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ANALYTIC CENTRAL PATH, SENSITIVITY ANALYSIS AND PARAMETRIC LINEAR PROGRAMMING

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ABSTRACT

In this paper we consider properties of the central path and the analytic center of the optimal face in the context of parametric linear programming. We first show that if the right-hand side vector of a standard linear program is perturbed, then the analytic center of the optimal face is one-side differentiable with respect to the perturbation parameter. In that case we also show that the whole analytic central path shifts in a uniform fashion. When the objective vector is perturbed, we show that the last part of the analytic central path is tangent to a central path defined on the optimal face of the original problem.

Key words: Parametric linear programming, sensitivity analysis, analytic central path.

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1 Introduction

Sensitivity analysis and parametric programming play a very important role in linear programming. In some cases, it is even more important to know how the optimal value or an optimal solution changes as a function of the input data rather than simply computing one of the optimal solutions. Indeed, the data of a given problem can never be absolutely accurate in real applications. Hence, it is crucial to keep track of how optimal solutions, or the optimal value, change if a part of the data changes. A recent book edited by Gal and Greenberg [10] is exclusively devoted to recent advances in sensitivity analysis and parametric programming. Several software packages exist, many of them developed in-house by large companies, to help understand a solutions characteristics. As an example we mention ANALYZE [7], which is currently being used by the Energy Information Administration, British Telecommunications, and Amoco Oil.

We make the distinction between the following two questions:

- How does an optimal solution behave with respect to data perturbations?
- How does the optimal objective value behave with respect to data perturbations?

The first question is usually asked when trying to *understand* an optimal solution. The second question is only concerned with how the optimal value relies on the data. Both questions are completely addressed from the classical basic solution perspective. The second question has been addressed from the vantage of a strictly complementary solution yielded by many interior point algorithms, known as the *analytic center optimal solution*, or simply the central optimal solution. Greenberg [8], Jansen, Roos and Terlaky [13] and Zhang [25] give reasons and examples why the central optimal solution is more desirable over a basic solution. The parametric analysis of the optimal value is developed from the central optimal solution perspective in [19]. Monteiro and Mehrotra [16] and Roos and Terlaky [20] independently developed an algorithm, using the central optimal solution, that completely describes the objective function along any single direction of change in either the cost coefficients or the right hand side vector. This algorithm also produces the unique optimal partition as the data is changed. Greenberg [9] shows that the question of how the objective function responds to simultaneous changes in cost coefficients and right hand side levels may be answered using the central optimal solution. All of the above analysis deals with information that is attainable from asking the second question.

Questions concerning how the central optimal solution relies on its data are addressed by Nunez and Freund [18] using a concept proposed by Renegar called the distance to ill-posedness. However, the bounds given by Nunez and Freund become arbitrarily bad as the central optimal solution is approached. The reason for this is that they do not distinguish which data elements are being perturbed.

In this paper we consider the perturbation in the objective coefficients and in the right hand side vector separately. In the case where the right hand side vector is perturbed, we show that the central optimal solution is infinitely differentiable with respect to the perturbation parameter. Moreover, the derivatives in the perturbation parameter are uniformly bounded along the central path. If the objective is perturbed, then the central optimal solution may change completely. But, it will be shown in this case that the central path will be tangential to the central path defined on the optimal face of the original problem.

The organization of this paper is as follows. In the next section we will discuss some basic properties related to the central path in the context of parametric linear programming. In Section 3 we discuss the properties of the central path when the right hand side is perturbed. In Section 4 we continue the discussion when the objective is perturbed. Finally we conclude the paper in Section 5.

2 Analytic central path and optimal solutions

Consider the following standard linear program:

$$
\begin{array}{lll}
(LP) & \text{minimize} & c^T x \\
 & \text{subject to} & Ax = b \\
 & & x \geq 0
\end{array}
$$

and its dual problem

$$
\begin{array}{lll}
(LD) & \text{maximize} & b^T y \\
 & \text{subject to} & A^T y + s = c \\
 & & s \geq 0
\end{array}
$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. The primal feasible polyhedron is denoted $\mathcal{P}$ and the dual feasible region is denoted $\mathcal{D}$. The relative interiors of these sets are denoted $\mathcal{P}^o$ and $\mathcal{D}^o$ and are $\{x : x \in \mathcal{P}, x > 0\}$ and $\{(y, s) : (y, s) \in \mathcal{D}, s > 0\}$, respectively. Furthermore, the primal and dual optimality sets are $\mathcal{P}^*$ and $\mathcal{D}^*$. Elements of $\mathcal{D}$ are referred to as (y, s), y, or s.

Assume that both (LP) and (LD) satisfy the Slater condition, i.e. $\mathcal{P}^o \neq \emptyset$ and $\mathcal{D}^o \neq \emptyset$. It is well known that under the Slater condition there exists the *analytic central path* for (LP) and (LD). This means that for all $\mu > 0$ there exist a unique $x(\mu) \in \mathcal{P}^o$ and $s(\mu) \in \mathcal{D}^o$, such that

$$x_i(\mu)s_i(\mu) = \mu \text{ for all } i = 1, 2, ..., n.$$

We call $(b, c) \in \mathbb{R}^m \times \mathbb{R}^n$ *rim data.* For any $A \in \mathbb{R}^{m\times n}$, (b, c) is *admissible* if (LP) and (LD) defined by A and (b, c) are both feasible. We say that $(\delta b, \delta c)$ is an admissible direction of change if there exists $\theta^* > 0$ such that $(b,c) + \theta(\delta b, \delta c)$ is admissible for all $\theta \in [0, \theta^*)$. Define

$$\begin{aligned}
\mathcal{H} &\equiv \{(\delta b, \delta c) : (\delta b, \delta c) \text{ is an admissible direction of change}\} \\
\mathcal{H}_b &\equiv \{\delta b : (\delta b, 0) \text{ is an admissible direction of change}\} \text{ and} \\
\mathcal{H}_c &\equiv \{\delta c : (0, \delta c) \text{ is an admissible direction of change}\}.
\end{aligned}$$

It is worth noting that if the $\text{rank}(A) = m$, then Slater's condition implies $\mathcal{H}_b = \mathbb{R}^m$. In general, when $\text{rank}(A) < m$ this may not be true. For example, take a standard transportation problem with equality constraints. A direction that implies an increase in demand without a corresponding increase in supply is not an admissible direction of change. In this paper however, we assume that $\text{rank}(A) = m$. Hence, $\mathcal{H}_b = \mathbb{R}^m$ and, due to Slater's condition, we also have $\mathcal{H}_c = \mathbb{R}^n$. When the perturbation direction is understood, the primal polyhedron for $b + \theta_b \delta b$ is denoted $\mathcal{P}_{\theta_b}$, and the dual polyhedron for $c + \theta_c \delta c$ is denoted $\mathcal{D}_{\theta_c}$.

Each of the admissible sets has a subset of interest which relies on the concept of the *optimal partition.* Define

$$\begin{aligned}
B &= \{i : \text{ there exists } x \in \mathcal{P}^* \text{ such that } x_i > 0\} \\
N &= \{1, 2, 3, \ldots, n\} \backslash B.
\end{aligned}$$

B and N form the optimal partition and

$$\begin{aligned}
\mathcal{P}^* &= \{x \in \mathcal{P} : x_N = 0\} \\
\mathcal{D}^* &= \{(y, s) \in \mathcal{D} : s_B = 0\},
\end{aligned}$$

where a set subscript is used to denote the sub-vector corresponding to indices contained in the set. Let $\mathcal{H}^1$, $\mathcal{H}_b^1$, and $\mathcal{H}_c^1$ be respective subsets of $\mathcal{H}$, $\mathcal{H}_b$, and $\mathcal{H}_c$, where if a direction of change is contained in one of these subsets, then the optimal partition remains intact for sufficiently small amounts of change. In other words, $\mathcal{H}^1$, $\mathcal{H}_b^1$, and $\mathcal{H}_c^1$ are the admissible directions of change for which the optimal sets are invariant. It is easy to see that

$$\mathcal{H}_b^1 = \text{coll}(A_B), \quad \mathcal{H}_c^1 = \text{row}(\begin{bmatrix} A_B, & A_N \\ 0, & I \end{bmatrix}), \quad \mathcal{H}^1 = \mathcal{H}_b^1 \times \mathcal{H}_c^1.$$

The perturbed problems of interest, where δb and δc are admissible directions and θ_c and θ_b are sufficiently small positive numbers, are

$$\begin{array}{lll}
(LP)' & \text{minimize} & (c + \theta_c \delta c)^T x \\
& \text{subject to} & Ax = b + \theta_b \delta b \\
& & x \geq 0,
\end{array}$$

and

$$
\begin{array}{lll}
(LD)' & \text{maximize} & (b+\theta_b \delta b)^T y \\
 & \text{subject to} & A^T y + s = c + \theta_c \delta c \\
 & & s \geq 0.
\end{array}
$$

The first question we are interested in is:

> *If δc and δb are admissible and Slater's condition holds when $\theta_c = \theta_b = 0$, then does Slater's condition hold for sufficiently small θ_c and θ_b?*

The answer is positive. We state this fact in the following proposition.

Proposition 2.1 *If (LP) and (LD) satisfy the Slater condition and δc and δb are admissible, then there exist $\theta_c^0 > 0$ and $\theta_b^0 > 0$ such that for all $0 \leq \theta_c \leq \theta_c^0$ and $0 \leq \theta_b \leq \theta_b^0$ the problems $(LP)'$ and $(LD)'$ also satisfy the Slater condition.*

The proof of this proposition is quite straightforward, and is omitted here.

A next natural question is:

> *As θ_c and θ_b approach 0, can we conclude that the central path of $(LP)'$ and $(LD)'$ continuously approaches the central path of (LP) and (LD)?*

This, however, is not true in general since the optimal face can change drastically with the slightest perturbation. This can be seen in Figure 2.

This rules out the possibility that the analytic central paths change smoothly in the limit. However, if the optimal solution sets of (LP) and (LD) are viewed as a point-to-set mapping, operating on $\{(b,c) : (b,c) \text{ is admissible}\}$, then we find that this map is closed.

Proposition 2.2 *Let*

$$OPT(b,c) = \{(x,s) : x \textit{ optimal to } (LP) \textit{ and } s \textit{ optimal to } (LD)\}.$$

If $(x^k, s^k) \in OPT(b^k, c^k)$ and $x^k \to x$, $s^k \to s$, $b^k \to b$ and $c^k \to c$, then

$$(x,s) \in OPT(b,c).$$

Proof.

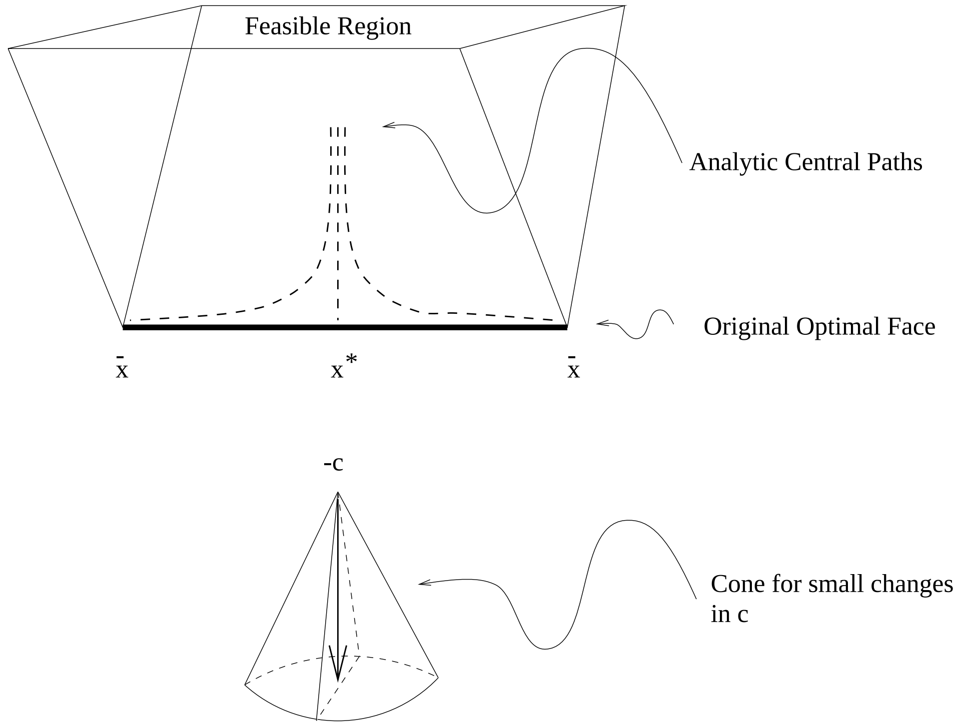

Figure 1: The vector in the center of the cone is $-c$. If the perturbed vector lies in the cut through the cone, then the central optimal solution remains intact. Hence, $\delta c \in \mathcal{H}_c^1$. Otherwise, the new central optimal solution is located at one of the $\bar{x}$'s, and $\delta c \in \mathcal{H}_c \backslash \mathcal{H}_c^1$.

By definition, if $(x^k, s^k) \in OPT(b^k, c^k)$, then the following KKT conditions are satisfied:

$$\left[\begin{array}{l} Ax^k = b^k \\ x^k \geq 0 \\ A^T y^k + s^k = c^k \\ s^k \geq 0 \\ (s^k)^T x^k = 0. \end{array}\right.$$

Letting $k \to \infty$ we obtain

$$\left[\begin{array}{l} Ax = b \\ x \geq 0 \\ A^T y + s = c \\ s \geq 0 \\ s^T x = 0, \end{array}\right.$$

which shows that $(x, s) \in OPT(b, c)$.

Q.E.D.

Although there is in general no "smooth shifting" of the analytic central path with respect to the perturbation parameters, we will see in Theorem 2.1 that if the perturbation parameters goes to zero faster than μ, then the central path still terminates at the analytic center of the optimal face of the original problem.

Without loss of generality, assume that δb and δc are chosen such that $(LP)'$ and $(LD)'$ satisfy Slater's condition for $\theta_c = 1$ and $\theta_b = 1$. Let x^1 and (y^1, s^1) be interior solutions to $(LP)'$ and $(LD)'$.

For $0 < \theta_c < 1$ and $0 < \theta_b < 1$, define

$$x(\theta_c, \theta_b, \mu) \text{ and } (y(\theta_c, \theta_c, \mu), s(\theta_c, \theta_b, \mu))$$

to be solutions of the following system, which defines the analytic central paths for $(LP)'$ and $(LD)'$,

$$\left[\begin{array}{l} Ax = b + \theta_b \delta b \\ A^T y + s = c + \theta_c \delta c \\ XSe = \mu e. \end{array}\right. \tag{2.1}$$

Since its introduction into mathematical programming the path of analytic centers has been known to be analytic not only in μ, $\mu > 0$, but also in θ_c and θ_b, see [22]. Hence, $x(\theta_c, \theta_b, \mu)$ and $s(\theta_c, \theta_b, \mu)$ are completely analytic functions when $\mu > 0$.

The next lemma shows that the union of all level sets of the perturbed problems is bounded. Let $\{(\theta_c^k, \theta_b^k, \mu^k)\}$ be a sequence such that $(\theta_c^k, \theta_b^k, \mu^k) \downarrow (0,0,0)$ as $k \uparrow \infty$. The tacit assumption that $(\theta_c^k, \theta_b^k, \mu^k) < (1,1,1)$ is made to assure that Slater's condition holds. For each $M \geq 0$, define

$$\mathcal{L}_M^k \equiv \{(x, s) \in \mathcal{P}_{\theta_b^k} \times \mathcal{D}_{\theta_c^k} : x^T s \leq M\}.$$

Lemma 2.1 is an extension of the boundedness result found in [19].

Lemma 2.1 *For all $M \geq 0$,*

$$\bigcup_k \mathcal{L}_M^k$$

is bounded.

Proof. Fix k and $(x^k, s^k) \in \mathcal{P}^o_{\theta_b^k} \times \mathcal{D}^o_{\theta_c^k}$. Choose $(x, s) \in \mathcal{L}_M^k$. Then $s^k - s \in \text{row}(A)$ and $x^k - x \in \text{null}(A)$. So

$$0 = (x - x^k)^T (s - s^k) = x^T s - x^T s^k - (x^k)^T s + (x^k)^T s^k.$$

Non-negativity yields

$$x_i s_i^k \leq x^T s^k \leq x^T s + (x^k)^T s^k \leq M + (x^k)^T s^k.$$

Hence,

$$x_i \leq \frac{M + (x^k)^T s^k}{s_i^k}.$$

Similarly,

$$s_i \leq \frac{M + (x^k)^T s^k}{x_i^k}.$$

So for any fixed k and $M \geq 0$, $\mathcal{L}_M^k$ is bounded.

Let $\bar{\mu} > 0$ and define,

$$s^k = s(0, 0, \bar{\mu}) + \theta_c^k \delta c \;\text{ and }\; x^k = x(0, 0, \bar{\mu}) + \theta_b^k A^T (AA^T)^{-1} \delta b.$$

For positive $\bar{\mu} > 0$, $x(0, 0, \bar{\mu})$ and $s(0, 0, \bar{\mu})$ are positive and hence there exists a natural number K^1 such that for all $k \geq K^1$, $s^k > 0$ and $x^k > 0$. Since $A^T y(\bar{\mu}, 0, 0) + s^k = c + \theta_c^k \delta c$ and $Ax^k = b + \theta_b^k \delta b$, it follows that $(x^k, s^k) \in \mathcal{P}^o_{\theta_b^k} \times \mathcal{D}^o_{\theta_c^k}$, for all $k \geq K^1$. Furthermore, since $(\theta_c^k, \theta_b^k) \downarrow (0, 0)$ as $k \uparrow \infty$, we know that $\{(x^k, s^k) : k \geq K^1\}$ is bounded away from zero. Hence both

$$\frac{M + (x^k)^T s^k}{s_i^k} \;\text{ and }\; \frac{M + (x^k)^T s^k}{x_i^k}$$

are bounded for all $k \geq K^1$. This implies that

$$\bigcup_{k \geq K^1} \mathcal{L}_M^k$$

is bounded. Since

$$\bigcup_{1 \leq k < K^1} \mathcal{L}_M^k$$

is a finite union of bounded sets, the result follows.

Q.E.D.

Lemma 2.1 has the consequence that for any sequence $\{(\theta_c^k, \theta_b^k, \mu^k)\}$ decreasing to zero, the sequences $\{x(\theta_c^k, \theta_b^k, \mu^k)\}$ and $\{s(\theta_c^k, \theta_b^k, \mu^k)\}$ have at least one cluster point.

The next lemma guarantees that if the perturbation parameters decrease to zero at the same rate as μ decreases to zero, then $\{x(\theta_c, \theta_b, \mu)\}$ and $\{(y(\theta_c, \theta_b, \mu), s(\theta_c, \theta_b, \mu))\}$ terminate in the relative interior of the original optimal sets. We remark that this result also follows from Theorem 4 in [15].

Lemma 2.2 *If $\theta_c = O(\mu)$ and $\theta_b = O(\mu)$ and $\mu \downarrow 0$, then any cluster point of $x(\theta_c, \theta_b, \mu)$ will be contained in the relative interior of the optimal face of (LP), and any cluster point of $(y(\theta_c, \theta_b, \mu), s(\theta_c, \theta_b, \mu))$ will be contained in the relative interior of the optimal face of (LD).*

Proof.

Denote $x = x(\theta_c, \theta_b, \mu)$ and $s = s(\theta_c, \theta_b, \mu)$. Let x^* and s^* be two optimal solutions of (LP) and (LD) that are strictly complementary, i.e. $x^* + s^* > 0$. Observe that

$$[\theta_b x^1 + (1-\theta_b)x^* - x]^T[\theta_c s^1 + (1-\theta_c)s^* - s] = 0.$$

Using that $x^T s = n\mu$ we have,

$$\begin{aligned}(1-\theta_b)(x^*)^T s + (1-\theta_c)x^T s^* &= \theta_c\theta_b(x^1)^T s^1 + \theta_b(1-\theta_c)(x^1)^T s^* + \theta_c(1-\theta_b)(x^*)^T s^1 \\ &\quad + n\mu - \theta_b(x^1)^T s - \theta_c x^T s^1 \\ &\leq \theta_c\theta_b(x^1)^T s^1 + \theta_b(1-\theta_c)(x^1)^T s^* + \theta_c(1-\theta_b)(x^*)^T s^1 + n\mu.\end{aligned}$$

Using that $\theta_c = O(\mu)$ and $\theta_b = O(\mu)$ it follows that

$$(x^*)^T s = O(\mu) \text{ and } x^T s^* = O(\mu).$$

As $\mu = x_i s_i$, the above relation implies that there exists a constant, say $\Gamma > 0$, independent of μ, such that

$$x_i^* \leq \Gamma x_i \text{ and } s_i^* \leq \Gamma s_i$$

for all i and all $\mu > 0$. Since x^* and s^* are strictly complementary, it follows that any cluster point of the sequence $\{(x(\theta_c, \theta_b, \mu), s(\theta_c, \theta_b, \mu))\}$ will be strictly strictly complementary as well.

Q.E.D.

The next result shows that if θ_c and θ_b go to zero faster than μ, then $x(\theta_c, \theta_b, \mu)$ and $s(\theta_c, \theta_b, \mu)$ still converges to the analytic center of the optimal faces of the original problem.

Theorem 2.1 *Let $\hat{x}$ be the analytic center of the optimal face for (LP) and $(\hat{y}, \hat{s})$ be the analytic center of the optimal face for (LD). If $\theta_c = o(\mu)$ and $(\theta_b, \mu) \downarrow (0,0)$, then*

$$\lim_{\mu \downarrow 0} x(\theta_c, \theta_b, \mu) = \hat{x}.$$

If $\theta_b = o(\mu)$ and $(\theta_c, \mu) \downarrow (0,0)$, then

$$\lim_{\mu \downarrow 0} (y(\theta_c, \theta_b, \mu), s(\theta_c, \theta_b, \mu)) = (\hat{y}, \hat{s}).$$

Proof.

Consider a sequence $\{(\theta_c^k, \theta_b^k, \mu^k) : k = 1, 2, ...\}$ with

$$\lim_{k\uparrow\infty} \theta_b^k = \lim_{k\uparrow\infty} \mu^k = 0, \text{ and } \lim_{k\uparrow\infty} \theta_c^k/\mu^k = 0.$$

Denote $x^k = x(\theta_c^k, \theta_b^k, \mu^k)$ and $s^k = s(\theta_c^k, \theta_b^k, \mu^k)$.

To prove $\lim_{k\uparrow\infty} x^k = \hat{x}$, we consider the sequence $\hat{x}^k$, with $\hat{x}^k$ being optimal to

$$\begin{array}{ll} \text{minimize} & (c^T x/\mu^k) - \sum_{i=1}^n \log x_i \\ \text{subject to} & Ax = b + \theta_b^k \delta b. \end{array}$$

The boundedness shown in Lemma 2.1 implies that we may assume without loss of generality the existence of

$$\lim_{k\uparrow\infty} x^k \text{ and } \lim_{k\uparrow\infty} \hat{x}^k.$$

It is clear that both of these limits are feasible to (LP). Assume for the sake of attaining a contradiction that

$$\lim_{k\uparrow\infty} (\hat{x}^k - x^k) \neq 0.$$

Let $F_k(x) = (c^T x/\mu^k) - \sum_{i=1}^n \log x_i$. It follows by the optimality of x^k and $\hat{x}^k$ respectively that

$$F_k(\hat{x}^k) \leq F_k(x^k) \leq F_k(\hat{x}^k) - \frac{\theta_c^k}{\mu^k}(\delta c)^T x^k + \frac{\theta_c^k}{\mu^k}(\delta c)^T \hat{x}^k. \tag{2.2}$$

Observe now that x^k and $\hat{x}^k$ are bounded by Lemma 2.1, and that $F_k(x)$ is strongly convex. Moreover, the eigenvalues of the Hessian matrix of F_k are uniformly bounded from zero. Hence, if $\lim_{k\uparrow\infty}(\hat{x}^k - x^k) \neq 0$ then for all natural numbers, N, there exists $k \geq N$, such that

$$F_k(x^k) \geq F_k(\hat{x}^k) + \epsilon.$$

This contradicts (2.2) since $\theta_c^k/\mu^k \downarrow 0$. So we must have

$$\lim_{k\uparrow\infty} (\hat{x}^k - x^k) = 0.$$

In other words, we have shown

$$\lim_{k\uparrow\infty} (x(\theta_c^k, \theta_b^k, \mu^k) - x(0, \theta_b^k, \mu^k)) = 0. \tag{2.3}$$

One can also show that

$$\lim_{k\uparrow\infty} (x(0, \theta_b^k, \mu^k) - x(0, 0, \mu^k)) = 0. \tag{2.4}$$

(See e.g. Theorem 3.3 in Section 3.)

Because,

$$\lim_{k\uparrow\infty} x(0,0,\mu^k) = \hat{x},$$

and (2.3) and (2.4) imply that

$$\lim_{k\uparrow\infty} (x(\theta_c^k, \theta_b^k, \mu^k) - x(0,0,\mu^k)) = 0,$$

we have,

$$\lim_{k\uparrow\infty} x^k = \hat{x}.$$

Similarly, on the dual side we can show that

$$\lim_{k\uparrow\infty} (y^k, s^k) = (\hat{y}, \hat{s})$$

if $\theta_b^k = o(\mu^k)$ and $(\theta_c^k, \mu^k) \downarrow (0,0)$.

Q.E.D.

3 Central path under perturbation: Change the right hand side

In this and the next section we consider the situation in which perturbation occurs exclusively in b or c. These cases allow a certain type of continuity of either the primal or the dual central path. The fact that the optimal partition is monotone if either the b or c vector is perturbed was proven by Adler and Monteiro [1] and is also found in [2] by Berkelaar, Roos and Terlaky. For completeness, we present this result below in Propositions 3.1 and 3.2, which will be used in later analysis. Before mentioning this result, we note that the optimal partition changes only in a finite number of break points (see, e.g., [1]). Hence, the existence of an open interval for which the optimal partition remains constant is guaranteed.

Proposition 3.1 *Let (B,N) be the optimal partition of (LP) and (LD). Let (B',N') be the optimal partition of (LP)′ and (LD)′ with the perturbation parameter θ_c (assuming that $\theta_b = 0$). Then, for sufficiently small θ_c we have*

$$N \subseteq N' \text{ and } B' \subseteq B.$$

Similarly, we have:

Proposition 3.2 *Let (B,N) be the optimal partition of (LP) and (LD). Let (B',N') be the optimal partition of (LP)′ and (LD)′ with the perturbation parameter θ_b (assuming that $\theta_c = 0$). Then, for sufficiently small θ_b we have*

$$N' \subseteq N \text{ and } B \subseteq B'.$$

It turns out that the perturbation on the right-hand side vector b has a different effect from perturbing the objective vector c. This section considers only the perturbation of b.

3.1 The central optimal solution

In order to understand how the analytic central path is affected by perturbing b, the focus of this subsection is on how the limit point of the analytic central path — the analytic center of the optimal face — reacts to such perturbation. This makes sense because in general there can be multiple optimal solutions, but the analytic center of the optimal face is unique and representable in term of the problem data. As it is shown in [13], [21], and [25], optimal solutions in the relative interior of the optimal face carry more information than an arbitrary vertex optimal solution.

According to Proposition 2.1, the existence of the analytic central path is guaranteed if the perturbation is within a certain region. This means that the analytic center of the optimal face (we shall call it the *central optimal solution* hereafter) is a well defined function in terms of the perturbation parameters.

As in the previous section, consider

$$\begin{array}{lll} (LP)_{\theta_b} & \text{minimize} & c^T x \\ & \text{subject to} & Ax = b + \theta_b \delta b \\ & & x \geq 0 \end{array}$$

and the corresponding dual,

$$\begin{array}{lll} (LD)_{\theta_b} & \text{maximize} & (b + \theta_b \delta b)^T y \\ & \text{subject to} & A^T y + s = c \\ & & s \geq 0. \end{array}$$

Due to Proposition 2.1 we know that there exists $\theta_b^0 > 0$ such that for all $0 \leq \theta_b \leq \theta_b^0$, $(LP)_{\theta_b}$ and $(LD)_{\theta_b}$ have a primal-dual analytic central path. Dropping the dependence on θ_c in this subsection, for each such θ_b let the analytic central path be

$$\{(x(\theta_b, \mu), (y(\theta_b, \mu), s(\theta_b, \mu)) : \mu > 0\}.$$

Hence, the analytic centers of the primal-dual optimal faces are

$$x^*(\theta_b) = \lim_{\mu \to 0} x(\theta_b, \mu)$$

and

$$s^*(\theta_b) = \lim_{\mu \to 0} s(\theta_b, \mu).$$

In Nunez and Freund [18] a bound is given for the quantity

$$\|(x(\theta_b,\mu), y(\theta_b,\mu), s(\theta_b,\mu)) - (x(0,\mu), y(0,\mu), s(0,\mu))\|$$

using a condition number of (*LP*) and (*LD*). Unfortunately, their bound tends to infinity as μ goes to zero. Later in the paper we shall derive a bound on this quantity by a different approach.

Before continuing, we make a few notational conventions concerning differentials. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable, then

$$D^k_{x_i} f(x)$$

is the vector whose j^{th} component is, $\frac{\partial^k f_j}{\partial x_i^k}$, where f_j is the j^{th} component function. Of particular interest in this section is $D^k_{\theta_b} x^*(\theta_b)$. This notation is extended as follows,

$$D^k_{\theta_b} x^*(0^+) = \lim_{\theta_b \downarrow 0} D^k_{\theta_b} x^*(\theta_b), \text{ and}$$

$$D^k_{\theta_b^+} x^*(0) = \lim_{\theta_b \downarrow 0} \frac{D^{k-1}_{\theta_b} x^*(\theta_b) - D^{k-1}_{\theta_b} x^*(0)}{\theta_b}.$$

Otherwise, if a real valued single variable function is of concern, say g, the derivatives are denoted by g', g'', ..., $g^{(k)}$. The right sided derivatives and limiting derivatives are written, g'_+, g''_+, ..., $g^{(k)}_+$ and $g'(x_0^+)$, $g''(x_0^+)$, ..., $g^{(k)}(x_0^+)$, respectively.

Our attention now turns back to differentiating the central optimal solution. Let the optimal partition for (*LP*) and (*LD*) be B and N and the optimal partition for $(LP)_{\theta_b}$ and $(LD)_{\theta_b}$ be B' and N'. By Proposition 3.2 we know that

$$B \subseteq B'.$$

Denote

$$\mathcal{I} = B' \setminus B.$$

It is clear that for $i \in N \setminus \mathcal{I} = N'$ that

$$x^*_{N'}(\theta_b) = 0$$

for sufficiently small θ_b. This shows that

$$D^k_{\theta_b^+} x^*_{N'}(0) = 0, \tag{3.1}$$

for all k. Furthermore, if $\delta b \in \mathcal{H}^1_b$, then $x^*_{N'}(\theta_b) = 0$ for θ_b in a neighborhood of zero, so that

$$D^k_{\theta_b} x^*_{N'}(0) = 0,$$

for all k.

The differentiability of $x^*_{B'}(\theta_b)$ is considered in two cases, depending on whether or not $\delta b \in \mathcal{H}^1_b$. If $\delta b \in \mathcal{H}^1_b$ then the complete analyticity of $x^*(\theta_b)$ is established in the next theorem. The result is a direct consequence of the general implicit function theorem, see [5].

Theorem 3.1 *If $\delta b \in \mathcal{H}_b^1$ then x^* is locally analytic along δb.*

Proof. As previously mentioned, $\delta b \in \mathcal{H}_b^1$ implies $\delta b \in \operatorname{coll}(A_B)$. So there exists a full row-rank matrix $\bar{A}_B$ such that $\{x : A_B x = b + \theta \delta b\} = \{x : \bar{A}_B x = \bar{b} + \theta \bar{\delta b}\}$. Then the analytic center solution of the unperturbed problem is the unique solution to

$$\begin{aligned} \bar{A}_B x_B &= \bar{b} + \theta \bar{\delta b} && (3.2) \\ \bar{A}_B^T y + s_B &= 0 && (3.3) \\ X_B s_B &= e, && (3.4) \end{aligned}$$

when $\theta = 0$. Denote this solution by $(x_B^*, \bar{s}_B, \bar{y})$. Define $\phi : \mathbb{R}^{2|B|+m'+1} \to \mathbb{R}^{2|B|+m'}$ as

$$\phi(x_B, y, s_B, \theta) = \begin{bmatrix} \bar{A}_B x_B - \theta \bar{\delta b} - \bar{b} \\ \bar{A}_B^T y + s_B \\ X_B s_B - e \end{bmatrix}.$$

Then ϕ is analytic in an open neighborhood of $(x_B^*, \bar{y}, \bar{s}, 0)$ and $\phi((x_B^*, \bar{y}, \bar{s}, 0)) = 0$. Since the Jacobian of $\phi((x_B^*, \bar{y}, \bar{s}, 0))$ with respect to (x_B, y, s) is non-singular, the implicit function theorem implies that x_B is an analytic function of θ in some sufficiently small neighborhood of zero.

Q.E.D.

Notice that in the proof, if δb had not been in the $\operatorname{coll}(A_B)$, then the implicit function theorem is not valid since the needed row reduction would not have been possible. Furthermore, Theorem 3.1 implies that if we are concerned with a direction of change that does not immediately alter the partition, then not only is $x^*(\theta_b)$ of class $\mathcal{C}^\infty$, but $x^*(\theta_b)$ has a power series expansion. Differentiating (3.2), (3.3) and (3.4) with respect to θ_b we have,

$$\left\{ \begin{aligned} \bar{A}_B(D^1_{\theta_b} x_B^*) &= \bar{\delta b} \\ \bar{A}_B^T(D^1_{\theta_b} y) + D^1_{\theta_b} s_B &= 0 \\ S_B(D^1_{\theta_b} x_B^*) + X_B(D^1_{\theta_b} s_B) &= 0. \end{aligned} \right. \tag{3.5}$$

Noticing that this is a non-singular system of linear equations in $D^1_{\theta_b} x_B^*$, $D^1_{\theta_b} s_B$, and $D^1_{\theta_b} y_B$, we have the following,

$$\left\{ \begin{aligned} D^1_{\theta_b} x_B^*(\theta_b) &= X_B^2 \bar{A}_B^T (\bar{A}_B X_B^2 \bar{A}_B^T)^{-1} \bar{\delta b}, \\ D^1_{\theta_b} s_B(\theta_b) &= \bar{A}_B^T (\bar{A}_B X_B^2 \bar{A}_B^T)^{-1} \bar{\delta b}, \text{ and} \\ D^1_{\theta_b} y(\theta_b) &= (\bar{A}_B X_B^2 \bar{A}_B^T)^{-1} \bar{\delta b}. \end{aligned} \right. \tag{3.6}$$

Using (3.5), we are able to recursively establish the higher order derivatives. Define

$$\Omega^k \equiv \sum_{i=1}^{k-1} \binom{k}{i} (D^i_{\theta_b} S_B)(D^{k-i}_{\theta_b} x_B).$$

Then for $k \geq 2$ we have,

$$\left\{ \begin{array}{rcl} \bar{A}_B(D^k_{\theta_b} x^*_B) & = & 0 \\ \bar{A}^T_B(D^k_{\theta_b} y) + D^k_{\theta_b} s_B & = & 0 \\ S_B(D^k_{\theta_b} x^*_B) + X_B(D^k_{\theta_b} s_B) & = & \Omega^k, \end{array} \right.$$

which implies

$$\left\{ \begin{array}{rcl} D^k_{\theta_b} x^*_B(\theta_b) & = & X_B\Omega^k - X_B\bar{A}^T_B(\bar{A}_B X^2_B \bar{A}^T_B)^{-1}\bar{A}_B X_B \Omega^k \\ D^k_{\theta_b} s_B(\theta_b) & = & \bar{A}^T_B(\bar{A}_B X^2_B \bar{A}^T_B)^{-1}\bar{A}_B X_B \Omega^k \\ D^k_{\theta_b} y(\theta_b) & = & -(\bar{A}_B X^2_B \bar{A}^T_B)^{-1}\bar{A}_B X_B \Omega^k. \end{array} \right. \tag{3.7}$$

Hence,

$$x^*(\theta_b) = \sum_{k=0}^{\infty} \frac{(D^k_{\theta_b} x^*(0))}{k!}(\theta_b)^k, \tag{3.8}$$

for θ_b sufficiently close to zero, where $D^k_{\theta_b} x^*_N(0) = 0$. Notice that this power series expansion is easy to calculate since it requires only one matrix factorization. Hence, a quick attempt at inferring where the new central optimal solution can be done with high order extrapolation methods.

We now consider the case when $\delta b \in \mathcal{H}_b \backslash \mathcal{H}^1_b$. The next lemma, due to Güler [12], concerning the limiting derivatives along the central path is needed.

Lemma 3.1 *Let B and N be the optimal partition for (LP) and (LD). Let $x(\mu)$ and $s(\mu)$ be the primal-dual central path. Then both $D^k_\mu x(0^+)$ and $D^k_\mu s(0^+)$ exist and the following equalities hold,*

$$D^k_\mu x(0^+) = D^k_{\mu^+} x(0)$$

and

$$D^k_\mu s(0^+) = D^k_{\mu^+} s(0).$$

Moreover,

$$D^1_{\mu^+} x_N(0) > 0 \text{ and } D^1_{\mu^+} s_B(0) > 0.$$

Proof.

See Güler [12].

Q.E.D.

The formula for the kth derivative of a composition function is integral in completing our analysis; see [6]. Let $h(x) = f(g(x))$, where both f and g are $\mathcal{C}^\infty$ on some suitable neighborhoods. Then $h^{(k)}(x)$ is

$$\sum_{m=1}^{k} \sum \frac{k!}{j_1! j_2! \ldots j_k!} \cdot \left[\left(\frac{g'(x)}{1!}\right)^{j_1} \left(\frac{g''(x)}{2!}\right)^{j_2} \cdots \left(\frac{g^{(k)}(x)}{k!}\right)^{j_k} \right] \cdot \frac{d^m f}{dy^m}(g(x)), \tag{3.9}$$

where the second sum is taken over all non-negative integer solutions of $\sum\limits_{i=1}^{k} ij_i = k$ and $\sum\limits_{i=1}^{k} j_i = m$. The following lemma provides some differential properties used to complete the desired results pertaining to $x_B(\theta_c, \theta_b, 0)$.

Lemma 3.2 *Let $f : [0, \mu^*) \to [0, \theta^*)$ be an increasing, $\mathcal{C}^\infty$, function on $(0, \mu^*)$, such that, $f^{(k)}(0^+)$ exists for all $k \geq 1$, and $f'(0^+) > 0$. Then $g = f^{-1}$ exists and has the property that $g^{(k)}(0^+)$ exists for all $k \geq 1$. Furthermore, if f is continuous at zero, then f is $\mathcal{C}^\infty$ on $[0, \mu^*)$ and g is $\mathcal{C}^\infty$ on $[0, \theta^*)$.*

Proof. Let f be as above. The general inverse function theorem, see [17], establishes that $g = f^{-1}$ exists and is $\mathcal{C}^\infty$ on $(0, \theta^*)$. Since

$$g'(\theta) = \frac{1}{f'(g(\theta))} \tag{3.10}$$

for all $\theta \in (0, \theta^*)$, we immediately have

$$g'(0+) = \frac{1}{f'(g(0+))}$$

which completes the first statement when $k = 1$.

By definition of the inverse function we have

$$\theta = f(g(\theta)) \tag{3.11}$$

for $\theta \in (0, \theta^*)$ and so we can apply (3.9) on (3.11) for all $k > 1$. Observe that by this formula we can express $g^{(k)}(\theta)$ in terms of $g'(\theta)$, $g''(\theta)$, $g'''(\theta)$, ..., $g^{(k-1)}(\theta)$, and $f'(g(\theta))$, $f''(g(\theta))$, $f'''(g(\theta))$, ..., $f^{(k)}(g(\theta))$. So by a simply induction argument we conclude that $g^{(k)}(0+)$ exists for all k.

Now, suppose that f is continuous at zero, and let $t(\mu)$ be from the mean value theorem. Then

$$\frac{f(\mu) - f(0)}{\mu} = f'(t(\mu)) \text{ for all } \mu \in (0, \mu^*).$$

Then using the assumption that $f'(0+)$ exists implies

$$\lim_{\mu \to 0^+} \left|\frac{f(\mu) - f(0)}{\mu} - f'(\mu)\right| = \lim_{\mu \to 0^+} |f'(t(\mu)) - f'(\mu)| = 0.$$

So $f'_+(0) = f'(0+)$ and we have that f' is continuous at zero. Repeated applications of the mean value theorem give that f is $\mathcal{C}^\infty$ on $[0, \mu^*)$. Since f continuous at zero implies g is continuous at zero, a similar argument shows g is $\mathcal{C}^\infty$ on $[0, \theta^*)$. **Q.E.D.**

We now prove that if $\delta b \in \mathcal{H}_b \backslash \mathcal{H}_b^1$, then the primal central optimal solution is infinitely, continuously, one-sided differentiable with respect to the perturbation parameter θ_b.

Theorem 3.2 *$D^k_{\theta_b} x^*(0^+)$ exists for all natural numbers k and $D^k_{\theta_b} x^*(0^+) = D^k_{\theta_b^+} x^*(0)$*

Proof. The case when $i \in N'$ has already been considered (see (3.1)). Let $i \in B \cup \mathcal{I}$, and consider the following linear program which has an optimal value of zero:

$$\min\{\theta_b : A_B z_B + A_{\mathcal{I}} z_{\mathcal{I}} - \delta b \theta_b = b, z_B \geq 0, z_{\mathcal{I}} \geq 0, \theta_b \geq 0\}.$$

Let $\{(z_B(\mu), z_{\mathcal{I}}(\mu), \theta_b(\mu)) : \mu \geq 0\}$ be the central path. Then

$$z_B(\mu) = x_B(\theta_b(\mu)) \ , \ z_{\mathcal{I}}(\mu) = x_{\mathcal{I}}(\theta_b(\mu))$$

and continuity implies

$$z_B(\mu) \to x_B(0) > 0 \ , \ z_{\mathcal{I}}(\mu) \to x_{\mathcal{I}}(0) = 0 \ , \ \theta_b(\mu) \to 0,$$

as $\mu \downarrow 0$. Hence, the optimal partition is B and $\mathcal{I} \cup \{\theta_b\}$.

Lemma 3.1 implies that $\theta'_{b+}(0) > 0$, and hence there exists some interval, say $[0, \mu^*)$, where $\theta_b(\mu)$ is invertible. Denote the inverse by $\mu(\theta_b)$ and let the corresponding interval be $[0, \theta_b^*)$. Then for all $\theta_b \in (0, \theta_b^*)$ we have

$$z_B(\mu(\theta_b)) = x_B(\theta_b) \ , \ z_{\mathcal{I}}(\mu(\theta_b)) = x_{\mathcal{I}}(\theta_b)$$

Applying (3.9) we have

$$D^k_{\theta_b} x_i(\theta_b) = \sum_{m=1}^{k} \sum \frac{k!}{j_1! j_2! \ldots j_k!} \cdot \left[\left(\frac{\mu'(\theta_b)}{1!} \right)^{j_1} \left(\frac{\mu''(\theta_b)}{2!} \right)^{j_2} \cdots \left(\frac{\mu^{(k)}(\theta_b)}{k!} \right)^{j_k} \right] \cdot z_i^{(m)}(\mu(\theta_b))$$

for $i \in B \cup \mathcal{I}$. Lemmas 3.1 and 3.2 imply that the right hand side converges as $\theta_b \downarrow 0$. So $D^k_{\theta_b} x^*(0^+)$ exists. Since z and μ are continuous functions, $x^*(\theta_b)$ is continuous, and the mean value theorem is used as in the proof of Lemma 3.2 to conclude the result.

Q.E.D.

Two immediate corollaries are in order. We use an extension of the big–O notation, viz. $u(t) = \Theta(w(t))$ if $u(t)$ and $w(t)$ are positive seqeunces, and the ratios $u(t)/w(t)$ and $w(t)/u(t)$ are both bounded.

Corollary 3.1 *There hold*

$$x_B^*(\theta_b) - x_B^*(0) = O(\theta_b), \quad x_{\mathcal{I}}^*(\theta_b) = \Theta(\theta_b) \text{ and } x_{N'}^*(\theta_b) = 0$$

for sufficiently small $\theta_b > 0$.

Corollary 3.2 *There hold*

$$x_{N'}(\theta_b, \mu) = \Theta(\mu) \textit{ and } s_B(\theta_b, \mu) = \Theta(\mu)$$

for sufficiently small $\theta_b > 0$.

Proof.

Due to $x_{N'}(\theta_b, \mu) - x^*_{N'}(\theta_b) = \Theta(\mu)$ and $x^*_{N'}(\theta_b) = 0$ for small $\theta_b > 0$, it follows that

$$x_{N'}(\theta_b, \mu) = \Theta(\mu).$$

By Corollary 3.1 we know that

$$x_B(\theta_b, \mu) = \Theta(1)$$

and so

$$s_B(\theta_b, \mu) = \Theta(\mu).$$

Q.E.D.

3.2 Bounds along the analytic central path

Bounds are now developed for the first derivatives of $x(\theta_b, \mu)$, $\mu > 0$, and $x^*(\theta_b)$. Bounding $D^1_{\theta_b} x(\theta_b, \mu)$, $\mu > 0$, is considered first. Similar to (3.5) we have

$$\begin{aligned} A(D^1_{\theta_b} x(\theta_b, \mu)) &= \delta b \\ A^T(D^1_{\theta_b} y) + D^1_{\theta_b} s &= 0 \\ S(D^1_{\theta_b} x(\theta_b, \mu)) + X(D^1_{\theta_b} s) &= 0, \end{aligned}$$

which implies

$$D^1_{\theta_b} x(\theta_b, \mu) = X^2 A^T (A X^2 A^T)^{-1} \delta b. \tag{3.12}$$

For a given full row-rank matrix A, Dikin [4] showed that the following condition number χ_A is finite (also independently rediscovered by Stewart [23] and Todd [24]):

$$\chi_A := \sup\{\|(ADA^T)^{-1} AD\| : D \text{ is positive diagonal}\}.$$

Applying this result to (3.6) and (3.12), we have the following result:

Theorem 3.3 *For any positive μ it follows that*

$$\|D^1_{\theta_b} x(\theta_b, \mu)\| \leq \chi_A \|\delta b\|. \tag{3.13}$$

Furthermore, if $\delta b \in \mathcal{H}^1_b$, then

$$\|D^1_{\theta_b} x^*(0)\| \leq \chi_{\bar{A}} \|\bar{\delta b}\|. \tag{3.14}$$

Otherwise $\theta_b \in \mathcal{H}_b \backslash \mathcal{H}^1_b$ and

$$\|D^1_{\theta_b^+} x^*(0)\| \leq \chi_{\bar{A}} \|\bar{\delta b}\|. \tag{3.15}$$

Proof.

Both (3.13) and (3.14) follow immediately from the definition of χ_A. The bound in (3.15) follows since Theorem 3.2 implies $D^1_{\theta_b^+} x^*(0) = D^1_{\theta_b} x^*(0^+)$, and the right side of this equality is bounded by $\chi_{\bar{A}} \|\bar{\delta b}\|$ because of (3.14).

Q.E.D.

4 Central path under perturbation: Change the objective

Let us review Figure 1. One may observe that if the perturbation on the objective function is small, then the analytic central path can be misled by the difference, in the sense that the central path $x(\theta_c, \mu)$ first approaches the analytic center x^* in the optimal face of (*LP*) ignoring the difference between c and $c + \theta_c \delta c$. Then, it realizes that the direction is wrong and makes a sharp turn and moves further towards $\bar{x}$. The last movement is nearly parallel to the analytic central path defined on the optimal face of (*LP*), leading from x^* to $\bar{x}$.

The purpose of this section is to show that this indeed happens. Throughout this section it is assumed that $\theta_b = 0$, and we eliminate the argument of θ_b. Hence, $x(\theta_c, \theta_b, \mu)$ is referred to by $x(\theta_c, \mu)$.

Consider the following system:

$$\left[\begin{array}{l} A_B x^*_B = b \\ A^T_B y^* + s^*_B = (\delta c)_B \\ X^*_B s^*_B = \nu e_B \\ x^*_B \geq 0, \; s^*_B \geq 0, \end{array}\right. \tag{4.1}$$

where the * notation is used because x^*_B is contained in the optimal set of (*LP*). This system has unique solutions $x^*_B(\nu)$ and $s^*_B(\nu)$ for every given positive ν. In particular, if $\delta c = 0$ and $\nu = 1$, then $(x^*_B(1), 0)$ is the central optimal solution for (*LP*). In general, $(x^*_B(\nu), 0)$ corresponds to the

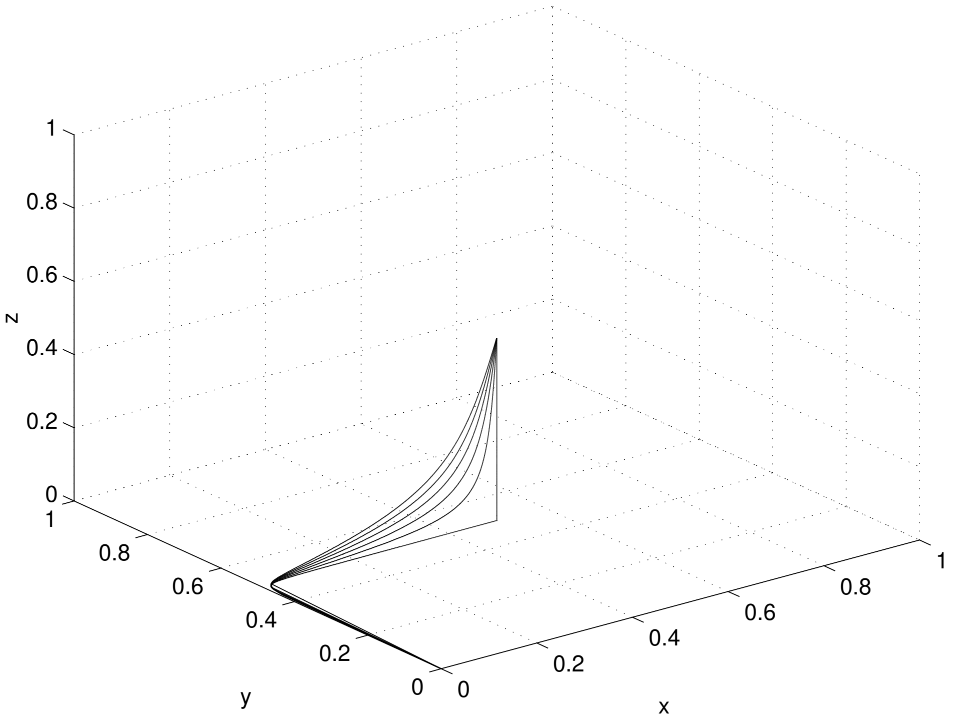

Figure 2: The vertical line is the central path for $\min\{z : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$. The path contained in the x-y plane is the central path for $\min\{\frac{1}{4}x + \frac{5}{10000}y : 0 \leq x \leq 1, 0 \leq y \leq 1\}$. The remaining paths correspond to $\min\{z + \frac{\theta_c}{4}x + \frac{5\theta_c}{10000}y : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$, where $\theta_c = 1, 0.8, 0.6, 0.4, 0.2$.

central path defined on the optimal face of the original problem. As $\nu \to 0$, this central path will lead us to the central optimal solution of the perturbed problem. In the case that $B' = B$, the path reduces to a single point.

Now consider the primal-dual central path for $(LP)'$ and $(LD)'$ (assuming $\theta_b = 0$). The following equation must be satisfied:

$$\left[\begin{array}{l} Ax(\theta_c, \mu) = b \\ A^T y(\theta_c, \mu) + s(\theta_c, \mu) = c + \theta_c \delta c \\ X(\theta_c, \mu) s(\theta_c, \mu) = \mu e. \end{array}\right.$$

If we let $(\hat{y}, \hat{s})$ denote the central optimal solution for (LD), then it follows that $A^T \hat{y} + \hat{s} = c$, $\hat{s} \geq 0$ and $\hat{s}_B = 0$. Let

$$\bar{y} = (y(\theta_c, \mu) - \hat{y})/\theta_c \text{ and } \bar{s} = (s(\theta_c, \mu) - \hat{s})/\theta_c.$$

It is clear that $(\bar{y}, \bar{s})$ satisfies $A^T \bar{y} + \bar{s} = \delta c$ and $X_B(\theta_c, \mu)\bar{s}_B = \frac{\mu}{\theta_c} e_B$.

Let ν be a fixed positive constant and $\mu = \theta_c \nu$. We have

$$\left[\begin{array}{l} A_B x_B(\theta_c, \theta_c \nu) = b - A_N x_N(\theta_c, \theta_c \nu) \\ (A_B)^T \bar{y} + \bar{s}_B = (\delta c)_B \\ X_B(\theta_c, \theta_c \nu) \bar{s}_B = \nu e_B. \end{array}\right.$$

Due to Proposition 3.1 we know that

$$\lim_{\theta_c \to 0} x_N(\theta_c, \theta_c \nu) = 0.$$

Because the equations (4.1) have a unique solution $x_B(\nu)$ it follows that

$$\lim_{\theta_c \to 0} x_B(\theta_c, \theta_c \nu) = x_B^*(\nu).$$

This proves the following result:

Theorem 4.1 *For any given parameter $\nu > 0$, it holds that*

$$\lim_{\theta_c \to 0} x(\theta_c, \theta_c \nu) = (x_B^*(\nu), 0).$$

Loosely speaking, the above theorem states that as the perturbation parameter θ_c tends to zero, the last part of the central path of the perturbed problem gets arbitrarily close to the "central path" directly defined on the optimal face of the original problem.

There are connections between Theorem 4.1 above and Theorem 4.13 in Bonnans and Potra [3]. However, Bonnans and Potra considered a single shifted analytic center in a specific algorithmic framework, whereas our result is algorithm independent and concerns the "central path" defined on the optimal face of the original problem.

The situation described by Theorem 4.1 is depicted in Figure 2.

5 Concluding remarks

In this paper we carried out an investigation on how the analytic central path and the central optimal solution (analytic center of the optimal set) react to the changes in the right-hand side vector b and the objective vector c. These issues are important in the context of sensitivity analysis and parametric programming.

It turned out that the change caused by b can be quite different from that caused by c. In the former case, we proved that the central optimal solution has one-sided differentiability with respect to the perturbation parameter. The whole central path also has a smooth and uniformly bounded shift in terms of the perturbation parameter. Perturbation in the objective vector, however, can cause a drastic change in the central path. However, in this case we showed that every element of the central path defined directly on the optimal face is a cluster point of the (perturbed) whole central path, when the perturbation and centrality parameters tend to zero.

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