1 Introduction

Empirical studies have emphasized the role played by social networks in communicating valuable information that is dispersed within the society (see e.g. Granovetter 1974, Rogers and Kincaid 1981, Coleman 1966, Rogers and Shoemaker 1971). The information may concern stock market tips, job openings, the quality of products ranging from cars to computers, and new medical advances, among other things.[1] While agents who participate in communication networks receive various kinds of benefits, they also incur costs in forming and maintaining links with other agents to obtain the benefits. Such costs could be in terms of time, effort and money.

In this paper, we present an approach to network formation based on the notion that social networks are formed by individual decisions that trade off the costs of forming links against the potential benefits of doing so. We suppose that once an agent i forms a link with another agent j they can both share information. One example of this type of link formation is a telephone call. The caller typically pays the telephone company, but both the parties, the caller and the receiver can share information. We suppose that a link with another agent allows access to the benefits available to the latter via his own links. Thus individual links generate externalities. A distinctive aspect of our model is that the costs of link formation are incurred only by the person who initiates the link. This allows us to formulate the network formation process as a non-cooperative game.[2]

We model the idea of imperfect reliability in terms of a positive probability that a link fails to transmit information. From an empirical point of view, imperfect reliability is a plausible assumption. Consider, for example, the network of people who are in contact via telephones. Suppose that agent i incurs a cost – in terms of effort – and calls agent j. It is quite possible that he may not get through to j because the latter may

[1]Boorman (1975) provides an early model of information flow in networks in the context of job search. Baker and Iyer (1991) analyze the impact of communication networks for stock market volatility, while Bala and Goyal (1998a) study information diffusion in fixed networks.

[2]The model is applicable in cases where links are durable, and must be established at the outset of the game by incurring a fixed cost of c. Once in place, the links provide a stochastic flow of benefits to the agents. This specification allows us to abstract from complex timing issues which would arise in a dynamic game of information sharing.

not be available at that time. In this setting, we examine the effect of imperfect reliability of links on the nature of stable and efficient networks. Our notion of stability is a non-cooperative one, and requires that agents play according to a Nash equilibrium.

As the topic exhibits significant analytic difficulty, we consider a relatively simple two-parameter model which attempts to capture the costs and benefits from link formation. Each agent is assumed to possess some information of value 1 to other agents, and a link between the agents allows this information to be transferred. Each link formed by an agent costs an amount $c > 0$. The reliability of a link is measured by a parameter $p \in (0, 1)$. Here, p is the probability that an established link between i and j "succeeds", i.e. allows information to flow between the agents, while $1 - p$ is the probability that it "fails". Moreover, link reliability across different pairs of agents is assumed to be independent. An agent's strategy is to choose the subset of agents with whom he forms links. The choices of all the agents specifies a directed network which permits information flows between them. As $p < 1$, the network formed by the agents choices is stochastic, since one or more links may fail. In a realization of the network, agent i obtains the information of all the agents with whom he has a path (i.e. either a link or a chain of links) in the realization. The agent's payoff is his expected benefit over all realizations less his costs of link formation.

Our findings on stability may be summarized as follows. If agents find it at all worthwhile to form links in a Nash network, then that network must be *connected* (Proposition 3.1). In other words, the network ensures that every agent obtains information with positive probability from every other agent. Our main result (Proposition 3.3) concerns the distinction between what we call *minimally connected* networks and *super-connected* networks. A minimally connected network is one with a unique path between any two agents. In such a network, every link is 'critical' to communication between the agents. Such a network imposes very strong demands upon reliability, because if even one link fails, there will be at least two agents who will not obtain information possessed by the other agent. In a super-connected network, on the other hand, every link is non-critical.[3] Proposition 3.3 shows that if $c < p$ then for large n, every Nash network will be super-connected.

[3]For example, a star network, where every agent communicates through a central agent, is minimally connected. A wheel network, where agents are arranged in a circle, is super-connected since the network remains connected, after any single link is deleted.

We also consider efficient networks. We find that if costs are very low or very high, or if links are highly reliable then there is virtually no conflict between efficiency and stability. However, for intermediate ranges of costs and link reliability, Nash networks may be "under-connected" relative to the social optimum. Note that links between agents create positive externalities for other agents. Hence if Nash networks are typically super-connected in large societies, this should be even more true for efficient networks. This is in fact the case: in Proposition 4.3 we prove that efficient networks are super-connected for a much larger range of cost values.

The present paper is part of the recent literature on network formation, see e.g., Bala and Goyal (1999), Dutta and Mutuswami (1997), Goyal (1993), Jackson and Wolinsky (1996) and Watts (1997). This work studies issues of *stability* and *efficiency* in network formation. Here stability refers to the networks which are consistent with agents' incentives – in terms of costs and benefits – to create links with other agents, while efficiency deals with the kinds of networks which maximize some measure of social surplus.

In this paper, we extend the model of network formation presented in Bala and Goyal (1999). In that paper, links are assumed to be perfectly reliable; in other words, the network formed is deterministic. We also analyzed the case of information decay: agents' payoffs decrease geometrically according to a parameter δ, in the distance to other agents. The main point of the present paper is that imperfect reliability has very different effects on network formation as compared to information decay. Specifically, in our earlier paper we showed that with information decay, minimally connected networks (notably the star) are Nash for a large range of c and δ, *independently* of the size of the society. By contrast, our results in the present paper show that minimally connected networks are increasingly replaced by super-connected networks as n increases. Thus, imperfect reliability creates radically different incentives for agents as compared to information decay. We also find that similar differences arise in the case of efficient networks.

Some of the previous work has also considered network reliability. Jackson and Wolinsky's paper provides a discussion of network formation when links formed by agents can break down with positive probability. In a broader perspective, Chwe (1995) presents a model of strategic reliability in communication. His approach defines communication protocols (which are somewhat related to networks since they allow for transmitting messages between agents) and he studies questions of incentive compatibility and efficiency of protocols in games of incomplete information. Our focus, on the other hand, is upon the properties of networks which arise endogeneously from agents' choices in a normal form game of information sharing.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 considers Nash networks, while Section 4 studies efficiency. Section 5 concludes.

2 The Model

Let $N = \{1, \ldots, n\}$ be a set of agents and let i and j be typical members of this set. We shall assume throughout that $n \geq 3$. Each agent is assumed to possess some information of value to other agents. He can augment his information by communicating with other people; this communication takes resources, time and effort and is made possible via the setting up of *pair-wise* links. Agents form links simultaneously in our model.[4] However, such links are not fully reliable: they can fail to transmit information with positive probability.

A *strategy* of agent $i \in N$ is a vector $g_i = (g_{i,1}, \ldots, g_{i,i-1}, g_{i,i+1}, \ldots, g_{i,n})$ where $g_{i,j} \in \{0, 1\}$ for each $j \in N \backslash \{i\}$. We say agent i has a *link* with j if $g_{i,j} = 1$. A link between agents i and j potentially allows for two-way (symmetric) flow of information. Throughout the paper we restrict our attention to pure strategies. The set of all strategies of agent i is denoted by $\mathcal{G}_i$. Since agent i has the option of forming or not forming a link with each of the remaining $n-1$ agents, the number of strategies of agent i is clearly $|\mathcal{G}_i| = 2^{n-1}$. The set $\mathcal{G} = \mathcal{G}_1 \times \cdots \times \mathcal{G}_n$ is the strategy space of all the agents. A strategy profile $g = (g_1, \ldots, g_n)$ can be represented as a *directed network*. Figure 1 below provides an example with $n = 3$ agents:

[4]Another possibility would be to allow agents to form links sequentially. In such a game, the precise incentives to form and dissolve links will differ. However, we believe that some of the main properties of Nash and efficient networks that we identify in the simultaneous move setting – e.g., super-connectedness – should still obtain.

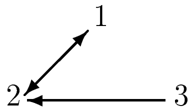

Figure 1

Here agent 2 has formed links with agents 1 and 3, agent 1 has a link with agent 2 while agent 3 does not link up with any other agent. The link $g_{i,j} = 1$ is represented by an *edge* starting at j with the arrowhead pointing at i. It can be seen that there is a one-to-one correspondence between the set of all directed networks with n vertices and the set of strategies $\mathcal{G}$.

For $g \in \mathcal{G}$, define $\mu_i^d(g) = |\{k \in N \mid g_{i,k} = 1\}|$. Here, $\mu_i^d(g)$ is the number of links formed by agent i. To describe information flows in the network, we introduce the notion of the *closure* of g. This is a non-directed network denoted $h = \mathrm{cl}(g)$, and defined by $h_{i,j} = \max\{g_{i,j}, g_{j,i}\}$ for each i and j in N.[5] Pictorially, the closure of a network corresponds to replacing each directed edge of g by a non-directed one.[6] Each link $h_{i,j} = 1$ *succeeds* (i.e. allows information to flow) with probability $p \in (0,1)$ and *fails* (does not permit information flow) with probability $1 - p$. Furthermore, the success or failure of different links are assumed to be independent. Thus h may be regarded as a random network. Formally, we say that h' is a *realization* of h (denoted $h' \subset h$) if for every i we have $h'_{i,j} \leq h_{i,j}$ for $j \neq i$.[7] Given $h' \subset h$, let $L(h')$ be the total number of links in h'. Under the hypothesis of independence, the probability of network h' being realized is

$$\lambda(h' \,|\, h) = p^{L(h')}(1-p)^{L(h)-L(h')}. \tag{2.1}$$

For $h' \subset h$ we say there is a *path* between i and j in h' if either $h'_{i,j} = 1$ or there exists a non-empty set of agents $\{i_1, \ldots, i_m\}$ distinct from each other and from i and j such that $h'_{i,i_1} = \cdots = h'_{i_m,j} = 1$. Define $\mathbf{1}_i(j; h')$ to equal 1 if there is a path between i and j in h' and to equal 0 otherwise. We suppose that i observes an agent in h' if and only if there is a path between that agent and i in h'. A network g is said to be connected if there is a path in $h = cl(g)$, between any two agents i and

[5] An anonymous referee has suggested an alternative specification: $h_{i,j} = \min\{g_{i,j}, g_{j,i}\}$. In this case, a link is formed if and only if both individuals choose to form a link. We discuss this alternative specification below in footnote 11.

[6] The network h can be regarded as non-directed because $h_{i,j} = h_{j,i}$ for each i and j.

[7] The network h' should also be regarded as non-directed. Hence, we implictly assume that $h'_{j,i} = h'_{i,j}$ for all $j \neq i$.

j. A network is called empty is it has no links. A set $C \subset N$ is called a component of g is there is a path in h for every pair of agents i and j, and there is no strict superset $C'0$ of C, for which this is true. The geodesic distance $d(i, j; h)$ between two agents i and j is the number of links in the shortest path between them in h.

We can define $\mu_i(h') \equiv \sum_{j \neq i} \mathbf{1}_i(j; h')$ as the total number of people that agent i observes in the realization h'. We assume that each link formed by agent i costs $c > 0$, while each agent that i observes in a realization of the network yields a benefit of $V > 0$. Without loss of generality, we set $V = 1$.[8]

Given the strategy-tuple g, define the function $B_i(h)$ for agent i as

$$B_i(h) = \sum_{h' \subset h} \lambda(h' \,|\, h)\mu_i(h'). \tag{2.2}$$

where $h = \text{cl}(g)$. The probability that the network h' is realized is $\lambda(h' \mid h)$, in which case agent i accesses the information of $\mu_i(h')$ agents in total. Hence $B_i(h)$ is i's *expected benefit* from the random network h. Using (2.2), we define i's (expected) payoff from the strategy-tuple g as

$$\Pi_i(g) = B_i(h) - \mu_i^d(g)c = \sum_{h' \subset h} \lambda(h' \,|\, h)\mu_i(h') - \mu_i^d(g)c. \tag{2.3}$$

The first term in (2.3) is i's expected benefit from the network, while the second is i's cost of forming links, which is incurred at the outset. The expected benefit and the payoff of agent i can be expressed in a different form, which is also useful. Substituting $\mu_i(h') = \sum_{j \neq i} \mathbf{1}_i(j; h')$ in equation (2.2), we get:

$$\begin{aligned} B_i(h) &= \sum_{h' \subset h} \lambda(h' \,|\, h) \sum_{j \neq i} \mathbf{1}_i(j; h') \\ &= \sum_{j \neq i} \Big[\sum_{h' \subset h} \lambda(h' \,|\, h)\mathbf{1}_i(j; h') \Big] = \sum_{j \neq i} p_i(j; h). \end{aligned} \tag{2.4}$$

where $p_i(j; h) = \sum_{h' \subset h} \lambda(h' \mid h)\mathbf{1}_i(j; h')$ is the *probability* that i observes j in the random network h. From (2.4), using (2.3) we obtain

$$\Pi_i(g) = B_i(h) - \mu_i^d(g)c = \sum_{j \neq i} p_i(j; h) - \mu_i^d(g)c. \tag{2.5}$$

[8] For simplicity, we assume a linear specification of the payoffs in the present paper. This implies that the value of additional information is constant. We should expect that the marginal value of information declines as more information becomes available. We believe that this simplification is not crucial for our results. For a study of network formation under a fairly general class of payoff functions and with perfectly reliable links, see Bala and Goyal (1999).

Applying either (2.3) or (2.5) to the network in Figure 1, we calculate $\Pi_i(g) = p + p^2 - c$, $2p - 2c$, and $p + p^2$ for agents i = 1,2 and 3 respectively. As information is assumed to flow in both directions of a link, agent 3 gets an expected benefit of $p + p^2$ without forming any links. Hence, the payoffs allow for significant "free riding" in link formation.

Given a network $g \in \mathcal{G}$, let g_{-i} denote the network obtained when all of agent i's links are removed. The network g can be written as $g = g_i \oplus g_{-i}$ where the '$\oplus$' indicates that g is formed as the union of the links in g_i and g_{-i}. The strategy g_i is said to be a *best response* of agent i to g_{-i} if

$$\Pi_i(g_i \oplus g_{-i}) \geq \Pi_i(g_i' \oplus g_{-i}), \quad \text{for all } g_i' \in \mathcal{G}_i. \tag{2.6}$$

The set of all of agent i's best responses to g_{-i} is denoted $BR_i(g_{-i})$. Furthermore, a network $g = (g_1, \ldots, g_n)$ is said to be a *Nash network* if $g_i \in BR_i(g_{-i})$ for each i, i.e. agents are playing a Nash equilibrium. A *strict* Nash network is one where agents get a strictly higher payoff with their current strategy than they would with any other strategy. We shall define our welfare measure in terms of the sum of payoffs of all agents. Formally, let $W : \mathcal{G} \to \mathcal{R}$ be defined as $W(g) = \sum_{i=1}^{n} \Pi_i(g)$ for $g \in \mathcal{G}$. A network g is *efficient* if $W(g) \geq W(g')$ for all $g' \in \mathcal{G}$. An efficient network is one which maximizes the total expected value of information made available to the agents, less the aggregate cost of communication.

We say that $g \in \mathcal{G}$ is *essential* if $g_{i,j} = 1$ implies $g_{j,i} = 0$. We note that if $g \in \mathcal{G}$ is either a Nash network or an efficient network, then g must be essential. The argument underlying the above observation is as follows. If $g_{i,j} = 1$ then by the definition of Π_j agent j pays an additional cost of c from setting $g_{j,i} = 1$ as well, while neither he nor any one else gets any benefit from it. Hence if g is not essential, it cannot be either Nash or efficient.[9]

We start with the following intuitive property of the benefit function $B_i(\cdot)$ which is useful for our analysis.

[9]The payoff function (2.3) assumes that the links $g_{i,j} = 1$ and $g_{j,i} = 1$ are perfectly correlated. The above observation is a consequence of this assumption. An alternative assumption is that $g_{i,j} = 1$ and $g_{j,i} = 1$ are independent: in this case the link $h_{i,j} = 1$ succeeds with probability $q = 1-(1-p)^2 = 2p-p^2$. We briefly discuss the impact of the alternative assumption in Section 3.

Lemma 2.1 *Suppose $p \in (0,1)$. Let $g^o \in \mathcal{G}^a$ be a network with $h^o = \mathrm{cl}(g^o)$, and suppose there are two agents i and j such that $g^o_{i,j} = g^o_{j,i} = 0$. Let g be the same as g^o except for an additional link $g_{i,j} = 1$ and let $h = \mathrm{cl}(g)$. Then for all agents m, $B_m(h) \geq B_m(h^o)$. The inequality is strict if $m = i$ or $m = j$.*

The proof of this lemma is omitted. This observation is actually more general than stated, since it also implies that an agent m's benefit is non-decreasing in the addition of any number of links. The following lemma describes some properties of the payoff function.

Lemma 2.2 *The payoff function $\Pi_i(g)$ is a polynomial $\sum_{k=0}^{L(h)} a_i^k p^k$, where the coefficient $a_i^o = -\mu_i^d(g)c$ and $a_i^1 = |\{j \mid h_{i,j} = 1\}|$.*

The proof is given in the appendix.

We shall say that a network is empty if it contains no links, and that is complete if there exists a link between every pair of agents. The empty network is denoted by g^e, while the complete network is denoted by g^c. The star architecture is prominent in this literature: denote a star by g^s, where, for a fixed "central agent" n (say), we have $h^s_{n,j} = 1$ for all $j \neq n$ and $h^s_{j,k} = 0$ for all $j \neq n$, $k \neq n$. In a line network g^l, we have $h^l_{i,i-1} = 1$, for $i = 2, 3, \ldots, n$. In a wheel network g^w, we have agents arranged around a circle, i.e., $h^w_{1,n} = 1$, and $h^w_{i,i-1} = 1$ for all $i = 2, 3, \ldots, n$ and $h^w_{i,j} = 0$, otherwise.

Two networks have the same architecture if one network can be obtained from the other by permuting the strategies of agents in the other network. Thus, in a society with n agents there are n networks with the star architecture.

3 Nash networks

We are interested in describing Nash networks for the above model as a function of the link success parameter p and the cost c of link formation.

We start by noting an interesting implication of the assumption that link formation is one-sided. By the definition of payoffs, although one agent bears the cost of a link, both agents potentially obtain the benefits associated with it. This asymmetry in payoffs is relevant for defining the architecture of the network. As an illustration, we note that there are now three types of 'star' networks, depending upon which agents bear the costs of the links in the network. For a society with $n = 5$ agents, Figures 2a-2c (left to right) illustrate these types. Figure 2a shows an *inward-pointing* or ip-star, Figure 2b an *outward-pointing* or op-star and Figure 2c depicts a *mixed-type* or mt-star.

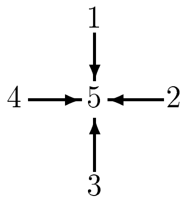

Fig 2a: Inward-pointing

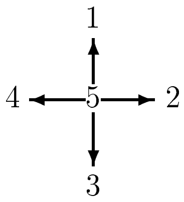

Figure 2b: Outward-pointing

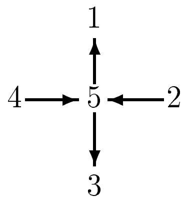

Figure 2c: Mixed-type

We calculate the payoff obtained by an agent in a star g^s using (2.5). Consider the central agent n. Clearly, $p_n(j; h^s) = p$ for each $j \neq n$, so that $B_n(h^s) = \sum_{i \neq n} p_n(i; h^s) = (n-1)p$. If $i \neq n$ then $p_i(n; h^s) = p$ while $p_i(j; h^s) = p^2$ for each $j \notin \{n, i\}$. Hence $B_i(h^s) = p + (n-2)p^2$. From this, we obtain:

$$\Pi_n(g^s) = (n-1)p - \mu_n^d(g^s)c, \quad \Pi_i(g^s) = p + (n-2)p^2 - \mu_i^d(g^s)c, \ i \neq n. \tag{3.1}$$

To illustrate the range of networks that can be supported as Nash networks, we provide a complete characterization of Nash networks for a society of three agents.

Example 1: Fix $n = 3$. We find that the empty network, the star, and the complete network are the possible Nash architectures. In particular, the empty network is Nash if $c \geq p$. The inward-pointing star and the mixed-type star are Nash if $\{(p, c) \mid 0 < p < 1, \ p + p^2 - 2p^3 \leq c \leq p\}$, while the outward pointing star is Nash if $\{(p, c) \mid 0 < p < 1, \ p + p^2 - 2p^3 \leq c \leq p + p^2\}$. The complete network is Nash if $0 < c \leq p + p^2 - 2p^3$. Figure 3 displays the parameter regions where different networks are Nash.

— Figure 3 somewhere here —

The computations involved in establishing the regions give an idea of some of the general considerations involved in studying equilibrium networks and so we present them here.

First, let g be an inward-pointing star, i.e. $g_{3,1} = g_{3,2} = 1$.[10] Then using (3.1), agent 3's payoff is given by $\Pi_3(g) = 2p - 2c$, while if he does not form a link, his payoff is 0. Thus, he is playing a best response if and only if $p \geq c$. Furthermore, $\Pi_i(g) = p + p^2$ for $i = 1,2$. We now calculates agent 1's expected payoff if he deviates and forms an additional link with agent 2. In any realization h' where $h'_{1,2} = 1$ (which occurs with probability p) he obtains 2's information. Furthermore, in any realization h' where $h'_{1,2} = 0$ (probability $(1-p)$) he still obtains 2's information if the links $h'_{1,3} = h'_{2,3} = 1$. The probability of such a realization is $(1-p)p^2$. Hence, the probability that 1 obtains 2's information is $p + (1-p)p^2$. By symmetry, the same holds for 1 vis-a-vis 3, so that 1's payoff by deviating is $2(p + (1-p)p^2) - c$. It is, therefore, not worthwhile to deviate if and only if $c \geq 2(p + (1-p)p^2) - (p + p^2) = p + p^2 - 2p^3$. Thus, the inward-pointing star is Nash in the region $\{(p,c) \mid 0 < p < 1,\ p + p^2 - 2p^3 \leq c \leq p\}$. Next, it can be checked that a mixed-type star $g_{3,1} = 1$, $g_{2,3} = 1$ is Nash for the same region identified for the inward-pointing star.

The situation is somewhat different if g is the outward-pointing star $g_{1,3} = g_{2,3} = 1$. Here, agent 3 is obviously playing a best response. For agent 1 we have $\Pi_1(g) = p + p^2 - c$. This is better than forming no link at all provided $c < p + p^2$. Furthermore it dominates forming an additional link with agent 2 if $p + p^2 - c \geq 2(p + (1-p)p^2) - 2c$ or $c \geq p + p^2 - 2p^3$. Hence, the outward-pointing star is Nash in the region $\{(p,c) \mid 0 < p < 1,\ p + p^2 - 2p^3 \leq c \leq p + p^2\}$.

Furthermore, it is easily seen that the empty network is Nash if and only if $c \geq p$. In the case of complete networks, there are two possibilities. (a) $g_{1,2} = g_{2,3} = g_{3,1} = 1$ and (b) $g_{1,2} = g_{1,3} = g_{2,3} = 1$. In both cases, it can be verified that a complete network is Nash if and only if $0 < c \leq p + p^2 - 2p^3$. Interestingly, the empty network and the complete network are Nash in the non-empty region $\{(p,c) \mid 0 \leq p \leq 1/2, p \leq c \leq p + p^2 - 2p^3\}$, while no star is Nash. Here the cost is sufficiently high to ensure that it is not worthwhile to form a link if no one else does. However, once a link is established, it pays to form another to ensure greater reliability because p is low and costs are not too high.

□

[10] For brevity in describing a network, we only identify the links $g_{i,j} = 1$.

We now derive some general properties of Nash networks.[11] The following result shows that if communication occurs at all in a Nash network, then every agent communicates with every other agent with strictly positive probability.

Proposition 3.1 *Let $g \in \mathcal{G}^a$ be a Nash network. Then it is either connected or empty.*

The intuition for this result is as follows: consider a non-empty network g. Suppose that is not connected; let there be two components C and C' with $|C| \geq |C'|$. Also let $|C| > 1$. Then there exists an agent $i \in C$ who forms a link with some other agent $k \in C$. Since g is Nash, this link must yield a non-negative marginal payoff. Now consider some agent $j \in C'$. By definition, there is no path between j and component C, in g. Suppose that agent j forms a link with k. The proof proceeds by showing that the marginal payoff to j from such a link strictly exceeds the marginal payoff that i gets from the link with k. Since the latter is non-negative, this implies that g cannot be a Nash equilibrium.[12] We now provide the proof.

Proof: Consider a non-empty Nash network $g \in \mathcal{G}^a$ which is not connected. As g is non-empty, there exist agents i and m such that $g_{i,m} = 1$. Let $h = \mathrm{cl}(g)$ and note that $h_{i,m} = 1$. Suppose g^o is the network without the link $g_{i,m} = 1$, ceteris paribus. Define $h^o = \mathrm{cl}(g^o)$. We observe that each realization $\hat{h} \subset h$ is one of two types: either $\hat{h} = h'$ for some $h' \subset h^o$ (if the link $h_{i,m} = 1$ fails) or $\hat{h} = h' \oplus h_{i,m}$ for some $h' \subset h^o$, where $h' \oplus h_{i,m}$ denotes the network where the link $h_{i,m} = 1$ is also present. Moreover, by independence we have $\lambda(\hat{h} \,|\, h) = (1-p)\lambda(h' \,|\, h^o)$ in the former case, and $\lambda(\hat{h} \,|\, h) = p\lambda(h' \,|\, h^o)$ in the latter case. It follows that the marginal payoff $\Pi_i(g) - \Pi_i(g^o)$ to agent i from the link $g_{i,m} = 1$

[11] Under the alternative specification, $h_{i,j} = \min\{g_{i,j}, g_{j,i}\}$, the distinction between different types of stars cannot arise in equilibrium, since a link is only formed if both agents involved agree to the link. In this sense, there are fewer networks that can be candidates for Nash equilibrium. However the alternative specification introduces an additional aspect of coordination: it is worthwhile for agent i to form a costly link with agent j only if the latter also wants to form a link. This suggests that, when costs of forming links are small, there will exist a relatively large number of equilibrium architectures – including some partially connected ones – corresponding to varying levels of successful coordination between pairs of agents. For example, in the above example with $n = 3$, if $c < p$ then the partially connected network with $g_{1,2} = 1$, $g_{2,3} = 0$, and $g_{1,3} = 0$, is a Nash network under the alternative specification, while it is not a Nash network under our formulation. See Dutta, van den Nouweland and Tijs(1998), for a study of the alternative formulation.

[12] It is worth emphasizing that this argument exploits the fact that link formation is one-sided; hence we only have to check the incentives of individual players to form or delete links.

is given by

$$\Big[p \sum_{h' \subset h^o} \lambda(h' \,|\, h^o) \mu_i(h' \oplus h_{i,m}) \quad + \quad (1-p) \sum_{h' \subset h^o} \lambda(h' \,|\, h^o) \mu_i(h')\Big] - c - \sum_{h' \subset h^o} \lambda(h' \,|\, h^o) \mu_i(h')$$
$$= \quad p \sum_{h' \subset h^o} \lambda(h' \,|\, h^o) \Big[\mu_i(h' \oplus h_{i,m}) - \mu_i(h')\Big] - c. \qquad (3.2)$$

Since g is Nash, the expression in (3.2) is non-negative.

If g is not connected, there must exist an agent j such that $\mathbf{1}_i(j; h) = 0$. Since $\mathbf{1}_i(m; h) = 1$ this also implies $g_{j,m} = g_{m,j} = 0$. We consider the marginal payoff obtained by agent j if, starting from the network g, he forms an additional link $g_{j,m} = 1$ with agent m, ceteris paribus. This is given by $\Pi_j(g \oplus g_{j,m}) - \Pi_j(g)$. Using arguments analgous to the case of agent i, we can write the marginal payoffs of agent j, as follows:

$$\Big[p \sum_{h' \subset h} \lambda(h' \,|\, h) \mu_j(h' \oplus h_{j,m}) \quad + \quad (1-p) \sum_{h' \subset h} \lambda(h' \,|\, h) \mu_j(h')\Big] - c - \sum_{h' \subset h} \lambda(h' \,|\, h) \mu_j(h')$$
$$= \quad p \sum_{h' \subset h} \lambda(h' \,|\, h) \Big[\mu_j(h' \oplus h_{j,m}) - \mu_j(h')\Big] - c. \qquad (3.3)$$

We now procced to show that (3.3) is strictly greater than (3.2); thus it is profitable for j to deviate from his strategy in g, by forming an additional link with m.

There are two kinds of h' networks, given h: those where the link $g_{i,m}$ succeeds and those in which it fails. Using the assumption that the probability of success is independent across links, we can rewrite the R.H.S of (3.3) as follows:

$$p\Big[p \sum_{h' \subset h^o} \lambda(h' \,|\, h^o)[\mu_j(h' \oplus h_{j,m} \oplus h_{i,m}) - \mu_j(h' \oplus h_{i,m})]$$
$$+(1-p) \sum_{h' \subset h^o} \lambda(h' \,|\, h^o)[\mu_j(h' \oplus h_{j,m}) - \mu_j(h')]\Big] - c \qquad (3.4)$$

$$= p\Big[p \sum_{h' \subset h^0} \lambda(h' \,|\, h^o)[\mu_j(h' \oplus h_{j,m} \oplus h_{i,m}) - \mu_j(h')]$$
$$+(1-p) \sum_{h' \subset h^o} \lambda(h' \,|\, h^o)[\mu_j(h' \oplus h_{j,m}) - \mu_j(h')]\Big] - c \qquad (3.5)$$

where the last expression is obtained by noting that $\mu_j(h' \oplus h_{i,m}) = \mu_j(h')$, when $h' \subset h^o$.

Consider the term $[\mu_j(h' \oplus h_{j,m} \oplus h_{i,m}) - \mu_j(h')]$. Note that if some agent k is accessed by i in $h' \subset h^o$, *due* to the link $g_{i,m}$ then certainly this agent is accessed by j in $h' \oplus h_{j,m} \oplus h_{i,m}$. Moreover, since agent j has only possible path with m, this access must be through $g_{j,m}$. Thus it follows that:

$$\mu_j(h' \oplus h_{j,m} \oplus h_{i,m}) - \mu_j(h') \geq \mu_i(h' \oplus h_{i,m}) - \mu_i(h') \tag{3.6}$$

Note also that if $h' \subset h^o$ is empty then $h' \oplus h_{i,m} \oplus h_{j,m}$ allows for agent j to access i in addition. Thus there are $h' \subset h^o$ for which the inequality in (3.5) is strict. Summarizing, we have that:

$$\sum_{h' \subset h^0} \lambda(h' \,|\, h^o)[\mu_j(h' \oplus h_{j,m} \oplus h_{i,m}) - \mu_j(h')] > \sum_{h' \subset h^o} \lambda(h' \,|\, h^o)[\mu_i(h' \oplus h_{i,m}) - \mu_i(h')] \tag{3.7}$$

Consider next the term $[\mu_j(h' \oplus h_{j,m}) - \mu_j(h')]$. If some agent k is contacted by i in $h' \oplus h_{i,m}$, *due* to the link $g_{i,m}$, then it follows that this same agent k is also accessed by j in the network $h' \oplus h_{j,m}$, *due* to the link $g_{j,m}$. Hence, for $h' \subset h^o$,

$$\mu_j(h' \oplus h_{j,m}) - \mu_j(h') \geq \mu_i(h' \oplus h_{i,m}) - \mu_i(h') \tag{3.8}$$

Summarizing, we have that:

$$\sum_{h' \subset h^o} \lambda(h' \,|\, h^o)[\mu_j(h' \oplus h_{j,m}) - \mu_j(h')] \geq \sum_{h' \subset h^o} \lambda(h' \,|\, h^o)[\mu_i(h' \oplus h_{i,m}) - \mu_i(h')] \tag{3.9}$$

Using (3.7) and (3.9) in (3.5), we can deduce that:

$$\begin{aligned} p\Big[p \sum_{h' \subset h^0} &\lambda(h' \,|\, h^o)[\mu_j(h' \oplus h_{j,m} \oplus h_{i,m}) - \mu_j(h')] \\ +(1-p) \sum_{h' \subset h^o} &\lambda(h' \,|\, h^o)[\mu_j(h' \oplus h_{j,m}) - \mu_j(h')]\Big] - c \\ > p \sum_{h' \subset h^o} &\lambda(h' \,|\, h^o)\Big[\mu_i(h' \oplus h_{i,m}) - \mu_i(h')\Big] - c. \end{aligned} \tag{3.10}$$

We know from (3.2) that the right hand side of (3.10) is non-negative. Then using (3.3), (3.5) and (3.10), we can infer that it is strictly profitable for player j to form a link with player m. This completes the proof.

□

Our next result provides conditions under which some familiar architectures are Nash.

Proposition 3.2 *Let the payoffs be given by (2.3). (a) Given $p \in (0,1)$ there exists $c(p) > 0$ such that a complete network g^c is (strict) Nash for all $c \in (0, c(p))$. (b) Given $c \in (0,1)$ there exists $p(c) \in (c,1)$ such that $p \in (p(c),1)$ implies that all types of stars (inward-pointing, outward-pointing and mixed-type) are Nash. If $n \geq 4$, they are in fact strict Nash. (c) Given $c \in (1,(n-1))$ there exists $p(c) < 1$ such that $p \in (p(c),1)$ implies that the outward-pointing star is (strict) Nash. (d) The empty network is (strict) Nash for all $c > p$.*

Proof: We begin with (a). Let $g = g_i \oplus g_{-i}$ be a complete network and suppose that agent i has one or more links in his strategy g_i. Let g^o be a network where some of these links have been deleted, ceteris paribus. From Lemma 2.1 we get $B_i(h^o) < B_i(h)$ where $h^o = \text{cl}(g^o)$ and $h = \text{cl}(g)$. It follows that if $c = 0$ then g_i is a strict best response for agent i. By continuity, there exists $c_i(p) > 0$ for which g_i is a strict best response for all $c \in (0, c_i(p))$. Statement (a) follows by setting $c(p) = \min_i c_i(p)$ over all agents i who have one or more links in their strategy g_i.

For (b), choose $p(c) \in (c,1)$ to satisfy $(1-p) + (n-2)(1-p^2) < c$ for all $p \in (p(c),1)$. In what follows, fix $p \in (p(c),1)$. Let g be a mixed-type star and let agent n (say) be the "central agent" of the star. Consider a agent $i \neq n$ for whom $g_{i,n} = 1$. By (3.1) i's payoff is $p + (n-2)p^2 - c$. If he forms no link at all, he obtains a payoff of 0. Since $p > c$ it is worthwhile for him to form at least one link. Next, if i deletes his link with n and forms it with an agent $j \notin \{i,n\}$ instead, ceteris paribus, his payoff can be calculated to be $p + p^2 + (n-3)p^3 - c$ which is dominated by forming one with n. Hence if he forms one link, we can assume he forms it with agent n. Moreover, by forming $k \geq 2$ links, his payoff is bounded above by $(n-1) - kc$. Subtracting the payoff from the star, his maximum incremental payoff from two or more links is no larger than $(1-p) + (n-2)(1-p^2) - c$ which is negative, by choice of p. Hence i's best response is to maintain a single link with agent n. For agent n, if $g_{j,n} = 0$, then $p > c$ implies it is worthwhile for n to form a link with agent j. Thus, the mixed-type star is Nash. Similar arguments apply for the inward-pointing and outward-pointing stars.[13]

For part (c), note that $c \in (1,n)$ implies $c > p$. Hence the inward-pointing star and the mixed-type star cannot be Nash. However, the outward-pointing star g^s can be supported.

[13]Note that if $n \geq 4$, then agent $i \neq n$ has a *strict* incentive to form a link with n rather than $j \notin \{i,n\}$, but only a weak one if $n = 3$. Hence the mixed-type and outward-pointing stars are strict Nash for $n \geq 4$ but only Nash when $n = 3$. On the other hand, the inward-pointing star is strict Nash for all $n \geq 3$.

From (3.2), the payoff of agent $i \neq n$ is $\Pi_i(g^s) = p + (n-2)p^2 - c$. Given $c \in (0,1)$ there clearly exists $p(c) \in (c, 1)$ such that $p \in (p(c), 1)$ implies $\Pi_i(g^s) > 0$, so that i will form at least one link in his best response. Arguments analogous to (b) above establish that $p(c)$ may be additionally chosen to ensure that i will not wish to form more than one link for any $p \in (p(c), 1)$.

For part (d), if $c > p$ and no other agent forms a link, it will not be worthwhile for an agent to form a link. Hence the empty network is strict Nash.

□

The main result of this section shows that if $c < p$, then for large societies, every link in a Nash network is 'redundant'! By way of motivation, we provide an example concerning the stability of the star in large societies.

Example 2: Let $c \in (0,1)$ and $p \in (c,1)$ be fixed. Suppose g^s is an inward-pointing star with agent n as the "central" agent. We set $h^s = \mathrm{cl}(g^s)$. From (3.1), agent 1's (say) payoff is $\Pi_1(g^s) = p + (n-2)p^2$. We consider what happens when agent 1 forms a link with agent 2, ceteris paribus. Denote the resulting network as g with $h = \mathrm{cl}(g)$. Agent 1 can now obtain n's information in two ways: when the link $h^s_{n,1} = h_{n,1} = 1$ succeeds (probability p) and when it fails (probability $(1-p)$), provided the links $h^s_{1,2} = h^s_{2,n} = 1$ succeed (probability p^2). Hence $p_1(n; h) = p + (1-p)p^2 = p + p^2 - p^3$. Similar arguments show that $p_1(2; h) = p + p^2 - p^3$, and that $p_1(j; h) = (p + p^2 - p^3)p = p^2 + p^3 - p^4$ for $j \in \{3, \ldots, n-1\}$. Hence $B_1(h) = \sum_{j \neq 1} p_1(j; h) = 2(p + p^2 - p^3) + (n-3)(p^2 + p^3 - p^4)$ and $\Pi_1(g) = 2p + (n-1)p^2 + (n-5)p^3 - (n-3)p^4 - c$. Furthermore, $\Pi_1(g) - \Pi_1(g^s) = p + p^2 + (n-5)(p^3 - p^4) - 2p^4 - c$. Since $p < 1$, we have $p^3 - p^4 > 0$. It follows that there exists an integer $\bar{n}$ such that $\Pi_1(g) - \Pi_1(g^s) > 0$ for all $n \geq \bar{n}$, i.e. the inward-pointing star is not Nash.

□

Proposition 3.2 shows that for fixed c, if p is sufficiently close to 1 then the inward-pointing star is (strict) Nash. However, Example 2 above reveals that p is not independent of n: no matter how close p is to 1, for sufficiently large societies the inward-pointing star *cannot* be supported as a Nash network. The intuition for the above finding is as follows: in a star, the solitary link between agent 1 and the central agent is crucial for him to obtain benefits from the rest of society. Furthermore, the loss of benefits due to the failure of the critical link with the central agent becomes unboundedly large as n increases. From some point onwards, it becomes worthwhile for agent 1 to establish another route to the remaining agents in order to recover the foregone benefits, at which point the star is no longer Nash.

Formally, we say that a network g is *minimally connected* if $h = \text{cl}(g)$ is connected, and there is a unique path in h between any two agents. Likewise, we shall say that in a network g a link $g_{i,j}$ is *critical* if it constitutes the unique path between agents i and j in g. Thus in a minimally connected network, every link is critical. Correspondingly, a link g_{ij} is redundant in the network g if there is a path in the network $g - g_{ij}$. We shall say that a network g is *super-connected* if every link in g is redundant.[14] A minimally connected network is very demanding in terms of reliability: even if one link fails, two non-empty subsets of agents will no longer be able to transmit information to each other. On the other hand, super-connected networks have redundancy built in: even when a link happens to fail, all agents may still be able to fully communicate amongst themselves. Stated differently, for every agent there are at least two paths to some other agent in the society. Note that the star and the line are minimally connected, while the wheel and the complete network are super-connected.[15]

The above classification leads to the question: how important is redundancy in Nash networks? Is it the case that agents rely on single paths for communicating with others, or do they allow for multiple pathways? The following result provides an answer to this question.

Proposition 3.3 *Suppose $c < p(1 - p^{n/2})$. If $g \in \mathcal{G}^a$ is a non-empty Nash network, then it must be super-connected.*

The proof of this result makes use of a couple of lemmas. We first state these results.

Lemma 3.1 *Let g be a connected network and let $h = \text{cl}(g)$. The payoff of agent i is $\Pi_i(g) \geq \sum_{k=1}^{n-1} d_i^k p^k - \mu_i^d(g)c$ where d_i^k is the number of agents at geodesic distance k from agent i in the network h.*

[14] Sometimes it will be convenient to refer to h rather than g to be minimally or super-connected.

[15] We are indebted to the editor for suggesting this stronger notion of connectedness as well as for proposing Propositions 3.3 and 4.3.

Lemma 3.2 *Let $\sum_{k=1}^{t} a^k p^k$ be a polynomial where the coefficients $\{a^k\}$ are integers satisfying (1) a^k is a non-negative integer for each $k \in \{1, \ldots, t\}$, (2) $\sum_{k=1}^{t} a^k = t$ and (3) $a^k \geq 1$ for some $k \in \{2, \ldots, t\}$ implies $a^{\bar{k}} \geq 1$ for all $\bar{k} \in \{1, \ldots, k-1\}$. Then*

$$\sum_{k=1}^{t} a^k p^k \geq \sum_{k=1}^{t} p^k. \tag{3.11}$$

for all $p \in (0, 1)$.

The proof of these lemmas is given in the appendix. We now provide a proof of the proposition.

Proof of Proposition 3.3: The proof is by contradiction. We know from Proposition 3.1 that g is connected. Suppose there exists a link $g_{i,j}$ in g which is critical. Then $g - g_{ij}$ has two components, C and C', with $i \in C$ and $j \in C'$. Let $|C| = n_1$ and $|C'| = n_2$. Suppose, without loss of generality, that $n_1 \geq n_2$. Then it follows that $n_1 \geq n/2$. Let $r \in C$, be an agent furthest away from j in g. Since agent j has only one link with agents in C, it follows that $d(j, r; g) \geq 2$. Moreover, since $k, j \in C$, it also follows that there is at least one path between k and r, which does not involve g_{ij}.

We now suppose that starting from the network g, agent j forms an additional link with agent r, ceteris paribus. Denote the new network as g^o and put $h^o = \mathrm{cl}(g^o)$. There are now two paths between j and each agent $m \in C$: via the link $h^o_{i,j} = h_{i,j} = 1$ and via the link $h^o_{j,r} = 1$, using a path between r and m in h (which does not involve the link $h_{i,j} = 1$). Thus even if the link $h_{i,j} = 1$ fails (with probability $(1-p)$) agent j can still obtain the information of m if the link $h^o_{j,r} = 1$ succeeds, as do all the links in the path between r and m. It follows that $p_j(m; h^o) \geq p_j(m; h) + (1-p)p^{d(j,m;h')}$, where $h' \subset h^o$ is the network where all links in h^o except $h^o_{i,j} = h_{i,j} = 1$ are present.[16] Note that $d(j, m; h') \leq n - 1$ for all $m \neq j$ since there is at least one path between r and m which is independent of the link between i and j. Summing over all agents $m \neq j$ we get

$$B_j(h^o) \geq B_j(h) + (1-p) \sum_{m \in C} p^{d(j,m;h')}. \tag{3.12}$$

[16] The inequality can be strict. This happens if, for instance, the link $h_{i,j} = 1$ succeeds, some other links on the path from j to m fail, while the link $h^o_{j,r} = 1$ and all links on the path from r to m succeed.

We now note that $\sum_{m \neq j} p^{d(j,m;h')} = \sum_{k=1}^{n_1} d_j^k(h')p^k$, where $d_j^k(h')$ is the number of agents at distance k from agent j in the network h'. Furthermore, since h' is connected, $\sum_{k=1}^{n_1} d_j^k(h') = n_1$. Also, if there are agents at distance $k \geq 2$ from agent j in h' there clearly must be agents at all lower distances as well, i.e. $d_j^k(h') \geq 1$ implies $d_r^{k'}(h') \geq 1$ for all $1 \leq k' < k$. As conditions (1)-(3) of Lemma 3.2 are satisfied, we have $\sum_{k=1}^{n_1} d_j^k(h')p^k \geq \sum_{k=1}^{n_1} p^k$. Using (3.12) and the above inequality, we obtain

$$B_j(h^o) \geq B_j(h) + (1-p)\sum_{k=1}^{n_1} d_j^k(h')p^k \geq B_j(h) + (1-p)\sum_{k=1}^{n_1} p^k = B_j(h) + p.(1-p^{n_1}). \quad (3.13)$$

Since j's additional link with r costs c, we get $\Pi_j(g^o) - \Pi_j(g) \geq p(1-p^{n_1}) - c$. This expression is larger than 0 provided $p(1-p^{n_1}) > c$. Since $n_1 \geq n/2$, it follows that $p(1-p^{n_1}) \geq p(1-p^{n/2})$. Thus this condition is certainly satisfied under the hypothesis of the proposition. It follows that agent i can obtain a strictly higher payoff by deviating from his strategy in g, contradicting the assumption that g is Nash.

$\square$

We now interpret the significance of Proposition 3.3. For a fixed c and p, with $c < p$, as the size of a society increases, an equilibrium network has the property that every link in it is redundant. This shows that minimally connected networks, such as the star, for example, are not stable and therefore should not be observed in large societies.

Several comments are in order concerning Proposition 3.3. First, we would ideally like the proposition to be complemented by a result which shows that for each n, Nash networks exist in all regions of the parameter space. Due to the formidable computational difficulties, we have been unable to answer this question fully, though our investigations for small values of n lead us to conjecture that this is true.[17]

Second, it seems plausible that an result analogous to the above proposition should hold when $c > 1$. However, as the computations for this are more involved, we do not pursue this issue further in this paper.

Next, we also note that the assumption that the links $g_{i,j} = 1$ and $g_{j,i} = 1$ are perfectly correlated plays a role in the above result, by ensuring that if i forms a link with j, then j has no incentive to form a link with i. An alternative assumption is that if $\min\{g_{i,j}, g_{j,i}\} = 1$

[17] In the paper we focus on pure strategies only. We note that the network formation game we examine is a finite game, and so existence of equilibrium in mixed strategies follows directly from standard results in game theory.

then the link $h_{i,j} = 1$ succeeds with probability $q = 1 - (1-p)^2 = 2p - p^2$. While this may alter the parameter regions where specific networks are Nash, the intuition and the main results of the paper should still hold.[18]

Finally, it is interesting to contrast this model with the one developed in our earlier paper (Bala and Goyal, 1999). In that paper, the payoff of agent i in a network g is given by

$$\Pi_i(g) = \sum_{\{j:\mathbf{1}_i(j;h)=1\}} \delta^{d(i,j;h)} - \mu_i^d(g)c. \tag{3.14}$$

where $\delta \in (0,1)$ is a parameter which measures "information decay" and $h = \mathrm{cl}(g)$. Here, the network g (and h) are deterministic. However, the value of information obtained from another agent decays geometrically based on the geodesic distance to that agent. In general, Nash networks in that model are all minimally connected. For instance, for all $n \geq 3$, star networks (of all types) are Nash in the region $\{(\delta, c) \mid \delta \in (0,1), \delta - \delta^2 < c < \delta\}$. This contrasts sharply with the finding of Proposition 3.3, if $c < p$, then for large n, the star network (or any other minimally connected network) is not Nash. Thus, the presence of uncertainty creates very different kinds of incentives in network formation as compared to the model specified by (3.14).

4 Efficient Networks

We now turn to the study of networks which are optimal from a social viewpoint. Our emphasis will be on the relationship between Nash networks and efficient ones as p and c are allowed to vary over the parameter space. Due to the difficulty of the topic, however, our analysis will be fairly limited. The welfare function $W : \mathcal{G} \rightarrow \mathcal{R}$ is taken to be the sum of payoffs, i.e. $W(g) = \sum_{i=1}^n \Pi_i(g) = \sum_{i=1}^n (B_i(h) - \mu_i^d(g)c)$ where $h = \mathrm{cl}(g)$. Recall that a network g is said to be *efficient* if $W(g) \geq W(g')$ for all $g' \in \mathcal{G}$. We restrict ourselves to networks in the set $\mathcal{G}^a$. In the analysis of efficiency, this is without loss of generality.

Let g be a network and let $h = \mathrm{cl}(g)$. From Lemma 2.2, each $B_i(h) = \sum_{k=0}^{n-1} a_i^k p^k$ where $a_i^1 = |\{j \mid h_{i,j} = 1\}|$. Thus, each link $g_{i,j} = 1$ contributes 1 each to the coefficient a_i^1 and a_j^1. Since the total number of links in h is $L(h)$, we get $\sum_{i=1}^n a_i^1 = 2L(h)$. On the other

[18] The notion of multiple paths between agents has to be suitably extended so that if $\min\{g_{i,j}, g_{j,i}\} = 1$ then i and j are said to have multiple paths with each other.

hand, $\sum_{i=1}^{n} \mu_i^d(g) = \sum_{i=1}^{n} |\{j \mid g_{i,j} = 1\}| = L(h)$. Thus, the welfare function $W(g)$ can be expressed as a polynomial

$$W(g) = 2L(h)p + \sum_{k=2}^{L(h)} a^k p^k - L(h)c. \tag{4.1}$$

for some coefficients $\{a^k\}$. In particular, (4.1) indicates that the welfare properties of efficient networks depend only upon their non-directed features. This is a consequence of the linearity of payoffs in the costs of link formation. In what follows, our analysis is in terms of h rather than g.

Example 3: Fix $n = 3$. There are four possible architectures: (a) the empty network h^e. (b) a single link network h^n given by $h^n_{1,2} = 1$. (c) the star network h^s given by $h^s_{1,2} = h^s_{1,3} = 1$. (d) the complete network h^c given by $h^c_{1,2} = h^c_{2,3} = h^c_{3,1} = 1$. Figure 4 depicts the parameter regions where different networks are efficient.

— Figure 4 somewhere here —

We compute $W(h^e) = 0$, $W(h^n) = 2p - c$, $W(h^s) = 4p + 2p^2 - 2c$ and $W(h^c) = 6(p + p^2 - p^3) - 3c$. If $c > 2p$ then $W(h^e) > W(h^n)$. Likewise, if $c < 2p + 2p^2$ then $W(h^s) > W(h^n)$. Since these two regions cover the entire parameter space, h^n can never be efficient.

For the remaining three networks, straightforward calculations show that the empty network h^e is efficient in the region

$$\{(p, c) \mid p \in (0, 1/2),\ c > 2p + 2p^2 - 2p^3\} \cup \{(p, c) \mid p \in [1/2, 1),\ c > 2p + p^2\}. \tag{4.2}$$

the star h^s is efficient in the region

$$\{(p, c) \mid p \in [1/2, 1),\ 2p + 4p^2 - 6p^3 < c < 2p + p^2\}. \tag{4.3}$$

while the complete network h^c is efficient in the region

$$\{(p, c) \mid p \in (0, 1/2),\ c < 2p + 2p^2 - 2p^3\} \cup \{(p, c) \mid p \in [1/2, 1),\ c < 2p + 4p^2 - 6p^3\}. \tag{4.4}$$

□

We observe that there exist points (p, c) where two or more networks with a different number of links can simultaneously be efficient. (For example, at a point (p, c) where

$c = 2p + 4p^2 - 6p^3$ for $p \in [1/2, 1)$, the star and the complete network are both efficient, even though the former has two links while the latter has three). The result below shows for general n that such points are "rare". Specifically, the number of links in an efficient network is generically constant. Second, if we take $p = 0.8$ (say), we see that the number of links in an efficient network is non-increasing in c. This is also true more generally.

Proposition 4.1 *(a) For almost all values of $(p, c) \in (0,1) \times (0, \infty)$, if h and h^o are efficient networks, then $L(h) = L(h^o)$. (b) For fixed $p \in (0,1)$, the number of links in an efficient network is a non-increasing function of $c \in (0, \infty) \backslash V$, where V is a finite set.*

Example 3 also shows that when costs are very low or very high the efficient network is the complete network and the empty network, respectively. Moreover, when links are highly reliable, the star is efficient. These properties hold for general n.

Proposition 4.2 *(a) Given $p \in (0,1)$, there exists $c_2(p) > c_1(p) > 0$ such that the complete network g^c is efficient for all $c \in (0, c_1(p))$ (b) the empty network g^e is efficient for all $c > c_2(p)$. (c) Given $c \in (0, n)$ there exists $p(c) < 1$ such that the star network is efficient for all $p \in (p(c), 1)$.*

The arguments used here are analogous to those in Bala and Goyal (1999). We provide a proof in the appendix.

A comparison of Proposition 3.2 with the above result shows that there are regions of the parameter space where the conflict between efficiency and stability is not severe. Specifically, if the cost of link formation is very low or very high, or the reliability parameter p is close to 1, then efficient networks are also Nash. At the same time, a comparison of Figure 3 and Figure 4 for $n = 3$ shows that there are parameter regions where efficient networks are not Nash. For example, the region where the complete network is Nash is a strict subset of the region where it is efficient. This is to be expected, as additional links generate significant benefits for other agents by raising the overall reliability of the network, which are not taken into account in Nash behavior. The fact that Nash networks may be "underconnected" relative to the social optimum also affords a contrast with the decay model of (3.14), where Bala and Goyal show that efficient networks are Nash for most of the parameter space.

Our final result provides a parallel to Proposition 3.3 on Nash networks. It shows that for large societies, efficient networks are super-connected.

Proposition 4.3 *Suppose $c < 2p(1 - p^{n/2})$. Then an efficient network is super-connected.*

Proof: Suppose that h^o is efficient and not connected. Then there are two agents i and j such that there is no path between them in h^o, so that $p_i(j; h^o) = p_j(i; h^o) = 0$. Let h be the network formed when a link $h_{i,j} = 1$ is added, ceteris paribus. Then $p_i(j; h) = p_j(i; h) = p$. Moreover, from Lemma 2.1 all other agents payoffs either stay the same or increase. Hence, welfare increases by at least $2p - c$, which is strictly positive under the hypothesis that $c < 2p(1 - p^{n/2}))$, thus contradicting the supposition that h^o is efficient.

We now show that h^o must in fact be super-connected. The proof is by contradiction. Suppose this is not true, i.e., there exists a link $h_{i,j}$ in h which is critical. Then $h - h_{ij}$ has two components, C and C', with $i \in C$ and $j \in C'$. Let $|C| = n_1$ and $|C'| = n_2$. Suppose, without loss of generality, that $n_1 \geq n_2$. Then it follows that $n_1 \geq n/2$. Let $r \in C$, be an agent furthest away from j in g. Since agent j has only one link with agents in C, it follows that $d(j, r; g) \geq 2$. Moreover, since $k, j \in C$, it also follows that there is at least one path between k and r, which does not involve g_{ij}.

We now suppose that starting from the network g, agent i forms an additional link with agent r, ceteris paribus. Proceeding as in Proposition 3.3, agent j's expected benefit increases by at least $p(1 - p^{n_1})$. Similarly, the payoff of each of the agents $k \in C$ increases. Moreover, by Lemma 2.1, every other agent's payoff is non-decreasing in this link. The lower bound to the total increase is $p(1-p^{n_1})$. Hence, welfare rises by at least $2p(1-p^{n_1})-c$; this expression is strictly positive, by hypothesis. This contradicts the supposition that h is efficient.

$\square$

We see that if $c < 2p$, efficiency requires the presence of redundant links as n becomes large. In particular, while star networks are efficient for all n, (as demonstrated in Proposition 4.2) they require larger and larger values of p to maximize social welfare as the society expands in size.

We would like a result which shows that efficient networks are either empty or connected, as is the case with Nash networks. However, this does not seem to be an easy question to settle. Proposition 4.3 goes some distance by showing a significant parameter region where efficient networks must be super-connected. We have been unable to develop a better (and more precise) bound for c than that stated in the proposition. This is also the main difficulty in deriving a general result on connectedness of efficient networks.

It is also possible to show, via a simple continuity argument, that for high levels of reliability, an efficient network is either connected or empty. We briefly sketch the argument here: consider the model with full reliability, i.e., $p = 1$. The welfare of a minimally connected network is given by $(n - c)(n - 1)$, while the welfare from a network with $k \geq 2$ (minimal) components is given by $\sum_{i=1}^{k}(n_i - c)(n_i - 1)$, where n_i is the number of agents in component i. It is easily seen that the former is strictly greater than the latter, so long as $n > c$. Finally, note that the welfare from an empty network is 0. Thus so long as $n > c$, the minimally connected network provides a strictly higher welfare than every other network. Similarly, it can be shown that if $n < c$ then the empty network provides a strictly higher welfare than every other network. From the payoff expression (2.5), and the definition of welfare function in (4.1), it follows that the welfare function is continuous with respect to the reliability parameter p. Thus for values of p close to 1, an efficient network is either connected or empty.

We note that unlike the case with Nash networks, existence of an efficient network is not a problem as the domain of the welfare function $W(\cdot)$ is a finite set. Moreover, we see that the super-connectedness of efficient networks has been demonstrated for twice the range of c values that was shown for Nash networks. As we are concerned with total welfare, and the addition of a link provides strictly positive expected benefits to at least two agents, this is to be expected. In a very loose sense, it suggests that having redundant links is even more important for efficiency as compared to stability.

Finally, it is worthwhile to contrast the above result with the result of the information decay model (3.14). Proposition 5.5 in Bala and Goyal (1999) shows that the star is the uniquely efficient network in the region $\delta - \delta^2 < c < 2\delta + (n-2)\delta^2$. As with Nash networks, it affords a sharp contrast to what we find here.

5 Conclusion

We consider a non-cooperative model of social communication in networks where communication is costly and not fully reliable. We show that Nash networks, provided they are not empty, ensure that every agent communicates with every other agent with positive probability. If the society is large, and link formation costs are moderate, Nash networks for the most part must be 'super-connected', i.e. every link will be redundant and thus there will be more than one (distinct) path of communication between any pair of agents. This redundancy arises in response to the need for ensuring reliability in communication. Our findings contrast with the properties of a deterministic model of information decay, where Nash networks typically involve unique paths between agents.

We also study efficient networks. We show that if costs are very low or very high, or if links are highly reliable then there is virtually no conflict between efficiency and stability. However, for intermediate ranges of costs and link reliability, Nash networks may be under-connected relative to the social optimum. As with Nash networks, if the society is large, and link formation costs are moderate, efficient networks will typically have redundant links to increase reliability.

6 Appendix

Proof of Lemma 2.2: Recall from (2.2) that $B_i(h) = \sum_{h' \subset h} \lambda(h' \mid h)\mu_i(h')$ where $\lambda(h' \,|\, h) = p^{L(h')}(1-p)^{L(h)-L(h')}$. Hence $B_i(h)$ potentially involves powers of p upto degree $L(h)$. Moreover, since $\mu_i(h') > 0$ requires $L(h') > 0$, all non-zero terms in $B_i(h)$ involve p^q for some $q \geq 1$. Hence

$$B_i(h) \;=\; \sum_{k=1}^{L(h)} a_i^k p^k. \tag{6.5}$$

for some coefficients $\{a_i^k\}$. It follows that $\Pi_i(g) = B_i(h) - \mu_i^d(g)c = \sum_{k=0}^{L(h)} a_i^k p^k$ is also a polynomial of degree at most $L(h)$, with $a_i^o = -\mu_i^d(g)c$. Next, suppose that $h_{i,j} = 1$. We characterize the probability $p_i(j;h)$ that i observes j. From (2.4) this is given by $p_i(j;h) = \sum_{h' \subset h} \lambda(h' \mid h)\mathbf{1}_i(j;h')$. Consider the event $E = \{h' \subset h \mid h'_{i,j} = 1\}$. Clearly, the probability of E is p, and if E occurs then i observes j. If E does not occur (with probability $1-p$) then i may still observe j in a realization h' where there is a path between i and j involving two or more links. However, the probability of such a realization is of the form $(1-p)^{k_1}p^{k_2}$ where $k_1 \geq 1$ and $k_2 \geq 2$. Hence, such an event can only contribute terms of degree 2 or higher to $p_i(j;h)$. A similar argument shows that if $h_{i,j} = 0$ then $p_i(j;h)$ can only have terms involving p^2 or higher. Thus each j for which $h_{i,j} = 1$ contributes p (and possibly terms of higher degree) to $p_i(j;h)$ and each j for which $h_{i,j} = 0$ contributes either 0 or terms of degree higher than 1 to $p_i(j;h)$. The claim that $a_i^1 = |\{j \mid h_{i,j} = 1\}|$ follows from the above observation in conjunction with (2.4).

□

Proof of Lemma 3.1: Suppose the geodesic distance between j and i is $d(i,j;h)$. Since g is connected, there is at least one path between each agent j and agent i. Consider the shortest such path. Agent i can communicate with agent j if and only if all $d(i,j;h)$ links on the path between j and i succeed. The probability of this event is $p^{d(i,j;h)}$. Hence $p_i(j;h) \geq p^{d(i,j;h)}$ and $B_i(h) \geq \sum_{j \neq i} p_i(j;h) \geq \sum_{j \neq i} p^{d(i,j;h)}$. Since g is connected, $1 \leq d(i,j;h) \leq n-1$ for all j. If there are d_i^k agents at distance k from i, the coefficient of p^k in $B_i(h)$ will be d_i^k. The result follows.

□

Proof of Lemma 3.2: We show $\sum_{k=1}^{t}(a^k - 1)p^k \geq 0$ which is equivalent to (3.11). The proof is by induction. If $t = 1$, condition (1) and (2) imply $a^1 = 1$ so that (3.11) is trivially satisfied. Suppose for some $t \geq 1$, $\sum_{k=1}^{t}(a^k - 1)p^k \geq 0$ for all $\{a^k\}$ satisfying (1)-(3). Consider the case $t+1$, i.e. the polynomial $\sum_{k=1}^{t+1}(a^k - 1)p^k$ where $\{a^k\}$ satisfy (1)-(3). If $a^{t+1} \geq 1$, then (3) implies that $a^k \geq 1$ for all $k < t+1$. Since $\sum_{k=1}^{t+1} a^k = t+1$ from (2), we get $a^k = 1$ for all k and $\sum_{k=1}^{t+1}(a^k - 1)p^k = 0$. Suppose instead that $a^{t+1} = 0$. Then (2) implies $\sum_{k=1}^{t} a^k = t+1$. From (1), this means $a^{k'} \geq 2$ for some $k' \in \{1,\ldots,t\}$. Define $b^k = a^k$ for all $k \neq k'$ and $b^k = a^{k'} - 1$. Clearly, $\{b^k\}$ satisfy (1) and (3), while by definition of $\{b^k\}$ we have $\sum_{k=1}^{t} b^k = t$. Hence we can apply the induction step to get $\sum_{k=1}^{t}(b^k - 1)p^k \geq 0$. Since $k' \leq t$ and $p \in (0,1)$ we have $p^{k'} - p^{t+1} \geq 0$. Hence $\sum_{k=1}^{t+1}(a^k - 1)p^k = \sum_{k=1}^{t}(b^k - 1)p^k + p^{k'} - p^{t+1} \geq 0$ as required.

□

Proof of Proposition 4.1: Let h and h^o be two networks such that $L(h) \neq L(h^o)$. Using (4.1), the set of points (p,c) where $W(h) = W(h^o)$ satisfies the equation $c = 2p + Q(p)/(L(h) - L(h^o))$ for some polynomial $Q(p)$ which involves only terms of degree 2 or higher. The graph of this polynomial has Lebesgue measure 0 in $\mathcal{R}^2$ (see Halmos(1974), Exercise 4, page 145). Let U be the (finite) union of the graphs of all polynomials generated by pairs h and h^o satisfying $L(h) \neq L(h^o)$, intersected with $(0,1) \times (0,\infty)$. The result follows since U has Lebesgue measure 0.

We now consider part (b). Let

$$V = \{c \in (0, \infty) \mid W(h) = W(h^o) \text{ for } h, h^o \text{ satisfying } L(h) \neq L(h^o)\}. \tag{6.6}$$

Since the set of all networks is a finite set, V is also finite. It follows that the number of links in an efficient network is a well-defined number on the set V^c. Fix $c \in V^c$ and suppose h is efficient. Then $W(h) > W(h^o)$ for all h^o such that $L(h) \neq L(h^o)$. In particular this also holds for all h^o such that $L(h) > L(h^o)$. If $c' \in V^c$ satisfies $c' < c$, then clearly $W(h) > W(h^o)$ continues to hold for all h^o such that $L(h) > L(h^o)$.

□

Proof of Proposition 4.2: If $c = 0$ then Lemma 2.1 implies that welfare is uniquely maximized at a complete network. Part (a) follows by continuity. Part (b) follows trivially because the welfare of any non-empty network is negative for c sufficiently large. For part (c), choose $p(c)$ to ensure that $n(n-1)(1-p^{n-1}) < c$ for all $p \in (p(c), 1)$. Let h be an efficient network and suppose that C is a component of it containing at least two agents. Assume that C is minimal, i.e. that there is a unique path between any two agents in C. Let $q = |C|$. Clearly, $q \leq n$. Using Lemma 3.1, $B_i(h) = \sum_{k=1}^{q-1} d_i^k p^k$ where d_i^k is the number of agents in C at distance k from agent i. Since $p^k \geq p^{q-1}$ for each $k \leq q-1$, we have $B_i(h) \geq (q-1)p^{q-1}$. Furthermore the contribution to social benefit from the agents in C is at least $q(q-1)p^{q-1}$. Since the maximum expected benefit of any agent in C is $q(q-1)$, the addition of any links between agents in C can raise total expected benefit by no more that $q(q-1)(1-p^{q-1}) \leq n(n-1)(1-p^{n-1}) < c$ by choice of p. Hence, any component of an efficient network must be minimally connected. Within the class of networks whose components are minimally connected, the welfare function coincides with the one in the model of information decay, where payoffs are specified by (3.14). The result then follows from Proposition 5.5 of Bala and Goyal.

□

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