \tilde{x}) = 0$, which implies $Q(x^* - \tilde{x}) = 0$, i.e. $Qx^* = Q\tilde{x}$, since Q is positive semidefinite. Using that $Qx^* = Q\tilde{x}$ we conclude that $c^T x^* = c^T \tilde{x}$ and $b^T y^* = b^T \tilde{y}$. $\square$

The set of optimal solutions of (QP) and (QD) can be characterized by *maximal complementary solutions* and the corresponding *tripartition* (Güler and Ye [16]). Let us define

$$\begin{aligned}
\mathcal{B} &= \{\, i \,:\, x_i > 0 \text{ for an optimal solution } x \text{ of (QP) } \},\\
\mathcal{N} &= \{\, i \,:\, s_i > 0 \text{ for an optimal solution } (x, y, s) \text{ of (QD) } \},\\
\mathcal{T} &= \{1, \ldots, n\} \setminus (\mathcal{B} \cup \mathcal{N}).
\end{aligned}$$

This tripartition of the index set is denoted by $\pi := (\mathcal{B}, \mathcal{N}, \mathcal{T})$. A maximal complementary solution (x, y, s) is a solution for which

$$x_i > 0 \Longleftrightarrow i \in \mathcal{B}, \quad s_i > 0 \Longleftrightarrow i \in \mathcal{N};$$

the existence of such a solution is a consequence of the convexity of the optimal sets of (QP) and (QD). Güler and Ye [16] showed that interior point methods (as Carpenter et al. [4] and Vanderbei [34]) generate a maximal complementary solution (in the limit). In LP the set $\mathcal{T}$ is empty, which leads to the existence of a strictly complementary solution. We define the *support* of a vector v as follows

$$\sigma(v) := \{\, i \,:\, v_i > 0 \,\}.$$

Hence, given a maximal complementary solution (x^*, y^*, s^*) we have $\mathcal{B} = \sigma(x^*)$ and $\mathcal{N} = \sigma(s^*)$.

In this paper we restrict to CQPs with perturbations in the right–hand side. Results for perturbation of the objective vector are easily obtained by dualizing the parametric problem, and are therefore omitted. We heavily use the tripartition of the problem, as induced by a maximal complementary solution. Boot [3] and Bank et al. [2] both considered $\mathcal{N}$ and $\mathcal{T}$ together, leading to less strong results and loss of information.

The parametric problem is given by

$$(\mathrm{QP}_\lambda) \quad \min_x \left\{ c^T x + \frac{1}{2} x^T Q x \;:\; Ax = b + \lambda \Delta b,\; x \geq 0 \right\},$$

where $\Delta b \in \mathbb{R}^m$ is a given vector of variation. For $\lambda \in \mathbb{R}$ we denote the tripartition by $\pi_\lambda = (\mathcal{B}_\lambda, \mathcal{N}_\lambda, \mathcal{T}_\lambda)$ and a maximal complementary solution by $(x^{(\lambda)}, y^{(\lambda)}, s^{(\lambda)})$. The dual is

$$(\mathrm{QD}_\lambda) \quad \max_{x,y,s} \left\{ (b + \lambda \Delta b)^T y - \frac{1}{2} x^T Q x \;:\; A^T y + s - Qx = c,\; s \geq 0 \right\}.$$

The optimal value of (QP_λ) and (QD_λ) is denoted by $\phi(\lambda)$, with the convention that

$$\begin{aligned}
\phi(\lambda) &= \infty \quad \text{if } (\mathrm{QP}_\lambda) \text{ is infeasible},\\
\phi(\lambda) &= -\infty \quad \text{if } (\mathrm{QP}_\lambda) \text{ is feasible and unbounded},
\end{aligned}$$

so $\phi : \mathbb{R} \to \mathbb{R} \cup \{-\infty, \infty\}$. Since the feasible set of (QD_λ) does not depend on λ, (QD_λ) will be either feasible or infeasible for all λ. We denote by Λ the set of values λ for which (QD_λ) has a bounded optimal value. Since we assumed that (QP) and (QD) have optimal solutions it follows that $\Lambda \neq \phi$. The first lemma gives some basic facts on $\phi(\lambda)$.

Lemma 2.2 *(i) The optimal value function $\phi(\lambda)$ is convex on Λ;*
(ii) Λ is a (possibly unbounded) closed interval.

Proof: (i) Let $\lambda_1, \lambda_2 \in \Lambda$ and $\alpha \in (0,1)$ be given and define $\lambda_\alpha := \alpha\lambda_1 + (1-\alpha)\lambda_2$. Then we have

$$\begin{aligned}
\phi(\lambda_\alpha) &= \alpha\left((b+\lambda_1\Delta b)^T y^{(\lambda_\alpha)} - \frac{1}{2}(x^{(\lambda_\alpha)})^T Q x^{(\lambda_\alpha)}\right) + \\
&\quad (1-\alpha)\left((b+\lambda_2\Delta b)^T y^{(\lambda_\alpha)} - \frac{1}{2}(x^{(\lambda_\alpha)})^T Q x^{(\lambda_\alpha)}\right) \\
&\leq \alpha\phi(\lambda_1) + (1-\alpha)\phi(\lambda_2),
\end{aligned}$$

where the inequality holds since the feasible set of (QD_λ) is independent of λ.
(ii) From the convexity of $\phi(\lambda)$ it follows that $\phi(\lambda)$ is continuous. Thus Λ is an interval. The closedness of this interval will be proved by showing that the complement of the domain of ϕ is an open subset of the real line. Let λ belong to the complement of the domain of ϕ. This means that (QD_λ) is unbounded. This is equivalent to the existence of a vector z, such that

$$A^T z \leq 0, \ \ (b + \lambda\Delta b)^T z > 0.$$

Fixing z and considering λ as a variable, the set of all λ satifying $(b+\lambda\Delta b)^T z > 0$ is an open interval. For all λ in this interval (QD_λ) is unbounded. Hence, the domain of ϕ is open. This concludes the proof. $\square$

3 Analysis

In this section we will show that the optimal value function $\phi(\lambda)$ is piecewise quadratic, where the 'pieces' correspond to intervals where the tripartition is constant. Moreover, we show how $\phi(\lambda)$ can be computed.

3.1 Basic properties

The first lemma shows that the tripartition is constant on certain intervals.

Lemma 3.1 *Let λ_0 and λ_1 be such that $\pi_{\lambda_0} = \pi_{\lambda_1} = \pi$. Then $\pi_\lambda = \pi$ for all $\lambda \in [\lambda_0, \lambda_1]$.*

Proof: Without loss of generality we assume that $\lambda_0 = 0$ and $\lambda_1 = 1$. Let $(x^{(0)}, y^{(0)}, s^{(0)})$ and $(x^{(1)}, y^{(1)}, s^{(1)})$ be maximal complementary solutions for the respective problems. We define for $\lambda \in (0,1)$

$$\begin{aligned}
x(\lambda) &:= (1-\lambda)x^{(0)} + \lambda x^{(1)}, \\
y(\lambda) &:= (1-\lambda)y^{(0)} + \lambda y^{(1)}, \qquad (1)\\
s(\lambda) &:= (1-\lambda)s^{(0)} + \lambda s^{(1)}.
\end{aligned}$$

It is easy to see that $Ax(\lambda) = b + \lambda \Delta b$ and $x(\lambda) \geq 0$. Also

$$A^T y(\lambda) + s(\lambda) - Qx(\lambda) = (1-\lambda)c + \lambda c = c,$$

and $s(\lambda) \geq 0$. So $(x(\lambda), y(\lambda), s(\lambda))$ is feasible for (QP_λ) and (QD_λ). Since $\pi_0 = \pi_1$, we have $x(\lambda)^T s(\lambda) = 0$, hence the proposed solution is optimal for (QP_λ) and (QD_λ). Using the support of $x(\lambda)$ and $s(\lambda)$, $\mathcal{B}$ and $\mathcal{N}$ respectively, and their complement $\mathcal{T}$, this implies

$$\mathcal{B} \subseteq \mathcal{B}_\lambda, \;\; \mathcal{N} \subseteq \mathcal{N}_\lambda \;\text{ and }\; \mathcal{T} \supseteq \mathcal{T}_\lambda. \tag{2}$$

We will show that equality holds. Assuming to the contrary that $\mathcal{T} \supset \mathcal{T}_\lambda$, there exists a maximal complementary solution $(x^{(\lambda)}, y^{(\lambda)}, s^{(\lambda)})$ of (QP_λ) and (QD_λ) such that

$$(x^{(\lambda)})_i + (s^{(\lambda)})_i > 0 \;\text{ for some }\; i \in \mathcal{T}, \; i \notin \mathcal{T}_\lambda. \tag{3}$$

Let us now define for $\epsilon > 0$

$$\begin{aligned} \bar{x}(\epsilon) &:= x^{(1)} + \epsilon(x^{(1)} - x^{(0)}), \\ \bar{y}(\epsilon) &:= y^{(1)} + \epsilon(y^{(1)} - y^{(0)}), \\ \bar{s}(\epsilon) &:= s^{(1)} + \epsilon(s^{(1)} - s^{(0)}). \end{aligned}$$

For some $\overline{\epsilon} > 0$ small enough it holds

$$\bar{x}(\overline{\epsilon})_{\mathcal{B}} > 0, \; \bar{x}(\overline{\epsilon})_{\mathcal{N} \cup \mathcal{T}} = 0, \qquad \bar{s}(\overline{\epsilon})_{\mathcal{N}} > 0, \; \bar{s}(\overline{\epsilon})_{\mathcal{B} \cup \mathcal{T}} = 0, \tag{4}$$

from which it follows that the proposed solutions are optimal for $(\mathrm{QP}_{1+\overline{\epsilon}})$ and $(\mathrm{QD}_{1+\overline{\epsilon}})$. Finally, we define

$$\begin{aligned} \tilde{x}^{(1)} &:= \tfrac{\overline{\epsilon}}{1-\lambda+\overline{\epsilon}} x^{(\lambda)} + \tfrac{1-\lambda}{1-\lambda+\overline{\epsilon}} \bar{x}(\overline{\epsilon}), \\ \tilde{y}^{(1)} &:= \tfrac{\overline{\epsilon}}{1-\lambda+\overline{\epsilon}} y^{(\lambda)} + \tfrac{1-\lambda}{1-\lambda+\overline{\epsilon}} \bar{y}(\overline{\epsilon}), \\ \tilde{s}^{(1)} &:= \tfrac{\overline{\epsilon}}{1-\lambda+\overline{\epsilon}} s^{(\lambda)} + \tfrac{1-\lambda}{1-\lambda+\overline{\epsilon}} \bar{s}(\overline{\epsilon}), \end{aligned}$$

which are feasible in (QP_1) and (QD_1). Also,

$$(\tilde{x}^{(1)})^T \tilde{s}^{(1)} = \frac{\overline{\epsilon}}{1-\lambda+\overline{\epsilon}} \frac{1-\lambda}{1-\lambda+\overline{\epsilon}} \left((x^{(\lambda)})^T \bar{s}(\overline{\epsilon}) + \bar{x}(\overline{\epsilon})^T s^{(\lambda)} \right).$$

Since (2) and (4) imply

$$(x^{(\lambda)})^T \bar{s}(\overline{\epsilon}) = (x^{(\lambda)})^T_{\mathcal{N}} \bar{s}(\overline{\epsilon})_{\mathcal{N}} = 0$$

and

$$(s^{(\lambda)})^T \bar{x}(\overline{\epsilon}) = (s^{(\lambda)})^T_{\mathcal{B}} \bar{x}(\overline{\epsilon})_{\mathcal{B}} = 0,$$

$(\tilde{x}^{(1)}, \tilde{y}^{(1)}, \tilde{s}^{(1)})$ is optimal for (QP_1) and (QD_1). However, if (3) would hold, we would have a solution of (QP_1) and (QD_1) with either $(\tilde{x}^{(1)})_i > 0$ or $(\tilde{s}^{(1)})_i > 0$ for $i \in \mathcal{T}$, contradicting the definition of $(\mathcal{B}, \mathcal{N}, \mathcal{T})$. Thus we conclude $\mathcal{T}_\lambda = \mathcal{T}$. Using (2) the lemma follows. □

Lemma 3.1 implies the following corollary on the behavior of the optimal value function on intervals.

Corollary 3.2 *If $\pi_{\lambda_0} = \pi_{\lambda_1} = \pi$ then*
(i) on $[\lambda_0, \lambda_1]$, $\phi(\lambda)$ is the quadratic function given by

$$\begin{aligned}\phi(\lambda) &= \phi(0) + \frac{\lambda - \lambda_0}{\lambda_1 - \lambda_0}\left((b + \lambda_0 \Delta b)^T (y^{(\lambda_1)} - y^{(\lambda_0)}) + c^T (x^{(\lambda_1)} - x^{(\lambda_0)})\right) + \\ & \frac{1}{2}\left(\frac{\lambda - \lambda_0}{\lambda_1 - \lambda_0}\right)^2 \Delta b^T (y^{(\lambda_1)} - y^{(\lambda_0)});\end{aligned}$$

(ii) $\phi(\lambda)$ is linear on $[\lambda_0, \lambda_1]$ if and only if $Qx^{(\lambda_0)} = Qx^{(\lambda_1)}$; in this case

$$\phi(\lambda) = \phi(0) + \frac{\lambda - \lambda_0}{\lambda_1 - \lambda_0}\, c^T (x^{(\lambda_1)} - x^{(\lambda_0)}).$$

Proof: (i) Assuming $\lambda_0 = 0$ and $\lambda_1 = 1$ it follows from the proof of Lemma 3.1 that $(x(\lambda), y(\lambda), s(\lambda))$ defined in (1) is optimal in (QD_λ) for $\lambda \in (0, 1)$. Hence

$$\begin{aligned}\phi(\lambda) &= (b + \lambda \Delta b)^T y(\lambda) - \frac{1}{2} x(\lambda)^T Q x(\lambda) \\ &= \phi(0) + \lambda \Delta b^T y^{(0)} + \lambda b^T (y^{(1)} - y^{(0)}) + \lambda^2 \Delta b^T (y^{(1)} - y^{(0)}) - \\ & \lambda (x^{(1)} - x^{(0)})^T Q x^{(0)} - \frac{1}{2}\lambda^2 (x^{(1)} - x^{(0)})^T Q (x^{(1)} - x^{(0)}).\end{aligned}$$

Note that

$$\begin{aligned}A(x^{(1)} - x^{(0)}) &= \Delta b \\ A^T (y^{(1)} - y^{(0)}) + s^{(1)} - s^{(0)} &= Q(x^{(1)} - x^{(0)}).\end{aligned}$$

Multiplying the second equation with $x^{(1)} - x^{(0)}$ respectively with $x^{(0)}$, and using the first, gives

$$\Delta b^T (y^{(1)} - y^{(0)}) = (x^{(1)} - x^{(0)})^T Q (x^{(1)} - x^{(0)}) \tag{5}$$
$$b^T (y^{(1)} - y^{(0)}) = (x^{(0)})^T Q (x^{(1)} - x^{(0)}). \tag{6}$$

So we may write $\phi(\lambda)$ as

$$\phi(\lambda) = \phi(0) + \lambda \Delta b^T y^{(0)} + \frac{1}{2}\lambda^2 \Delta b^T (y^{(1)} - y^{(0)}).$$

Furthermore,

$$\begin{aligned}\Delta b^T y^{(0)} &= (x^{(1)} - x^{(0)})^T A^T y^{(0)} = (x^{(1)} - x^{(0)})^T (c - s^{(0)} + Qx^{(0)}) \\ &= (x^{(1)} - x^{(0)})^T c + b^T (y^{(1)} - y^{(0)}).\end{aligned}$$

So, alternatively we have

$$\phi(\lambda) = \phi(0) + \lambda (b^T (y^{(1)} - y^{(0)}) + c^T (x^{(1)} - x^{(0)})) + \frac{1}{2}\lambda^2 \Delta b^T (y^{(1)} - y^{(0)}). \tag{7}$$

The required equality now easily follows.
(ii) Assuming again $\lambda_0 = 0$ and $\lambda_1 = 1$, we have from (i) that $\phi(\lambda)$ is linear on $[0, 1]$ if and only if $\Delta b^T (y^{(1)} - y^{(0)}) = 0$. Using (5) this is equivalent to

$$(x^{(1)} - x^{(0)})^T Q (x^{(1)} - x^{(0)}) = 0$$

which holds if and only if $Q(x^{(1)} - x^{(0)})$. From (6) and (7) it follows that

$$\phi(\lambda) = \phi(0) + \lambda\, c^T(x^{(1)} - x^{(0)}).$$

□

With the results obtained so far we can prove the next theorem, giving a characterization of the optimal value function in terms of tripartitions.

Theorem 3.3 *(i) The interval Λ can be partitioned in a finite set of subintervals such that the tripartition is constant on a subinterval.*
(ii) The optimal value function $\phi(\lambda)$ is continuous, convex and piecewise quadratic on Λ.

Proof: (i) Since the number of possible tripartitions is finite and the number of elements of Λ is infinite it follows from Lemma 3.1 that Λ can be partitioned into (open) subintervals on which the tripartition is constant, while it is different in the singletons in between the subintervals.
(ii) Corollary 3.2 implies that on each subinterval defined by a tripartition the function $\phi(\lambda)$ is quadratic. Since $\phi(\lambda)$ is convex (Lemma 2.2) it is continuous. □

To illustrate the results sofar we give a small example.

Example 3.4 Let the CQP be defined by

$$Q = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad A = (2\ \ 1), \quad b = 2.$$

The vector of variation is taken to be $\Delta b = 1$. The next table gives maximal complementary solutions and the corresponding tripartitions as a function of λ.

	x_1	x_2	s_1	s_2	$\mathcal{B}$	$\mathcal{N}$	$\mathcal{T}$
$\lambda = -2$	0	0	> 0	> 0	ϕ	$\{1, 2\}$	ϕ
$-2 < \lambda < -1$	$-\frac{1}{2}\lambda - 1$	0	0	$-\frac{1}{2}\lambda$	$\{1\}$	$\{2\}$	ϕ
$\lambda = -1$	1	0	0	0	$\{1\}$	ϕ	$\{2\}$
$\lambda > -1$	$1 + \lambda/3$	$\lambda)/3$	0	0	$\{1, 2\}$	ϕ	ϕ.

Hence, $\phi(\lambda)$ is given by

$$\begin{aligned} -2 \le \lambda \le -1 \qquad & \phi(\lambda) = \frac{1}{4}\lambda^2 + \lambda + 1, \\ \lambda \ge -1 \qquad & \phi(\lambda) = \frac{1}{6}\lambda^2 + \lambda + 1. \end{aligned}$$

Note that the problem is infeasible for $\lambda < -2$. It is easy to check that $\phi(\lambda)$ is differentiable at $\lambda = 0$, although the tripartition changes in this point. ■

3.2 Derivatives and intervals

Let us call the *transition-points* of $\phi(\lambda)$ those points λ where the tripartition changes; in between the transition-points are the *curvilinearity intervals* or *pieces*. We will show that the transition-points occur exactly where the *first* or *second order derivative* of the optimal value function does not exist. Moreover, we prove that when taking maximal complementary solutions for two parameter values in a curvilinearity interval, a convex combination of these solutions is a maximal complementary solution for the corresponding parameter value (cf. Lemma 3.1). We first deal with the differentiability of $\phi(\lambda)$.

Lemma 3.5 *If $\phi(\cdot)$ is differentiable in λ then the derivative $\phi'(\lambda)$ is given by $\Delta b^T y$, where y is any optimal solution of (QD$_\lambda$). Otherwise, the left and right first order derivatives $\phi'_-(\lambda)$ and $\phi'_+(\lambda)$ are given by*

$$\phi'_-(\lambda) = \min_{x,y,s} \Big\{ \Delta b^T y \;\; : \;\; Ax = b + \lambda \Delta b,\ x_{\mathcal{B}_\lambda} \geq 0,\ x_{\mathcal{N}_\lambda \cup \mathcal{T}_\lambda} = 0,$$
$$A^T y + s - Qx = c,\ s_{\mathcal{N}_\lambda} \geq 0,\ s_{\mathcal{B}_\lambda \cup \mathcal{T}_\lambda} = 0 \Big\}, \qquad (8)$$

$$\phi'_+(\lambda) = \max_{x,y,s} \Big\{ \Delta b^T y \;\; : \;\; Ax = b + \lambda \Delta b,\ x_{\mathcal{B}_\lambda} \geq 0,\ x_{\mathcal{N}_\lambda \cup \mathcal{T}_\lambda} = 0,$$
$$A^T y + s - Qx = c,\ s_{\mathcal{N}_\lambda} \geq 0,\ s_{\mathcal{B}_\lambda \cup \mathcal{T}_\lambda} = 0 \Big\}. \qquad (9)$$

Proof: Let $\lambda \in \operatorname{int}(\Lambda)$ and $\epsilon \in \mathbb{R}$ sufficiently small. Then it holds for *any* optimal solution $(x(\lambda)$,$y(\lambda)$,$s(\lambda))$ of (QD$_\lambda$) that

$$\begin{aligned} \phi(\lambda+\varepsilon) &= (b+(\lambda+\varepsilon)\Delta b)^T y^{(\lambda+\varepsilon)} - \frac{1}{2}(x^{(\lambda+\varepsilon)})^T Q x^{(\lambda+\varepsilon)} \\ &\geq (b+\lambda\Delta b)^T y(\lambda) - \frac{1}{2}x(\lambda)^T Q x(\lambda) + \varepsilon \Delta b^T y(\lambda) \\ &= \phi(\lambda) + \varepsilon \Delta b^T y(\lambda), \end{aligned}$$

so the right and left derivatives at λ satisfy

$$\begin{aligned} \phi'_+(\lambda) &= \lim_{\varepsilon \downarrow 0} \frac{\phi(\lambda+\varepsilon)-\phi(\lambda)}{\varepsilon} \geq \Delta b^T y(\lambda), \\ \phi'_-(\lambda) &= \lim_{\varepsilon \uparrow 0} \frac{\phi(\lambda+\varepsilon)-\phi(\lambda)}{\varepsilon} \leq \Delta b^T y(\lambda). \end{aligned}$$

If $\phi(\lambda)$ is differentiable at λ then we necessarily have $\phi'(\lambda) = \Delta b^T y(\lambda)$ for any optimal solution $y(\lambda)$ of (QD$_\lambda$). Otherwise, note that the objective value of each y feasible in (QD$_\lambda$) as a function of λ is linear with slope $\Delta b^T y$. Since $\phi(\cdot)$ is assumed to be not differentiable in λ there must be different optimal solutions with different slope. We will restrict to proving (8), the proof of (9) is analogous. Let us consider a maximal complementary solution $(x^*(\epsilon), y^*(\epsilon), s^*(\epsilon))$ of (QD$_{\lambda-\epsilon}$). Let $\epsilon > 0$ be chosen to assure that $\phi(\lambda)$ is differentiable in $\lambda - \epsilon$. Furthermore, let us denote $x(\lambda)$, $y(\lambda)$ and $s(\lambda)$, the optimal solution used in the upper part of the proof, by x, y and s. It easily follows that $\phi'(\lambda-\epsilon) = \Delta b^T y^*(\epsilon)$. Since $\phi'(\cdot)$ is increasing ($\phi(\cdot)$ is convex) we conclude that

$$\phi'(\lambda-\epsilon) = \Delta b^T y^*(\epsilon) \leq \phi'_-(\lambda) \leq \Delta b^T y.$$

Taking the limit of $\epsilon \uparrow 0$, it holds

$$\Delta b^T y^* \leq \phi'_-(\lambda) \leq \Delta b^T y, \tag{10}$$

where y^* denotes the limit of $y^*(\epsilon)$ when $\epsilon \uparrow 0$. Since, $\phi(\lambda)$ is continuous, we conlude that y^* is optimal in (QD_λ). Thus, it is feasible in (8). From (10) we conclude that y^* should be optimal in (8), but y was already an optimal solution of (8). Thus, $y^* = y$ which completes the proof. For (9) an analogous argument holds. $\square$

The next result shows a property of strictly complementary solutions of (8) and (9), that will be needed later on.

Lemma 3.6 *Assume that $\lambda = 0$ is a transition-point of the optimal value function with tripartition $\pi_0 = (\mathcal{B}_0, \mathcal{N}_0, \mathcal{T}_0)$; further assume that the (open) curvilinearity interval to the right of zero contains $\lambda = 1$ with tripartition $\pi_1 = (\mathcal{B}_1, \mathcal{N}_1, \mathcal{T}_1)$. Let (x^*, y^*, s^*) be a strictly complementary solution of (9) with $\lambda = 0$. Then, $\sigma(x^*) \subseteq \mathcal{B}_1$ and $\sigma(s^*) \subseteq \mathcal{N}_1$.*

Proof: On $(0, 1]$ the tripartition is constant, hence convex combinations of maximal complementary solutions for two values in this interval are optimal in between these two values (cf. (1)). Taking limit to zero implies the existence of $(\underline{x}, \underline{y}, \underline{s})$, optimal in (QP_0) and (QD_0), with $\sigma(\underline{x}) \subseteq \mathcal{B}_1$, $\sigma(\underline{s}) \subseteq \mathcal{N}_1$ and $\Delta b^T \underline{y} = \phi'_+(0) = \Delta b^T y^*$. Since (x^*, y^*, s^*) and $(\underline{x}, \underline{y}, \underline{s})$ are both optimal in (QP_0) and (QD_0), Lemma 2.1 implies

$$(x^*)^T s^{(1)} = (x^*)^T (c + Qx^{(1)} - A^T y^{(1)}) = c^T \underline{x} + \underline{x}^T Q x^{(1)} - \underline{x}^T A^T y^{(1)} = \underline{x}^T s^{(1)} = 0,$$

and

$$\begin{aligned} (x^{(1)})^T s^* &= (x^{(1)})^T (c + Qx^* - A^T y^*) = (x^{(1)})^T (c + Q\underline{x}) - (b + \Delta b)^T y^* \\ &= (x^{(1)})^T (c + Q\underline{x}) - (b + \Delta b)^T \underline{y} = (x^{(1)})^T (c + Q\underline{x} - A^T \underline{y}) = (x^{(1)})^T \underline{s} = 0. \end{aligned}$$

Consider now for $\epsilon \in (0, 1)$

$$x(\varepsilon) = (1 - \varepsilon) x^* + \varepsilon\, x^{(1)}, \quad y(\varepsilon) = (1 - \varepsilon) y^* + \varepsilon\, y^{(1)}, \quad s(\varepsilon) = (1 - \varepsilon) s^* + \varepsilon\, s^{(1)},$$

then $x(\varepsilon)$ and $s(\varepsilon)$ are feasible and complementary in $(\mathrm{QP}_\varepsilon)$ and $(\mathrm{QD}_\varepsilon)$, hence optimal. So it holds $\sigma(x^*) \subseteq \mathcal{B}_1$ and $\sigma(s^*) \subseteq \mathcal{N}_1$. $\square$

Obviously, a similar result can be obtained for the interval to the left of a transition-point, using the solution of (8).

Theorem 3.7 *The transition-points of the optimal value function occur exactly where its first or second order derivative does not exist.*

Proof: Suppose to the contrary that the optimal value function is quadratic on say $[-1,1]$, while the tripartition is π_0 for $\lambda = 0$, π_{-1} on $[-1,0)$ and π_1 on $(0,1]$. Since the optimal value function is differentiable in $\lambda = 0$, it follows from the proof of Lemma 3.6 that $x(\varepsilon) := (1-\varepsilon)x^{(-1)} + \varepsilon x^{(0)}$ is optimal in $(\mathrm{QP}_{\varepsilon-1})$ for $\varepsilon \in (0,1)$, while $x(\varepsilon) := (1-\varepsilon)x^{(0)} + \varepsilon x^{(1)}$ is optimal in $(\mathrm{QP}_{\varepsilon})$ for $\varepsilon \in (0,1)$. Since the derivative of the optimal value function is linear on $[-1,1]$ it follows that $(x^{(-1)}+x^{(1)})/2$ is optimal in (QP_0). This implies $\mathcal{B}_1 \cup \mathcal{B}_{-1} \subseteq \mathcal{B}_0$. Similarly one shows $\mathcal{N}_1 \cup \mathcal{N}_{-1} \subseteq \mathcal{N}_0$. Combining this result with Lemma 3.6 it follows $\mathcal{B}_1 = \mathcal{B}_{-1} = \mathcal{B}_0$ and $\mathcal{N}_1 = \mathcal{N}_{-1} = \mathcal{N}_0$ contradicting the assumption. $\square$

The next theorem states that on a curvilinearity interval, a convex combination of two maximal complementary solutions is a maximal complementary solution on this interval again.

Theorem 3.8 *Let* $(x^{(0)}, y^{(0)}, s^{(0)})$ *and* $(x^{(1)}, y^{(1)}, s^{(1)})$ *denote some maximal complementary solutions of* (QP_0), (QD_0) *and* (QP_1),(QD_1) *respectively. For* $\lambda \in [0,1]$

$$\begin{aligned} x(\lambda) &:= (1-\lambda)x^{(0)} + \lambda x^{(1)} \\ y(\lambda) &:= (1-\lambda)y^{(0)} + \lambda y^{(1)} \\ s(\lambda) &:= (1-\lambda)s^{(0)} + \lambda s^{(1)} \end{aligned}$$

are maximal complementary solutions of (QP_λ) *and* (QD_λ) *if and only if* $\pi(\lambda)$ *is independent of* λ, *for all* λ *in the closed interval* $[0,1]$.

Proof: $[\Rightarrow]$: Let $0 < \lambda_1, \lambda_2 < 1$ be arbitrary. Consider the optimal solutions $(x(\lambda_1), y(\lambda_1), s(\lambda_1))$ and $(x(\lambda_2), y(\lambda_2), s(\lambda_2))$. We want to show that $\pi(\lambda_1) = \pi(\lambda_2)$. To this end, suppose $x_i(\lambda_1) = 0$, this means that $(x^{(1)})_i = (x^{(0)})_i = 0$. So $x_i(\lambda_2) = 0$. Thus $\mathcal{N}_{\lambda_1} \cup \mathcal{T}_{\lambda_1} = \mathcal{N}_{\lambda_2} \cup \mathcal{T}_{\lambda_2}$, which indicates that $\mathcal{B}_{\lambda_1} = \mathcal{B}_{\lambda_2}$. Now suppose that $s_k(\lambda_1) = 0$, which means that $(s^{(0)})_k = (s^{(1)})_k = 0$. Thus, $s_k(\lambda_2) = 0$. So, $\mathcal{B}_{\lambda_1} \cup \mathcal{T}_{\lambda_1} = \mathcal{B}_{\lambda_2} \cup \mathcal{T}_{\lambda_2}$. Hence, $\mathcal{N}_{\lambda_1} = \mathcal{N}_{\lambda_2}$. Combining these results, we must also have $\mathcal{T}_{\lambda_1} = \mathcal{T}_{\lambda_2}$, which completes the proof.
$[\Leftarrow]$: See the proof of Lemma 3.1. $\square$

To obtain the tripartition in a neighboring curvilinearity interval we have to solve an auxiliary QP problem. This is contrary to the LP case, where (see Adler and Monteiro [1], Jansen et al. [20]) the (tri)partition follows as a byproduct of computing the left and right derivatives (8) and (9). The next lemma shows how we can obtain the tripartition in the neighboring curvilinearity interval.

Lemma 3.9 *Consider the situation as in Lemma 3.6. Let* (x^*, s^*) *belong to a strictly complementary solution of (9) for* $\lambda = 0$, *and define* $\mathcal{B} := \sigma(x^*)$, $\mathcal{N} := \sigma(s^*)$, $\mathcal{T} := \{1, \ldots, n\} \setminus (\mathcal{B} \cup \mathcal{N})$. *Consider*

$$\min_{\xi,\rho,\eta} \left\{ -\Delta b^T \eta + \xi^T Q \xi \;:\; A\xi = \Delta b,\; A^T \eta + \rho - Q\xi = 0,\; \xi_{\mathcal{N}} = 0,\; \rho_{\mathcal{B}} = 0, \xi_{\mathcal{T}} \geq 0, \rho_{\mathcal{T}} \geq 0 \right\}, \tag{11}$$

and let (ξ^*, ρ^*, η^*) belong to a maximal complementary solution of (11). Then $\mathcal{B}_1 = \mathcal{B} \cup \{i : \xi_i^* > 0\}$, $\mathcal{N}_1 = \mathcal{N} \cup \{i : \rho_i^* > 0\}$, and $\mathcal{T}_1 = \{1, \ldots, n\} \setminus (\mathcal{B}_1 \cup \mathcal{N}_1)$.

Proof: It is easy to check that for a feasible solution of (11)

$$-\Delta b^T \eta + \xi^T Q \xi = \xi^T (Q\xi - A^T \eta) = \xi^T \rho = \xi_{\mathcal{T}}^T \rho_{\mathcal{T}} \geq 0.$$

The dual of (11) is given by

$$\max_{\delta,\xi,\gamma,\zeta} \Big\{ \Delta b^T \delta - \xi^T Q \xi \; : \; A\zeta = \Delta b,\ A^T \delta + \gamma + Q\zeta - 2Q\xi = 0,\ \gamma_{\mathcal{B}} = 0,\ \zeta_{\mathcal{N}} = 0,\ \gamma_{\mathcal{T}} \geq 0,\ \zeta_{\mathcal{T}} \geq 0 \Big\},$$

and for a feasible solution it holds

$$\Delta b^T \delta - \xi^T Q \xi = \delta^T A \zeta - \xi^T Q \xi = -\zeta^T \gamma - (\zeta - \xi)^T Q (\zeta - \xi) \leq 0.$$

Consequently, the optimal value of (11), if it exists, is zero. Consider the assignment

$$\xi = \zeta = x^{(1)} - x^*,\ \rho = \gamma = s^{(1)} - s^*,\ \eta = \delta = y^{(1)} - y^*,$$

which satisfies the first two linear equalities in (11). Using the fact that $\mathcal{B} \subseteq \mathcal{B}_1$ and $\mathcal{N} \subseteq \mathcal{N}_1$ (Lemma 3.6) it follows

$$\xi_{\mathcal{N}} = x_{\mathcal{N}}^{(1)} - x_{\mathcal{N}}^* = 0, \quad \xi_{\mathcal{T}} = x_{\mathcal{T}}^{(1)} - x_{\mathcal{T}}^* = x_{\mathcal{T}}^{(1)} \geq 0,$$

and

$$\rho_{\mathcal{B}} = s_{\mathcal{B}}^{(1)} - s_{\mathcal{B}}^* = 0, \quad \rho_{\mathcal{T}} = s_{\mathcal{T}}^{(1)} - s_{\mathcal{T}}^* = s_{\mathcal{T}}^{(1)} \geq 0,$$

and so $\xi_{\mathcal{T}}^T \rho_{\mathcal{T}} = 0$, implying that the assignment is an optimal solution. The fact that $(x^{(1)}, s^{(1)})$ is maximal complementary implies that the assignment must be maximal complementary in (11) as well, implying the result. □

In case a parameter value is not a transition-point of the optimal value function the curvilinearity interval must be computed. This can be done by solving appropriate LP problems.

Lemma 3.10 *Let $(\underline{\lambda}, \overline{\lambda})$ be an interval on which the tripartition is constant and given by $\pi = (\mathcal{B}, \mathcal{N}, \mathcal{T})$, and such that $\pi_{\underline{\lambda}} \neq \pi$ and $\pi_{\overline{\lambda}} \neq \pi$. Then*

$$\underline{\lambda} = \min_{\lambda,x,y,s} \Big\{ \lambda \; : \; Ax - \lambda \Delta b = b,\ x_{\mathcal{B}} \geq 0,\ x_{\mathcal{N} \cup \mathcal{T}} = 0,\ A^T y + s - Qx = c,\ s_{\mathcal{N}} \geq 0,\ s_{\mathcal{B} \cup \mathcal{T}} = 0 \Big\}, \tag{12}$$

$$\overline{\lambda} = \max_{\lambda,x,y,s} \Big\{ \lambda \; : \; Ax - \lambda \Delta b = b,\ x_{\mathcal{B}} \geq 0,\ x_{\mathcal{N} \cup \mathcal{T}} = 0,\ A^T y + s - Qx = c,\ s_{\mathcal{N}} \geq 0,\ s_{\mathcal{B} \cup \mathcal{T}} = 0 \Big\}. \tag{13}$$

Proof: We prove (12). For any $\lambda \in (\underline{\lambda}, \overline{\lambda})$ the pair $(\lambda, x^{(\lambda)}, y^{(\lambda)}, s^{(\lambda)})$ is feasible in (12). If the minimum value of the LP problem in (12) would be smaller than $\underline{\lambda}$ then a linear combination of the corresponding solution and a maximal complementary solution at $\lambda \in (\underline{\lambda}, \overline{\lambda})$ would give a contradiction with the definition of $(\mathcal{B}, \mathcal{N}, \mathcal{T})$ at $\underline{\lambda}$. The proof of (13) is similar. □

It need not be the case that a strictly complementary solution of (12) or (13) gives the tripartition in the corresponding transition-point, as is the case in LP [1, 20]; only *some* of the primal and dual variables can be shown to be either in its $\mathcal{B}$ or $\mathcal{N}$ part. Instead, to compute the tripartition in the transition-point the CQP problem itself needs to be solved here, while using the information on some of the variables.

Example 3.11 Consider the following MV-model,

$$\min_x \left\{ \frac{1}{2} x^T Q x \ : \ Ax = b + \lambda \Delta b, \ x \geq 0 \right\}$$

where

$$Q = \begin{bmatrix} 3 & 3 & -1 \\ 3 & 11 & 23 \\ -1 & 23 & 75 \end{bmatrix}, \ A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \end{bmatrix}, \ b = \begin{pmatrix} 1 & 0 \end{pmatrix}, \ \Delta b = \begin{pmatrix} 0 & 1 \end{pmatrix},$$

In $\lambda = 3$ the primal optimal solution is unique, whereas the dual optimal solution is nonunique. The next table shows the maximal complementary solution and its tripartition in $\lambda = 3$ and the two basic solutions with their corresponding supports.

	x_1	x_2	x_3	s_1	s_2	s_3	y_1	y_2	$\mathcal{B}$	$\mathcal{N}$	$\mathcal{T}$
basic solution	0	1	0	0	0	4	-1	4	{2}	{3}	{1}
max. compl. sol.	0	1	0	2	0	2	-4	5	{2}	{1,3}	$\emptyset$
basic solution	0	1	0	4	0	0	-7	6	{2}	{1}	{3}

The reader can verify that the optimal value function is not continuously differentiable in $\lambda = 3$. ■

In the next section we will outline an algorithm for computing the optimal value function. We will also consider two special cases LP, and nondegenerate QP.

3.3 Computing the optimal value function

All the ingredients for the computation of the optimal value function have been obtained at this stage. The algorithm is outlined as follows (note: by 'solve' we mean 'compute a maximal complementary solution of')

1. Solve (QP$_\lambda$) for some initial λ_0;
2. Solve (13) to obtain $\overline{\lambda}$, a transition-point;
3. Solve (QP$_{\overline{\lambda}}$) to compute the tripartition in $\overline{\lambda}$;
4. Solve (9) to obtain the slope and a support;
5. Solve (11) to obtain the tripartition of the next interval and proceed with Step 2.

Evidently, if the problem in Step 2 or Step 4 is unbounded or infeasible the algorithm stops. The part of the optimal value function to the left of λ_0 can be computed analogously.

With respect to the complexity of the algorithm we mention that each subproblem can be solved in polynomial time with an interior point algorithm. However, even in LP there can be exponentially many transition-points (Murty [29]).

Let us consider two special cases of the algorithm.

Linear Programming

Since LP is a special case of QP, the algorithm is also applicable to LP. Putting $Q := 0$, the relation between our results and that of Jansen et al. [20] is not directly clear. We will show here how the LP problem of Jansen et al. [20] for finding the partition on a linearity interval can be obtained from the algorithm.

Let $\lambda = 0$ denote a transition-point and let its corresponding partition be denoted by $(\mathcal{B}_0, \mathcal{N}_0)$. To obtain the partition on the linearity interval to the right of $\lambda = 0$, we first need to solve the following LP problem.

$$\begin{aligned} \max_{x,y,s} \{\ \Delta b^T y \quad : \quad & Ax = b + \lambda \Delta b,\ x_{\mathcal{B}_0} \geq 0,\ x_{\mathcal{N}_0} = 0, \\ & A^T y + s = c,\ s_{\mathcal{N}_0} \geq 0,\ s_{\mathcal{B}_0} = 0\ \} \end{aligned} \tag{14}$$

Let (x^*, s^*) belong to a strictly complementary solution of (14). Define $\mathcal{B} := \sigma(x^*)$, $\mathcal{N} := \sigma(s^*)$ and $\mathcal{T} := \{1, \ldots, n\} \backslash (\mathcal{B} \cup \mathcal{N})$. The second problem we need to solve, to obtain the partition on the linearity interval to the right of $\lambda = 0$, is

$$\begin{aligned} \max_{x,y,s} \{\ \Delta b^T y \quad : \quad & Ax = \Delta b,\ x_{\mathcal{T}} \geq 0,\ x_{\mathcal{N}} = 0, \\ & A^T y + s = 0,\ s_{\mathcal{T}} \geq 0,\ \sigma_{\mathcal{B}} = 0\ \}, \end{aligned} \tag{15}$$

Note that the set $\mathcal{T}$ is not empty in general since $\mathcal{B}$ and $\mathcal{N}$ are defined as the *support* of x^* and s^* respectively. Solving (14) is independent of x. Therefore, we can delete the equalities and inequality in x. We obtain

$$\max_y \{\ \Delta b^T y \quad : \quad A^T y + s = c,\ s_{\mathcal{N}_0} \geq 0,\ s_{\mathcal{B}_0} = 0\ \} \tag{16}$$

This LP problem is exactly the primal problem of Jansen et al. [20]. The optimal value of (15) is $\Delta b^T y = c^T x = 0$ Therefore, we can as well minimize $c^T x$ for this problem. The optimal value is independent of y, thus we obtain

$$\min_x \{ \, c^T x \quad : \quad Ax = \Delta b, \; x_{\mathcal{N}_0} \geq 0 \, \}, \tag{17}$$

where we have used that the set $\mathcal{T}$ contains parts of $\mathcal{B}_0$ and $\mathcal{N}_0$.

It is not difficult to see that (16) and (17) are dual problems. The LP problems (16) and (17) are precisely those in the analysis of Jansen et al. [20]. In the algorithm for parametric LP we only need to solve one LP and obtain a strict complementary solution of this LP. This yields the optimal partition. A similar result holds for finding the partition in a breakpoint.

The resulting algorithm for LP consists of solving one LP problem for each partition. It will be clear that the amount of work to obtain the partitions is considerably less than in the case of CQP. ■

Nondegenerate Parametric CQP Problems

For nondegenerate CQP problems (we only need nondegeneracy in the transition-points) the algorithm also simplifies. In the case of nondegeneracy, the optimal value function is continuously differentiable (this follows from Lemma 3.5). Thus, solving (8) and (9) is not necessary, since the solutions in the transition-points are unique. Moreover, when we solve (12) and (13), we obtain the optimal tripartition in a transition-point, due to the uniqueness of the solutions. The amount of work decreases significantly for nondegenerate problems. In the case of nondegeneracy we only need to solve an LP to obtain the tripartition in a transition-point and a QP to obtain the tripartition on a curvilinearity interval. ■

If we compare the algorithm for LP and the algorithm for nondegenerate CQP problems to the general algorithm we see that in the algorithm for LP and the algorithm for nondegenerate CQP problems only two problems have to be solved per 'iteration', while in the algorithm for degenerate CQP we need four problems per 'iteration'[1]. This has to do with the 'difficulty' of a degenerate transition-point and the set $\mathcal{T}$. To deal with this transition-point in the case of degeneracy and without strict complementarity (set $\mathcal{T}$ is not empty), we need to solve three problems, instead of only one.

4 Application to mean–variance models

An important application of the parametric CQP problem is the computation of an efficient frontier for mean–variance models, introduced by Markowitz [26]. Given assets with expected return r_i and covariances v_{ij}, the problem is to find portfolios of the assets that have minimal variance given a level of total return, and maximal return given a level of total variance.

[1] By iteration we mean finding a transition-point and a (curvi)linearity interval.

Mathematically, if x_i is the proportion invested in asset i then the basic mean–variance problem is

$$\min_x \left\{ \frac{1}{2} x^T V x \ : \ e^T x = 1, \ r^T x = \lambda, \ Dx = d, \ x \geq 0 \right\},$$

where e is an all–one vector, and $Dx = d$ may represent additional constraints on the portfolios to be chosen (for instance those related to volatility of the portfolio). This problem can be viewed as a (right–hand side) parametric CQP problem, with parameter λ representing the total return of investment. The efficient frontier is then just the optimal value function[2].

Since the publication of Markowitz's monograph [26] in 1959, people have been interested in an analytic derivation of the efficient frontier. Markowitz considered the case of nondegeneracy and developed an algorithm to compute the efficient frontier using a simplex–type critical line algorithm. Merton [27] considered the problem under the assumption of positive definiteness of the covariance matrix V and under allowing short sales. Mathematically allowing short sales is equivalent to letting x be a free variable. In this case the efficient frontier is differentiable and quadratic. It can easily be shown that when x is free, the constrained CQP problem can be regarded as an unconstrained CQP. Merton's approach is used in many standard textbooks (e.g. [18, 33]). Szegő [33] and Vörös [35] examined the mean–variance model, where short sales are not allowed. They concluded that the efficient frontier is convex, and continuously differentiable. Later Vörös [36] showed, with a counterexample, that the efficient frontier is not always continuously differentiable, despite the conclusions in [33, 35]. Moreover, he showed that it is continuously differentiable when V is diagonal. Independently, Kriens and Van Lieshout [22] also considered the efficient frontier in the nondegenerate case (V is positive definite) and presented an analytic derivation. Until now, the degenerate case was not explicitly considered and dealt with. Existing methods are not able to handle degenaracy correctly.

The theoretical results in this paper can immediately be applied to this problem. This has some useful practical consequences. First, the presented algorithm can be applied to mean–variance models, even in the case of degeneracy, to derive the efficient frontier. Second, the efficient frontier is piecewise quadratic; the quadratic parts correspond to intervals where the set of assets that can occur in *some* efficient portfolio is the same, while there exist linearly varying 'maximal complementary portfolios' over such an interval. Maybe even more importantly, the tripartition shows which of the assets will never appear in an efficient portfolio given a specified return. Hence, it gives an unambiguous separation in important assets and not important assets. This accounts for more stability in our approach, as opposed to using bases. For LP Greenberg [15] has given some other practical applications where the (tri)partition is absolutely necessary.

Since we use optimal sets instead of just one optimal solution, we might choose among these solutions one that fits best a subsidiary goal, thereby alleviating a drawback of this model given by Michaud [28]. An advantage of using certain interior point methods for solving a mean–variance model, is that during the iterations feasible solutions can be obtained (so having the given level of return) with decreasing variance. Hence, the optimal (efficient) portfolio is obtained as the limit of a sequence of near–efficient portfolios, which can also be useful in practice if robust solutions are needed.

We conclude this paper with an example of a mean–variance model and its efficient frontier. It

[2] Note that in most textbooks the level of return is on the vertical axis instead of on the horizontal axis.

is taken from Vörös [36], who claims it to be a real–world example.

Example 4.1 Consider the following MV-model,

$$\min_x \left\{ x^T V x \ : \ e^T x = 1, \ r^T x = \lambda, \ x \geq 0 \right\}$$

where

$$V = \begin{bmatrix} 0.82 & -0.23 & 0.155 & -0.013 & -0.314 \\ -0.23 & 0.484 & 0.346 & 0.197 & 0.592 \\ 0.155 & 0.346 & 0.298 & 0.143 & 0.419 \\ -0.013 & 0.197 & 0.143 & 0.172 & 0.362 \\ -0.314 & 0.592 & 0.419 & 0.362 & 0.916 \end{bmatrix}$$

$$r = \begin{pmatrix} 1.78 & 0.37 & 0.237 & 0.315 & 0.49 \end{pmatrix}^T .$$

The next table shows the tripartitions, curvilinearity intervals and transition-points (rounded off to three digits).

	B	N	T
$\lambda = 0.237$	{3}	{1,2,4,5}	$\emptyset$
$0.237 < \lambda < 0.505$	{1,3,4}	{2,5}	$\emptyset$
$\lambda = 0.505$	{1,3,4}	{5}	{2}
$0.505 < \lambda < 0.520$	{1,2,3,4}	{5}	$\emptyset$
$\lambda = 0.520$	{1,2,4}	{5}	{3}
$0.520 < \lambda < 1.065$	{1,2,4}	{3,5}	$\emptyset$
$\lambda = 1.065$	{1,2,4}	{3}	{5}
$1.065 < \lambda < 1.145$	{1,2,4,5}	{3}	$\emptyset$
$\lambda = 1.145$	{1,2,5}	{3}	{4}
$1.145 < \lambda < 1.392$	{1,2,5}	{3,4}	$\emptyset$
$\lambda = 1.392$	{1,5}	{3,4}	{2}
$1.392 < \lambda < 1.780$	{1,5}	{2,3,4}	$\emptyset$
$\lambda = 1.780$	{1}	{2,3,4,5}	$\emptyset$

In this example the efficient frontier is continuously differentiable. Furthermore, only 13 of the 243 possible tripartitions occur. When $\lambda > 1.780$ or $\lambda < 0.237$, $(\mathrm{QP})_\lambda$ is infeasible, while $(\mathrm{QD})_\lambda$ is unbounded. ■

Although in the example the set $\mathcal{T}$ is empty on intervals, we remark that this is not true in general. Furthermore, the optimal value function need not be differentiable, as follows immediately from the LP case.

5 Concluding remarks

We have given a characterization of the optimal value function in degenerate quadratic programming using maximal complementary solutions and the induced tripartitions. A subject of further research is to incorporate this approach in an efficient (interior point) implementation, and to test its practical value. Our preliminary computations on QP models with up to 200 variables and constraints have shown this to be viable, where we used an implementation of a predictor-corrector primal-dual path-following interior point method in MATLAB 4.2, MATHWORKS, INC. to solve the linear and quadratic problems involved. Furthermore, a more close comparison of the procedure presented here with existing methods for mean–variance models is subject of future research. Finally, it is worthwile investigating whether ideas can be generalized to linear complementarity problems with sufficient matrices, quadratically constrained quadratic programming, convex programming and semi–definite programming.

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