LQ Control without Riccati Equations: Stochastic Systems

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April, 1999

Abstract

We study stochastic linear–quadratic (LQ) optimal control problems over an infinite horizon, allowing the cost matrices to be indefinite. We develop a systematic approach based on semidefinite programming (SDP). A central issue is the stability of the feedback control; and we show this can be effectively examined through the complementary duality of the SDP. Furthermore, we establish several implication relations among the SDP complementary duality, the (generalized) Riccati equation, and the optimality of the LQ control problem. Based on these relations, we propose a numerical procedure that provides a thorough treatment of the LQ control problem via SDP: it identifies a stabilizing feedback control that is optimal or determines that the problem possesses no optimal solution. For the latter case, we develop an ϵ-approximation scheme that is asymptotically optimal.

Key words: stochastic LQ control, semidefinite programming, complementary duality, mean-square stability, generalized Riccati equation.

AMS subject classification: 93E20, 90C25, 93D15.

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ECONOMETRIC INSTITUTE REPORT NO. 9920/A

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1 Introduction

Consider the following stochastic linear–quadratic (LQ) optimal control problem:

$$\text{(LQ)} \quad \min \quad J(x_0, u(\cdot)) := \mathsf{E}\int_0^{+\infty} [x(t)^T Q x(t) + u(t)^T R u(t)] dt \tag{1}$$

$$\text{s.t.} \quad dx(t) = [Ax(t) + Bu(t)]dt + [Cx(t) + Du(t)]dW(t) \tag{2}$$

$$x(0) = x_0 \in \Re^n.$$

Here and throughout the paper, A, B, C, D and Q, R are constant matrices, with Q and R being symmetric matrices; the superscript T denotes the transpose of matrices and vectors; $W(\cdot)$ is a one-dimensional standard Brownian motion (with $t \in [0, +\infty)$ and $W(0) = 0$), defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$; and $u(\cdot)$ denotes the (open-loop) control, which belongs to $L^2_{\mathcal{F}}(\Re^m)$, the space of all $\Re^m$-valued, $\mathcal{F}_t$-adapted measurable processes satisfying:

$$\mathsf{E}\int_0^{+\infty} \|u(t)\|^2 dt < +\infty.$$

Note that allowing multi-dimensional Brownian motion renders no essential difficulty to our results below.

Studies on LQ control problems have a long history that can be traced back to the pioneering works of Kalman [12] and Wonham [22] (also refer to [5, 6, 9] and the references therein). A primary tool in solving LQ control problems is the following (stochastic algebraic) Riccati equation, with the symmetric matrix P being the unknown:

$$A^T P + PA + Q + C^T PC - (PB + C^T PD)(R + D^T PD)^{-1}(B^T P + D^T PC) = 0. \tag{3}$$

Suppose P^* is a solution of the above equation with $R + D^T P^* D \succ 0$ (positive definite). Then, following the standard LQ control theory, we know that

$$u^*(t) = -(R + D^T P^* D)^{-1}(B^T P^* + D^T P^* C)x^*(t) \tag{4}$$

is an optimal state feedback control for (LQ).

The limitation of the standard theory, however, lies in the difficulty involved in solving the Riccati equation (3), in particular since the matrix inverse term also involves the unknown P. In fact, there is no guarantee for $R + D^T P^* D \succ 0$, except when $Q \succeq 0$ (nonnegative definite) and $R \succ 0$, in which case $P^* \succeq 0$ follows as a solution to (3); see [1, Corollary 5.1].

In the deterministic case, $R \succeq 0$ is in fact necessary for the LQ problem to be well-posed (cf. [24, Chapter 6, Proposition 2.4]), whereas the Riccati approach further requires the nonsingularity of R. The positive definiteness of R has been a starting point of most of the stochastic

LQ literature; see [5, 6, 9, 22] and the references therein. Recent studies (e.g., [8]) have, however, made a case for studying LQ problems in which R is singular or even indefinite in the stochastic case. Relevant applications range from portfolio selection [25] and contingent claim [13] to pollution control [8]. In particular, a singular or indefinite R may naturally occur in a class of problems in which the control affects the diffusion part of the system dynamics (i.e., $D \neq 0$ in (2)). For instance, the trade of a large block of an asset may affect its volatility; production control may affect not only the mean but also the standard deviation of (future) demands, as the forecast error, measured by the coefficient of variation – the ratio of standard deviation to mean, stays constant.

With an indefinite R, the matrix inverse in (3), and hence the Riccati equation itself, may not be well-defined in general. To overcome this difficulty, our objective here is to use semidefinite programming (SDP), instead of the Riccati equation, as a unifying tool to solve the stochastic LQ control problem, generally in the absence of the positive/nonnegative definiteness of the cost matrices R and Q.

A related issue that has to be treated with care, due to the infinite time horizon of (LQ) in (1), is the mean-square stability of the constructed control. It turns out that this stability issue can be systematically treated through studying the dual SDP problem.

In a recent work, [1], the relationship between the Riccati equation in (3) and the associated linear matrix inequalities (the latter relates intimately to SDP; refer to [23]) has been extensively studied under the assumption that $R + D^T PD \succ 0$ for some solution P of the Riccati equation (3). While this assumption is critical in the approach of [1], the validity of the assumption is hard to verify *a priori*, since P is unknown. Moreover, it can happen that when this assumption is not satisfied the corresponding LQ problem still possesses an optimal control; see our Example 6.1 below.

In contrast, our focus here is to develop a direct connection between the LQ control problem and the SDP, without having to solve the Riccati equation or to assume that $R + D^T PD$ is non-singular. Indeed, our objective is to develop a general approach to LQ control via SDP, or *LQ control without Riccati*.

Briefly, the rest of the paper is organized as follows. In §2, we introduce several regularity conditions relating to stability, and present the preliminaries for SDP. Our main results are presented in the next two sections: We establish first in §3, the connection between stability and the dual SDP; and then in §4, several implication relations among the optimality of LQ control problem, the complementary duality of the SDP, and a generalized version of the Riccati equation (3) involving a "pseudo" matrix inverse. A synthesis of these results is presented in §5, where we also outline a numerical procedure that provides a thorough treatment to the

LQ control problem via SDP. Several examples illustrating some of the key technical points in our approach are presented in §6. For problems that do not possess an attainable optimal control, an ϵ-approximation scheme is developed in §7, which achieves asymptotic optimality. Brief conclusions are summarized in §8.

2 Preliminaries

2.1 Regularity Conditions

Definition 2.1 (i) An open-loop control $u(\cdot)$ is called (*mean-square*) *stabilizing* at x_0, if the corresponding state $x(\cdot)$ of (2) with the initial state x_0 satisfies $\lim_{t\to+\infty} \mathsf{E}[x(t)^T x(t)] = 0$.

(ii) A feedback control $u(t) = Kx(t)$, where K is a constant matrix, is called *stabilizing*, if for every initial state x_0, the solution to the following equation

$$\begin{cases} dx(t) = (A + BK)x(t)dt + (C + DK)x(t)dW(t), \\ x(0) = x_0, \end{cases}$$

satisfies $\lim_{t\to+\infty} \mathsf{E}[x(t)^T x(t)] = 0$.

(iii) The system in (2) is called (mean-square) *stablizable* if there exists a stabilizing feedback control of the form $u(t) = Kx(t)$ where K is a constant matrix.

Definition 2.2 (i) An open-loop control $u(\cdot) \in L^2_{\mathcal{F}}(\Re^m)$ is called admissible (at x_0) if it is stabilizing at x_0. The set of all admissible controls at x_0 is denoted as $U_{ad}^{x_0}$.

(ii) An admissible pair $(x^*(\cdot), u^*(\cdot))$ is called *optimal* (at x_0) if $u^*(\cdot)$ achieves the infimum of $J(x_0, u(\cdot))$ over $u(\cdot) \in U_{ad}^{x_0}$.

(iii) The control problem (LQ) is called *well-posed* (at x_0) if

$$-\infty < \inf_{u(\cdot)\in U_{ad}^{x_0}} J(x_0, u(\cdot)) < +\infty;$$

(LQ) is called *attainable* (at x_0) if it is well-posed (at x_0) and there exists an optimal admissible control.

Unless explicitly stated otherwise, we shall assume throughout the paper that the system under consideration, (2), is mean-square stabilizable. Note that this is a very mild regularity condition; in particular, it is implied by the well-posedness of (LQ) when $Q \succ 0$ and $R \succeq 0$. Indeed, the well-posedness of (LQ) yields that there is at least one control whose cost is finite. As a result, under that control, $\lim_{t\to+\infty} \mathsf{E}[x(t)^T Q x(t)] = 0$. Thus the control must be stabilizing due to the nonsingularity of Q.

On the other hand, to appreciate why the admissible controls have to be stabilizing, consider in the case when $Q \succ 0$ and $R \succeq 0$. In order for the cost objective in (1) to be finite it is necessary (as shown above) that the corresponding control must be stabilizing. In general, a non-stabilizing control is not manageable even if its performance is good. Hence, it is better to exclude these ill-behaved controls from consideration.

Attainability of an optimal admissible control is another issue, even when the problem is well-posed. Consider the following simple (deterministic) LQ problem:

$$\begin{array}{ll} \min & \int_0^\infty x(t)^2 dt \\ \text{s.t.} & dx(t) = [-x(t) + u(t)]dt \\ & x(0) = 1. \end{array}$$

Clearly, this problem has an infimum value zero. However, there is no control that attains the zero cost. In general, deciding whether or not a problem has an attainable optimal control is as hard as solving the (LQ) problem, especially for problems with large sizes. A systematic approach is developed below, based on SDP.

2.2 Semidefinite Programming

Semidefinite programming (SDP) is a newly developed tool in optimization following the seminal work of Nesterov and Nemirovski [17]; see also Alizadeh [3]. For a general introduction to *linear matrix inequalities* (LMIs) and SDP, see e.g. [7, 23]. We also refer to [19] and the references therein for an extensive coverage on the (primal-dual) interior-point approach to SDP, a major computational method available for solving SDPs.

SDP is a special form of the so-called *conic optimization problem*, which is in essence to optimize a linear function over the intersection of two closed convex sets: one being an affine subspace and the other being a cone. The type of conic optimization is characterized by the choice of the cone. In SDP, we use the cone formed by positive semidefinite matrices in the linear space of symmetric matrices. Like a linear programming problem, an SDP problem can be cast in various ways. In its standard form, an SDP is the following convex optimization problem:

$$\begin{array}{lll} (SDP)_p & \min & \langle C, X \rangle \\ & \text{s.t.} & \langle A_i, X \rangle = b_i, \ \text{for } i = 1, ..., m \\ & & X \succeq 0 \end{array}$$

where C and A_i are $n \times n$ symmetric matrices, $b_i \in \Re^m$ is a vector, $i = 1, ..., m$, and $\langle X, Y \rangle := \sum_{i,j} X_{ij} Y_{ij}$ denotes the matrix inner-product.

The above SDP has an associated dual problem, which is an SDP problem itself:

$$\begin{array}{lll} (SDP)_d & \max & b^T y \\ & \text{s.t.} & \sum_{i=1}^m y_i A_i + Z = C \\ & & Z \succeq 0. \end{array}$$

A conic optimization problem is said to satisfy the *Slater condition* if its feasible region has a non-empty intersection with the interior of the cone. In our standard primal/dual SDP problems stated above, the Slater condition has the following characterization. For the primal problem $(SDP)_p$, the Slater condition is equivalent to the existence of a primal feasible solution X^0 such that $X^0 \succ 0$. Similarly, for the dual problem $(SDP)_d$, the Slater condition is to say that there is a dual feasible solution (y^0, Z^0) with $Z^0 \succ 0$.

For conic optimization problems, a well-developed duality theory exists; see e.g. [23, 15, 17]. Key points of the theory can be highlighted as follows:

- The *weak duality* always holds, i.e. any feasible solution to the primal (minimization) problem always possesses an objective value that is no less than the (dual) objective value of any dual feasible solution (the dual being a maximization problem).

- In contrast, the *strong duality* — that the optimal values of the primal and dual problems coincide — holds if there exists a pair of *complementary optimal solutions* X^* and (y^*, Z^*), namely, they satisfy $X^*Z^* = 0$. If, furthermore, it holds that $X^* + Z^* \succ 0$, then this pair of optimal solutions is called *strictly complementary*.

- Unlike linear programming, SDP may fail to satisfy the strong duality, let alone strict complementarity. It is known, however, that if the primal problem is feasible and the dual satisfies the Slater condition, then the primal problem must have a non-empty and compact optimal solution set. Moreover, if both the primal and the dual satisfy the Slater condition, then both problems must have non-empty and compact optimal solution sets, and the strong duality holds.

In the SDP literature, the Slater type regularity conditions are mostly assumed in order to avoid pathological cases. In Luo, Sturm and Zhang [15, 16], extensive analysis can be found addressing the issue of regularity in the context of duality theory, and the related issue of how to detect the duality status numerically.

The linkage between the LQ control problem and the SDP is best understood in the deterministic setting. Consider the deterministic version of (LQ) as follows:

$$
\begin{array}{ll}
\min & \int_0^\infty [x(t)^T Q x(t) + u(t)^T R u(t)] dt \\
\text{s.t.} & \dot{x}(t) = Ax(t) + Bu(t) \\
 & x(0) = x_0 \in \Re^n.
\end{array}
$$

Assume $Q \succeq 0$ and $R \succ 0$. Then, the optimal solution to the above problem is:

$$
u^*(t) = -R^{-1} B^T P^* x^*(t),
$$

where P^* is the solution to the Riccati equation:

$$Q + A^T P + PA - PBR^{-1}B^T P = 0.$$

It turns out that solutions to the Riccati equation can be found through solving the following SDP:

$$\begin{array}{ll} \max & \langle I, P\rangle \\ \text{s.t.} & \begin{bmatrix} R, & B^T P \\ PB, & Q + A^T P + PA \end{bmatrix} \succeq 0 \\ & P \in \mathcal{S}^{n\times n}, \end{array}$$

where $\mathcal{S}^{n\times n}$ denotes the space of n by n symmetric matrices. Note that the above is an SDP problem because the constraint set can be viewed as an intersection between a linear subspace and the cone of positive semidefinite matrices. The dual of this SDP problem is:

$$\begin{array}{ll} \min & \langle R, Z_b\rangle + \langle Q, Z_n\rangle \\ \text{s.t.} & I + Z_u^T B^T + BZ_u + Z_n A^T + AZ_n = 0 \\ & Z := \begin{bmatrix} Z_b, & Z_u \\ Z_u^T, & Z_n \end{bmatrix} \succeq 0. \end{array}$$

It is interesting to note that both the primal and dual SDP's above are well-defined even in the presence of the singularity of R and Q. In particular, there is no matrix inverse involved. This is the starting point of our approach developed below.

Along a similar line, the primal and dual SDP's for the stochastic problem in (LQ) can be written as follows:

$$\begin{array}{lll} \text{(P)} & \max & \langle I, P\rangle \\ & \text{s.t.} & \mathcal{L}(P) \succeq 0 \\ & & P \in \mathcal{S}^{n\times n}, \end{array}$$

where

$$\mathcal{L}(P) := \begin{bmatrix} R + D^T PD, & B^T P + D^T PC \\ PB + C^T PD, & Q + C^T PC + A^T P + PA \end{bmatrix}; \tag{5}$$

and

$$\begin{array}{lll} \text{(D)} & \min & \langle R, Z_B\rangle + \langle Q, Z_N\rangle \\ & \text{s.t.} & I + Z_U^T B^T + BZ_U + Z_N A^T + AZ_N \\ & & +CZ_N C^T + DZ_U C^T + CZ_U^T D^T + DZ_B D^T = 0 \\ & & Z := \begin{bmatrix} Z_B, & Z_U \\ Z_U^T, & Z_N \end{bmatrix} \succeq 0. \end{array}$$

In the above forms, (P) and (D) are said to satisfy the Slater condition if there exist primal and dual feasible solutions P^0 and Z^0 such that $\mathcal{L}(P^0) \succ 0$ and $Z^0 \succ 0$, respectively. On the other hand, a primal optimal solution P^* and a dual optimal solution Z^* are called complementary optimal solutions if $\mathcal{L}(P^*)Z^* = 0$. Furthermore, they are called strictly complementary if $\mathcal{L}(P^*) + Z^* \succ 0$.

As a least condition in order to apply the SDP approach, we assume throughout the paper that the feasible set of (P) is nonempty. This assumption is satisfied automatically if the Riccati equation (3) has a solution P^* with $R + D^T P^* D \succ 0$ (which is the key assumption in [1]) by virtue of the well-known Schur lemma. It is also satisfied when $Q \succeq 0$ and $R \succeq 0$. Therefore, without this assumption, the original LQ problem cannot be solved by either Riccati or SDP approach.

3 Stability

Since we require admissible controls to be stabilizing, we need first to address the stability issue, which, as it will become evident below, relates closely to the dual SDP.

The following result quoted from [1, Theorems 2.1, 5.2] will be used later.

Proposition 3.1 The following conditions are equivalent.

(i) System (2) is mean-square stabilizable.

(ii) Problem (D) satisfies the Slater condition.

(iii) There exists a matrix K and a symmetric matrix Y such that

$$(A + BK)Y + Y(A + BK)^T + (C + DK)Y(C + DK)^T \prec 0, Y \succ 0. \tag{6}$$

In this case the feedback $u(t) = Kx(t)$ is a stabilizing control.

(iv) There exists a matrix K such that for any X there exists a unique solution Y to the following equation

$$(A + BK)Y + Y(A + BK)^T + (C + DK)Y(C + DK)^T + X = 0. \tag{7}$$

Moreover, if $X \succ 0$ (resp. $X \succeq 0$) then $Y \succ 0$ (resp. $Y \succeq 0$). Furthermore, in this case the feedback $u(t) = Kx(t)$ is a stabilizing control.

(v) There exist a matrix X and a positive definite matrix $Y \succ 0$ such that

$$\begin{bmatrix} AY + YA^T + BX + X^T B^T & CY + DX \\ YC^T + X^T D^T & -Y \end{bmatrix} \prec 0. \tag{8}$$

In this case, the feedback $u(t) = XY^{-1}x(t)$ is a stabilizing control.

Note that the last equivalent condition above is an LMI condition, based on which the mean-square stabilizability can be verified numerically (cf. [7, 10]). Moreover, to check whether or not a given feedback control $u(t) = Kx(t)$ is stabilizing, it suffices to check if the LMI's in (6) have a feasible solution, which again can be carried out numerically.

Define the *stochastic Riccati operator*

$$F(P) := A^T P + PA + Q + C^T PC - (PB + C^T PD)(R + D^T PD)^+(B^T P + D^T PC). \quad (9)$$

Here, M^+ stands for the *pseudo-inverse* of a matrix M (refer to [18]). Note that when M is a positive semidefinite matrix, M^+ satisfies the following properties:

$$M^+ \succeq 0, \quad (M^+)^T = M^+, \quad M^+M = MM^+;$$

$$MM^+M = M, \qquad M^+MM^+ = M^+.$$

Clearly, the equation $F(P) = 0$ generalizes the classical Riccati equation (3), hence is called the *generalized Riccati equation.*

The following *extended Schur's lemma* [2] plays an important technical role.

Lemma 3.2 Let matrices $M = M^T, N$ and $S = S^T$ be given with appropriate dimensions. Then the following three conditions are equivalent:

(i) $M - NS^+N^T \succeq 0$, $S \succeq 0$, and $N(I - SS^+) = 0$.

(ii) $\begin{bmatrix} M & N \\ N^T & S \end{bmatrix} \succeq 0.$

(iii) $\begin{bmatrix} S & N^T \\ N & M \end{bmatrix} \succeq 0.$

Theorem 3.3 If a feasible solution of (P), P^*, is such that $F(P^*) = 0$, and the feedback control

$$u(t) = -(R + D^T P^* D)^+(B^T P^* + D^T P^* C)x(t) \quad (10)$$

is stabilizing, then there exist complementary optimal solutions of (P) and (D). In particular, P^* is optimal; and there exists a complementary dual optimal solution Z^*, such that $Z_N^* \succ 0$.

Proof. Denote $K := -(R + D^T P^* D)^+(B^T P^* + D^T P^* C)$. By the stability assumption of the control (10) and Proposition 3.1-(iv), the following equation

$$(A + BK)Y + Y(A + BK)^T + (C + DK)Y(C + DK)^T + I = 0$$

has a positive solution. Let it be $Y^* \succ 0$. Furthermore, let

$$Z_N^* = Y^*,\ Z_U^* = KZ_N^* \text{ and } Z_B^* = K(Z_U^*)^T. \tag{11}$$

By this construction, we can easily verify that

$$\begin{bmatrix} Z_B^*, & Z_U^* \\ (Z_U^*)^T, & Z_N^* \end{bmatrix} = \begin{bmatrix} I, & K \\ 0, & I \end{bmatrix} \begin{bmatrix} 0, & 0 \\ 0, & Z_N^* \end{bmatrix} \begin{bmatrix} I, & 0 \\ K^T, & I \end{bmatrix} \succeq 0.$$

Moreover, by direct verification, we know

$$I + (Z_U^*)^T B^T + BZ_U^* + Z_N^* A^T + AZ_N^* + CZ_N C^T + DZ_U^* C^T + C(Z_U^*)^T D^T + DZ_B^* D^T = 0.$$

Therefore, Z^* is a feasible solution of (D). Moreover, using extended Schur's lemma (Lemma 3.2), we have

$$\begin{aligned}
&\mathcal{L}(P^*) \begin{bmatrix} Z_B^*, & Z_U^* \\ (Z_U^*)^T, & Z_N^* \end{bmatrix} \\
= &\begin{bmatrix} I, & 0 \\ -K^T, & I \end{bmatrix} \begin{bmatrix} R + D^T P^* D, & 0 \\ 0, & F(P^*) \end{bmatrix} \begin{bmatrix} I, & -K \\ 0, & I \end{bmatrix} \begin{bmatrix} Z_B^*, & Z_U^* \\ (Z_U^*)^T, & Z_N^* \end{bmatrix} \\
= &\begin{bmatrix} I, & 0 \\ -K^T, & I \end{bmatrix} \begin{bmatrix} (R + D^T P^* D)(Z_B^* - K(Z_U^*)^T), & R(Z_U^* - KZ_N^*) \\ F(P^*)(Z_U^*)^T, & F(P^*)Z_N^* \end{bmatrix} \\
= &\begin{bmatrix} 0, & 0 \\ 0, & 0 \end{bmatrix}.
\end{aligned}$$

That is, P^* and Z^* are complementary solutions; in particular, P^* is optimal to (P) and Z^* is optimal to (D). The last statement of the theorem follows from the fact that $Z_N^* = Y^* \succ 0$. □

One should note that the assumption of the above theorem, namely, the control (10) is stabilizing, is *not* automatically valid, even in the case when $R + D^T P^* D \succ 0$; see Example 6.2.

The following result shows, however, that one can possibly obtain a stabilizing feedback control via a *dual* solution.

Theorem 3.4 Let $Z = \begin{bmatrix} Z_B, & Z_U \\ Z_U^T, & Z_N \end{bmatrix}$ be a feasible solution of (D) with $Z_N \succ 0$, then the feedback control $u(t) = Z_U Z_N^{-1} x(t)$ is stabilizing.

Proof. First of all, by feasibility of Z to (D) along with Schur's lemma we have

$$Z_B \succeq Z_U Z_N^{-1} Z_U^T.$$

Then,

$$\begin{aligned}
0 = \ & I + Z_U^T B^T + BZ_U + Z_N A^T + AZ_N + CZ_N C^T + DZ_U C^T + CZ_U^T D^T + DZ_B D^T \\
\succeq \ & I + Z_U^T B^T + BZ_U + Z_N A^T + AZ_N + CZ_N C^T + DZ_U C^T + CZ_U^T D^T + DZ_U Z_N^{-1} Z_U^T D^T \\
\succ \ & Z_U^T B^T + BZ_U + Z_N A^T + AZ_N + (CZ_N + DZ_U) Z_N^{-1} (Z_N C^T + Z_U^T D^T).
\end{aligned}$$

Applying Schur's lemma again, we conclude that Proposition 3.1-(v) holds with $X = Z_U$ and $Y = Z_N \succ 0$. Hence $u(t) = Z_U Z_N^{-1} x(t)$ is stabilizing. □

4 Optimality

Here we establish the relationship among the optimality of the original LQ problem, the primal/dual SDP problems, and the generalized Riccati equation.

Theorem 4.1 If (LQ) is attainable at any $x_0 \in \Re^n$, then (P) must have an optimal solution P^* satisfying $F(P^*) = 0$.

Proof. First, note that since (LQ) has a finite optimal value with respect to any initial value, it must have a quadratic representation

$$\inf_{u(\cdot)\in U_{ad}^{x_0}} J(x_0, u(\cdot)) = x_0^T M x_0, \quad \forall x_0 \in \Re^n;$$

see [4, p.21] (note that the proof there, which is for the deterministic case, readily extends to the stochastic case).

Suppose for the time being that the matrix M is a feasible solution of (P), the validity of which will be proved later. Let $u^*(\cdot)$ be the optimal control and $x^*(\cdot)$ be the corresponding state from the initial x_0. Then for any feasible solution P of (P), applying Itô's formula yields

$$\begin{aligned}
& d(x^*(t)^T P x^*(t)) \\
= & \Big[(Ax^*(t) + Bu^*(t))^T P x^*(t) + x^*(t)^T P (Ax^*(t) + Bu^*(t)) \\
& + (Cx^*(t) + Du^*(t))^T P (Cx^*(t) + Du^*(t))\Big] dt + \{...\} dW(t) \\
= & \Big[x^*(t)^T (A^T P + PA + C^T PC) x^*(t) + 2u^*(t)^T (B^T P + D^T PC) x^*(t) + u^*(t)^T D^T P D u^*(t)\Big] dt \\
& + \{...\} dW(t).
\end{aligned}$$

Integrating the above over $[0, \infty)$, taking expectations, and using the fact that $\mathsf{E}[x^*(t)^T P x^*(t)] \to 0$ (as $u^*(\cdot)$ is stabilizing), we obtain:

$$\begin{aligned}
0 = \; & x_0^T P x_0 + \mathsf{E} \textstyle\int_0^\infty [x^*(t)^T (A^T P + PA + C^T PC) x^*(t) + 2u^*(t)^T (B^T P + D^T PC) x^*(t) \\
& + u^*(t)^T D^T P D u^*(t)] dt.
\end{aligned} \tag{12}$$

Completion of square yields:

$$\begin{aligned}
& J(x_0, u^*(\cdot)) \\
= \; & \mathsf{E} \int_0^\infty [x^*(t)^T Q x^*(t) + u^*(t)^T R u^*(t)] dt \\
= \; & x_0^T P x_0 + \\
& \mathsf{E} \int_0^\infty \Big\{[u^*(t) - Kx^*(t)]^T (R + D^T P D) [u^*(t) - Kx^*(t)] + x^*(t)^T F(P) x^*(t)\Big\} dt,
\end{aligned} \tag{13}$$

where $K := -(R+D^TPD)^+(B^TP+D^TPC)$. Since P is feasible to (P), we have $R+D^TPD \succeq 0$ and $F(P) \succeq 0$ by extend Schur's lemma. This means that

$$x_0^T M x_0 \equiv J(x_0, u^*(\cdot)) \geq x_0^T P x_0, \tag{14}$$

for any P feasible to (P). Hence M must be optimal to (P). On the other hand, taking $P = M$ in (13) and noting $J(x_0, u^*(\cdot)) = x_0^T M x_0$, we conclude that $x(t)^T F(M) x(t) = 0$ for all $t \in [0, \infty)$. Since x_0 can be chosen arbitrarily it follows that $F(M) = 0$. The desired result thus follows.

What remains is to show the primal feasibility of M. To this end we consider a perturbation on the problem (LQ). That is, we keep all the data A, B, C and D unchanged, and let $R_\epsilon = R + \epsilon I$ and $Q_\epsilon = Q + \epsilon I$ where $\epsilon > 0$ is a small positive number.

Under the perturbation ($\epsilon > 0$), the corresponding SDP's,

$$\begin{array}{lll} (P_\epsilon) & \max & \langle I, P\rangle \\ & \text{s.t.} & \begin{bmatrix} R+\epsilon I + D^TPD, & B^TP + D^TPC \\ PB + C^TPD, & Q + \epsilon I + C^TPC + A^TP + PA \end{bmatrix} \succeq 0 \\ & & P \in \mathcal{S}^{n\times n} \end{array}$$

and

$$\begin{array}{lll} (D_\epsilon) & \min & \langle R+\epsilon I, Z_B\rangle + \langle Q+\epsilon I, Z_N\rangle \\ & \text{s.t.} & I + Z_U^TB^T + BZ_U + Z_NA^T + AZ_N \\ & & \quad +CZ_NC^T + DZ_UC^T + CZ_U^TD^T + DZ_BD^T = 0 \\ & & \begin{bmatrix} Z_B, & Z_U \\ Z_U^T, & Z_N \end{bmatrix} \succeq 0 \end{array}$$

both satisfy the Slater condition (the former does because the feasible set of (P) is assumed to be nonempty, and the latter because of the mean-square stabilizability assumption and Proposition 3.1-(ii)), and therefore complementary optimal solutions exist ([23]). Observe that the feasible set of (D_ϵ) is independent of ϵ. Take any dual feasible solution Z^0. By weak duality, we have

$$\text{tr } P = \langle I, P\rangle \leq \langle R+\epsilon I, Z_B^0\rangle + \langle Q+\epsilon I, Z_N^0\rangle. \tag{15}$$

Let $\hat{P}$ be a feasible solution of (P), which exists by our assumption. Certainly, $\hat{P}$ is feasible to (P_ϵ) for all $\epsilon \geq 0$. Theorem 5.5 in [1] asserts that for $\epsilon > 0$, the unique optimal solution for (P_ϵ), denoted by P_ϵ^*, dominates any other feasible solutions. Hence, $\hat{P} \preceq P_\epsilon^*$ for all $\epsilon > 0$.

This, together with (15), implies in particular that P_ϵ^* are contained in a compact set, with $0 \leq \epsilon \leq \epsilon_0$ ($\epsilon_0 > 0$ is a pre-determined constant). Now, take a convergent subsequence such that

$$\lim_{i\to\infty} P_{\epsilon_i}^* = P_0^*$$

with $\epsilon_i \to 0$ as $i \to \infty$.

Clearly, P_0^* is a feasible solution of (P) since the feasible region of (P_ϵ) monotonically shrinks as $\epsilon \downarrow 0$. We now show that $P_0^* = M$. Define the perturbed cost function

$$J_\epsilon(x_0, u(\cdot)) = \mathsf{E} \int_0^\infty [x(t)^T Q_\epsilon x(t) + u(t)^T R_\epsilon u(t)] dt.$$

Similar to (13), we can show

$$\begin{aligned} J_\epsilon(x_0, u(\cdot)) = x_0^T P_\epsilon^* x_0 + \quad & \mathsf{E} \textstyle\int_0^\infty \{[u(t) - K_\epsilon x(t)]^T (R_\epsilon + D^T P_\epsilon^* D)[u(t) - K_\epsilon x(t)] \\ & + x(t)^T F_\epsilon(P_\epsilon^*) x(t)\} dt, \end{aligned}$$

for any $u(\cdot) \in U_{ad}^{x_0}$, where $K_\epsilon := -(R_\epsilon + D^T P_\epsilon^* D)^+ (B^T P_\epsilon^* + D^T P_\epsilon C)$ and F_ϵ is the "perturbed" Riccati operator with Q and R in (9) replaced by Q_ϵ and R_ϵ, respectively. In [1], Theorems 5.4 and 5.5, it is stated that if the problem (P_ϵ) is strictly feasible (which is the case here) with an optimal solution P_ϵ^*, then the corresponding feedback control $u(t) = K_\epsilon x(t)$ must be stabilizing (and hence admissible), and $P_{\epsilon_i}^*$ must be the unique optimal solution to (P_ϵ) satisfying the corresponding Riccati equation $F_\epsilon(P) = 0$. Thus $u(t) = K_\epsilon x(t)$ must be optimal, and

$$\inf_{u(\cdot) \in U_{ad}^{x_0}} J_\epsilon(x_0, u(\cdot)) = x_0^T P_\epsilon^* x_0,$$

which further yields

$$x_0^T P_{\epsilon_i}^* x_0 = \inf_{u(\cdot) \in U_{ad}^{x_0}} J_{\epsilon_i}(x_0, u(\cdot)) \geq \inf_{u(\cdot) \in U_{ad}^{x_0}} J(x_0, u(\cdot)) = x_0^T M x_0.$$

Letting $\epsilon_i \to 0$, we obtain

$$x_0^T P_0^* x_0 \geq x_0^T M x_0.$$

On the other hand, (14) gives rise to the opposite inequality since P_0^* is feasible to (P). Thus we have $M = P_0^*$. This establishes that M is indeed a primal feasible solution. $\square$

An important implication, which is the contrapositive of the above theorem is this: If (P) has no optimal solution, *or*, if (P) has optimal solutions but none of them satisfies the generalized Riccati equation $F(P) = 0$, then the LQ control problem (LQ) is *not* attainable at some x_0; in particular, it does not have any optimal feedback control. Note the condition of this statement can be easily verified numerically.

A natural question to ask at this point is whether or not the converse of the above statement is true. Namely, if (P) admits an optimal solution P^* satisfying $F(P^*) = 0$, then does (LQ) have an attainable optimal control? Recall that in the finite horizon case an optimal feedback control is represented as $u^*(t) = -(R + D^T P^* D)^{-1} (B^T P^* + D^T P^* C) x^*(t)$ (cf. [8, Theorem

3.2]). However in the present case $R+D^TP^*D$ may be singular, therefore we naturally expect that an optimal feedback control should be

$$u^*(t) = -(R+D^TP^*D)^+(B^TP^*+D^TP^*C)x^*(t). \tag{16}$$

The following result establishes that this control is indeed optimal if it is stabilizing. (Note that in the infinite horizon case, stability is needed.)

Theorem 4.2 If a feasible solution of (P), P^*, is such that $F(P^*) = 0$, and the feedback control $u^*(t)$ in (16) is stabilizing, then it must be optimal for (LQ).

Proof. For any admissible control $u(\cdot) \in U_{ad}^{x_0}$, an argument similar to that in proving (13) leads to

$$\begin{aligned}
& J(x_0, u(\cdot)) \\
= \; & \mathsf{E}\int_0^\infty [x(t)^TQx(t) + u(t)^TRu(t)]dt \\
= \; & x_0^TP^*x_0 \\
& +\mathsf{E}\int_0^\infty \left\{[u(t)-Kx(t)]^T(R+D^TP^*D)[u(t)-Kx(t)] + x(t)^TF(P^*)x(t)\right\} dt \\
= \; & x_0^TP^*x_0 \\
& +\mathsf{E}\int_0^\infty [u(t)-Kx(t)]^T(R+D^TP^*D)[u(t)-Kx(t)]dt,
\end{aligned} \tag{17}$$

where $K := -(R+D^TP^*D)^+(B^TP^*+D^TP^*C)$. Since $u^*(t) = Kx^*(t)$ is stabilizing, the above shows that $u^*(\cdot)$ must be optimal. □

As mentioned the stability of the control in (16) can be examined numerically via the LMIs as stipulated in Proposition 3.1-(iii). On the other hand, by Theorem 3.3, in order for this control to be stabilizing, it is *necessary* that there exist complementary solutions P^* and Z^* of (P) and (D), respectively, with $Z_N^* \succ 0$. It is interesting that under these (weaker) conditions, one can prove the existence of an explicitly representable optimal feedback control of (LQ) in (20) below (which is not necessarily in the same form as the control in (16)).

First we need a lemma.

Lemma 4.3 Suppose (P) and (D) have complementary optimal solutions, P^* and Z^*, respectively. Then, $F(P^*) = 0$.

Proof. We have the following decomposition:

$$\mathcal{L}(P^*) = \begin{bmatrix} I, & 0 \\ -K^T, & I \end{bmatrix} \begin{bmatrix} R+D^TP^*D, & 0 \\ 0, & F(P^*) \end{bmatrix} \begin{bmatrix} I, & -K \\ 0, & I \end{bmatrix}, \tag{18}$$

where $K := -(R + D^T P^* D)^+(B^T P^* + D^T P^* C)$. From the relation $\mathcal{L}(P^*)Z^* = 0$, it follows that

$$
\begin{aligned}
& \begin{bmatrix} R + D^T P^* D, & 0 \\ 0, & F(P^*) \end{bmatrix} \begin{bmatrix} I, & -K \\ 0, & I \end{bmatrix} \begin{bmatrix} Z_B^*, & Z_U^* \\ (Z_U^*)^T, & Z_N^* \end{bmatrix} \\
= & \begin{bmatrix} (R + D^T P^* D)(Z_B^* - K(Z_U^*)^T), & (R + D^T P^* D)(Z_U^* - K Z_N^*) \\ F(P^*)(Z_U^*)^T, & F(P^*) Z_N^* \end{bmatrix} \\
= & \begin{bmatrix} 0, & 0 \\ 0, & 0 \end{bmatrix}.
\end{aligned} \tag{19}
$$

Therefore,

$$
F(P^*)(Z_U^*)^T = 0, \qquad F(P^*) Z_N^* = 0
$$

and

$$
Z_U^* F(P^*) = 0, \qquad Z_N^* F(P^*) = 0.
$$

On the other hand, the dual nonnegativity constraint, together with the extended Schur's lemma, yields

$$
Z_B^* - Z_U^*(Z_N^*)^+(Z_U^*)^T \succeq 0,
$$

and

$$
Z_U^*(I - Z_N^*(Z_N^*)^+) = 0.
$$

Multiplying $F(P^*)$ on both sides of the dual equality constraint and making use of the above relations, we obtain

$$
\begin{aligned}
0 &= F(P^*)(I + C Z_N^* C^T + D Z_U^* C^T + C^T Z_U^* D + D Z_B^* D^T) F(P^*) \\
&\succeq F(P^*)(I + C Z_N^* C^T + D Z_U^* C^T + C^T Z_U^* D + D Z_U^*(Z_N^*)^+(Z_U^*)^T D^T) F(P^*) \\
&= F(P^*)^2 + F(P^*)[C Z_N^* + D Z_U^*](Z_N^*)^+[Z_N^* C^T + (Z_U^*)^T D^T] F(P^*) \\
&\succeq F(P^*)^2.
\end{aligned}
$$

This means, $F(P^*) = 0$. $\square$

Theorem 4.4 Assume that solving (P) and (D) yields complementary optimal solutions P^* and Z^*, with $Z_N^* \succ 0$. Then $F(P^*) = 0$, and (LQ) has an attainable optimal feedback control given by

$$
u^*(t) = Z_U^*(Z_N^*)^{-1} x^*(t). \tag{20}
$$

Proof. First, that $F(P^*) = 0$ is seen from Lemma 4.3 (even without the assumption $Z_N^* \succ 0$).

Next, for any feasible solution P of (P) and any (stabilizing) control $u(\cdot) \in U_{ad}^{x_0}$, along with the corresponding state $x(\cdot)$, an argument similar to the one that proved (13) leads to

$$
\begin{array}{rl}
& J(x_0, u(\cdot)) \\
= & \mathsf{E}\int_0^\infty [x(t)^T Q x(t) + u(t)^T R u(t)] dt \\
= & x_0^T P x_0 \\
& +\mathsf{E}\int_0^\infty \left\{ [u(t) - Kx(t)]^T (R + D^T P D)[u(t) - Kx(t)] + x(t)^T F(P) x(t) \right\} dt,
\end{array}
\tag{21}
$$

where $K := -(R + D^T P D)^+ (B^T P + D^T P C)$. Since P is feasible to (P), we have $F(P) \succeq 0$. This means that

$$
J(x_0, u(\cdot)) \geq x_0^T P x_0, \ \forall u(\cdot) \in U_{ad}^{x_0},
$$

for any P feasible to (P).

Note that at this point we cannot conclude that the control in (16) is optimal, as we do not know whether or not it is stabilizing, whereas (21) holds only for stabilizing controls. To get around, let us define a feedback control $u^*(t) = Z_U^* (Z_N^*)^{-1} x^*(t)$, which, following Theorem 3.4, is stabilizing. Hence (21) with $u(\cdot) = u^*(\cdot)$ and $P = P^*$ yields

$$
J(x_0, u^*(\cdot)) = x_0^T P^* x_0 + \mathsf{E}\int_0^\infty [u^*(t) - K^* x^*(t)]^T (R + D^T P^* D)[u^*(t) - K^* x^*(t)] dt, \tag{22}
$$

with $K^* := -(R + D^T P^* D)^+ (B^T P^* + D^T P^* C)$. Next, we show that

$$
\begin{array}{rl}
& [u^*(t) - K^* x^*(t)]^T (R + D^T P^* D)[u^*(t) - K^* x^*(t)] \\
\equiv & [u^*(t) - Z_U^* (Z_N^*)^{-1} x^*(t)]^T (R + D^T P^* D)[u^*(t) - Z_U^* (Z_N^*)^{-1} x^*(t)] \\
= & 0.
\end{array}
\tag{23}
$$

To this end, apply the complementary duality. From the relation $\mathcal{L}(P^*) Z^* = 0$, it follows that

$$
\begin{array}{rl}
& \begin{bmatrix} R + D^T P^* D, & 0 \\ 0, & F(P^*) \end{bmatrix} \begin{bmatrix} I, & -K^* \\ 0, & I \end{bmatrix} \begin{bmatrix} Z_B^*, & Z_U^* \\ (Z_U^*)^T, & Z_N^* \end{bmatrix} \\
= & \begin{bmatrix} (R + D^T P^* D)(Z_B^* - K^* (Z_U^*)^T), & (R + D^T P^* D)(Z_U^* - K^* Z_N^*) \\ F(P^*)(Z_U^*)^T, & F(P^*) Z_N^* \end{bmatrix} \\
= & \begin{bmatrix} (R + D^T P^* D) Z_B^* + (B^T P^* + D^T P^* C)(Z_U^*)^T, & (R + D^T P^* D) Z_U^* + (B^T P^* + D^T P^* C) Z_N^* \\ F(P^*)(Z_U^*)^T, & F(P^*) Z_N^* \end{bmatrix} \\
= & \begin{bmatrix} 0, & 0 \\ 0, & 0 \end{bmatrix}.
\end{array}
$$

Therefore,

$$
(R + D^T P^* D) Z_U^* = -(B^T P^* + D^T P^* C) Z_N^*. \tag{24}
$$

Repeatedly using the above identity, we get

$$\begin{aligned}
0 &= [u^*(t) - Z_U^*(Z_N^*)^{-1}x^*(t)]^T(R + D^TP^*D)[u^*(t) - Z_U^*(Z_N^*)^{-1}x^*(t)] \\
&= u^*(t)^T(R + D^TP^*D)u^*(t) - 2u^*(t)^T(R + D^TP^*D)Z_U^*(Z_N^*)^{-1}x^*(t) \\
&\quad +x^*(t)^T(Z_N^*)^{-1}(Z_U^*)^T(R + D^TP^*D)Z_U^*(Z_N^*)^{-1}x^*(t) \\
&= u^*(t)^T(R + D^TP^*D)u^*(t) + 2u^*(t)^T(B^TP^* + D^TP^*C)x^*(t) \\
&\quad +x^*(t)^T(Z_N^*)^{-1}(Z_U^*)^T(R + D^TP^*D)(R + D^TP^*D)^+(R + D^TP^*D)Z_U^*(Z_N^*)^{-1}x^*(t) \\
&= u^*(t)^T(R + D^TP^*D)u^*(t) + 2u^*(t)^T(B^TP^* + D^TP^*C)x^*(t) \\
&\quad +x^*(t)^T(P^*B + C^TP^*D)(R + D^TP^*D)^+(B^TP^* + D^TP^*C)x^*(t) \\
&= [u^*(t) - K^*x^*(t)]^T(R + D^TP^*D)[u^*(t) - K^*x^*(t)].
\end{aligned}$$

This proves (23). It then follows from (21) and (22) that

$$J(x_0, u^*(\cdot)) = x_0^TP^*x_0 \leq J(x_0, u(\cdot)), \qquad \forall u(\cdot) \in U_{ad}^{x_0}.$$

Hence, $u^*(\cdot)$ is optimal. □

A sufficient condition for Theorem 4.4 to hold is the *strict complementarity* of the SDP's (P) and (D), as the following proposition asserts.

Proposition 4.5 Suppose that (P) and (D) have strictly complementary optimal solutions P^* and Z^* respectively, i.e. $\mathcal{L}(P^*)Z^* = 0$ and $\mathcal{L}(P^*) + Z^* \succ 0$. Then, $Z_N^* \succ 0$.

Proof. Following Lemma 4.3 and (18) we have

$$\mathcal{L}(P^*) = (H^{-1})^T\text{diag}(R + D^TP^*D, 0)H^{-1}$$

where

$$H = \begin{bmatrix} I, & K \\ 0, & I \end{bmatrix} \text{ and } K = -(R + D^TP^*D)^+(B^TP^* + D^TP^*C).$$

Let $\bar{Z}^* := H^{-1}Z^*(H^{-1})^T$.

It is readily seen that $\bar{Z}_N^* = Z_N^*$. Moreover, diag $(R + D^TP^*D, 0) = H^T\mathcal{L}(P^*)H$ and $\bar{Z}^* = H^{-1}Z^*(H^{-1})^T$ are nonnegative definite and they are complementary to each other. We shall further show that they stay strictly complementary. To see this we first note that the range space of $\mathcal{L}(P^*)$, range$(\mathcal{L}(P^*))$, and the range space of Z^*, range(Z^*), form an orthogonal decomposition of the whole space. Clearly, range$(H^T\mathcal{L}(P^*)H)$ has the same dimension as that of range$(\mathcal{L}(P^*))$; and range$(\bar{Z}^*)$ has the same dimension as that of range(Z^*). Finally, range$(H^T\mathcal{L}(P^*)H)$ and range$(\bar{Z}^*)$ remain orthogonal to each other. Hence they span the whole space too. Therefore,

$$H^T\mathcal{L}(P^*)H + \bar{Z}^* \succ 0.$$

As we noted before, the last diagonal block of the above matrix is Z_N^*. Hence, $Z_N^* \succ 0$, as the proposition stipulates. $\square$

If P^* and Z^* are strictly complementary optimal solutions with

$$R + D^T P^* D \succ 0, \tag{25}$$

and $Z_N^* \succ 0$, then by (24), the feedback control in (20) coincides with the one in (10). But if $R + D^T P^* D$ is singular, then these two controls can indeed be *different*; see Example 6.1.

5 Synthesis

To summarize the results we have so far obtained, let us introduce the following statements:

(a) (LQ) is attainable at any $x_0 \in \Re^n$.

(b) (P) has an optimal solution P^* that satisfies:

- **(i)** the generalized stochastic Riccati equation $F(P) = 0$;
- **(ii)** the corresponding feedback control $u^*(t)$ of (16) is stabilizing.

(c) (P) and (D) have complementary optimal solutions P^* and Z^*, with $Z_N^* \succ 0$.

The following is a summary of our main results:

Theorem 5.1 The following implications hold:

- **(a)** $\Rightarrow$ **(b(i))**: Theorem 4.1.
- **(b)** $\Rightarrow$ **(a)**, with the control in (16) being optimal: Theorem 4.2.
- **(b)** $\Rightarrow$ **(c)**: Theorem 3.3.
- **(c)** $\Rightarrow$ **(a)**, with the control in (20) being optimal: Theorem 4.4.

Some remarks are in order. Among the above statements, **(a)** is a direct statement about the solution of the original problem (LQ); **(b)** and **(c)**, on the other hand, provide two computational approaches to (LQ) via SDP — note that they both imply **(a)** with the respective optimal feedback controls explicitly given. There are differences, however, between the two; in particular, they have different requirements, and lead to different controls. Computationally, **(c)** appears to have an edge over **(b)**, as most SDP solvers are based on primal-dual interior point methods. This implies that the iterative solutions produced by such a solver will likely to converge to the analytic centers of the primal and dual optimal sets respectively, which are known to be "maximally complementary" to each other. Therefore, if there is indeed any dual optimal solution with $Z_N^* \succ 0$, then the solver will return such a solution. In this respect, checking $Z_N^* \succ 0$ is much easier than verifying the stabilizing condition in **(b(ii))**.

Furthermore, Theorem 5.1 also reveals the relationship between **(b)** and **(c)**: **(b)** implies **(c)**, whereas **(c)** implies **(b(i))** via **(a)**. That **(c)** cannot imply the stabilizing condition in **(b(ii))** is in itself an interesting fact, which suggests that the two controls in (16) and (20) are in general intrinsically different. Even when the former is stabilizing, and hence both controls are optimal — since we then have **(b)** $\Rightarrow$ **(c)** $\Rightarrow$ **(a)** — they can still be different (except for the special case of (25), where **(b)** and **(c)** become equivalent); see Example 6.1.

If **(b(ii))** is known to be satisfied, then **(b)** is reduced to **(b(i))**. Consequently, **(a)**, **(b)** and **(c)** are equivalent. Hence we have the following result.

Corollary 5.2 Suppose that the control in (16) is stabilizing, Then, **(a)**, **(b)** and **(c)** are equivalent.

An important special case is when

$$Q \succ 0, \qquad R \succ 0. \tag{26}$$

In this case, (P) satisfies the Slater condition because $P = 0$ is a strictly feasible solution, and so does (D) because of the mean-square stabilizability assumption of the original LQ problem. Therefore, **(c)** holds. On the other hand, by [1], Corollary 5.1, the control in (16) must be stabilizing. Hence Corollary 5.2 yields that **(a)** and **(b)** must hold true as well.

Corollary 5.3 Assume that (26) holds. Then all the three statements **(a)**, **(b)**, and **(c)** are true.

Based on the results obtained earlier, it is possible to develop a computational procedure as follows to provide a complete treatment of the stochastic LQ control problem, and in particular to answer if the problem has an optimal feedback control representable as in (16) or in (20). The procedure only involves solving some LMIs, an SDP and its dual, for which numerical algorithms have been extensively developed (among others see [11, 20, 21]). Insofar as the complementary primal SDP solution satisfies the generalized Riccati equation (see Lemma 4.3), the procedure can also be viewed as a numerical approach to solving the Riccati equation.

Step 1. Check if the feasible set of (P) is nonempty (which is an LMI condition). If not, then stop: the LQ problem cannot be solved by either the SDP approach or by the Riccati equation; else continue.

Step 2. Check if (D) satisfies the Slater condition, which amounts to solving a system of strict LMIs. If not, then stop: the LQ problem is not mean-square stablizable according to Proposition 3.1-(ii) and hence ill-posed; else continue.

Step 3. At this point we know that (P) is feasible and (D) satisfies the Slater condition, and hence (P) has an optimal solution (see, e.g., [15, Theorem 5]). Check if there is any optimal solution of (P) that satisfies $F(P) = 0$. If not, then stop: the LQ problem has no attainable optimal feedback control according to Theorem 4.1; else continue.

Step 4. Check if the control in (16) is stabilizing (which can be checked by LMIs according to Proposition 3.1-(iii)). If yes, then stop: the control is optimal; else continue.

Step 5. Check if (P) and (D) have complementary optimal solutions P^* and Z^* with $Z_N^* \succ 0$. If yes, then stop: the control $u^*(t) = Z_U^*(Z_N^*)^{-1}x^*(t)$ is optimal; otherwise (LQ) cannot be solved by our SDP approach, neither can it be solved by any other existing method.

6 Examples

The first example below demonstrates how the LQ control problem can be solved by the SDP approach developed here, even in the presence of the singularity of $R + D^T P^* D$. It also shows that optimal stabilizing controls can be obtained by both SDP approaches in **(b)** and **(c)** of Theorem 5.1, leading to different optimal controls in (16) and (20), respectively.

Example 6.1 Let $m = n = 1$; $A = C = -1$, $B = D = 1$; $Q = 1$ and $R = -1$. Namely, the problem is this:

$$\begin{array}{ll} \min & \mathsf{E}\int_0^\infty [x(t)^2 - u(t)^2]dt \\ \text{s.t.} & dx(t) = [-x(t) + u(t)]dt + [-x(t) + u(t)]dW(t) \\ & x(0) = x_0. \end{array}$$

This system is mean-square stabilizable, as $u(t) = \alpha x(t)$ is stabilizing for any α with $|\alpha| < 1$. To see this, applying Itô's formula to the system (1) under the above feedback control, we obtain

$$d\mathsf{E}[x(t)^2] = (\alpha^2 - 1)\mathsf{E}[x(t)^2]dt, \quad \mathsf{E}[x(0)]^2 = x_0^2.$$

Hence

$$\mathsf{E}[x(t)^2] = e^{(\alpha^2-1)t}x_0^2, \tag{27}$$

which converges to 0 as $t \to +\infty$.

Now, the primal SDP is

$$\begin{array}{ll} \max & p \\ \text{s.t.} & \begin{bmatrix} -1+p, & 0 \\ 0, & 1-p \end{bmatrix} \succeq 0. \end{array}$$

The above has an optimal solution $p^* = 1$ (the only feasible solution), which also satisfies the generalized Riccati equation $F(p) = 1 - p = 0$. (Note that singularity occurs in this solution). The feedback control given by (16) reduces to $u^*(t) = 0$, which is stabilizing as shown above ($\alpha = 0 < 1$). Therefore, all the tests in Steps 1–4 of the computational procedure presented in §5 are passed. Consequently, $u^*(t) = 0$ is *one* optimal control of the LQ problem. Moreover, the corresponding objective value is

$$\mathsf{E}\int_0^\infty [x^*(t)]^2 dt = x_0^2 \int_0^\infty e^{-t} dt = x_0^2, \tag{28}$$

where the first equality is due to (27) with $\alpha = 0$.

Next, we can obtain additional — in fact, infinitely many more — optimal controls by virtue of **(c)**. Indeed, the dual SDP in this case is:

$$\begin{array}{ll} \min & -z_b + z_n \\ \text{s.t.} & 1 + z_b - z_n = 0, \\ & z := \begin{bmatrix} z_b, & z_u \\ z_u, & z_n \end{bmatrix} \succeq 0. \end{array}$$

It can be directly verified that the above has multiple optimal solutions:

$$(z_b, z_u, z_n) = (z_b, z_u, 1 + z_b),$$

parameterized by (z_u, z_b) with

$$z_b \geq 0, \quad z_u^2 \leq z_b(1 + z_b). \tag{29}$$

In particular, note that the above ensures $z_n = 1 + z_b > 0$. Furthermore, these (parameterized) solutions are all complementary to the primal optimal solution $p^* = 0$. Hence, the test in Step 5 of the numerical procedure is passed, which gives rise to (multiple) optimal controls

$$u^*(t) = z_u z_n^{-1} x^*(t) \equiv \alpha x^*(t).$$

Notice that the feedback gain α satisfies

$$|\alpha| = \frac{z_u}{1 + z_b} \leq \frac{\sqrt{z_b(1 + z_b)}}{1 + z_b} = \sqrt{\frac{z_b}{1 + z_b}} < 1$$

where the first inequality follows from (29). Therefore, these controls are indeed stabilizing by (27). Finally, the optimal cost corresponding to these controls is:

$$
\begin{aligned}
J = &\ \mathsf{E}\int_0^\infty (1-\alpha^2)[x(t)]^2 dt \\
= &\ x_0^2(1-\alpha^2)\int_0^\infty e^{(\alpha^2-1)t}dt \\
= &\ x_0^2,
\end{aligned}
$$

which coincides with (28).

The next example illustrates two points: First, when the primal SDP solution satisfies the Riccati equation (i.e., **(b(i))** holds), and moreover (25) holds, the resulting feedback control may still not be stabilizing (i.e., **(b(ii))** fails). When this does happen, the complementarity condition in **(c)** fails. Second, there indeed exist well-posed stochastic control problems which, however, do not have any attainable optimal control. This calls for approximation methods, which will be presented in the next section.

Example 6.2 Suppose $m = n = 1$; $A = C = 0$, $B = D = 1$; $Q = 4$ and $R = -1$. The control system is as follows:

$$
\begin{array}{ll}
\min & \mathsf{E}\int_0^\infty [4x(t)^2 - u(t)^2]dt \\
\text{s.t.} & dx(t) = u(t)dt + u(t)dW(t) \\
& x(0) = x_0.
\end{array}
$$

Consider a feedback control $u(t) = -kx(t)$. Applying Itô's lemma yields

$$d[x(t)]^2 = (k^2 - 2k)[x(t)]^2 - 2k[x(t)]^2 dW(t).$$

Clearly, such a feedback control is stabilizing if and only if $k^2 - 2k < 0$, or, $0 < k < 2$. In particular, this implies that $u(t) = -x(t)$ is stabilizing while $u(t) = -2x(t)$ is not.

For any $0 < k < 2$, it follows that $\mathsf{E}\int_0^\infty [x(t)]^2 dt = x_0^2/(2k-k^2)$. Therefore, the control $u(t) = -kx(t)$ has a cost

$$(4-k^2)\mathsf{E}\int_0^\infty [x(t)]^2 dt = (1+\frac{2}{k})x_0^2.$$

As $k \uparrow 2$ we see that the cost can be arbitrarily close to $2x_0^2$. Nevertheless, this optimum is not attainable when $x_0 \neq 0$.

In terms of the corresponding SDPs, the primal reads

$$
\begin{array}{ll}
\max & p \\
\text{s.t.} & \begin{bmatrix} -1+p, & p \\ p, & 4 \end{bmatrix} \succeq 0.
\end{array}
$$

This problem has only one feasible solution $p^* = 2$, which is necessarily the optimal solution. It clearly satisfies the Riccati equation:

$$4 + \frac{p^2}{1-p} = 0.$$

The control in (16) is hence:

$$u^*(t) = -\frac{p^*}{-1+p^*}x^*(t) = -2x^*(t),$$

which is *not* stabilizing as we discussed before.

Since **(b(ii))** does not hold, we expect **(c)** to fail as well, as **(b)** and **(c)** are equivalent under (25). So, let us now examine the dual:

$$\begin{array}{ll} \min & 4z_n - z_b \\ \text{s.t.} & 1 + 2z_u + z_b = 0, \\ & z := \begin{bmatrix} z_b, & z_u \\ z_u, & z_n \end{bmatrix} \succeq 0. \end{array}$$

This problem is strictly feasible, since the original LQ problem is stabilizable. For instance, $(z_b, z_u, z_n) = (1, -1, 2)$ is a strictly feasible solution. Hence ([23, Theorem 3.1]), the infimum of the dual objective value must coincide with the supremum of the primal objective value, which is 2. This means, should the dual optimal solution z^* exist, it must satisfy $4z_n^* - z_b^* = 2$, or $z_n^* = (2 + z_b^*)/4$. This, along with $z_u^* = -(1 + z_b^*)/2$, leads to

$$z_b^* z_n^* = z_b^* \cdot \frac{2 + z_b^*}{4} < \frac{(1 + z_b^*)^2}{4} = (z_u^*)^2,$$

which violates $z \succeq 0$. Consequently, the dual does not have an attainable optimal solution, and the complementary duality fails.

The third example below illustrates a situation opposite to Example 6.2: **(b(i))** fails while **(b(ii))** holds. Namely, when $R + D^T P^* D = 0$, the optimal primal solution P^* may *not* satisfy the generalized Riccati equation $F(P) = 0$ (and *vice versa*), even when the corresponding control is stabilizing. In this case, Theorem 4.1 shows that the LQ problem has no attainable optimal control.

Example 6.3 Let $m = n = 1$; $A = -1$, $B = 1$, $C = D = 0$; $Q = 1$ and $R = 0$. This is actually a deterministic system that is mean-square stabilizable, as $u(t) = 0$ is stabilizing. The corresponding SDP reads:

$$\begin{array}{ll} \max & p \\ \text{s.t.} & \begin{bmatrix} 0, & p \\ p, & 1 - 2p \end{bmatrix} \succeq 0. \end{array}$$

The above has a unique feasible solution $p^* = 0$, which is hence optimal too. However, the generalized Riccati equation in this case is $F(p) = 1 - 2p = 0$, which has a unique solution $p = \frac{1}{2}$. Therefore, the two solutions are completely different.

Moreover, notice that while **(b(i))** fails in this case, **(b(ii))** does hold: $u^*(t) = 0$ is indeed stabilizing. On the other hand, in view of Theorem 5.1, **(c)** $\Rightarrow$ **(a)** $\Rightarrow$ **(b(i))**, we expect **(c)** to fail. Indeed, the dual SDP is

$$\begin{array}{ll} \min & z_n \\ \text{s.t.} & 1 + 2z_u - 2z_n = 0, \\ & z := \begin{bmatrix} z_b, & z_u \\ z_u, & z_n \end{bmatrix} \succeq 0. \end{array}$$

This dual problem is strictly feasible, and it has an infimum equal to 0, which is the supremum of the primal. However, the dual optimal solution is not attainable, because whenever $z_n = 0$ we must have $z_u = 1/2$, and hence it is impossible to have $z \succeq 0$.

7 ϵ-Approximation

Examples 6.2 and 6.3 have illustrated that the LQ control problem could be well-posed, but still there exists no attainable optimal control. When this happens we propose to consider (LQ$_\epsilon$), obtained by keeping all the data A, B, C and D in (LQ) unchanged, and letting $R_\epsilon = R + \epsilon I$ and $Q_\epsilon = Q + \epsilon I$ with $\epsilon > 0$; such a perturbation was already considered in the proof of Theorem 4.1. Recall that the associated SDPs for (LQ$_\epsilon$) are

$$\begin{array}{lll} (\mathrm{P}_\epsilon) & \max & \langle I, P \rangle \\ & \text{s.t.} & \begin{bmatrix} R + \epsilon I + D^T P D, & B^T P + D^T P C \\ PB + C^T P D, & Q + \epsilon I + C^T P C + A^T P + PA \end{bmatrix} \succeq 0 \\ & & P \in \mathcal{S}^{n \times n} \end{array}$$

and

$$\begin{array}{lll} (\mathrm{D}_\epsilon) & \min & \langle R + \epsilon I, Z_B \rangle + \langle Q + \epsilon I, Z_N \rangle \\ & \text{s.t.} & I + Z_U^T B^T + B Z_U + Z_N A^T + A Z_N \\ & & \quad + C Z_N C^T + D Z_U C^T + C Z_U^T D^T + D Z_B D^T = 0 \\ & & \begin{bmatrix} Z_B, & Z_U \\ Z_U^T, & Z_N \end{bmatrix} \succeq 0. \end{array}$$

Assuming that the LQ is stablizable and (P) is feasible, both (P$_\epsilon$) and (D$_\epsilon$) satisfy the Slater condition.

Theorem 7.1 Suppose (LQ) is well-posed. Let $J_\epsilon^*(x_0)$ and $J^*(x_0)$ be the optimal values of (LQ$_\epsilon$) and (LQ), respectively. Then,

$$\lim_{\epsilon \downarrow 0} J_\epsilon^*(x_0) = J^*(x_0).$$

Proof. Let the optimal solution of (P_ϵ) be P_ϵ^*. In the proof of Theorem 4.1, we proved that

$$u^\epsilon(t) = -(R_\epsilon + D^T P_\epsilon^* D)^+ (B^T P_\epsilon^* + D^T P_\epsilon C) x^\epsilon(t) \tag{30}$$

is optimal for (LQ_ϵ), with the corresponding optimal objective value equal to $J_\epsilon^*(x_0) = x_0^T P_\epsilon^* x_0$.

Following the same argument as in the proof of Theorem 4.1, we know that P_ϵ^* is contained in a compact set, with $0 < \epsilon \le \epsilon_0$. Moreover, since by definition $J_\epsilon^*(x_0)$ decreases monotonically as $\epsilon \downarrow 0$, so does P_ϵ^*. Therefore, P_ϵ^* itself also converges as $\epsilon \downarrow 0$.

What remains is to show that $x_0^T P_0^* x_0$ is equal to the true infimum of (LQ), now denoted as $J^*(x_0)$. To this end, first note that

$$x_0^T P_\epsilon^* x_0 = J_\epsilon^*(x_0) \ge J^*(x_0),$$

where the inequality is due to the positive perturbation in (P_ϵ). Letting $\epsilon \to 0$, we obtain

$$x_0^T P_0^* x_0 \ge J^*(x_0).$$

On the other hand, since P_0^* is feasible to (P) (see the proof of Theorem 4.1), it follows from (21) that

$$J^*(x_0) \equiv \inf_{u(\cdot) \in U_{ad}^{x_0}} J(x_0, u(\cdot)) \ge x_0^T P_0^* x_0.$$

This completes the proof. $\square$

The above theorem says that the objective value achieved by the perturbed problem is asymptotically optimal. The next result is concerned with the asymptotic optimality of the feedback control.

Theorem 7.2 The feedback control $u^\epsilon(\cdot)$ constructed by (30) is asymptotically optimal for (LQ), namely,

$$\lim_{\epsilon \downarrow 0} J(x_0, u^\epsilon(\cdot)) = J^*(x_0).$$

Proof. Denote by $J_\epsilon(x_0, u(\cdot))$ the cost of the perturbed problem (LQ_ϵ) under an admissible control $u(\cdot) \in U_{ad}^{x_0}$ w.r.t. the initial state x_0. Then for any $\eta > 0$, there is an ϵ_0 such that when $0 < \epsilon < \epsilon_0$:

$$\begin{aligned} J^*(x_0) &\le J(x_0, u^\epsilon(\cdot)) \\ &\le J_\epsilon(x_0, u^\epsilon(\cdot)) \\ &= J_\epsilon^*(x_0) \\ &\le J^*(x_0) + \eta, \end{aligned}$$

where the last inequality is due to Theorem 7.1. This proves our claim. □

Consider Example 6.2. With perturbation, the corresponding primal SDP becomes

$$\begin{array}{ll} \max & p \\ \text{s.t.} & \begin{bmatrix} -1+\epsilon+p, & p \\ p, & 4+\epsilon \end{bmatrix} \succeq 0. \end{array}$$

Solving this problem yields

$$p_\epsilon^* = \frac{4+\epsilon+\sqrt{(4+\epsilon)^2+4(4+\epsilon)(-1+\epsilon)}}{2}.$$

Clearly, $p_\epsilon^* = 2 + O(\sqrt{\epsilon})$, and hence the optimal value of (P_ϵ), $p_\epsilon^* x_0^2$, converges to $2x_0^2$ as $\epsilon \downarrow 0$.

8 Conclusions

We have developed a systematic approach to the stochastic LQ control problem based on SDP, allowing indefinite cost matrices. We have shown that, in addition to its obvious computational advantages, the SDP provides critical qualitative information about the LQ control problem, in particular, regarding issues such as stability and optimality.

Among the three statements presented in §5, the strongest is **(b)**, which consists of two parts: (i) the optimal solution to the primal SDP satisfies the generalized Riccati equation, and (ii) the corresponding feedback control is stabilizing. It implies **(c)**: the SDP complementary duality, which, in turn, implies **(a)**: the existence of an optimal control to the LQ problem. Both **(b)** and **(c)** hence provide useful computational approaches to solving the LQ problem. On the other hand, **(b(i))** is implied by **(a)**; and this provides a computationally viable means — via checking whether the primal optimal solution satisfies the generalized Riccati equation — to identify LQ problems that do not possess an optimal solution. Furthermore, for such problems we have developed an ϵ-approximation scheme that yields asymptotic optimal solutions.

It remains an open problem whether our SDP approach might actually fail to find an optimal control (a counter-example), or this is simply impossible (an assertive proof). This gap is reflected in the last step of the computational procedure outlined in §5.

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