INVENTORY CONTROL AND REGENERATIVE PROCESSES: COMPUTATIONS

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ABSTRACT. This article is a continuation of the paper "Inventory Control and Regenerative processes: Theory" (cf. [3]) and presents closed form expressions for Laplace transforms associated with the cost functions of the classical single item inventory models with indivisible items, a fixed lead time and backlogging. For given instances of these inventory control models these Laplace transforms will be used as input functions in a newly developed Laplace transform inversion algorithm (cf. [5]) which yields highly accurate computational results of the associated cost functions.

1. Preliminaries and Notations

In a previous paper (cf. [3]) a framework for single item inventory control models based on the theory of regenerative processes is discussed. In that paper expressions for the average cost and related service measures are derived. Since these expressions are rather complicated it seems at first sight impossible to generate accurate calculations of the cost measures for instances of these inventory models. Fortunately, due to the convolution structure of the expressions both for indivisible and divisible items, it is relatively easy to derive elementary formulas for the Laplace transforms, respectively generating functions of the different functions involved in the cost measures. Applying now to these Laplace transforms the newly developed and numerically stable Laplace transform inversion algorithm (cf. [5]), we obtain, for given instances of the inventory models with an indivisible item, numerical results for the cost functions and some of the service measures. Observe, these results are accurate and of machine precision. In case inventory models with a divisible item are considered, we may want to use for numerical calculations the Fourier inversion method of Abate and Whitt (cf. [1], [2]). To keep the size of the paper limited only single item inventory models with an indivisible item will be discussed. In each section we first derive the Laplace transforms of the functions involved in the cost measures of a given model. Moreover, in the same subsection a numerical instance of this model is presented together with a plot of the average

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cost function and a service measure. Subsequently in the different subsections the periodic review (R,S), (R,s,S) and (R,s,nQ) models with a compound Poisson demand process and a fixed lead time are considered. Finally the continuous review (s,S) and (S,nQ) models with the same characteristics are discussed. Throughout this paper we assume that the demand process is a compound Poisson process with arrival rate $\lambda > 0$ and the cumulative individual demand distribution is denoted by F_Y. Although for continuous review models one may also use a more general compound renewal process (cf. [3]), the nice stochastic properties of a compound Poisson process and related to this the elementary expressions of the Laplace and Laplace-Stieltjes transforms of the total demand up to time $t > 0$ are the main reasons for using this process. Also for approximating convolution-like expressions it is helpful, although not necessary, to assume that the cumulative distribution function F_Y has a density f_Y. It should be clear that all the numerical examples satisfy this property and in particular we consider a Gamma cumulative distribution function for the individual demand of each customer. To introduce some short hand notation we observe for $h : [0,\infty) \to \mathbb{R}$, a function of bounded variation, that its Laplace transform is denoted by

$$L_h(\alpha) := \int_0^\infty \exp(-\alpha x) h(x) dx.$$

Moreover, its Laplace-Stieltjes transform is given by

$$LS_h(\alpha) = \int_{0-}^\infty \exp(-\alpha x) h(dx).$$

Clearly the parameter α is chosen in such a way that the above integrals are well defined. Also, for any function $f : \mathbb{R} \to \mathbb{R}$, vanishing on $(-\infty, 0]$, and a cumulative distribution function F on $[0,\infty)$ we introduce the convolution $f \star F : [0,\infty) \to \mathbb{R}$ given by

$$(f \star F)(x) := \int_{0-}^x f(x-y) F(dy) \tag{1.1}$$

Again we assume that f is chosen in such a way that $f \star F$ is well defined. It is now well known that

$$L_{f \star F}(\alpha) = L_f(\alpha) LS_F(\alpha) \tag{1.2}$$

and this important relation will be used throughout the rest of this paper. A special case of relation (1.1) is given by

$$\int_0^x F(y) dy = (I \star F)(x), \tag{1.3}$$

with $I(x) = x$ for every $x \geq 0$ and zero otherwise, and by relation (1.2) it is easy to see that

$$L_{I \star F}(\alpha) = \frac{LS_F(\alpha)}{\alpha^2}. \tag{1.4}$$

In what follows we present in the first three sections the necessary calculations for the periodic review models together with some instance of these models. In the remaining two sections we consider the continuous review models.

Periodic review models

As discussed in a previous paper (cf. [3]) the control rule of periodic review models is such that the inventory position is inspected after every fixed $R > 0$ time units and replenishment orders arrive at the facility L time units after the replenishment order was made. The positive constant L is called the lead time. The aggregate demand is described by a compound Poisson process $\mathbf{D} = \{\mathbf{D}(t) : t \geq 0\}$, with arrival rate $\lambda > 0$ and the costs are given by a nonnegative cost rate function $f : \mathbb{R} \to [0, \infty)$ together with ordering cost $K > 0$ for each replenishment order. To evaluate the average cost for periodic review models mainly two functions with domain $[0, \infty)$ need to be computed. For notational convenience these functions are denoted by g and h and they are given by

$$g(x) := \int_0^R (f \star F_{D(t+L)})(x)dt, \tag{1.5}$$

and

$$h(x) = \mathbb{E}\left(\int_0^{\sigma_1} f(x - \mathbf{D}(t+L))dt\right). \tag{1.6}$$

The random variable σ_1 in relation (1.6) represents a stopping time with respect to the demand process $\mathbf{D}$ and in each subsection its definition will be clear. We also observe that the function h shows up in the average cost computation of continuous review models. For the evaluation of the average number of items short, both within periodic and continuous review models, we often need to compute one particular function. This function is denoted by q_t and is given by

$$q_t(x) := \mathbb{E}(\max\{x - \mathbf{D}(t), 0\}) = (I \star F_{\mathbf{D}(t)})(x), \tag{1.7}$$

with I denoting the identity function $I(x) = x$ for every $x \geq 0$ and zero otherwise. In what follows these calculations are presented for the different periodic review models.

2. The (R, S) model and its computation

As derived in [3], the average cost $\Phi(R, S)$ for any (R, S) model is given by

$$\Phi(R, S) = \frac{K(1 - \exp(-\lambda R)) + \mathbb{E}\left(\int_0^R f(S - \mathbf{D}(t+L))dt\right)}{R}.$$

To evaluate the average cost for a given instance of the (R, S) model with R fixed, it is necessary to compute $g(x)$, where the function g is given by relation (1.5). By relation (1.2) it is easy to derive the Laplace transform L_g of g for functions f vanishing on $(-\infty, 0]$ and in the next lemma the Laplace transform L_g is derived for this special case.

Lemma 2.1. *If $f : \mathbb{R} \to [0, \infty)$ is a cost rate function satisfying $f(x) = 0$ for $x \leq 0$ with Laplace transform L_f then it follows that the Laplace transform L_g is given by*

$$L_g(\alpha) = \frac{L_f(\alpha)\exp(-\lambda L(1 - LS_{F_Y}(\alpha)))}{\lambda(1 - LS_{F_Y}(\alpha))}(1 - \exp(-\lambda R(1 - LS_{F_Y}(\alpha)))).$$

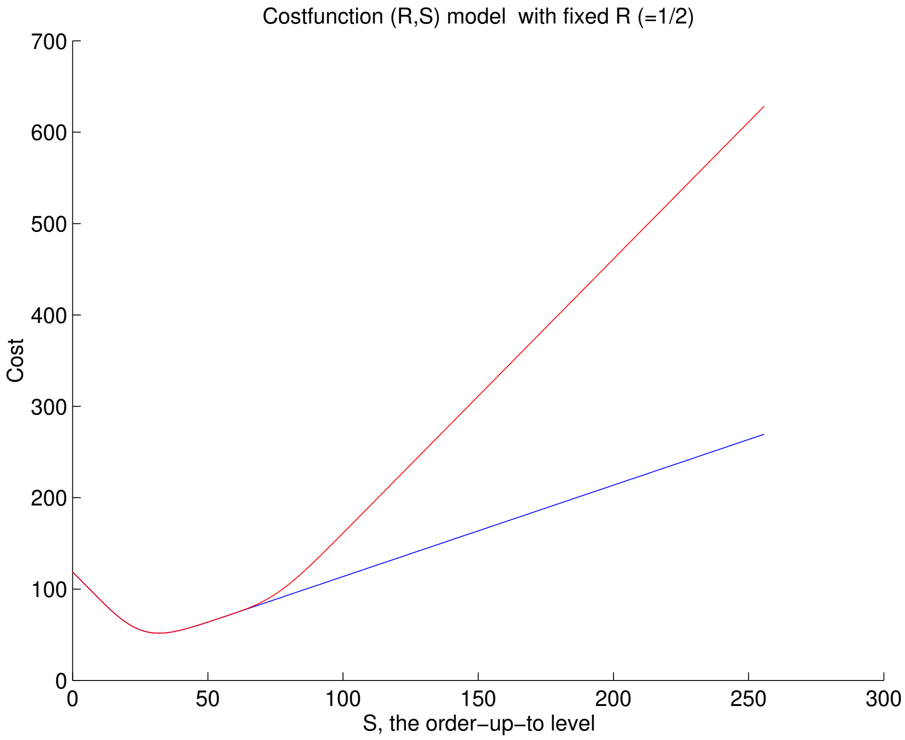

FIGURE 1. $\Phi(R,S)$ with parameters $K = 20, R = 1/2, L = 1/2, \lambda = 35, q = 50, p = 3, h_1 = 1,$ and $h_2 = 1,$ resp. $h_2 = 3$

Proof. By the definition of the function g and Fubini's theorem we obtain that

$$L_g(\alpha) = \int_0^R \int_0^\infty \exp(-\alpha x)(f \star F_{D(t+L)})(x)dxdt.$$

To evaluate the inner integral it follows by relation (1.2) that

$$\int_0^\infty \exp(-\alpha x)(f \star F_{D(t+L)})(x)dx = L_f(\alpha)LS_{F_{D(t+L)}}(\alpha).$$

Since $\mathbf{D}$ is a compound Poisson process with arrival rate $\lambda > 0$ and individual cumulative demand distribution F_Y, it is well-known for every $t > 0$ that

$$LS_{F_{D(t)}}(\alpha) = \exp(-\lambda t(1 - LS_{F_Y}(\alpha))).$$

This implies that the integral $\int_0^\infty \exp(-\alpha x)(f \star F_{D(t+L)})(x)dx$ equals

$$L_f(\alpha)\exp(-\lambda(t + L)(1 - LS_{F_Y}(\alpha)))$$

and by substituting this into the first integral the desired result follows. □

In case penalty costs are also introduced within the (R,S) model, these are mostly linear and this means that for some selected $p > 0$ the used cost rate function $f : I\!R \to I\!R$ satisfies $f(x) = -px$ for every $x \leq 0$. To reduce the case of linear penalty costs to the case of no penalty costs introduce the related cost rate function $f_1 : \mathbb{R} \to \mathbb{R}$ given by

$$f_1(x) = f(x) + px.$$

It is clear that the function f_1 vanishes on $(-\infty, 0]$. Moreover, by the above definition and the properties of the compound Poisson demand process we obtain that

$$g(x) = \int_0^R (f_1 \star F_{D(t+L)})(x)dt - p\int_0^R (x - \lambda\mu_Y(t+L))dt \tag{2.1}$$

with μ_Y denoting the expected individual demand. By Lemma 2.1 it is now possible to compute the Laplace transform of the first expression in relation (2.1) and this is given by

$$\frac{L_{f_1}(\alpha)\exp(-\lambda L(1 - LS_{F_Y}(\alpha)))}{\lambda(1 - LS_{F_Y}(\alpha))}(1 - \exp(-\lambda R(1 - LS_{F_Y}(\alpha)))), \tag{2.2}$$

with $L_{f_1}(\alpha) = L_f(\alpha) + p/\alpha^2$. Moreover, the second part of relation (2.1) equals

$$\frac{p\lambda\mu_Y}{2}((R+L)^2 - L^2) - pxR. \tag{2.3}$$

Hence it is easy to evaluate the function g by applying the numerical Laplace transform inversion algorithm to the Laplace transform in relation (2.2) and adding the elementary expression in relation (2.3). For completeness we also mention that the above procedure can also be applied to penalty costs given by the polynomial $-p(x)$ for every $x \leq 0$. In this case the cost rate function f_1 is given by $f_1(x) = f(x) + p(x)$ and this cost rate function vanishes on $(-\infty, 0]$. Moreover, it follows that

$$L_{f_1}(\alpha) = L_f(\alpha) + L_p(\alpha)$$

and for any compound Poisson demand process the expression

$$\int_0^R \mathbb{E}\left(p(x - \mathbf{D}(t+L))\right) dt$$

can be computed by means of some recurrent formulas.

Consider the piecewise linear cost rate function given by

$$f(x) = \begin{cases} -px & \text{if } x < 0 \\ h_1 x & \text{if } 0 \leq x \leq q \\ h_1 q + h_2(x - q) & \text{if } x \geq q \end{cases} \tag{2.4}$$

where q denotes a critical level of inventory, from which the inventory holding cost increases to h_2 per unit ($h_2 > h_1 > 0$). Using the previous results, we have computed the objective function $\Phi(R,S)$ of the (R,S) model, with cost rate function given by (2.4), $R = 1/2$ and gamma distributed individual demands with shape parameter 2.5 and scale parameter 2.5 (cf. [7]) and these calculations are shown in Figure 1. Moreover, in the same figure we have plotted $\Phi(R,S)$ with the cost rate function (2.4) for the special case of $h_1 = h_2$. The graph of $\Phi(R,S)$ with cost rate function given by (2.4) and $h_2 > h_1$ is naturally steeper, since the holding costs increase to $h_2 > h_1$ if the on hand inventory exceeds $q = 50$ items.

Figure 2 shows the costfunction $\Phi(R,S)$ with varying R and S, for the special piecewise linear costrate function f defined by equation (2.4). Observe, that in the special case of the (R,S) policy, for fixed R the cost function is convex in S. To determine the optimal R and S we can also use the built-in optimization routine of

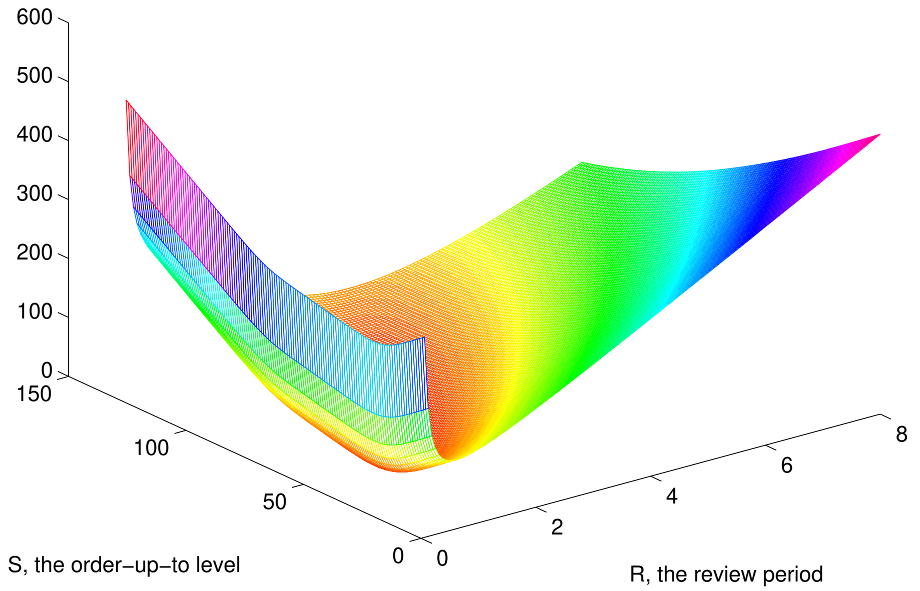

FIGURE 2. $\Phi(R,S)$ with parameters $K = 20, L = 1/2, \lambda = 35, q = 50, p = 3, h_1 = 1, h_2 = 3$

Matlab (version 5.2.0.3084). Hence, once we calculated the cost function, it is easy to find the optimal (R, S) policy.

Finally we compute for the (R, S) model the so-called average number β_2 of items short. As shown in [3] this is given by

$$\beta_2 := \frac{\mathbb{E}(\max\{\mathbf{D}(R+L)-S,0\}) - \mathbb{E}(\max\{\mathbf{D}(L)-S,0\})}{R}. \tag{2.5}$$

It is immediately clear that

$$\max\{\mathbf{D}(t)-S,0\} = \mathbf{D}(t) - S + \max\{S-\mathbf{D}(t),0\} \tag{2.6}$$

and this implies

$$\mathbb{E}(\max\{\mathbf{D}(t)-S,0\}) = \lambda\mu_Y t - S + \mathbb{E}(\max\{S-\mathbf{D}(t),0\}).$$

Hence by relation (2.5) the average number β_2 of items short can be computed by means of the Laplace transform inversion algorithm once we know the Laplace transform of the function q_t given by relation (1.7). Again this function has a convolution structure and by relation (1.4) it follows that

$$L_{q_t}(\alpha) = \frac{LS_{F_{D(t)}}(\alpha)}{\alpha^2} = \frac{\exp(-\lambda t(1-LS_{F_Y}(\alpha)))}{\alpha^2}. \tag{2.7}$$

For the same numerical example as before Figure 3 depicts a plot of the average number β_2 of items short. From Figure 3 it is easy to see that the β_2 service measure is extremely sensitive with respect to the value of S. This means that the

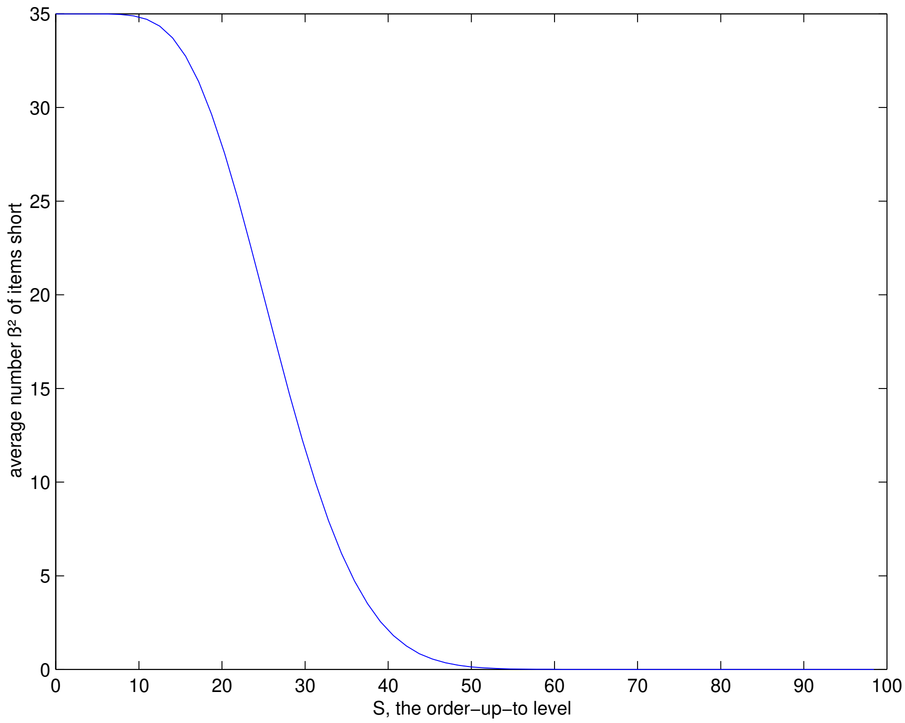

FIGURE 3. Average number β_2 of items short with parameters $R = 1/2, L = 1/2, \lambda = 35$ and shape resp. scale parameters $\alpha = \beta = 2.5$

approximation methods used within the inventory control literature (cf. [6]) will generate in general inaccurate predictions of the β_2 service measure.

3. The (R, s, S) model and its computation

As derived in [3], it follows that the average cost $\Phi(R, s, S)$ for any (R, s, S) model is given by

$$\Phi(R, s, S) = \frac{K + \mathbb{E}\left(\int_0^{\boldsymbol{\sigma}_1} f(S - \mathbf{D}(t + L))dt\right)}{U_0(S - s)R},$$

with $U_0 : [0, \infty) \to \mathbb{R}$ denoting the well known renewal function with a renewal in zero, belonging to the renewal sequence $\{\mathbf{D}(nR) : n \in \mathbb{N} \cup \{0\}\}$ and

$$\sigma_1 = R\tau, \tag{3.1}$$

with

$$\tau := \min\{n \in \mathbb{N} \cup \{0\} : \mathbf{D}(nR) > S - s\}.$$

To evaluate the average cost for a given instance of the (R, s, S) model with the decision variables R, s, S fixed, it is necessary to evaluate the renewal function U_0

and the function h defined by (1.6). It is well-known that the renewal function U_0 is given by

$$U_0(x) = \sum_{k=0}^{\infty} F_{D(R)}^{k*}(x) \tag{3.2}$$

and this yields by a standard argument that its Laplace transform L_{U_0} equals

$$L_{U_0}(\alpha) = \frac{1}{\alpha(1 - LS_{F_{D(R)}}(\alpha))} = \frac{1}{\alpha(1 - \exp(-\lambda R(1 - LS_{F_Y}(\alpha)))}$$

Hence by the Laplace transform inversion algorithm it is possible to calculate the value $U_0(S-s)$ and so for the computation of the average cost we only need to evaluate the function h.

Lemma 3.1. *For fixed values of R, s and S it follows for every $x \geq S - s$ that the function h introduced by relation* (1.6) *equals*

$$h(x) = \int_{0-}^{S-s} g(x-y) U_0(dy)$$

with U_0 and g given by relations (3.2), respectively (1.5).

Proof. By the definition of the random variable σ_1 given by relation (3.1) we obtain that the value $h(x)$ equals

$$h(x) = \sum_{n=1}^{\infty} \sum_{k=1}^{n} \mathbb{E}\left(\int_{(k-1)R}^{kR} f(x - \mathbf{D}(t+L)) \mathbf{1}_{\{\tau = n\}} dt \right).$$

Reversing first the summation signs and applying then Fubini's theorem reduces the above expression to

$$h(x) = \sum_{k=1}^{\infty} \int_{(k-1)R}^{kR} \mathbb{E}(f(x - \mathbf{D}(t+L)) \mathbf{1}_{\{\tau \geq k\}}) dt.$$

By the definition of the random variable τ and substituting $t \to t + (k-1)R$ the above integral simplifies to

$$\sum_{k=1}^{\infty} \int_{0}^{R} \mathbb{E}(f(x - \mathbf{D}(t + (k-1)R + L)) \mathbf{1}_{\{\mathbf{D}((k-1)R) \leq S-s\}}) dt$$

and by the stationary and independent increments of the compound Poisson process we obtain

$$h(x) = \sum_{k=1}^{\infty} \int_{0}^{R} \int_{0-}^{S-s} \mathbb{E}(f(x - y - \mathbf{D}(t+L))) F_{\mathbf{D}((k-1)R)}(dy) dt.$$

Applying again Fubini's theorem and using the definition of the renewal function U_0 it finally follows that

$$h(x) = \int_{0-}^{S-s} \int_{0}^{R} \mathbb{E}(f(x - y - \mathbf{D}(t+L))) dt U_0(dy)$$

and this proves the desired result. □

Clearly for linear penalty costs and using the same notation as in the section of the (R,S) model we obtain by Lemma 3.1 and relation (2.1) that the value $h(S)$ equals

$$\int_{0-}^{S-s} g_1(S-y)U_0(dy) + bU_0(S-s) - pR\int_{0-}^{S-s}(S-s-y)U_0(dy). \tag{3.3}$$

In relation (3.3) the constant b dependent on the given values of R, s, S, is given by

$$b = \frac{p\lambda\mu_Y}{2}((R+L)^2 - L^2) - pRS$$

while the function $g_1 : [0,\infty) \to I\!R$ is defined by

$$g_1(x) := \int_0^R (f_1 \star F_{D(t+L)})(x)dt.$$

By relation (3.3) and the expression for the average cost it follows that the value $\Phi(R,s,S)$ reduces to

$$\frac{b}{R} + \frac{K + \int_{0-}^{S-s} g_1(S-y)U_0(dy) - pR\int_{0-}^{S-s}(S-s-y)U_0(dy)}{RU_0(S-s)}. \tag{3.4}$$

To evaluate the function

$$x \longrightarrow \int_{0-}^{x}(x-y)U_0(dy)$$

we observe that it equals $(I\star U_0)(x)$ and hence by relation (1.4) its Laplace transform evaluated at α is given by

$$\frac{LS_{U_0}(\alpha)}{\alpha^2} = \frac{1}{\alpha^2(1-\exp(-\lambda R(1-LS_{F_Y}(\alpha)))}.$$

This enables us to compute this function also by the Laplace transform inversion algorithm. It now follows that

$$\int_{0-}^{S-s} g_1(S-y)U_0(dy) = g_1(S) + \int_{0+}^{S-s} g_1(S-y)U_0(dy)$$

and since $g_1(S)$ is computed in the section on the (R,S) model the last value in relation (3.4) to be calculated is given by

$$\int_{0+}^{S-s} g_1(S-y)U_0(dy) \tag{3.5}$$

In case the cumulative distribution function F_Y of the individual demand has a density f_Y it is well known that the renewal function U_0 is differentiable on $(0,\infty)$ with derivative $u_0(y)$ and so the expression in relation (3.5) reduces to

$$\int_{0}^{S-s} g_1(S-y)u_0(y)dy \tag{3.6}$$

Since for both functions g_1 and u_0 in the above "almost convolution integral" the Laplace transforms are easy to compute one can calculate them by means of the Laplace transform inversion algorithm. Observe in this algorithm we construct for both functions and given step size $h > 0$ two piecewise polynomials P_1 and P_2 given by

$$P_i(t) = \sum\nolimits_{j=0}^{n}\sum\nolimits_{k=0}^{n_i}\omega_{kj}^{(i)}p_k(\frac{t}{h}-j),\ i=1,2 \tag{3.7}$$

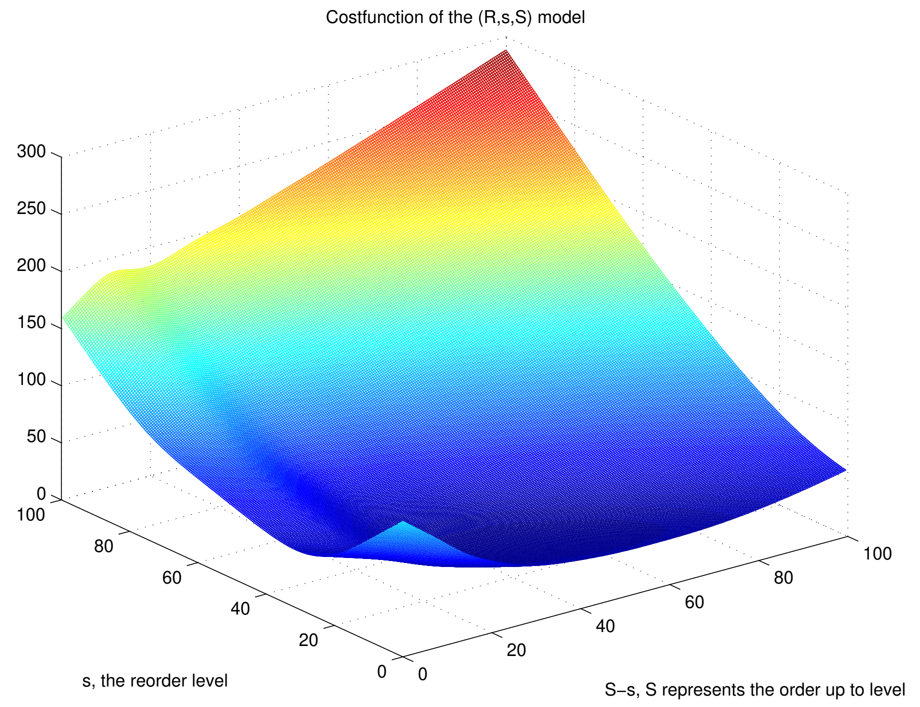

FIGURE 4. $\Phi(R, s, S)$ with parameters $K = 20$, $R = 1/2$, $L = 1/2$, $\lambda = 35$, $q = 50$, $p = 3$, $h_1 = 1$, $h_2 = 3$

with

$$p_k(\frac{t}{h} - j) = \phi_k(\frac{t}{h} - j) 1_{\{j \leq \frac{t}{h} \leq j+1\}}$$

and $\phi_k : [0, 1] \to R$ the kth Legendre polynomial on $[0, 1]$ [1] (cf. [4]). These piecewise polynomials serve as a finite dimensional representation of the functions to be approximated on $[0, n+1]$. In case the value $S - s = mh$ and $s = nh$ it follows that the integral $\int_0^{S-s} g_1(S - y) u_0(y) dy$ is approximately equal to

$$\int_0^{mh} P_1(nh - y) P_2(y) dy = h \int_0^m P_1((n - t)h) P_2(th) dt$$

and by substituting the expressions of $P_i, i = 1, 2$ given by relation (3.7) into this equality we obtain

$$h \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \sum_{j=0}^{m-1} \int_0^1 \omega^{(1)}_{k_1, n-1-j} \phi_{k_1}(1 - t) \omega^{(2)}{}_{k_2, j} \phi_{k_2}(t) dt. \tag{3.8}$$

It is well known (cf. [4]) that the relation $\phi_k(t) = (-1)^k \phi_k(1 - t)$ holds for every $0 \leq t \leq 1$ and that the set of Legendre polynomials $\{\phi_k : k \in N \cup \{0\}\}$ are

[1] By definition a Legendre polynomial is defined on [-1,1]. In this case we consider the scaled version on [0,1] by applying the mapping t→(t+1)/2

orthonormal with respect to the inner product given by

$$< f, g >= \int_0^1 f(t)g(t)dt$$

By these observations it follows that the value in relation (3.8) is equal to

$$h \sum_{k=0}^{\min\{n_1,n_2\}} (-1)^k \sum_{j=0}^{m-1} \omega_{k,n-1-j}^{(1)} {\omega^{(2)}}_{k,j}$$

and this yields a simple formula for the expression in relation (3.8). We consider the same numerical instance as given for the (R, S) model, with the piecewise linear cost rate function given by (2.4), and we plotted in Figure 4 for $R = 1/2$ fixed the average cost function with respect to the reorder level s and the difference $S - s$.

Finally we compute for the (R, s, S) model the average number β_2 of items short. As derived in [3], this is given by the expression:

$$\beta_2 = \frac{\mathbb{E}(\max\{\mathbf{D}(\sigma_1 + L) - S, 0\}) - \mathbb{E}(\max\{\mathbf{D}(L) - S, 0\})}{RU_0(S - s)} \tag{3.9}$$

By the same argument as in (2.6), the random variable $\max\{\mathbf{D}(\sigma_1 + L) - S, 0\} - \max\{\mathbf{D}(L) - S, 0\}$ equals

$$\mathbf{D}(\sigma_1 + L) - \mathbf{D}(L) + \max\{S - \mathbf{D}(\sigma_1 + L), 0\} - \max\{S - \mathbf{D}(L), 0\}. \tag{3.10}$$

Since $\mathbf{D}$ is a compound Poisson process and σ_1 is a stopping time with respect to $\mathbf{D}$ we obtain (cf. [3]) that the shifted stochastic process

$$\mathbf{D}_{\sigma_1} := \{\mathbf{D}(\sigma_1 + t) - \mathbf{D}(\sigma_1) : t \geq 0\}$$

is again a compound Poisson process independent of $\{\mathbf{D}(t) : t \leq \sigma_1\}$. By this observation it follows that

$$\mathbb{E}(\exp(-\alpha\mathbf{D}(\sigma_1 + L))) = \mathbb{E}(\exp(-\alpha\mathbf{D}(\sigma_1)))\exp(-\lambda L(1 - LS_{F_Y}(\alpha))) \tag{3.11}$$

By the same argument we obtain

$$\mathbb{E}(\mathbf{D}(\sigma_1 + L)) = \mathbb{E}(\mathbf{D}(\sigma_1)) + \mathbb{E}(\mathbf{D}(L)) = \mathbb{E}(\mathbf{D}(\sigma_1)) + \lambda L\mu_Y. \tag{3.12}$$

In order to compute $\mathbb{E}(\mathbf{D}(\sigma_1))$ we observe by the definition of σ_1 that

$$\mathbf{D}(\sigma_1) = S - s + \mathbf{B}(S - s), \tag{3.13}$$

with $\mathbf{B} = \{\mathbf{B}(t) : t \geq 0\}$ denoting the residual life process associated with the renewal sequence $\{\mathbf{D}(nR) : n \in \mathbb{N} \cup \{0\}\}$. It is well known for every $t > 0$ that

$$\mathbb{E}(t + \mathbf{B}(t)) = \mathbb{E}(\mathbf{D}(R))U_0(t) = \lambda R\mu_Y U_0(t)$$

and hence by relation (3.13) it follows

$$\mathbb{E}(\mathbf{D}(\sigma_1)) = \lambda R\mu_Y U_0(S - s). \tag{3.14}$$

This implies by relations (3.10), (3.12) and (3.14) that the average number β_2 of items short given by relation (3.9) equals

$$\beta_2 = \lambda\mu_Y + \frac{\mathbb{E}(\max\{S - \mathbf{D}(\sigma_1 + L), 0\}) - \mathbb{E}(\max\{S - \mathbf{D}(L), 0\})}{RU_0(S - s)}.$$

Observe by relation (1.7) the value $\mathbb{E}(\max\{x - \mathbf{D}(L), 0\}$ equals $q_L(S)$ and since the Laplace transform of the function q_L is given by relation (2.7), the value $q_L(S)$ can be easily computed by means of the Laplace transform inversion algorithm. Therefore, in order to evaluate β_2 it only remains to compute the expression

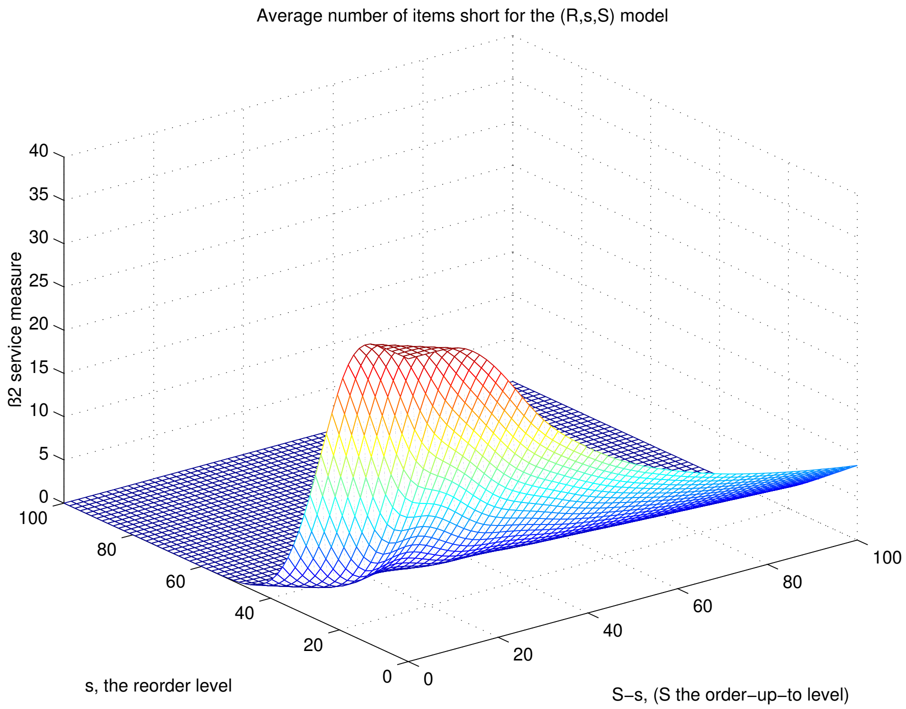

FIGURE 5. Average number β_2 of items short with parameters $R = 1/2, L = 1/2, \lambda = 35$ and shape resp. scale parameters $\alpha = \beta = 2.5$

$\mathbb{E}(\max\{x - \mathbf{D}(\sigma_1 + L), 0\})$, and to achieve this we will evaluate the two-dimensional Laplace transform of the function $q : [0, \infty) \times [0, \infty) \to \mathbb{R}$ given by

$$q(x, y) = \mathbb{E}(\max\{x - \mathbf{D}(\sigma(y) + L), 0\}),$$

with

$$\sigma(y) := \inf\{nR : \mathbf{D}(nR) > y,\ n \in \mathbb{N}\}. \tag{3.15}$$

It is easy to see that

$$\mathbb{E}(\max\{S - \mathbf{D}(\sigma_1 + L), 0\}) = q(S, S - s),$$

and to compute the two dimensional Laplace transform L_q of q given by the expression

$$L_q(\alpha, \beta) := \int_0^\infty \int_0^\infty \exp(-\alpha x - \beta y) q(x, y) dx dy, \tag{3.16}$$

we first focus our attention on the integral $\int_0^\infty \exp(-\alpha x) q(x, y) dx$. By the definition of q it is clear that

$$\int_0^\infty \exp(-\alpha x) q(x, y) dx = \int_0^\infty \exp(-\alpha x)(I \star F_{D(\sigma(y)+L)})(x) dx$$

and this implies by relation (1.4) that

$$\int_0^\infty \exp(-\alpha x) q(x, y) dx = \frac{\mathbb{E}(\exp(-\alpha \mathbf{D}(\sigma(y) + L)))}{\alpha^2}. \tag{3.17}$$

By relation (3.11), with $\sigma_1 = \sigma(S-s)$ replaced by $\sigma(y)$, it follows similarly

$$\mathbb{E}(\exp(-\alpha\mathbf{D}(\sigma(y)+L))) = \frac{\exp(-\lambda L(1-LS_{F_Y}(\alpha)))}{\alpha^2}\mathbb{E}(\exp(-\alpha\mathbf{D}(\sigma(y)))),$$

and so by relation (3.17) this yields

$$L_q(\alpha,\beta) = \frac{\exp(-\lambda L(1-LS_{F_Y}(\alpha)))}{\alpha^2}L_{q_\alpha}(\beta), \tag{3.18}$$

with $q_\alpha\colon [0,\infty) \to \mathbb{R}$ given by

$$q_\alpha(y) = \mathbb{E}(\exp(-\alpha\mathbf{D}(\sigma(y)))). \tag{3.19}$$

To evaluate $q_\alpha(y)$ we observe by the definition of the stopping time $\sigma(y)$ that it equals

$$\mathbb{E}\left(\exp(-\alpha(\mathbf{D}(\sigma(y))))1_{\{\mathbf{D}(R)>y\}}\right) + \mathbb{E}\left(\exp(-\alpha(\mathbf{D}(\sigma(y))))1_{\{\mathbf{D}(R)\leq y\}}\right). \tag{3.20}$$

Since $\mathbf{D}$ is a compound Poisson process it follows by the renewal argument that

$$\mathbb{E}\left(\exp(-\alpha(\mathbf{D}(\sigma(y)))) \mid \mathbf{D}(R)=x\right) = \mathbb{E}\left(\exp(-\alpha(x+\mathbf{D}(\sigma(y-x))))\right)$$

for every $0 \leq x \leq y$ and this implies by relation (3.20) that

$$q_\alpha(y) = \mathbb{E}\left(\exp(-\alpha\mathbf{D}(R))1_{\{\mathbf{D}(R)>y\}}\right) + (q_\alpha \star F_{\alpha,\mathbf{D}(R)})(y) \tag{3.21}$$

with

$$F_{\alpha,\mathbf{D}(R)}(y) := \int_{0-}^{y} \exp(-\alpha z)F_{D(R)}(dz).$$

It is now easy to verify that

$$\textstyle\int_0^\infty \exp(-\beta y)\mathbb{E}\left(\exp(-\alpha\mathbf{D}(R))1_{\{\mathbf{D}(R)>y\}}\right)dy \tag{3.22}$$

is given by the expression

$$\frac{LS_{F_{\mathbf{D}(R)}}(\alpha) - LS_{F_{\mathbf{D}(R)}}(\alpha+\beta)}{\beta}. \tag{3.23}$$

This implies by relations (3.21) and (1.2) that

$$L_{q_\alpha}(\beta) = \frac{LS_{F_{\mathbf{D}(R)}}(\alpha) - LS_{F_{\mathbf{D}(R)}}(\alpha+\beta)}{\beta} + L_{q_\alpha}(\beta)LS_{F_{\mathbf{D}(R)}}(\alpha+\beta).$$

Hence it follows that

$$L_{q_\alpha}(\beta) = \frac{LS_{F_{\mathbf{D}(R)}}(\alpha) - LS_{F_{\mathbf{D}(R)}}(\alpha+\beta)}{\beta(1-LS_{F_{\mathbf{D}(R)}}(\alpha+\beta))},$$

and substituting this expression into relation (3.18) finally yields

$$L_q(\alpha,\beta) = \frac{\exp(-\lambda L(1-LS_{F_Y}(\alpha)))}{\alpha^2}\frac{LS_{F_{D(R)}}(\alpha) - LS_{F_{D(R)}}(\alpha+\beta)}{\beta(1-LS_{F_{D(R)}}(\alpha+\beta))}. \tag{3.24}$$

The function q is obtained by inverting the Laplace transform given by relation (3.24) with the Laplace transform inversion algorithm for two dimensional Laplace transforms (cf. [5]).

4. The (R,s,nQ) model and its computation

As derived in [3], the average cost $\Phi(R,s,Q)$ for any (R,s,nQ) model is given by

$$\Phi(R,s,Q)=\frac{K(1-\mathbb{E}F_{D(R)}(Q\mathbf{U}))+\mathbb{E}\left(\int_0^R f(s+Q\mathbf{U}-\mathbf{D}(t+L))dt\right)}{R},$$

if the inventory position process $\mathbf{IP}=\{\mathbf{IP}(t):t\geq 0\}$ satisfies $\mathbf{IP}_0\stackrel{d}{=}s+Q\mathbf{U}$ (cf. [3]) and the random variable $\mathbf{U}$ is uniformly distributed on $(0,1]$ and independent of the demand process $\mathbf{D}$. Through the symmetry property of the uniform distribution ($\mathbf{U}\stackrel{d}{=}1-\mathbf{U}$) we obtain that the average cost equals

$$\frac{K(1-\mathbb{E}F_{D(R)}(Q\mathbf{U}))+\mathbb{E}\left(\int_0^R f(s+Q-(\mathbf{D}(t+L)+Q\mathbf{U}))dt\right)}{R},$$

and this expression is more convenient for calculation purposes since it has a convolution structure. To evaluate the average cost for a given instance of the (R,s,nQ) model with the decision variables R,s,Q fixed, it is necessary to evaluate the expressions

$$\mathbb{E}F_{D(R)}(Q\mathbf{U}) \tag{4.1}$$

and the expression

$$\mathbb{E}\left(\int_0^R f(s+Q-(\mathbf{D}(t+L)+Q\mathbf{U}))dt\right), \tag{4.2}$$

Observe that

$$\mathbb{E}\left(F_{D(R)}(Q\mathbf{U})\right)=\frac{1}{Q}\int_0^Q F_{D(R)}(x)dx=\frac{1}{Q}(I\star F_{D(R)})(Q),$$

and by relation (1.7) it follows that

$$\mathbb{E}\left(F_{D(R)}(Q\mathbf{U})\right)=\frac{1}{Q}\,q_R(Q). \tag{4.3}$$

Since the Laplace transform of q_R is given by relation (2.7), the value $q_R(Q)$ is easy to compute with the Laplace transform inversion algorithm and by relation (4.3) it follows that $\mathbb{E}\left(F_{D(R)}(Q\mathbf{U})\right)$ can be computed. Moreover, by the independence of the random variables $\mathbf{U}$ and $\mathbf{D}$ it follows that

$$\int_0^R \mathbb{E}f(x-Q\mathbf{U}-\mathbf{D}(t+L))dt=\mathbb{E}g(x-Q\mathbf{U}), \tag{4.4}$$

with the function $g\colon[0,\infty)\to\mathbb{R}$ given by relation (1.5). By this observation the next lemma is easy to prove.

Lemma 4.1. *For fixed values of R,s and Q and the function $g:[0,\infty)\to\mathbb{R}$ defined by (1.5) it follows for every $x\geq Q$ that*

$$\int_0^R \mathbb{E}f(x-Q\mathbf{U}-\mathbf{D}(t+L))dt=\frac{(I\star g)(x)-(I\star g)(x-Q)}{Q}. \tag{4.5}$$

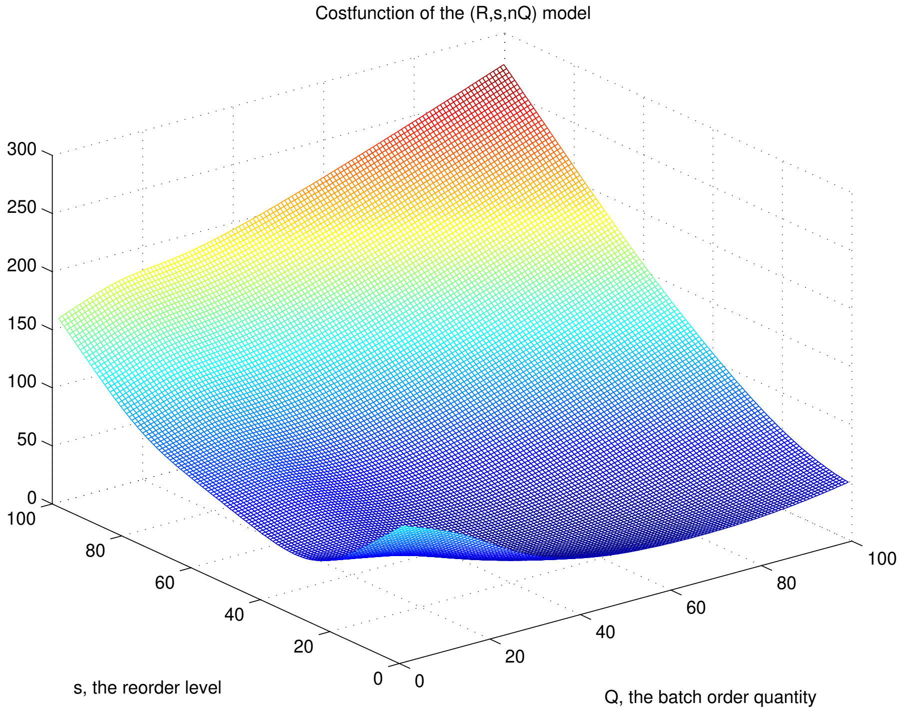

FIGURE 6. $\Phi(R,s,Q)$ parameters $K=20$, $R=1/2, L=1/2$, $\lambda=35$, $q=50$, $p=3$, $h_1=1$, $h_2=3$

Proof. Conditioning on the uniformly distributed random variable $\mathbf{U}$ we obtain by relation (4.4) that

$$\int_0^R \mathbb{E}f(x-Q\mathbf{U}-\mathbf{D}(t+L))dt = \frac{1}{Q}\int_0^Q g(x-u)du = \frac{1}{Q}\int_{x-Q}^x g(v)dv, \tag{4.6}$$

and the result follows immediately by relation (1.3). □

By Lemma 2.1 it is easy to verify that the Laplace transform of $(I\star g)$ equals

$$\frac{L_g(\alpha)}{\alpha} = \frac{L_f(\alpha)\exp(-\lambda L(1-LS_{F_Y}(\alpha)))}{\alpha\lambda(1-LS_{F_Y}(\alpha))}(1-\exp(-\lambda R(1-LS_{F_Y}(\alpha)))). \tag{4.7}$$

Hence the expression in (4.2) can be computed by inverting the Laplace transform given by (4.7) and substituting it into relation (4.5). We consider the same numerical instance as given for the (R,s,S) model, with the piecewise linear cost rate function given by (2.4), and we plotted in Figure 5 the average cost function $\Phi(R,s,Q)$ for $R=1/2$ with respect to the reorder level s and the batch order quantity Q.

Finally we compute for the (R,s,nQ) model the so-called average number β_2 of items short. As shown in [3] this is given by

$$\beta_2 := \frac{\mathbb{E}(\max\{\mathbf{D}(R+L)-s-Q\mathbf{U},0\}) - \mathbb{E}(\max\{\mathbf{D}(L)-s-Q\mathbf{U},0\})}{R} \tag{4.8}$$

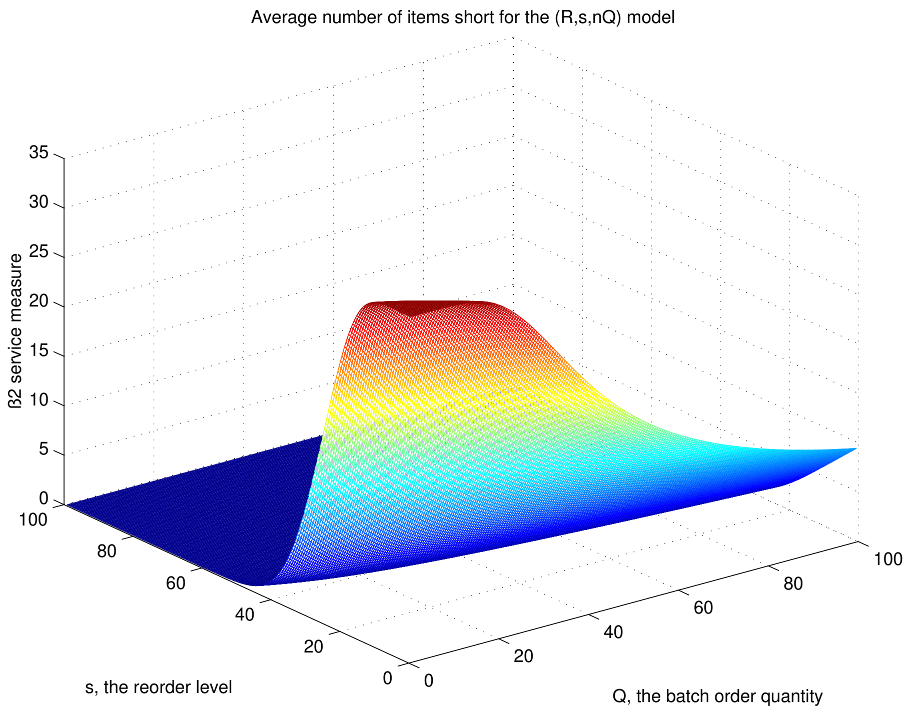

FIGURE 7. Average number β_2 of items short with parameters $R = 1/2, L = 1/2, \lambda = 35$ and shape resp. scale parameters $\alpha = \beta = 2.5$

It is immediately clear, using again the symmetry property of the uniformly distributed random variable that

$$\max\{\mathbf{D}(t) - s - Q\mathbf{U}, 0\} \stackrel{d}{=} \max\{\mathbf{D}(t) + Q\mathbf{U} - s - Q, 0\}$$

and the random variable $\max\{\mathbf{D}(t) + Q\mathbf{U} - s - Q, 0\}$ equals

$$\mathbf{D}(t) + Q\mathbf{U} - (s+Q) + \max\{s + Q - (\mathbf{D}(t) + Q\mathbf{U}), 0\}.$$

This implies that its expectation is given by

$$\mathbb{E}(\mathbf{D}(t) + Q\mathbf{U}) + \mathbb{E}(\max\{s + Q - (\mathbf{D}(t) + Q\mathbf{U}), 0\})$$

and hence the average number β_2 of items short equals

$$\lambda\mu_Y + \frac{\mathbb{E}(\max\{s + Q - (\mathbf{D}(R+L) + Q\mathbf{U}), 0\}) - \mathbb{E}(\max\{s + Q - (\mathbf{D}(L) + Q\mathbf{U}), 0\})}{R}$$

By the independence of the random variable $\mathbf{U}$ and the demand process $\mathbf{D}$ it follows that

$$\mathbb{E}(\max\{x - (\mathbf{D}(t) + Q\mathbf{U}), 0\} = \mathbb{E}\left(q_t(x - Q\mathbf{U})\right)$$

where q_t is given by relation (1.7). Because of the nice convolution structure of this formula, the average number β_2 of items short can be computed by means of the Laplace transform inversion algorithm once we can compute the expression $I\!E\,(q_t(x-Q\mathbf{U}))$. We can conclude in the same way as in Lemma 4.1 that

$$I\!E\,(q_t(x-Q\mathbf{U})) = \frac{(I \star q_t)(x) - (I \star q_t)(x-Q)}{Q} \tag{4.9}$$

and by relation (1.4) and (1.7) the Laplace transform of $(I \star q_t)$ is given by

$$\frac{1}{\alpha} L_{q_t}(\alpha) = \frac{\exp(-\lambda t(1 - LS_{F_Y}(\alpha)))}{\alpha^3}. \tag{4.10}$$

Hence $I\!E\,(q_t(x-Q\mathbf{U}))$ can be calculated by inverting the Laplace transform (4.10) and substituting it into (4.9).

Continuous review models

As discussed in a previous paper (cf. [3]) the control rule of continuous review models is such that the inventory position is inspected at every moment in time and replenishment orders arrive at the facility L time units after the replenishment order was made. The aggregate demand is described by a compound Poisson process $\mathbf{D} = \{\mathbf{D}(t) : t \geq 0\}$, with arrival rate $\lambda > 0$. Note that, as we mentioned in the introduction, although for continuous review models one may also use a more general compound renewal process (cf. [3]) the nice stochastic properties of a compound Poisson process and related to this the elementary expressions for the Laplace and Laplace-Stieltjes transforms of the total demand up to time t are the main reasons for using this process. The costs are given by a nonnegative cost rate function $f : I\!R \to [0,\infty)$ together with ordering cost $K > 0$ for each replenishment order. To evaluate the average cost for continuous review models mainly the functions g_t and h are needed, where h was already defined by relation (1.6) and g_t: $[0,\infty) \to I\!R$ is given by

$$g_t(x) := (f \star F_{D(t)})(x). \tag{4.11}$$

This function is obviously related to the function g, defined by (1.5) for the periodic review models.

5. The (s,S) model and its computation

As derived in [3], it follows that the average cost $\Phi(s,S)$ for any (s,S) model is given by

$$\Phi(s,S) = \frac{K + I\!E\left(\int_0^{\sigma_1} f(S - \mathbf{D}(t+L))dt\right)}{I\!E(\mathbf{T}_1)U_0(S-s)}$$

with $U_0 : [0,\infty) \to I\!R$ denoting the well known renewal function with a renewal in zero, belonging to the renewal sequence $\{\sum_{k=0}^{n} \mathbf{Y}_k : n \in I\!N\}$ and

$$\sigma_1 = \min\{t > 0 \,:\, \mathbf{D}(t) > S - s\}. \tag{5.1}$$

Since the aggregated demand is described by a compound Poisson process with arrival rate $\lambda > 0$, we obtain that $I\!E\mathbf{T}_1 = 1/\lambda$. To evaluate the average cost for a given instance of the (s,S) model with the decision variables s, S fixed, it is

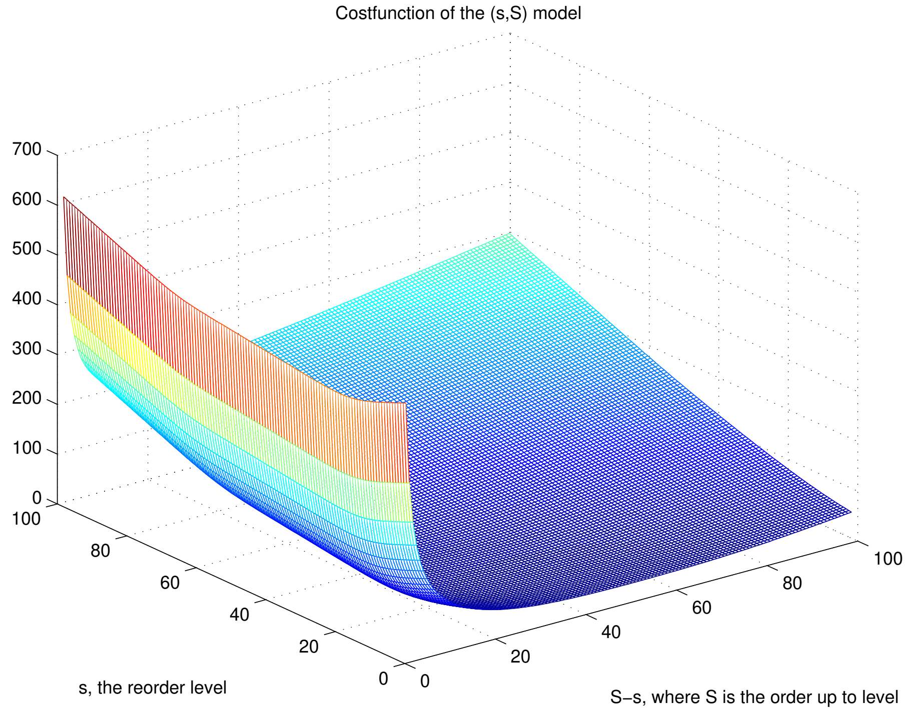

FIGURE 8. $\Phi(s,S)$ parameters $K = 20$, $L = 1/2$, $\lambda = 35$, $q = 50$, $p = 3$, $h_1 = 1$, $h_2 = 3$)

necessary to evaluate the renewal function U_0 and the function $h : [0,\infty) \to I\!R$ given by relation (1.6). It is well-known that the renewal function U_0 is given by

$$U_0(x) = \sum_{k=0}^{\infty} F_Y^{k*}(x) \tag{5.2}$$

and this yields by a standard argument that its Laplace transform L_{U_0} equals

$$L_{U_0}(\alpha) = \frac{1}{\alpha(1 - LS_{F_Y}(\alpha))}.$$

Hence by the Laplace transform inversion algorithm it is possible to calculate the value $U_0(S-s)$ and so for the computation of the average cost we only need to evaluate the function h.

Lemma 5.1. *For fixed values of s and S it follows for every $x \geq S - s$ that the function h defined by relation* (1.6) *equals*

$$h(x) = \frac{1}{\lambda}\int_{0-}^{S-s} g_L(x-y)U_0(dy)$$

with U_0 and g_L given by relations (5.2) respectively (4.11).

Proof. By the definition of the random variable σ_1 given by relation (5.1) we obtain

$$\int_0^{\infty} I\!E\left(\mathbf{1}_{\{D(t)\leq S-s\}} f(S - \mathbf{D}(t+L))\right) dt.$$

Conditioning on the number of arrival of customers the previous expression reduces to

$$h(x) = \sum_{k=0}^{\infty} \int_0^{\infty} \int_{0-}^{S-s} \mathbb{E} f(S - x - \mathbf{D}(L)) F_Y^{k\star}(dx) \mathbb{P}\{\mathbf{N}(t) = k\} dt.$$

and by Fubini's theorem this yields that $h(x)$ equals

$$\sum_{k=0}^{\infty} \int_{0-}^{S-s} \mathbb{E} f(S - x - \mathbf{D}(L)) \int_0^{\infty} \mathbb{P}\{\mathbf{N}(t) = k\} dt F_Y^{k\star}(dx).$$

Since the expected time between arrivals is $1/\lambda$ we obtain that

$$\int_0^{\infty} \mathbb{P}\{\mathbf{N}(t) = k\} dt = \mathbb{E} \int_{\mathbf{S}_k}^{\mathbf{S}_{k+1}} dt = \frac{1}{\lambda},$$

and hence it follows that

$$h(x) = \frac{1}{\lambda} \sum_{k=0}^{\infty} \int_{0-}^{S-s} \mathbb{E} f(S - x - \mathbf{D}(L)) F_Y^{k\star}(dx).$$

Applying now the monotone convergence theorem, this yields

$$h(x) = \frac{1}{\lambda} \int_{0-}^{S-s} \mathbb{E} f(S - x - \mathbf{D}(L)) U_0(dx)$$

and the desired result is verified. □

Clearly for linear penalty costs and using the same notation as in the section of the (R, S) model we obtain by Lemma 5.1 and relation (2.1) that the value $h(S)$ equals

$$\frac{1}{\lambda} \left(\int_{0-}^{S-s} g_L^1(S - y) U_0(dy) + b U_0(S - s) - p \int_{0-}^{S-s} (S - s - y) U_0(dy) \right). \tag{5.3}$$

In relation (3.3) the constant b dependent on the given values of s, S is given by

$$b = p \lambda \mu_Y 2L - pS,$$

while the function g_L^1 is given by g_L with f_1 replacing f in relation (4.11). For the numerical evaluation of these expressions one can proceed in the same way as for the (R, s, S) model.

Finally we compute for the (s, S) model the average number β_2 of items short. As derived in [3], this is given by the expression:

$$\beta_2 = \lambda \frac{\mathbb{E}(\max\{\mathbf{D}(\sigma_1 + L) - S, 0\}) - \mathbb{E}(\max\{\mathbf{D}(L) - S, 0\})}{U_0(s - S)} \tag{5.4}$$

Since the stochastic process $\mathbf{D}$ is a compound Poisson process and σ_1 is a stopping time with respect to $\mathbf{D}$ we obtain (cf. [3]) that the shifted stochastic process

$$\mathbf{D}_{\sigma_1} := \{\mathbf{D}(\sigma_1 + t) - \mathbf{D}(\sigma_1) : t \geq 0\}$$

is again a compound Poisson process independent of $\{\mathbf{D}(t) : t \leq \sigma_1\}$. Moreover, by the definition of σ_1 it follows that

$$\mathbf{D}(\sigma_1) = S - s + \mathbf{B}(S - s), \tag{5.5}$$

with $\mathbf{B} = \{\mathbf{B}(t) : t \geq 0\}$ denoting the residual life process associated with the renewal sequence $\{\sum_{k=0}^{n} \mathbf{Y}_k : n \in \mathbb{N} \cup \{0\}\}$. By these observations one can derive

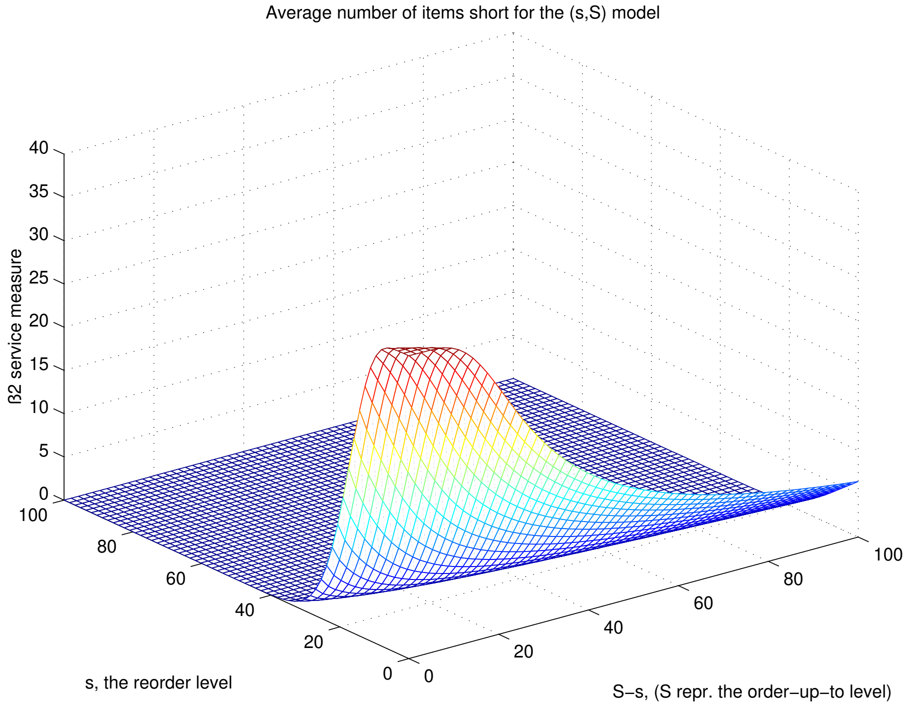

FIGURE 9. Average number β_2 of items short with parameters $L = 1/2, \lambda = 35$ and shape resp. scale parameters $\alpha = \beta = 2.5$

similarly as for the (R, s, S) model that the average number β_2 of items short given by relation (5.4) equals

$$\beta_2 = \lambda\mu_Y + \lambda\frac{I\!E(\max\{S - \mathbf{D}(\sigma_1 + L), 0\}) - I\!E(\max\{S - \mathbf{D}(L), 0\})}{U_0(S - s)}.$$

By relation (1.7) the value $I\!E(\max\{x - \mathbf{D}(L), 0\}$ equals $q_L(S)$ and was already computed for the (R, s, S) model. In order to evaluate β_2 it only remains to compute the expression $I\!E(\max\{x - \mathbf{D}(\sigma_1 + L), 0\})$, and to achieve this we will evaluate the two-dimensional Laplace transform of the function $q : [0, \infty) \times [0, \infty) \to I\!R$ given by

$$q(x, y) = I\!E(\max\{x - \mathbf{D}(\sigma(y) + L), 0\}),$$

with

$$\sigma(y) := \inf\{t \geq 0 : \mathbf{D}(t) > y\}. \tag{5.6}$$

It is easy to see that

$$I\!E(\max\{S - \mathbf{D}(\sigma_1 + L), 0\}) = q(S, S - s).$$

The two dimensional Laplace transform of the function q was already computed for the (R, s, S) model, with a different underlying renewal sequence. We only have to keep in mind that the underlying renewal sequence changes from $\{\mathbf{D}(R), \mathbf{D}(2R), \dots\}$

to $\{\mathbf{Y}_1, \mathbf{Y}_1 + \mathbf{Y}_2, \dots\}$ and so we obtain the expression

$$L_q(\alpha,\beta) = \frac{\exp(-\lambda L(1 - LS_{F_Y}(\alpha)))}{\alpha^2} \frac{LS_{F_Y}(\alpha) - LS_{F_Y}(\alpha+\beta)}{\beta(1 - LS_{F_Y}(\alpha+\beta))}. \tag{5.7}$$

The function q is now obtained by inverting the Laplace transform given by relation (5.7) with the Laplace transform inversion algorithm for two dimensional Laplace transforms (cf. [5]).

6. The (s, nQ) model and its computation

As derived in [3], it follows that the average cost $\Phi(s,Q)$ for any (s,nQ) model is given by

$$\Phi(s,Q) = \frac{K(1 - \mathbb{E}(F_Y(Q\mathbf{U}))) + \mathbb{E}\left(\int_0^{\mathbf{T}_1} f(s + Q\mathbf{U} - \mathbf{D}(t+L))dt\right)}{\mathbb{E}\mathbf{T}_1},$$

if the inventory position process $\mathbf{IP} = \{\mathbf{IP}(t) : t \geq 0\}$ satisfies $\mathbf{IP}_0 \stackrel{d}{=} s + Q\mathbf{U}$ (cf. [3]), the random variable $\mathbf{U}$ is uniformly distributed on $[0,1)$ and independent of the demand process $\mathbf{D}$. Moreover, the random variable $\mathbf{T}_1$ represents the arrival moment of the first customer. Since it is assumed in this section that the aggregate demand process is a compound Poisson process with arrival rate $\lambda > 0$, one can simplify the above formula for the average cost. By the definition of the random variable $\mathbf{T}_1$ it follows that

$$\mathbb{E}\left(\int_0^{\mathbf{T}_1} f(s + Q\mathbf{U} - \mathbf{D}(t+L))dt\right) = \mathbb{E}\left(\int_0^{\infty} f(s + Q\mathbf{U} - \mathbf{D}(t+L))1_{\{D(t)=0\}}dt\right).$$

By the stationary and independent increments of a compound Poisson process the previous expression reduces to

$$\mathbb{E}\left(\int_0^{\infty} f(s + Q\mathbf{U} - \mathbf{D}(t) - \tilde{\mathbf{D}}(L))1_{\{D(t)=0\}}dt\right),$$

with $\tilde{\mathbf{D}}(L) \stackrel{d}{=} \mathbf{D}(L)$ and $\tilde{\mathbf{D}}(L)$ independent of $\mathbf{D}(t)$. Since the random variable $\mathbf{U}$ is independent of $\mathbf{D}$ this yields

$$\mathbb{E}\left(\int_0^{\infty} f(s + Q\mathbf{U} - \tilde{\mathbf{D}}(L))1_{\{D(t)=0\}}dt\right) = \frac{1}{\lambda}\mathbb{E}f(s + Q\mathbf{U} - \mathbf{D}(L))$$

and so we obtain by the symmetry property of the uniform distribution that the average cost $\Phi(s,Q)$ reduces to

$$\lambda K(1 - \mathbb{E}(F_Y(Q\mathbf{U}))) + \mathbb{E}\left(f(s + Q - (Q\mathbf{U} + \mathbf{D}(L)))\right).$$

To evaluate the average cost for a given instance of the (s,nQ) model with the decision variables s, Q fixed, it is necessary to evaluate the expressions

$$\mathbb{E}\left(F_Y(Q\mathbf{U})\right) \tag{6.1}$$

and the expression

$$\mathbb{E}\left(f(x - (Q\mathbf{U} + \mathbf{D}(L)))\right). \tag{6.2}$$

Observe by relation (1.3) we obtain that

$$\mathbb{E}\left(F_Y(Q\mathbf{U})\right) = \frac{1}{Q}\int_0^Q F_Y(x)dx = \frac{1}{Q}(I \star F_Y)(Q),$$

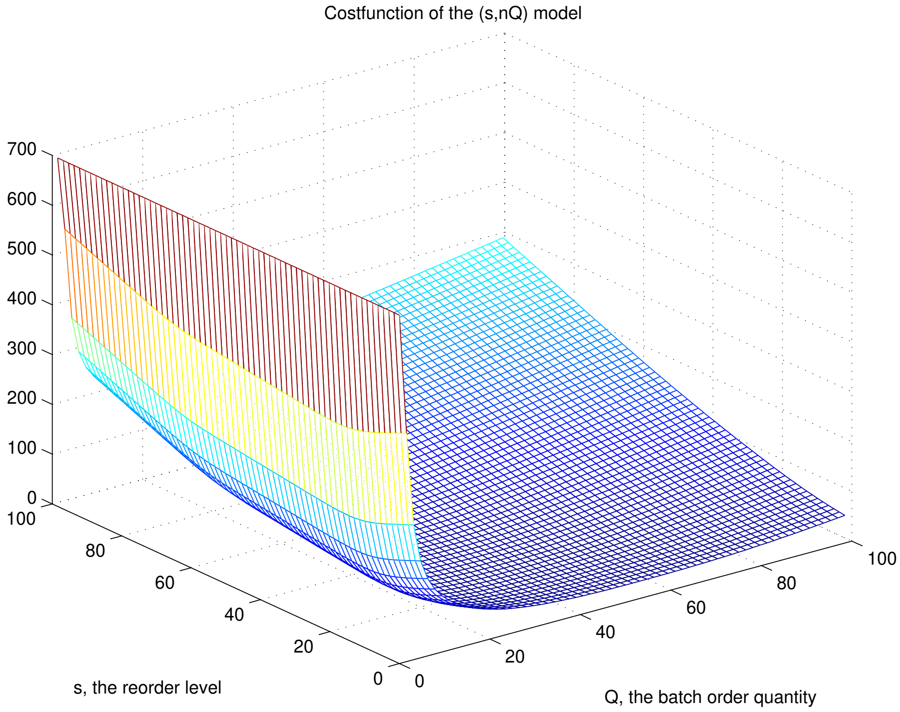

FIGURE 10. $\Phi(s,Q)$ with parameters $K = 20$, $L = 1/2$, $\lambda = 35$, $q = 50$, $p = 3$, $h_1 = 1$, $h_2 = 3$

and since by relation (1.4) and the Laplace transform inversion algorithm it is easy to compute $(I \star F_Y)(Q)$, it only remains to evaluate the expression (6.2). By the independence of the random variable $\mathbf{U}$ and the demand process $\mathbf{D}$ it follows for every $x \geq Q$ that

$$\mathbb{E}\left(f(x - (Q\mathbf{U} + \mathbf{D}(L)))\right) = \mathbb{E}g_L(x - Q\mathbf{U}), \tag{6.3}$$

with $g_L\colon [0,\infty) \to \mathbb{R}$ given by relation (4.11). It is now easy to verify the next result.

Lemma 6.1. *For fixed values of R, s and Q it follows for every $x \geq Q$ that*

$$\mathbb{E}\left(f(x - (Q\mathbf{U} + \mathbf{D}(L)))\right) = \frac{(I \star g_L)(x) - (I \star g_L)(x - Q)}{Q}. \tag{6.4}$$

with the function $g_L : [0,\infty) \to \mathbb{R}$ given by relation (4.11).

Proof. Similarly as for the (R, s, nQ) model one obtains by relation (6.3) that

$$\mathbb{E}\left(f(x - (Q\mathbf{U} + \mathbf{D}(L)))\right) = \frac{1}{Q}\int_{x-Q}^{x} g_L(v)dv$$

and the result follows immediately by relation (1.3). □

By relation (1.4) we immediately obtain that the Laplace transform of $(I \star g_L)$ is given by the expression

$$\frac{1}{\alpha} L_{g_L}(\alpha) = \frac{L_f(\alpha)\exp(-\lambda L((1 - LS_{F_Y}(\alpha))))}{\alpha} \tag{6.5}$$

and so we can compute $\mathbb{E}(f(x - (Q\mathbf{U} + \mathbf{D}(L))))$ by inverting the Laplace transform given by (6.5) and substituting it into relation (6.4). We consider the same numerical instance as given for the (R, s, nQ) model, with the piecewise linear cost rate function given by (2.4), and we plotted in Figure 10 the average cost function $\Phi(s, Q)$ with respect to the reorder level s and the batch order quantity Q.

Finally we compute for the (s, nQ) model the so-called average number β_2 of items short. As shown in [3] the value β_2 is given by

$$\lambda\left(\mathbb{E}(\max\{\mathbf{D}(\mathbf{T}_1 + L) - s - Q\mathbf{U}, 0\}) - \mathbb{E}(\max\{\mathbf{D}(L) - s - Q\mathbf{U}, 0\})\right) \tag{6.6}$$

It is immediately clear, using again the symmetry property of the uniformly distributed random variable $\mathbf{U}$ that

$$\max\{\mathbf{D}(t) - s - Q\mathbf{U}, 0\} \stackrel{d}{=} \max\{\mathbf{D}(t) + Q\mathbf{U} - s - Q, 0\}$$

and the random variable $\max\{\mathbf{D}(t) + Q\mathbf{U} - s - Q, 0\}$ equals

$$\mathbf{D}(t) + Q\mathbf{U} - (s + Q) + \max\{s + Q - (\mathbf{D}(t) + Q\mathbf{U}), 0\},$$

hence its expectation is given by

$$\mathbb{E}(\mathbf{D}(t) + Q\mathbf{U}) + \mathbb{E}(\max\{s + Q - (\mathbf{D}(t) + Q\mathbf{U}), 0\}).$$

In order to compute now the average number β_2 of items short we only have to evaluate the expressions

$$\mathbb{E}(\max\{s + Q - (\mathbf{D}(L) + \mathbf{Y}_1 + Q\mathbf{U}), 0\})$$

and

$$\mathbb{E}(\max\{s + Q - (\mathbf{D}(L) + Q\mathbf{U}), 0\}).$$

Observe by the independence of the random variable $\mathbf{U}$ and the process $\mathbf{D}$ that

$$\mathbb{E}(\max\{x - (\mathbf{D}(L) + \mathbf{Y}_1 + Q\mathbf{U}), 0\} = \mathbb{E}\left(q_L(x - \mathbf{Y}_1 - Q\mathbf{U})\right) \tag{6.7}$$

and

$$\mathbb{E}(\max\{x - (\mathbf{D}(L) + Q\mathbf{U}), 0\} = \mathbb{E}\left(q_L(x - Q\mathbf{U})\right) \tag{6.8}$$

with q_L given by relation (1.7). By the previous observations and since

$$\mathbf{D}(\mathbf{T}_1 + L) \stackrel{d}{=} \mathbf{D}(L) + \mathbf{Y}_1$$

with $\mathbf{D}(L)$ independent of $\mathbf{Y}_1$, we obtain that the average number β_2 of items short, given by relation (6.6), equals

$$\lambda\mu_Y + \lambda\left(\mathbb{E}\left(q_L(s + Q - \mathbf{Y}_1 - Q\mathbf{U})\right) - \mathbb{E}\left(q_L(s + Q - \mathbf{Y}_1 - Q\mathbf{U})\right)\right).$$

It only remains now to compute the values in the relations (6.7) and (6.8). Proceeding the same way as in Lemma 6.1 it follows that

$$\mathbb{E}\left(q_L(x - Q\mathbf{U})\right) = \frac{(I \star q_L)(x) - (I \star q_L)(x - Q)}{Q} \tag{6.9}$$

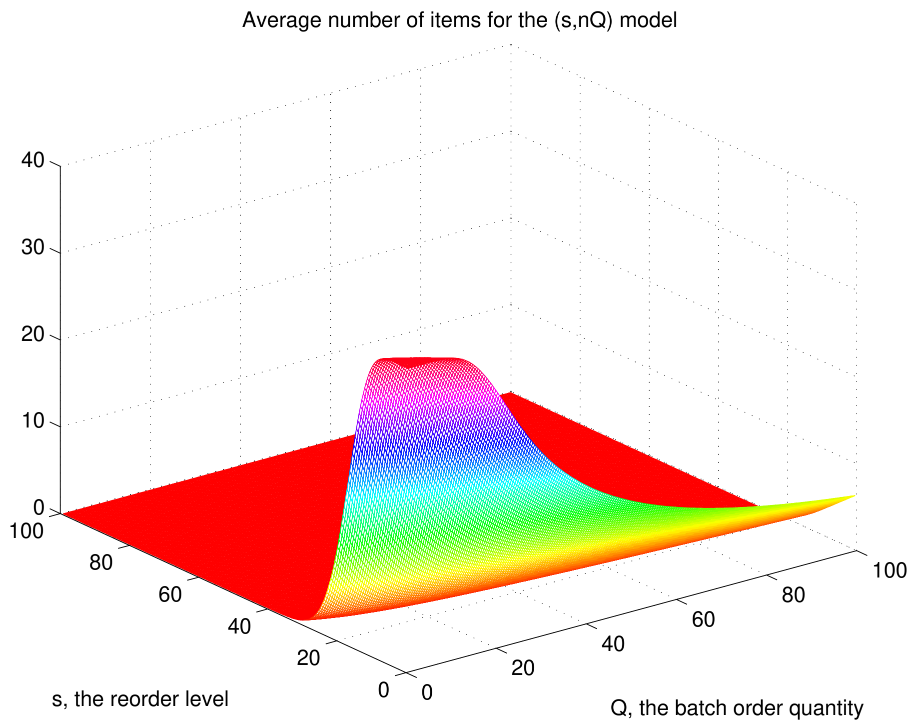

FIGURE 11. Average number β_2 of items short with parameters $L = 1/2, \lambda = 35$ and shape resp. scale parameters $\alpha = \beta = 2.5$

and by relations (2.7) and (1.4) the Laplace transform of the function $x \longrightarrow (I \star q_L)(x)$ is given by

$$\frac{1}{\alpha} L_{q_L}(\alpha) = \frac{LS_{F_{D(L)}}(\alpha)}{\alpha^3}.$$

By this observation and relation (6.9) it is now clear that the expression in (6.8) can be computed by means of the Laplace transform inversion algorithm. Moreover, similarly as for relation (6.9) it is easy to show that $\mathbb{E}\left(q_L(x - \mathbf{Y}_1 - Q\mathbf{U})\right)$ equals

$$\frac{1}{Q}\left(\int_0^x \mathbb{E} q_L(z - \mathbf{Y}_1)dz - \int_0^{x-Q} \mathbb{E} q_L(z - \mathbf{Y}_1)dz\right). \tag{6.10}$$

Since $q_L(x) = (I \star F_{D(L)})(x)$ we obtain that

$$\mathbb{E} q_L(x - \mathbf{Y}_1) = (q_L \star F_{Y_1})(x) = (I \star F_{D(L)} \star F_{Y_1})(x).$$

and this implies by relation (1.3) that

$$\int_0^x \mathbb{E} q_L(z - \mathbf{Y}_1)dz = (I^{2\star} \star F_{D(L)} \star F_{Y_1})(x).$$

Hence by relations (1.2) and (1.4) the Laplace transform of the function $x \longrightarrow \int_0^x \mathbb{E} q_L(z - \mathbf{Y}_1)dz$ is given by

$$L_{I^{2\star}}(\alpha) LS_{F_{D(L)} \star F_{Y_1}} = \frac{LS_{F_{D(L)}}(\alpha) LS_{F_{Y_1}}(\alpha)}{\alpha^3}, \tag{6.11}$$

and this yields by relation (6.10) that the function $x \longrightarrow \mathbb{E}\,(q_L(x - Q\mathbf{U} - \mathbf{Y}_1))$ can be computed by means of the Laplace transform inversion algorithm. For the same numerical example as before Figure 11 depicts a plot of the average number β_2 of items short. From Figure 11 it is easy to see that the β_2 service measure is extremely sensitive with respect to the values of Q and s.

7. Conclusions

It is not just the analysis of the single item inventory models which can be done in a uniform and compact way (cf. [3]) but also the computation of their cost functions and the related service measures. All the expressions for the cost functions of the related inventory models and the expressions for the service measures can be seen as convolutions of four basic functions. These basic functions have elementary Laplace transforms and due to the convolution structure, the Laplace transforms of the overall cost measures are also straightforward. Hence all the measures can be computed by means of the Laplace transform inversion algorithm (cf. [5]). The resulting functions for the cost measures are of machine precision and they were plotted in Matlab. These plots give a good intuition for the behaviour and sensitivity of these functions. More specifically, we observed for instance that the plots for the β_2 service measure for the different inventory models is extremely sensitive with respect to the value of the decision variables. This means that the approximation methods used within the inventory control literature (cf. [6]) will generate in general inaccurate predictions of the β_2 service measure.

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