Robust Pooling for Contracting Models with Asymmetric Information

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Abstract

We consider principal-agent contracting models between a seller and a buyer with single-dimensional private information. The buyer's type follows a continuous distribution on a bounded interval. We present a new modelling approach where the seller offers a menu of finitely many contracts to the buyer. The approach distinguishes itself from existing methods by pooling the buyer types using a partition. That is, the seller first chooses the number of contracts offered and then partitions the set of buyer types into subintervals. All types in a subinterval are pooled and offered the same contract by the design of our menu.

We call this approach robust pooling and apply it to utility maximisation and cost minimisation problems. In particular, we analyse two concrete problems from the literature. For both problems we are able to express structural results as a function of a single new parameter, which remarkably does not depend on all instance parameters. We determine the optimal partition and the corresponding optimal menu of contracts. This results in new insights into the (sub)optimality of the equidistant partition. For example, the equidistant partition is optimal for a family of instances for one of the problems. Finally, we derive performance guarantees for the equidistant and optimal partitions for a given number of contracts. For the considered problems the robust pooling approach has good performances with only a few contracts.

Keywords: mechanism design, asymmetric information, robust pooling, optimal partitioning, performance guarantees

1 Introduction

In principal-agent contracting problems, a principal wants to persuade an agent to perform a certain action and uses financial incentives to do so. Both parties are individually rational and only want to improve their own situation. We consider contracting problems where the principal is a seller of a certain product and the agent is a potential buyer. Thus, the seller desires either to initiate new trade with the buyer or to change the existing buyer's order quantity to a more beneficial order. In order to do so, the seller offers a contract to the buyer, describing the order quantity (the action) and a side payment (the incentive). The contract design must balance the value of the contract for both parties, since the buyer can refuse a disadvantageous contract.

The complexity of the contracting problem increases significantly when the buyer has private information on his valuation of contracts, i.e., there is information asymmetry. In terms of Mechanism Design, the buyer's private information is represented by so-called *types*. That is, the buyer's identity is an element of a known set of types $\mathcal{P}$ and specified by a probability distribution on $\mathcal{P}$. The distribution of types is assumed to be common knowledge, in particular also to the seller. We consider the case where the buyer has single-dimensional private information, represented by the type $p \in \mathcal{P}$.

In case of information asymmetry, the seller offers a menu of contracts, typically one contract for each of the possible buyer types. First, the optimal menu is determined by solving a certain optimisation problem, which we will discuss in later sections. Second, this menu is offered to the buyer. Finally, the buyer either chooses to accept a contract of the menu or refuses the offer, depending on what is most beneficial for the buyer. Note that the buyer can lie about his true type and choose any contract, which complicates the seller's optimisation process.

The modelling of the buyer types $\mathcal{P}$ is crucial for the contracting problem. In the Mechanism Design literature there are two typical choices. First, we have the classical *discrete model*: a finite discrete set $\mathcal{P} = \{p_1, \ldots, p_K\} \subseteq \mathbb{R}$ for some $K \in \mathbb{N}_{\geq 1}$ (discrete distribution). Here, the menu consists of K contracts, one for each type. Hence, the buyer chooses from a finite number of contracts. Second, we have the classical *continuous model*: a bounded interval $\mathcal{P} = [\underline{p}, \bar{p}] \subseteq \mathbb{R}$ with $\bar{p} > \underline{p}$ (continuous distribution). Here, the menu is a function that maps every type to a contract. In other words, infinitely many contracts are offered to the buyer.

Our goal is to design and analyse a model that combines aspects of both the discrete and continuous models. For this model, the buyer's type is continuously distributed on $\mathcal{P} = [\underline{p}, \bar{p}] \subseteq \mathbb{R}$ with $\bar{p} > \underline{p}$, but only finitely many contracts are offered. The main motivation for this approach is that offering finitely many contracts is often preferred in practice, as such menus are easier to communicate and implement. Furthermore, having only finitely many outcomes to offering the menu allows decision makers to include contracting in more complex company-wide scenario-based analyses. The discrete and continuous approaches are not suitable for achieving this goal, which we will later discuss in more detail. This combination of the discrete and continuous approaches has received limited attention in the literature, which we will review in the next section.

We present a modelling approach which we call *robust pooling* in order to achieve the stated goal. For the robust pooling model, the buyer's type lies in a bounded interval $[\underline{p}, \bar{p}]$, but only finitely many contracts are offered. First, the seller chooses the number of contracts $K \in \mathbb{N}_{\geq 1}$ that will be offered. Second, he partitions the interval $[\underline{p}, \bar{p}]$ into K subintervals denoted by $[\underline{p}_k, \bar{p}_k]$ for $k \in \{1, \dots, K\}$. Third, he designs a menu of K contracts with a single contract intended for each subinterval $[\underline{p}_k, \bar{p}_k]$. Finally, he offers the menu to the buyer, as usual. Note that, technically, the subintervals $[\underline{p}_k, \bar{p}_k]$ should not intersect, i.e., they should be half-open except for the last subinterval. However, using closed subintervals does not affect the results and simplifies notation.

Our modelling approach has two fundamental properties: pooling of types and robustness. First, the (discrete) *pooling* property refers to offering finitely many contracts, and thus offering the same contract to multiple types, *by design.* Second, the (continuous) *robustness* property means that each type $p \in \mathcal{P}$ accepts a contract from the menu and that this choice is correctly reflected in the model (for example in the objective function). In other words, the menu specifies an intended contract for each type and each type chooses its intended contract. Consequently, the buyer always accepts a contract from the menu, making the menu robust to the buyer's private information. In our case, for each $k \in \{1, \dots, K\}$ it is for all types in $[\underline{p}_k, \bar{p}_k]$ most beneficial to choose the k-th contract.

In our approach, the seller must decide on a partition scheme, i.e., the number of contracts and an appropriately corresponding partition of $[\underline{p}, \bar{p}]$. The robust pooling model enables us to determine the effect of different partition schemes, since our model handles an arbitrary number of contracts and any partition in a natural way. Due to the robustness property, we can evaluate the use of different schemes in a fair way by directly comparing the resulting objective values of the model.

Such a fair comparison is not possible with the classical discrete approach, since varying the number of contracts also implies changing the distribution of the buyer's type, effectively changing which scenarios could happen. Moreover, if the discrete distribution is actually an approximation of a continuous distribution, then the discrete approach is generally not robust. The classical continuous approach does not pool types by design and is therefore also unsuitable.

As already hinted, there are several aspects of the robust pooling model to analyse. First, what is the complexity of the model? In particular, can we identify conditions under which the model can be solved efficiently? Second, can we quantify the performance of partition schemes? A natural choice for a partition is the equidistant partition, where $[\underline{p}, \bar{p}]$ is partitioned into subintervals of equal length. However, is the equidistant partition the best possible partition? Also, offering infinitely many contracts (the continuous approach) results in the best possible objective value and is partition independent. When using our approach, how many contracts should be offered to guarantee, say, 95% of this best possible value?

We continue with a literature review of related modelling techniques and contracting problems.

1.1 Connection to the literature

For a general reference for the classical discrete and continuous modelling approaches, see for example Laffont and Martimort (2002). To our knowledge, a combination of the discrete and continuous approaches, such as our robust pooling model, has received limited attention in the literature. Bergemann et al. (2011) consider a linear-quadratic model based on Mussa and Rosen (1978), but with limited communication between the seller and the buyer. The limited communication implies that only a menu with a limited number of contracts can be offered. Their approach is effectively a restricted form of the classical continuous approach, where the menu is restricted to have finitely many contracts. The resulting model satisfies our desired pooling and robustness properties. They are able to reformulate the problem into a mean square minimisation problem and apply Quantisation theory (Lloyd-Max conditions) to determine the optimal menu of contracts and the optimal partition scheme. In particular, they show that compared to offering infinitely many contracts the loss in performance is of the order $\Theta(1/K^2)$ when using K optimal contracts.

The same modelling approach is used in Wong (2014), who analyses a more general version of the non-linear pricing problem in Bergemann et al. (2011). He determines general results on the loss of performance when offering K optimal contracts. Among other results, he proves that the loss in performance is of the order $\Theta(1/K^2)$ under more general assumptions than Bergemann et al. (2011).

We shall refer to the modelling approach used in Bergemann et al. (2011) and Wong (2014) as the *limited variety* model. In general, our robust pooling model is more restrictive than the limited variety model, since we partition (pool) types into subintervals a priori. Nevertheless, we use our robust pooling approach for the following reasons.

First, the robust pooling model has an added benefit regarding information extraction. The seller can extract private information from the buyer by observing the buyer's chosen contract. Recall that by design the k-th contract is chosen by all types in $[\underline{p}_k, \bar{p}_k]$. Thus, after observing the buyer's choice, the seller can narrow down the buyer's type to one of the subintervals of the partition. Since the used partition is a decision made by the seller, he is able to control the accuracy of said identification in a natural and intuitive way. In general, the limited variety model cannot guarantee such structured information extraction.

Second, an implicit goal of offering a limited number of contracts is to have a simple mechanism. Partitioning $[\underline{p}, \bar{p}]$ using a certain heuristic (e.g., equidistantly or according to some 'square-root' rule) is simple and intuitive, and could have a decent performance. That is, the formulation promotes the experimentation with partition schemes. Moreover, it could be that the loss in performance by restricting to subintervals is negligible compared to the loss by offering a limited number of contracts. Of course, it has to be researched whether this is the case.

The robust pooling model is also related to Robust Optimisation (see Ben-Tal et al. (2009)). That is, our model can be interpreted as a Robust Optimisation variant for the discrete model, where each subinterval $[\underline{p}_k, \bar{p}_k]$ is the so-called uncertainty set of type p_k. This will be further discussed when we have formalised the model.

In the recent years, there has been an increase in the application of Robust Optimisation to Mechanism Design models in the literature. For examples, see Aghassi and Bertsimas (2006), Bandi and Bertsimas (2014), Bergemann and Morris (2005) and Pınar and Kızılkale (2016). The main focus lies on making contracting models robust to the distribution of the buyer's type, i.e., it only depends on which types can occur and not on any probabilities. To our knowledge, Robust Optimisation has not been applied to obtain a model similar to our robust pooling model.

1.2 Contribution

Our contributions are as follows. We present a new modelling approach for contracting problems called robust pooling. Our approach distinguishes itself from the classical discrete and continuous models by having a continuous distribution for the buyer's type and offering a menu with finitely many contracts. Its two fundamental properties are pooling of types by design and robustness. Compared to the limited variety model, we use a partition to pool types a priori. We restrict the analysis to single-dimensional types, but the robust pooling principle can be applied to more general settings.

We show that under certain assumptions robust pooling models have a simplified reformulation or can even be solved efficiently. The assumptions required for this analysis are frequently used in the literature. Our analysis also results in new insights into the robustness of the classical discrete modelling approach when a discrete distribution is used as an approximation for a buyer's type with a continuous distribution. These results are derived for both utility maximisation and cost minimisation variants of the contracting problem, which are not equivalent under the made assumptions.

In the robust pooling approach, the seller must decide on a partition scheme, i.e., the number of contracts and an appropriately corresponding partition of $[\underline{p}, \bar{p}]$. Due to the robustness property, we can compare the performance of different partition schemes in a straightforward and fair way. However, these performances are difficult, if not impossible, to determine analytically in general. Therefore, we consider two specific problems. The first problem is based on a decreasing marginal utility for the buyer as his order quantity increases. It is a generalisation of the linear-quadratic model considered in Wong (2014). As such, we can relate our results to his. The second problem uses the classical Economic Order Quantity setting. It is the robust pooling variant of, for example, Corbett and De Groote (2000) and Voigt and Inderfurth (2011). To our knowledge, robust pooling has not been applied before to this setting.

For both problems, we derive closed-form formulas for the optimal menu and corresponding optimal objective value for any number of contracts and any partition. We show that structural results and performance measures can be expressed by functions of a single new parameter. Remarkably, this parameter does not depend on all instance parameters, implying families of instances with the same structure. Furthermore, we determine the optimal partition scheme, either analytically or numerically, depending on the problem. In particular, this leads to new insights into the (sub)optimality of the equidistant partition. Finally, we give performance guarantees for the equidistant and optimal partitions.

The robust pooling approach has good performances with only a few contracts for the two considered problems. For example, offering 3 contracts with an optimised partition has a performance guarantee of at most a 4% relative gap compared to offering infinitely many contracts. This validates the concept of our robust pooling approach.

The remainder of this paper is organised as follows. In Section 2 we consider robust pooling models in the context of utility maximisation. In Section 2.1-2.3 we analyse the model under certain assumptions. These results are applied in Sections 2.4-2.6 to the mentioned problem based on a decreasing marginal utility. In Section 3 we perform a similar analysis to cost minimisation models and apply it to the mentioned problem based on the Economic Order Quantity setting in Section 3.2. Finally, we conclude our findings in Section 4.

2 Contracting for maximising utility

In this section we consider principal-agent contracting models in the setting of utility maximisation. We first formalise the robust pooling model in Section 2.1. In Sections 2.2-2.3 we reformulate, analyse, and solve the model under certain assumptions. Finally, in Sections 2.4-2.6 we consider a concrete problem based on decreasing marginal utilities and analyse the performance of partitioning schemes in detail.

2.1 The model

The principal is a seller of products and wants to initiate trade with the agent, referred to as the buyer. The seller desires to enter a contractual agreement with the buyer to provide the goods. However, the buyer does not share all his information with the seller, complicating the design of a contract. Therefore, the seller uses Mechanism Design to construct a menu of contracts such that the buyer can be persuaded to order at the seller.

A contract is given by an order quantity $x \in \mathbb{R}_{\geq 0}$ and a side payment $z \in \mathbb{R}$ from the buyer to the seller. That is, the contract effectively specifies how many units of product the buyer receives and for which price. The buyer can refuse any contract, but we assume he acts individually rational and accepts an offered contract if this is most beneficial to himself.

The buyer has private information, which we assume can be represented by a single parameter $p \in \mathbb{R}_{\geq 0}$. Let $\phi_B(x|p)$ be the utility of order quantity x for the buyer with private parameter p. Likewise, $\phi_S(x)$ is the seller's utility for order quantity x. By default there is no contract (no trade) between the seller and the buyer, resulting in a default utility of zero for the buyer. Therefore, a contract (x, z) is accepted by the buyer if its net utility is non-negative:

$$\phi_B(x|p) - z \geq 0.$$

This is called the Individual Rationality (IR) constraint. The difficulty in designing a suitable contract is that the private utility parameter p is not shared with the seller. We assume that the parameter p follows a continuous distribution with strictly positive density function $\omega : [\underline{p}, \bar{p}] \to \mathbb{R}_{>0}$ on the interval $[\underline{p}, \bar{p}] \subseteq \mathbb{R}_{\geq 0}$ with $\bar{p} > \underline{p}$. This distribution is known to the seller. Each $p \in [\underline{p}, \bar{p}]$ is called a (buyer) *type*.

Instead of offering a single contract, the seller designs a menu consisting of $K \in \mathbb{N}_{\geq 1}$ contracts for the buyer to choose from. The number of contracts K is a decision made by the seller and plays a central role in the results to come. We define $\mathcal{K} = \{1, \ldots, K\}$. Next, the seller partitions $[\underline{p}, \bar{p}]$ into K subintervals $[\underline{p}_k, \bar{p}_k]$ with $\bar{p}_k > \underline{p}_k$. We call this a *proper K-partition*. Finally, the seller constructs K contracts, where contract (x_k, z_k) is designed for subinterval $[\underline{p}_k, \bar{p}_k]$ for each $k \in \mathcal{K}$. The contracts are determined by solving the following optimisation problem:

$$\max_{x,z} \quad \sum_{k\in\mathcal{K}} \left(\int_{\underline{p}_k}^{\bar{p}_k} \omega(p)\mathrm{d}p \right) (\phi_S(x_k) + z_k), \tag{2.1}$$

$$\text{s.t.} \quad \phi_B(x_k|p_k) - z_k \geq 0, \qquad \forall\, p_k \in [\underline{p}_k, \bar{p}_k], k \in \mathcal{K}, \tag{2.2}$$

$$\phi_B(x_k|p_k) - z_k \geq \phi_B(x_l|p_k) - z_l, \qquad \forall\, p_k \in [\underline{p}_k, \bar{p}_k], k, l \in \mathcal{K}, \tag{2.3}$$

$$x_k \geq 0, \qquad \forall\, k \in \mathcal{K}.$$

We refer to this model as the *robust pooling model*. Constraints (2.2) specify that contract (x_k, z_k)

must be individually rational for the buyer with respect to all corresponding types $p_k \in [\underline{p}_k, \bar{p}_k]$. Constraints (2.3) are the Incentive Compatibility (IC) constraints. These ensure that the buyer with type $p_k \in [\underline{p}_k, \bar{p}_k]$ prefers and chooses the intended contract (x_k, z_k) over all the other contracts in the menu. Recall that the buyer chooses the most beneficial contract for himself from the menu. To be precise, we need the following assumption, which is conventional in the Mechanism Design literature. If the IC constraint (2.3) where type $p_k \in [\underline{p}_k, \bar{p}_k]$ compares contract (x_k, z_k) to contract (x_l, z_l) holds with equality, then type p_k is indifferent between contracts (x_k, z_k) and (x_l, z_l). In this case, we assume that the seller can convince the buyer to choose contract (x_k, z_k).

Thus, a buyer with type $p_k \in [\underline{p}_k, \bar{p}_k]$ always chooses contract (x_k, z_k) by design of the menu. This is related to the well-known Revelation Principle (see Laffont and Martimort (2002) and Myerson (1982)). This principle states that without loss of optimality the seller can restrict his design to incentive-compatible direct coordination mechanisms and obtain a truthful choice of contract by the buyer. In other words, for a buyer with type $p_k \in [\underline{p}_k, \bar{p}_k]$ it is a weakly-dominant strategy to choose contract (x_k, z_k).

With this insight, we return to the robust pooling model. Notice that

$$\omega_k \equiv \int_{\underline{p}_k}^{\bar{p}_k} \omega(p)\mathrm{d}p \in (0, 1] \tag{2.4}$$

for $k \in \mathcal{K}$ defines the probability ω_k that the buyer's type lies in $[\underline{p}_k, \bar{p}_k]$ and consequently that the buyer chooses contract (x_k, z_k). The seller's objective (2.1) is to maximise his own expected net utility, which is the weighted sum of his valuation $\phi_S(x_k)$ of the order quantity x_k and the received side payment z_k.

The robust pooling model has a strong connection to Robust Optimisation models (see Ben-Tal et al. (2009)). Our model has finitely many decision variables and infinitely many constraints. Furthermore, for $k \in \mathcal{K}$ the interval $[\underline{p}_k, \bar{p}_k]$ can be interpreted as the so-called uncertainty interval for p_k. Thus, the robust pooling model can be seen as a Robust Optimisation variant of the classical discrete model with K uncertain parameters $p_1, \ldots, p_K$. We will not require Robust Optimisation techniques in the following sections. However, these techniques can be useful to analyse more complex robust pooling models.

To conclude, the robust pooling model pools the possible buyer types p into finitely many subintervals, enabling the seller to offer finitely many contracts in the menu. Furthermore, the contracts are robust by design, meaning that the buyer will always accept a contract from the menu for any possible type $p \in [\underline{p}, \bar{p}]$ and this choice is correctly reflected in the objective function.

We emphasize that the number of contracts K and the proper K-partition of $[\underline{p}, \bar{p}]$ are decisions made by the seller. Therefore, if solving the robust pooling model is sufficiently easy, we can focus on quantifying the effect of the number of contracts and the chosen partition. For example, how many contracts should be offered to obtain 90% of the maximum possible expected net utility? Also, the equidistant partition is a natural standard choice, but is it also optimal?

In order to answer such questions, we need to make assumptions and consider explicit models, as a general approach seems impossible. The first assumption is on the buyer's utility function.

Assumption 1. The buyer's utility function is $\phi_B(x|p) = \psi(x) + p\chi(x)$, where the functions $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and $\chi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ do not depend on the type p. Moreover, χ is non-decreasing and non-negative.

Assumption 1 is common in the Mechanism Design literature and allows us to determine structural results. In Section 2.2 we derive an equivalent and simpler formulation for the robust pooling model. The performed analysis is used in Section 2.3 to show a certain equivalence between our robust pooling model and two other models in the literature. Finally, in Sections 2.4-2.6 we analyse a model in detail where the buyer has a decreasing marginal utility for the products. In particular, we focus on the performance of partition schemes in those sections.

From this point onwards, we denote a menu of contracts by (x, z), where $x = (x_1, \ldots, x_K)$ and $z = (z_1, \ldots, z_K)$. A single contract is denoted by (x_k, z_k) for $k \in \mathcal{K}$. Also, we use ω_k defined by (2.4) in the objective instead of the integral notation.

2.2 Reformulation and analysis

Under Assumption 1, we make a change of variables by splitting the side payment into two parts:

$$z_k = \psi(x_k) + y_k,$$

where y_k will replace z_k as decision variable. Substitution of this definition leads to an equivalent model with simplified constraints:

$$\begin{aligned}
\max_{x,y} \quad & \sum_{k\in\mathcal{K}} \omega_k \left(\phi_S(x_k) + \psi(x_k) + y_k\right), && \\
\text{s.t.} \quad & p_k\chi(x_k) - y_k \geq 0, && \forall\, p_k \in [\underline{p}_k, \bar{p}_k], k \in \mathcal{K}, \quad (2.5)\\
& p_k\chi(x_k) - y_k \geq p_k\chi(x_l) - y_l, && \forall\, p_k \in [\underline{p}_k, \bar{p}_k], k, l \in \mathcal{K}, \quad (2.6)\\
& x_k \geq 0, && \forall\, k \in \mathcal{K}.
\end{aligned}$$

The benefit of this formulation is that several utility functions can be analysed as one model: different choices of ϕ_S and ψ can lead to the same function $\phi_S + \psi$. Furthermore, if $\phi_S + \psi$ is concave and χ linear, this formulation is concave, has linear constraints, and can be solved efficiently.

We continue with the first structural result for the robust pooling model. Lemma 1 essentially identifies an embedded dual shortest path problem as in Rochet and Stole (2003) and Vohra (2012).

Lemma 1. *Under Assumption 1, for any feasible x it is optimal to set*

$$y_k = \underline{p}_k\chi(x_k) - \sum_{i=1}^{k-1}(\bar{p}_i - \underline{p}_i)\chi(x_i) \qquad \forall\, k \in \mathcal{K}. \tag{2.7}$$

Proof. By combining several essential constraints, we can show that many constraints are superfluous and that some must hold with equality. See Appendix A.1 for the details. □

Lemma 1 allows us to eliminate the variable y (or z) and obtain an optimisation problem in terms of x. However, in order to do so, we need to be able to express the feasible region in terms of x. This is shown in Lemma 2.

Lemma 2. *Under Assumption 1, any x is feasible if and only if $0 \leq x_1 \leq \cdots \leq x_K$.*

Proof. Use Lemma 1, combine the IC constraints, and use that χ is non-decreasing. The details are in Appendix A.1. □

In Mechanism Design, the buyer's type is often related to efficiency: type $p_k > p_l$ gets more utility from a fixed order quantity than type p_l, i.e., $\phi_B(x|p_k) \geq \phi_B(x|p_l)$ for all $x \geq 0$. Thus, type p_k is more efficient. Lemma 2 shows that the order quantities are weakly ordered in terms of the corresponding type's efficiency: a less efficient type is offered a lower or equal order quantity than a more efficient type.

We can now combine our results to get an equivalent and much simpler formulation for the robust pooling model under our assumptions, see Theorem 3. Notice in particular that the reformulation has finitely many linear constraints.

Theorem 3. *Under Assumption 1, the robust pooling model with infinitely many constraints is equivalent to the following problem with finitely many and linear constraints:*

$$\max_{0\leq x_1\leq\cdots\leq x_K} \sum_{k\in\mathcal{K}} \omega_k \left(\phi_S(x_k) + \psi(x_k) + \Big(\underline{p}_k - (\bar{p}_k - \underline{p}_k) \sum_{i=k+1}^{K} \frac{\omega_i}{\omega_k} \Big) \chi(x_k) \right). \tag{2.8}$$

Proof. By Lemma 1 we can substitute the optimal formula (2.7) for y into the optimisation model. By Lemma 2 we conclude that the IR and IC constraints hold if and only if $0 \leq x_1 \leq \cdots \leq x_K$. This leads to the equivalent optimisation problem

$$\max_{0\leq x_1\leq\cdots\leq x_K} \sum_{k\in\mathcal{K}} \omega_k \Big(\phi_S(x_k) + \psi(x_k) + \underline{p}_k \chi(x_k) - \sum_{i=1}^{k-1} (\bar{p}_i - \underline{p}_i)\chi(x_i) \Big),$$

which can be rewritten into to formulation of the theorem by collecting the terms of x_k. □

The computational complexity of (2.8) depends on the shape of $\phi_S + \psi$ and χ. Furthermore, (2.8) allows for specialised (numerical) solvers, since the feasible region is independent of $\phi_S + \psi$ and χ.

We make additional assumptions to determine a family of explicitly solvable robust pooling models. That is, we are able to derive explicit formulas for the optimal menu of contracts. The following assumptions are a balance between the generality of the model and the brevity of the analysis, and can be weakened up to a certain extent to obtain similar results.

Assumption 2. The buyer has zero utility for ordering zero units of product: $\phi_B(0|p) = 0$ for all $p \in [\underline{p}, \bar{p}]$, implying $\psi(0) = \chi(0) = 0$.

Assumption 3. The function $\phi_S + \psi$ is strictly concave and differentiable on $\mathbb{R}_{\geq 0}$. The function χ is given by $\chi(x) = x$.

Assumption 4. The distribution on the private parameter p is uniform: $\omega(p) = 1/(\bar{p} - \underline{p})$, so $\omega_k = (\bar{p}_k - \underline{p}_k)/(\bar{p} - \underline{p})$ for all $k \in \mathcal{K}$.

Assuming $\phi_B(0|p) = 0$ (Assumption 2) ensures that there is no side payment if a contract specifies no trade, i.e., $x_k = 0$ implies $z_k = 0$. This is in line with the default situation, where the absence of trade implies zero utility for the buyer. Assumption 3 is needed to make (2.8) a concave maximisation problem that can be solved efficiently, for example using interior-point or cutting-plane methods (see Bertsekas (2015) and Boyd and Vandenberghe (2004)). Finally, the uniformity of p (Assumption 4) is significantly restrictive, but allows for a manageable exact analysis with closed-form formulas.

With the imposed additional structure, we can solve (2.8) exactly, see Theorem 4.

Theorem 4. *Under Assumptions 1-4, the robust pooling model is equivalent to the following concave problem:*

$$\max_{0 \le x_1 \le \cdots \le x_K} \sum_{k \in \mathcal{K}} \frac{\bar{p}_k - \underline{p}_k}{\bar{p} - \underline{p}} \Big(\phi_S(x_k) + \psi(x_k) + (\bar{p}_k + \underline{p}_k - \bar{p}) x_k \Big).$$

The optimal order quantities are given by

$$x_k = \begin{cases} 0 & \text{if } k < k^* \\ \left(\frac{\mathrm{d}}{\mathrm{d}x}(\phi_S + \psi)\right)^{-1} (\bar{p} - \bar{p}_k - \underline{p}_k) & \text{if } k^* \le k \le \hat{k} \\ \infty & \text{if } k > \hat{k} \end{cases},$$

and satisfy $0 < x_{k^*} < \cdots < x_{\hat{k}} < \infty$. *Here, the index of the first non-zero order quantity is*

$$k^* = \min \left\{ K + 1, \min \left\{ k \in \mathcal{K} : \bar{p}_k + \underline{p}_k - \bar{p} > -\frac{\mathrm{d}}{\mathrm{d}x} (\phi_S + \psi)(0) \right\} \right\},$$

and the index of last finite order quantity is

$$\hat{k} = \max \left\{ 0, \max \left\{ k \in \mathcal{K} : \bar{p}_k + \underline{p}_k - \bar{p} < -\lim_{x \to \infty} \frac{\mathrm{d}}{\mathrm{d}x} (\phi_S + \psi)(x) \right\} \right\}.$$

Proof. We specialise Theorem 3 under the stated assumptions and solve the resulting concave optimisation problem. See Appendix A.1 for the details. □

Theorem 4 defines two indices k^* and $\hat{k}$. Typically, the index $\hat{k}$ of last finite order quantity satisfies $\hat{k} = K$ and can be omitted. If $\hat{k} < K$ then the optimal objective value is ∞, which is unrealistic and indicates that the utility functions should be reconsidered. On the other hand, the index k^* of the first non-zero order quantity plays an essential role as we shall see in Section 2.4.

If $k^* > 1$ then $(x_k, z_k) = (0, 0)$ for $k < k^*$, i.e., there is no trade with types $p \in [\underline{p}, \underline{p}_{k^*})$.

In conclusion, we have shown how to reformulate and solve certain robust pooling models for maximising utility. In particular, the analysis allows us to prove in Section 2.3 that the robust pooling model under Assumption 1 is equivalent to other models from the literature. After that intermezzo, we continue in Section 2.4 to analyse the effect of the chosen partition for a model based on decreasing marginal utility. This model satisfies Assumptions 1-4, implying that we can apply Theorem 4.

2.3 Equivalences to other models

The structure of the reformulated robust pooling model (2.8) might be recognised by those familiar with either classical discrete contracting models or the limited variety model of Bergemann et al. (2011) and Wong (2014). In fact, under Assumption 1 there is an equivalence between these three models. We formalise and discuss this further in this section.

2.3.1 Pooling and robustness implies partitioning

In the robust pooling approach we partition $[\underline{p}, \bar{p}]$ to obtain pooling of types, i.e., a menu with finitely many contracts. As mentioned in Section 1, the limited variety model of Bergemann et al. (2011) and Wong (2014) achieves robustness and pooling without partitioning $[\underline{p}, \bar{p}]$ a priori. The limited variety model simply restricts the menu to include finitely many contracts. Thus, their approach is more general than our robust pooling. However, in this section we show that under Assumption 1 both approaches are equivalent, i.e., they have the same optimal solution.

Consider a menu of K contracts (x_k, z_k) that satisfies the pooling and robustness properties. Consequently, each type $p \in [\underline{p}, \bar{p}]$ chooses a contract from the menu. Without loss of generality, each contract is chosen by some types. Let $\hat{p}_k$ be the most inefficient type that chooses contract (x_k, z_k) for $k \in \mathcal{K}$. By changing the index of the contracts, we have $\underline{p} = \hat{p}_1 < \ldots < \hat{p}_K \leq \bar{p}$ without loss of generality. This implies that

$$
\begin{aligned}
\phi_B(x_k|\hat{p}_k) &\geq z_k, && \forall\, k \in \mathcal{K}, \\
\phi_B(x_k|\hat{p}_k) - \phi_B(x_l|\hat{p}_k) &\geq z_k - z_l, && \forall\, k, l \in \mathcal{K}.
\end{aligned}
\tag{2.9}
$$

We will prove that, in fact, types $p \geq \hat{p}_k$ prefer contract (x_k, z_k) over contracts (x_l, z_l) with $l < k$ by verifying that the respective IR and IC constraints hold. By adding (2.9) for $k, l \in \mathcal{K}$ and by Assumption 1, we have

$$\begin{aligned} 0 &\leq \phi_B(x_k|\hat{p}_k) - \phi_B(x_l|\hat{p}_k) + \phi_B(x_l|\hat{p}_l) - \phi_B(x_k|\hat{p}_l) \\ &= (\hat{p}_k - \hat{p}_l)(\chi(x_k) - \chi(x_l)), \qquad \forall\, k, l \in \mathcal{K}. \end{aligned}$$

Therefore, $\chi(x_k) \geq \chi(x_l)$ for $l < k$, since $\hat{p}_k > \hat{p}_l$ by definition. Using these results, we obtain for all $k \in \mathcal{K}$ that

$$\phi_B(x_k|p) = \psi(x_k) + p\chi(x_k) \geq \psi(x_k) + \hat{p}_k\chi(x_k) = \phi_B(x_k|\hat{p}_k) \geq z_k, \qquad \forall\, p \geq \hat{p}_k,$$

and

$$\begin{aligned} \phi_B(x_k|p) - \phi_B(x_l|p) &= \psi(x_k) - \psi(x_l) + p(\chi(x_k) - \chi(x_l)) \\ &\geq \psi(x_k) - \psi(x_l) + \hat{p}_k(\chi(x_k) - \chi(x_l)) \\ &= \phi_B(x_k|\hat{p}_k) - \phi_B(x_l|\hat{p}_k) \geq z_k - z_l, \qquad \forall\, l < k, p \geq \hat{p}_k. \end{aligned}$$

These inequalities correspond to IR and IC constraints. They imply that types $p \geq \hat{p}_k$ prefer contract (x_k, z_k) over contracts (x_l, z_l) with $l < k$. Using the definition of $\hat{p}_k$, we conclude that contract (x_k, z_k) must be chosen by types $\{p \in [\underline{p}, \bar{p}] : \hat{p}_k \leq p < \hat{p}_{k+1}\}$.

Thus, under Assumption 1 any menu that satisfies the pooling and robustness properties effectively partitions $[\underline{p}, \bar{p}]$ and pools the respective types, exactly as our robust pooling approach.

2.3.2 Robustness of the discrete approach

Suppose the buyer's type p follows a continuous distribution on $[\underline{p}, \bar{p}]$ and the seller wants to offer only finitely many contracts in the menu. Of course, our robust pooling approach is designed for this task, but can we also apply the classical discrete approach? That is, can the seller select K representatives from $[\underline{p}, \bar{p}]$, assign appropriate probabilities to the representatives, apply the classical discrete approach, and achieve the same robust result as our robust pooling approach? In this section we show that the discrete approach can be robust under Assumption 1.

First, if a discrete model satisfies Assumption 1 and is robust we conclude that it must be equivalent to the robust pooling model as shown in Section 2.3.1. Second, we prove that under Assumption 1 the robust pooling model is equivalent to a specifically constructed discrete model. The proofs of Lemmas 1 and 2 show that many constraints are redundant. Of all IR constraints (2.5) only $\underline{p}_1\chi(x_1)-y_1 \geq 0$ is needed. Of all IC constraints (2.6) we need $\underline{p}_k\chi(x_k)-y_k \geq \underline{p}_k\chi(x_{k-1})-y_{k-1}$ for all $k \in \mathcal{K}$ and the constraint $x_1 \leq \cdots \leq x_K$. The non-decreasing x can be enforced by replacing it with the IC constraints $\underline{p}_k\chi(x_k) - y_k \geq \underline{p}_k\chi(x_{k+1}) - y_{k+1}$ for all $k \in \mathcal{K}$. Adding a few more redundant IR and IC constraints, gives the following equivalent optimisation problem:

$$\begin{aligned}
\max_{x,y} \quad & \sum_{k\in\mathcal{K}} \omega_k \left(\phi_S(x_k) + \psi(x_k) + y_k\right), && \\
\text{s.t.} \quad & \underline{p}_k\chi(x_k) - y_k \geq 0, && \forall\, k \in \mathcal{K}, \\
& \underline{p}_k\chi(x_k) - y_k \geq \underline{p}_k\chi(x_l) - y_l, && \forall\, k, l \in \mathcal{K}, \\
& x_k \geq 0, && \forall\, k \in \mathcal{K}.
\end{aligned}$$

This is the classical discrete variant for the contracting problem, where each subinterval $[\underline{p}_k, \bar{p}_k]$ is represented by its most inefficient type $\underline{p}_k$ and this type has probability ω_k.

To conclude, the discrete model satisfying Assumption 1 has a hidden robustness provided that the representative of each subinterval $[\underline{p}_k, \bar{p}_k]$ is its most inefficient type $\underline{p}_k$ and this type has probability ω_k. Consequently, a *robust* discrete model using two types to approximate $[\underline{p}, \bar{p}]$ should not choose the extreme types $\underline{p}$ and $\bar{p}$ as representatives, since the contract for type $\bar{p}$ will be chosen with probability zero. Hence, effectively only a single contract is used.

2.4 Decreasing marginal utility problem

In this section we consider a specific contracting model that fits our robust pooling setting of Section 2.1 and can be analysed in detail. The model is based around the concept that the marginal utility of a product decreases for the buyer as the order quantity increases.

For order quantity $x \in \mathbb{R}_{\geq 0}$ the buyer's marginal utility of an additional product is given by $p - rx^n$ for some fixed parameters $r, n \in \mathbb{R}_{>0}$. Here, $p \in [\underline{p}, \bar{p}] \subseteq \mathbb{R}_{\geq 0}$ with $\bar{p} > \underline{p}$ is the private parameter of the buyer, as introduced in Section 2.1. This leads to the following utility function for the buyer:

$$\phi_B(x|p) = \int_0^x (p - ru^n)\mathrm{d}u = -\tfrac{1}{n+1} r x^{n+1} + px \equiv \psi(x) + p\chi(x).$$

The buyer's utility function is strictly concave in x and is negative for large order quantities. Therefore, the buyer has a finite individually optimal order quantity. For example, this could be the case if excess products are difficult to dispose of.

The seller's utility function is linear in the order quantity: $\phi_S(x) = Px$, where $P \in \mathbb{R}_{>0}$ is a fixed parameter. Therefore, the seller simply wants to sell as many products as possible. Consequently, ordering no products leads to zero utility for both the seller and the buyer.

For the entire section, we assume that the distribution of p is uniform, i.e., we make Assumption 4. The seller designs a menu of contracts using the robust pooling methodology described in Section 2.1. We refer to this problem as the *Decreasing Marginal Utility* (DMU-n) problem.

First, we derive the optimal solution and optimal objective value in Section 2.4.1. Second, we show how to express relative performance measures as 1-dimensional functions in Section 2.4.2. Finally, we discuss properties of the optimal partition in Section 2.4.3. These results hold the DMU-n problem for any $n \in \mathbb{R}_{>0}$ and are applied in Sections 2.5 and 2.6 for $n = 1$ and $n = 2$, respectively. Note that the DMU-1 model is essentially the same model as in Wong (2014).

2.4.1 Optimal solution and objective value

Since the DMU-n model satisfies Assumptions 1-4, we can invoke Theorem 4 to obtain the following corollary.

Corollary 5. *For given $K \in \mathbb{N}_{\geq 1}$ and proper K-partition of $[\underline{p}, \bar{p}]$, the optimal solution for the DMU-n problem is given by*

$$x_k = \begin{cases} 0 & \text{if } k < k^* \\ \sqrt[n]{\frac{P + \bar{p}_k + \underline{p}_k - \bar{p}}{r}} & \text{if } k \geq k^* \end{cases} \qquad \text{and} \qquad z_k = \underline{p}_k x_k - \tfrac{1}{n+1} r x_k^{n+1} - \sum_{i=1}^{k-1} (\bar{p}_i - \underline{p}_i) x_i.$$

where $k^* = \min\{k \in \mathcal{K} : P - \bar{p} + \bar{p}_k + \underline{p}_k > 0\}$. Thus, trade occurs for types $p \in [\underline{p}_{k^*}, \bar{p}]$. Also, $0 = x_1 = \cdots = x_{k^*-1} < x_{k^*} < \cdots < x_K < \infty$ and $z_1 = \cdots = z_{k^*-1} = 0$.

The corresponding optimal objective value Γ_K *is*

$$\Gamma_K = \frac{n}{n+1} \frac{1}{\sqrt[n]{r}} \sum_{k=k^*}^{K} \frac{\bar{p}_k - \underline{p}_k}{\bar{p} - \underline{p}} \left(P - \bar{p} + \bar{p}_k + \underline{p}_k\right)^{\frac{n+1}{n}}. \tag{2.10}$$

Proof. We apply Theorem 4 where

$$\frac{\mathrm{d}}{\mathrm{d}x}(\phi_S + \psi)(x) = P - rx^n \qquad \Longrightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\phi_S + \psi)(0) = P.$$

Since $P - \bar{p} + \bar{p}_K + \underline{p}_K = P + \underline{p}_K > 0$, we get $k^* = \min\{k \in \mathcal{K} : P - \bar{p} + \bar{p}_k + \underline{p}_k > 0\}$. Furthermore, since ψ decreases super-linearly we have $\hat{k} = K$. In other words, all contracts are sensible ($x_k, z_k < \infty$) and at least one contract instigates trade ($x_K > 0$). The results now follow from Theorem 4. □

The optimal objective value Γ_K is the main focus in the results to come. Notice that Γ_1 is independent of any partition, since there is no partition for a single contract ($K = 1$). For a given instance, Γ_1 is the lowest possible expected utility for the seller when using robust pooling. Furthermore, $k^* = 1$ for $K = 1$, since $P - \bar{p} + \bar{p}_K + \underline{p}_K = P + \underline{p}_K = P + \underline{p} > 0$. Thus, the optimal objective value for $K = 1$ simplifies to

$$\Gamma_1 = \frac{n}{n+1} \frac{1}{\sqrt[n]{r}} \left(P + \underline{p}\right)^{\frac{n+1}{n}}.$$

Likewise, using infinitely many contracts, i.e., letting $K \to \infty$ using sensible partitions, also leads to an objective value independent of any partition. We denote this value by Γ_∞, which is the highest possible expected utility for the seller when using robust pooling:

$$\Gamma_\infty = \frac{n}{n+1} \frac{1}{\sqrt[n]{r}} \int_{p^*}^{\bar{p}} \frac{1}{\bar{p} - \underline{p}} \left(P - \bar{p} + 2p\right)^{\frac{n+1}{n}} \mathrm{d}p,$$

where $p^* = \min\{p \in [\underline{p}, \bar{p}] : P - \bar{p} + 2p \geq 0\}$ is the continuous version of k^*. That is, p^* is the threshold for which the optimal order quantity is non-zero. To be precise, we have

$$p^* = \max\{\underline{p}, \tfrac{1}{2}(\bar{p} - P)\}.$$

Therefore, Γ_∞ can be written as

$$\Gamma_\infty = \begin{cases} \frac{n}{n+1} \frac{n}{2n+1} \frac{1}{2(\bar{p}-\underline{p})} \frac{1}{\sqrt[n]{r}} \left((P + \bar{p})^{\frac{2n+1}{n}} - \left(P - \bar{p} + 2\underline{p}\right)^{\frac{2n+1}{n}} \right) & \text{if } p^* = \underline{p} \\ \frac{n}{n+1} \frac{n}{2n+1} \frac{1}{2(\bar{p}-\underline{p})} \frac{1}{\sqrt[n]{r}} \left(P + \bar{p}\right)^{\frac{2n+1}{n}} & \text{if } p^* = \frac{1}{2}(\bar{p} - P) \end{cases}.$$

Notice that Γ_K is the composite midpoint rule for numerical integration applied to the integrand of Γ_∞. In other words, determining the optimal partition for robust pooling is equal to choosing the optimal partition for the composite midpoint rule. For more details on this numerical integration, see for example Dragomir et al. (1998) and Kirmaci (2004). Therefore, we could apply results from numerical integration to obtain performance guarantees for Γ_K compared to Γ_∞. In particular, this insight implies that loss in performance (the difference between Γ_K and Γ_∞) is of the order $\mathcal{O}(1/K^2)$. This is in line with the results of Bergemann et al. (2011) and Wong (2014). However, by analysing the performance of robust pooling in more detail, we can determine the achieved performances exactly.

2.4.2 Performance measures

For a given partition, we would like to compare the optimal objective value Γ_K for different number of contracts K. In order to do so, it is useful to redefine the partition as follows:

$$\underline{p}_k = \underline{p} + \delta_{k-1}(\bar{p} - \underline{p}) \qquad \text{and} \qquad \bar{p}_k = \underline{p} + \delta_k(\bar{p} - \underline{p}),$$

where $\delta_0 = 0$, $\delta_k \in [0,1]$ for $k = 1, \ldots, K-1$, and $\delta_K = 1$. Notice that δ_0 corresponds to $\underline{p}$ and δ_K to $\bar{p}$. Furthermore, δ_k for $k = 1, \ldots, K-1$ encode the chosen points to partition $[\underline{p}, \bar{p}]$. Thus, a proper K-partition satisfies $0 = \delta_0 < \cdots < \delta_K = 1$. We denote the partition by $\Delta = \{\delta_0, \ldots, \delta_K\}$. Substitution of this definition in (2.10) gives

$$\Gamma_K = \frac{n}{n+1} \frac{1}{\sqrt[n]{r}} \sum_{k=k^*}^{K} (\delta_k - \delta_{k-1}) \left(P + \underline{p} + (\delta_k + \delta_{k-1} - 1)(\bar{p} - \underline{p})\right)^{\frac{n+1}{n}}. \tag{2.11}$$

With this reformulated expression, we can for example consider the improvement of offering K contracts compared to a single contract:

$$\frac{\Gamma_K}{\Gamma_1} = \sum_{k=k^*}^{K} (\delta_k - \delta_{k-1}) \left(1 + (\delta_k + \delta_{k-1} - 1)\frac{\bar{p} - \underline{p}}{P + \underline{p}}\right)^{\frac{n+1}{n}}.$$

The parameter $(\bar{p} - \underline{p})/(P + \underline{p})$ plays a central role in all the following analysis. We call this parameter the *instance parameter* $\alpha \in \mathbb{R}_{>0}$:

$$\alpha = \frac{\bar{p} - \underline{p}}{P + \underline{p}}.$$

We will see that all structural results can be expressed in terms of α, i.e., it captures the essence of the instance. Returning to improvement Γ_K/Γ_1, we get

$$\frac{\Gamma_K}{\Gamma_1} = \sum_{k=k^*}^{K} (\delta_k - \delta_{k-1}) \Big(1 + (\delta_k + \delta_{k-1} - 1)\alpha\Big)^{\frac{n+1}{n}}.$$

In terms of α, we have $k^* = \min\left\{k \in \mathcal{K} : \delta_k + \delta_{k-1} > \frac{\alpha-1}{\alpha}\right\}$, since

$$\begin{aligned} & P - \bar{p} + \bar{p}_k + \underline{p}_k > 0 && \iff && P + \underline{p} + (\delta_k + \delta_{k-1} - 1)(\bar{p} - \underline{p}) > 0 \\ \iff \quad & 1 + (\delta_k + \delta_{k-1} - 1)\alpha > 0 && \iff && \delta_k + \delta_{k-1} > \tfrac{\alpha-1}{\alpha}. \end{aligned}$$

Thus, if $0 < \alpha \leq 1$ any partition satisfies $k^* = 1$, i.e., all contracts instigate trade between the seller and buyer.

It is now straightforward to determine the following bounds on the relative improvement for any $K > 1$ and proper K-partition:

$$\lim_{\alpha \to 0} \frac{\Gamma_K - \Gamma_1}{\Gamma_1} = 0 \qquad \text{and} \qquad \lim_{\alpha \to \infty} \frac{\Gamma_K - \Gamma_1}{\Gamma_1} = \infty.$$

Hence, for any arbitrarily large relative improvement there exists an instance that exceeds this relative improvement. In particular, this holds for two contracts and any proper 2-partition.

It is useful to introduce a normalisation factor ν:

$$\nu = \tfrac{n+1}{n} \sqrt[n]{r} (\bar{p} - \underline{p})^{-\frac{n+1}{n}}.$$

This leads to the more manageable formula

$$\nu \Gamma_K = \sum_{k=k^*}^{K} (\delta_k - \delta_{k-1}) \left(\tfrac{1}{\alpha} + \delta_k + \delta_{k-1} - 1\right)^{\frac{n+1}{n}}. \tag{2.12}$$

The normalisation factor ν will cancel out in relative performance measures, allowing us to use (2.12) in these expressions.

In similar vein, we can express Γ_∞ in terms of α. First, we focus on p^* and realise that

$$\underline{p} \geq \tfrac{1}{2}(\bar{p} - P) \qquad \iff \qquad P + \underline{p} - (\bar{p} - \underline{p}) \geq 0 \qquad \iff \qquad 1 - \alpha \geq 0.$$

Thus, we have

$$p^* = \begin{cases} \underline{p} & \text{if } \alpha \leq 1 \\ \frac{1}{2}(\bar{p} - P) & \text{if } \alpha > 1 \end{cases}.$$

For $0 < \alpha \leq 1$ this leads to $p^* = \underline{p}$ and

$$\nu\Gamma_\infty|_{0<\alpha\leq 1} = \frac{n}{2(2n+1)}\left((\tfrac{1}{\alpha}+1)^{\frac{2n+1}{n}} - (\tfrac{1}{\alpha}-1)^{\frac{2n+1}{n}}\right). \tag{2.13}$$

Similarly, for $\alpha > 1$ we have $p^* = \frac{1}{2}(\bar{p} - P)$ and

$$\nu\Gamma_\infty|_{\alpha>1} = \frac{n}{2(2n+1)}(\tfrac{1}{\alpha}+1)^{\frac{2n+1}{n}}. \tag{2.14}$$

Notice that for $\alpha = 1$ (2.13) and (2.14) give the same value, as expected. Furthermore, realise that $\nu\Gamma_K$ and $\nu\Gamma_\infty$ are completely determined by α. However, for a fixed α the values Γ_K and Γ_∞ can take on any value in $(0, \infty)$ by changing the parameter r.

The main benefit of robust pooling is the finite number of contracts in the menu. However, limiting the number of contracts will typically come at the cost of having a lower expected utility for the seller. Therefore, the main performance measure of interest is the *pooling performance* Γ_K/Γ_∞, which measures the fraction of expected utility achieved by offering K contracts in terms of the maximum obtainable expected utility Γ_∞.

With the above analysis, we can express relative performance measures as 1-dimensional functions of α. Hence, we are able to make graphs of performance measures in terms of α and determine performance bounds. This requires us to make $n \in \mathbb{R}_{>0}$ explicit and choose a partition scheme (see Sections 2.5 and 2.6). Before we do so, we determine general properties of an optimal partition for the DMU-n problem.

2.4.3 Properties of an optimal partition

A partition is equidistant if it partitions $[\underline{p}, \bar{p}]$ into equally sized subintervals. That is, we have $\delta_k^{\text{equi}} = k/K$, or equivalently $\delta_{k+1}^{\text{equi}} - \delta_k^{\text{equi}} = 1/K$ for $k = 1, \ldots, K-1$. The equidistant partition Δ^{equi} is a natural default choice, especially in numerical integration literature. However, is it the optimal partition for the DMU-n problem, i.e., does it maximise Γ_K?

First of all, one should realise that the optimality of partitions is not affected by the normalisation factor ν and thus only depends on the instance parameter α. Consequently, we can work with $\nu\Gamma_K$ to simplify notation. For the equidistant partitioning, (2.12) becomes

$$\nu\Gamma_K^{\text{equi}} = \tfrac{1}{K}\sum_{k=k^*}^{K}\left(\tfrac{1}{\alpha} + \tfrac{2k-1}{K} - 1\right)^{\frac{n+1}{n}} . \tag{2.15}$$

With the equidistant partition, the index k^* can be determined as follows:

$$\begin{aligned}
\delta_k^{\text{equi}} + \delta_{k-1}^{\text{equi}} > \tfrac{\alpha-1}{\alpha} \quad &\iff \quad \frac{2k-1}{K} > \tfrac{\alpha-1}{\alpha} \quad \iff \quad k > \tfrac{1}{2}\left(1 + K\left(\tfrac{\alpha-1}{\alpha}\right)\right) \\
&\implies \quad k^* = \max\left\{1, \left\lfloor 1 + \tfrac{1}{2}\left(1 + K\left(1 - \tfrac{1}{\alpha}\right)\right)\right\rfloor\right\} .
\end{aligned}$$

It is useful to specify this range of k^* with respect to α, which depends on the parity of K. For K odd we have $k^* \in \{1, \dots, \frac{1}{2}(K+1)\}$ and

$$k^* = \begin{cases} 1 & \text{if } \alpha \in \left(0, \frac{K}{K-1}\right) \\ k & \text{if } \alpha \in \left[\frac{K}{K-(2k-3)}, \frac{K}{K-(2k-1)}\right) \text{ for some } k \in \{2, \dots, \frac{1}{2}(K-1)\} \\ \frac{1}{2}(K+1) & \text{if } \alpha \in \left[\frac{K}{2}, \infty\right) \end{cases} .$$

For K even the range is $k^* \in \{1, \dots, \frac{1}{2}K + 1\}$ and

$$k^* = \begin{cases} 1 & \text{if } \alpha \in \left(0, \frac{K}{K-1}\right) \\ k & \text{if } \alpha \in \left[\frac{K}{K-(2k-3)}, \frac{K}{K-(2k-1)}\right) \text{ for some } k \in \{2, \dots, \frac{1}{2}K\} \\ \frac{1}{2}K+1 & \text{if } \alpha \in [K, \infty) \end{cases} .$$

Naturally, these properties do not hold for a partition in general.

Intuitively, if $k^* \geq 3$ for the equidistant partition, there is an inefficiency in the corresponding optimal menu of contracts. For $k^* \geq 3$ we are offering the contract $(x, z) = (0, 0)$ multiple times. These duplicate contracts in the menu are pointless and can be used more efficiently by changing them. The next lemma confirms this intuition.

Lemma 6. *For any $K \in \mathbb{N}_{\geq 1}$ an optimal partition must satisfy $0 = \delta_0 < \delta_1 < \cdots < \delta_{K-1} < \delta_K = 1$ and $k^* \in \{1, 2\}$.*

Proof. The proof first ensures $k^* \in \{1, 2\}$ by shifting partition points δ_k to the right and then perturbs (ϵ-shifts) coinciding partition points to improve the partition. The details are given in

Appendix A.2. □

Corollary 7. *For $K \in \mathbb{N}_{>3}$ the equidistant partition is suboptimal if $\alpha \geq K/(K-3)$.*

Proof. For $K > 3$ and for $\alpha \geq K/(K-3)$ we have $k^* \geq 3$, which is suboptimal by Lemma 6. □

Corollary 7 does not prove or disprove whether the equidistant partition can be optimal at all. In the next sections, Sections 2.5 and 2.6, we consider the DMU-n problem for $n = 1$ and $n = 2$ and provide the answer to this question. Note that a general formula for the optimal partition seems impossible. The difficulty in finding the optimal partition becomes clearer in the next sections.

2.5 Application to the DMU-1 problem

In this section we specialise the results of Section 2.4 to the DMU-1 problem ($n = 1$). Here, the buyer has a linearly decreasing marginal utility for the products, which leads to a quadratic utility function ϕ_B. The DMU-1 problem is essentially the same as the considered model in Wong (2014). We extend and complete his analysis by considering all possible instances, relating structural results to the instance parameter α, and also evaluating the performance of the equidistant partition.

As we will see, the DMU-1 problem is special compared to other DMU-n problems in the sense that the optimal partition has an exceptional structure and is relatively straightforward to determine. We will first derive the optimal partition for DMU-1 in Section 2.5.1. In Section 2.5.2 we analyse the performance of the equidistant and optimal partitions in terms of the number of contracts K.

2.5.1 Optimal partition

Recall that for any proper K-partition of $[\underline{p}, \bar{p}]$ the normalised optimal objective value is given by (2.12), which for the DMU-1 problem is

$$\nu\Gamma_K = \sum_{k=k^*}^{K} (\delta_k - \delta_{k-1}) \left(\tfrac{1}{\alpha} + \delta_k + \delta_{k-1} - 1\right)^2,$$

where $k^* = \min\{k \in \mathcal{K} : \delta_k + \delta_{k-1} > \frac{\alpha-1}{\alpha}\}$. We will now optimise the partition to maximise Γ_K. By Lemma 6, we know that the optimal partition satisfies $0 = \delta_0 < \cdots < \delta_K = 1$ and $k^* \in \{1, 2\}$. The following theorem gives the optimal partition for the DMU-1 problem.

Theorem 8. *For $K \in \mathbb{N}_{\geq 1}$ the optimal partition Δ^{opt} for the DMU-1 problem is given by*

$$\delta_k^{\text{opt}} = \begin{cases} \frac{k}{K} & \text{if } \alpha < \frac{K}{K-1} \\ 1 - \frac{K-k}{2K-1}\left(\frac{1}{\alpha} + 1\right) & \text{if } \alpha \geq \frac{K}{K-1} \end{cases} \qquad \text{for } k \in \{2, \ldots, K-1\}.$$

Hence, for $\alpha < K/(K-1)$ the equidistant partition is optimal and all contracts instigate trade ($k^* = 1$). For $\alpha \geq K/(K-1)$ the equidistant partition is suboptimal and a single contract instigates no trade ($k^* = 2$).

Proof. We fix k^* either to 1 or 2 and maximize Γ_K by setting its gradient to zero. The resulting systems of equalities are linear and can easily be solved. The final step is to check the feasibility of the obtained solutions. See Appendix A.3 for the details. □

The result of Theorem 8 is quite remarkable: for $\alpha < K/(K-1)$ the equidistant partition is the optimal partition. From Corollary 7 we know that the equidistant partition is not always optimal. Therefore, we expected that the equidistant partition is never optimal or optimal in the limit, e.g., for $\alpha \to 0$ or $\alpha \to \infty$. However, the equidistant partition is optimal for any instance satisfying $\alpha < K/(K-1)$. For $\alpha \geq K/(K-1)$ it turns out that we effectively only have to optimise δ_1, since it is optimal to partition the remaining subinterval $[\delta_1, 1]$ equidistantly. This fact can be verified from the formula or the stationarity conditions mentioned in the proof of Theorem 8.

We will show that the optimal objective value Γ_K^{opt} approximates Γ_∞ with an almost correctly shaped function of α when using the optimal partition. This does not hold for the equidistant partition, which gives additional insights into why it is sometimes suboptimal. The details are as follows. For $0 < \alpha \leq 1$ the normalised objective value $\nu\Gamma_\infty$ is given by (2.13), which simplifies to

$$\nu\Gamma_\infty|_{0<\alpha\leq 1} = \tfrac{1}{6}\left((\tfrac{1}{\alpha}+1)^3 - (\tfrac{1}{\alpha}-1)^3\right) = \tfrac{1}{\alpha^2} + \tfrac{1}{3}. \tag{2.16}$$

For $0 < \alpha < K/(K-1)$, which implies $0 < \alpha \leq 1$, we use (2.12) for the equidistant partition:

$$\nu\Gamma_K^{\text{equi}}|_{0<\alpha<\frac{K}{K-1}} = \tfrac{1}{K}\sum_{k=1}^{K}\left(\tfrac{1}{\alpha} + \tfrac{2k-1}{K} - 1\right)^2 = \tfrac{1}{\alpha^2} + \tfrac{1}{3}(1 - \tfrac{1}{K^2}).$$

Hence, $\nu\Gamma_K^{\text{equi}}$ is of the correct order $\Theta(\alpha^{-2})$ for $0 < \alpha \leq 1$ compared to $\nu\Gamma_\infty$, but there is an error in the constant. Now consider $\alpha \geq K/(K-1)$. For $\alpha > 1$, which is implied by $\alpha \geq K/(K-1)$, (2.14) simplifies to

$$\nu\Gamma_\infty|_{\alpha>1} = \tfrac{1}{6}(\tfrac{1}{\alpha}+1)^3. \tag{2.17}$$

This is not of the same order as $\nu\Gamma_K^{\text{equi}}$, which is $\Theta(\alpha^{-2})$ for any $\alpha > 0$. The optimal partition satisfies

$$\delta_k^{\text{opt}} - \delta_{k-1}^{\text{opt}} = \frac{1}{2K-1}\left(\frac{1}{\alpha}+1\right) = \frac{1}{2K-1}\left(\frac{P+\bar{p}}{\bar{p}-\underline{p}}\right),$$
$$P+\underline{p}+(\delta_k^{\text{opt}}+\delta_{k-1}^{\text{opt}}-1)(\bar{p}-\underline{p}) = \left(1-\frac{2K-2k+1}{2K-1}\right)(P+\bar{p}) = \frac{2(k-1)}{2K-1}(P+\bar{p}).$$

Therefore, the corresponding optimal objective value (2.11) is

$$\Gamma_K^{\text{opt}}|_{\alpha\geq\frac{K}{K-1}} = \frac{1}{2r}\frac{(P+\bar{p})^3}{\bar{p}-\underline{p}}\sum_{k=2}^{K}\frac{(2(k-1))^2}{(2K-1)^3} = \frac{1}{2r}\frac{(P+\bar{p})^3}{\bar{p}-\underline{p}}\frac{2(K-1)K}{3(2K-1)^2},$$

or when normalised:

$$\nu\Gamma_K^{\text{opt}}|_{\alpha\geq\frac{K}{K-1}} = \frac{2(K-1)K}{3(2K-1)^2}\left(\tfrac{1}{\alpha}+1\right)^3. \tag{2.18}$$

Again, we see that the term $(1/\alpha+1)^3$ is correct, but there is an error in the coefficient. Thus, in both cases Γ_K^{opt} approximates Γ_∞ with almost correctly shaped functions of α. In particular, the formulas show that the approximation converges to Γ_∞ as $K\to\infty$, as should be the case.

If the equidistant partition is not optimal, the optimal partition points δ_k^{opt} deviate from the equidistant values δ_k^{equi}. Before completing the analysis, we expected that $\delta_k^{\text{opt}} < \delta_k^{\text{equi}}$ and $\delta_k^{\text{opt}} > \delta_k^{\text{equi}}$ can both occur. However, this is not the case, as explained in the next corollary.

Corollary 9. *For $K \in \mathbb{N}_{\geq 1}$ the optimal partition for the DMU-1 problem satisfies $\delta_0^{\text{opt}} = 0$, $\delta_K^{\text{opt}} = 1$, and*

$$\delta_k^{\text{equi}} = \tfrac{k}{K} \leq \delta_k^{\text{opt}} < \tfrac{K+k-1}{2K-1} \qquad \textit{for } k\in\{1,\ldots,K-1\}.$$

Proof. For $\alpha < K/(K-1)$ this is trivial as the optimal partition is equidistant. For $\alpha \geq K/(K-1)$ we have

$$\delta_k^{\text{opt}} = \tfrac{K+k-1}{2K-1} - \tfrac{K-k}{2K-1}\tfrac{1}{\alpha} \in \left[\tfrac{k}{K}, \tfrac{K+k-1}{2K-1}\right).$$

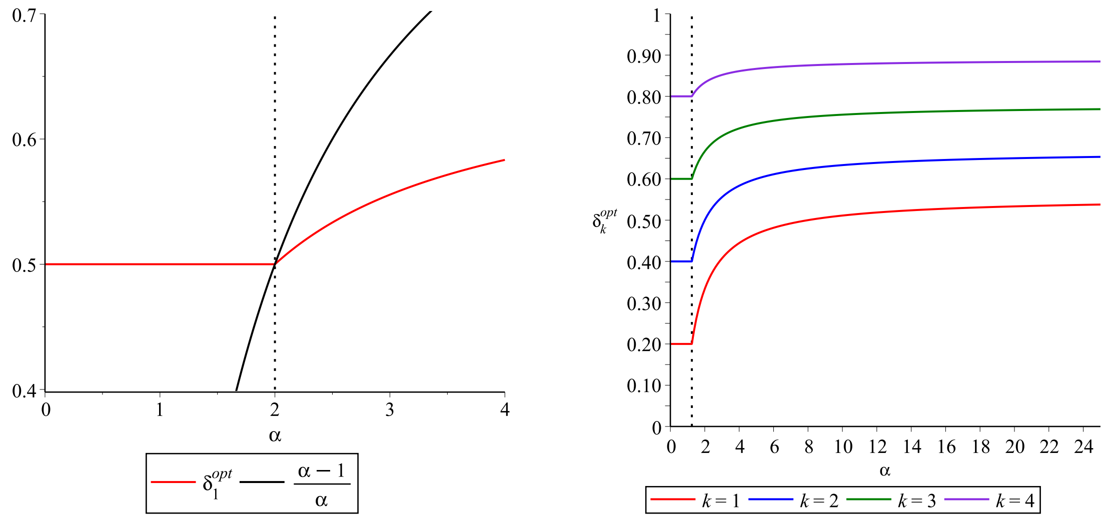
Combining these properties gives the desired result. □

Thus, Corollary 9 shows that the optimal partition points δ_k^{opt} always deviate to the right (larger values). A possible explanation is that for $\alpha \geq K/(K-1)$ we have $k^* = 2$ for the optimal partition. In other words, one contract instigates no trade: $(x_1, z_1) = (0, 0)$. If α increases we have observed before that k^* increases for the equidistant partition. Since $k^* > 2$ is suboptimal by Lemma 6, we must have $\delta_1^{\text{opt}} > \delta_1^{\text{equi}}$ in order to prevent $k^* > 2$. Given δ_1^{opt}, the remaining subinterval $[\delta_1^{\text{opt}}, 1]$ is partitioned equidistantly to obtain δ_k^{opt} for $k = 2, \ldots, K-1$. Thus, since $\delta_1^{\text{opt}} > \delta_1^{\text{equi}}$ we also get $\delta_k^{\text{opt}} > \delta_k^{\text{equi}}$ for $k = 2, \ldots, K-1$.

Figure 1a shows the optimal partition for $K = 2$ in terms of α. The two curves are δ_1^{opt} in red and $(\alpha - 1)/\alpha$ in black. Left of $\alpha = 2$ (the dotted line) the optimal partition satisfies $k^* = 1$ and all contracts instigate trade. Right of $\alpha = 2$ there is no trade with the most inefficient types $p \in [\underline{p}, \underline{p}_{k^*})$. The transition in formulas of δ_1^{opt} is continuous at the breakpoint $\alpha = 2$. Furthermore, this transition occurs exactly when the equidistant partition switches from $k^* = 1$ to $k^* = 2$, i.e., when $\delta_1^{\text{equi}} = (\alpha - 1)/\alpha$, as seen in the proof of Theorem 8.

The optimal partition for $K = 5$ is illustrated in Figure 1b. For $\alpha \geq 5/4$, notice that as α increases, the seller refuses the 20% most inefficient (lowest) types p which rapidly increases to 45%, with 55% as limit.

To conclude, we have determined the optimal partition for DMU-1 and all relevant values can again be expressed in terms of α. Therefore, we can compare the performance of the equidistant and optimal partitions, which is the topic of the next section.



(a) For $K = 2$ contracts.

(b) For $K = 5$ contracts.

Figure 1: DMU-1: optimal partition Δ^{opt} in terms of α.

2.5.2 Performance of partition schemes

We compare two partition schemes: the equidistant partition Δ^{equi} and the optimal partition Δ^{opt}. As mentioned in Section 2.4.2 the main performance measure of interest is the pooling performance Γ_K/Γ_∞. For the DMU-1 problem, Γ_∞ is given by (2.16) and (2.17). For the equidistant partition, we have

$$\nu\Gamma_K^{\text{equi}} = \tfrac{1}{K}\sum_{k=k^*}^{K}\left(\tfrac{1}{\alpha} + \tfrac{2k-1}{K} - 1\right)^2,$$

where $k^* = \max\left\{1, \left\lfloor 1 + \frac{1}{2}\left(1 + K\left(1 - \frac{1}{\alpha}\right)\right)\right\rfloor\right\}$. This allows us to express $\Gamma_K^{\text{equi}}/\Gamma_\infty$ in terms of α. For $0 < \alpha < K/(K-1)$ the optimal partition is equal to the equidistant partition, but for $\alpha \geq K/(K-1)$ we have (2.18) and the pooling performance

$$\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty}\Big|_{\alpha \geq \frac{K}{K-1}} = \frac{4(K-1)K}{(2K-1)^2} = 1 - \frac{1}{(2K-1)^2}.$$

Notice that this pooling performance is constant with respect to $\alpha \geq K/(K-1)$. Figure 2 shows the pooling performance for $K \in \{1, 2, 3\}$ for the equidistant and optimal partitions. By inspection of the graphs, we conclude that the infimum of $\Gamma_K^{\text{equi}}/\Gamma_\infty$ is reached for $\alpha \to \infty$ and the infimum of $\Gamma_K^{\text{opt}}/\Gamma_\infty$ is attained for each $\alpha \geq K/(K-1)$. We have $\lim_{\alpha\to\infty} \Gamma_K^{\text{equi}}/\Gamma_\infty = 1 - 1/K^2$. This implies that the following lower bounds are tight:

$$\frac{\Gamma_K^{\text{equi}}}{\Gamma_\infty} \geq 1 - \frac{1}{K^2} \qquad \text{and} \qquad \frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty} \geq 1 - \frac{1}{(2K-1)^2}.$$

For several values of K, the performance guarantees are listed in Table 1.

We observe that $\Gamma_1/\Gamma_\infty \to 0$ as $\alpha \to \infty$, i.e., offering a single robustly pooled contract can perform arbitrarily bad compared to offering infinitely many contracts. However, offering two contracts with the equidistant partition always achieves at least 75% of the maximum obtainable expected utility. For the optimal partition this is 88%. The reason is as follows. A large α can be interpreted as having a large uncertainty of the buyer's efficiency, i.e., a large interval $[\underline{p}, \bar{p}]$. In order to obtain a high expected utility, the seller wants to offer different contracts to inefficient and efficient types. This is why for $k^* > 1$ the seller refuses to trade with the most inefficient types (with $p \in [\underline{p}, \underline{p}_{k^*})$). For $K = 1$, a single contract, the seller cannot make a distinction between efficient and inefficient types and always instigates trade with the buyer ($k^* = 1$). In contrast, for $K \geq 2$ the seller can refuse inefficient types ($k^* > 1$ for α large enough).

Thus, it is essential for the seller to be able to refuse the most inefficient types when there is a high uncertainty in the buyer's efficiency. This is especially noticeable for the optimal partition: for α large enough (such that $k^* = 2$) inefficient types are refused, resulting in a constant pooling performance onwards.

Finally, notice that the optimal partition greatly outperforms the equidistant partition for large values of α. In particular, Table 1 shows that the seller can achieve the same performance guarantee with far fewer contracts when using the optimal partition. For example, for a guarantee of 96% the seller has to offer 3 contracts with the optimal partition and 5 contracts with the equidistant partition. For either partition, good performances can be achieved with only a few contracts, which validates the robust pooling approach.

Wong (2014) restricts his analysis to instances with $\alpha \geq K/(K-1)$ (such that $k^* = 2$) and determines the corresponding optimal partition and its pooling performance. Thus, our results extend and complete the analysis of DMU-1. In particular, by considering all possible instances, we observe the remarkable optimality of the equidistant partition for each $\alpha < K/(K-1)$.

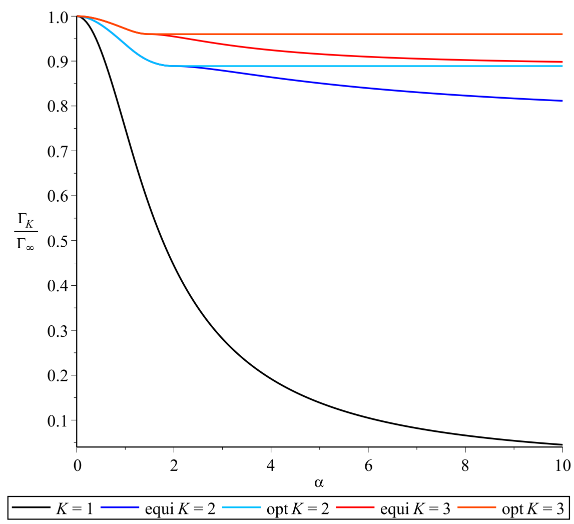


Figure 2: DMU-1: the pooling performance Γ_K/Γ_∞ for the equidistant and optimal partitions as functions of the instance parameter α.

K	Equidistant LB	Optimal LB
1	0	0
2	0.7500	0.8888
3	0.8888	0.9600
4	0.9375	0.9795
5	0.9600	0.9876
6	0.9722	0.9917
∞	1	1

Table 1: DMU-1: lower bounds for the pooling performance Γ_K/Γ_∞ for the equidistant and optimal partitions.

2.6 Application to the DMU-2 problem

To illustrate the special structure of the DMU-1 problem, we perform a similar analysis for the DMU-2 problem ($n = 2$). The buyer has a quadratically decreasing marginal utility for the products. Hence, for $0 < x < 1$ the marginal utility is higher than for the DMU-1 problem, but lower for $x > 1$. In Section 2.6.1 we show the complexity of finding closed-form formulas for the optimal partition. However, we can optimise the partition using numerical methods. In Section 2.6.2 we determine the performance of the equidistant and optimised partitions. Keep in mind that all values shown with four digits are truncated or rounded.

2.6.1 Optimal partition

For the DMU-2 problem, the normalised optimal objective value is given by

$$\nu\Gamma_K = \sum_{k=k^*}^{K} (\delta_k - \delta_{k-1}) \left(\tfrac{1}{\alpha} + \delta_k + \delta_{k-1} - 1\right)^{\frac{3}{2}},$$

where $k^* = \min\{k \in \mathcal{K} : \delta_k + \delta_{k-1} > \frac{\alpha-1}{\alpha}\}$. Again, we know by Lemma 6 that the optimal partition must be strictly ordered and must satisfy $k^* \in \{1, 2\}$. In Lemma 10 we determine the optimal partition for $K = 2$ contracts.

Lemma 10. *For the DMU-2 problem and $K = 2$, the optimal partition is*

$$\delta_1^{\text{opt}} = \begin{cases} \frac{1}{30}\left(\sqrt{36\frac{1}{\alpha^2} - 15} + 15 - 6\frac{1}{\alpha}\right) & \text{if } \alpha < \alpha^{\text{trans}} \\ 1 - \frac{2}{5}(\frac{1}{\alpha} + 1) & \text{if } \alpha \geq \alpha^{\text{trans}} \end{cases},$$

where $\alpha^{\text{trans}} \approx 1.5371$. Furthermore, δ_1^{opt} satisfies the tight bounds $0.3397 < \delta_1^{\text{opt}} < \frac{3}{5}$. Hence, for $\alpha < \alpha^{\text{trans}}$ all contracts instigate trade ($k^* = 1$). For $\alpha \geq \alpha^{\text{trans}}$ a single contract instigates no trade ($k^* = 2$).

Proof. Similar to the proof of Theorem 8, we condition on k^* being 1 or 2. However, determining the global maximum is more difficult. We refer to Appendix A.4 for the details. □

In Figure 3a the δ_1^{opt} for $K = 2$ is shown in red. Left of α^{trans} (the dotted line) we have $k^* = 1$ and on the right $k^* = 2$. As detailed in the proof, the difficulty of determining δ_1^{opt} is the existence of two local maxima, denoted by δ_1^+ and δ_1^*. As shown, the optimal partition jumps discontinuously from δ_1^+ to δ_1^* at α^{trans}. Figure 3a illustrates this jump and the coexistence of the local maxima δ_1^+ (shown in cyan) and δ_1^* (in blue) for $\alpha \in [3/2, 2/5\sqrt{15}]$. Comparing this figure with Figure 1a, it is clear that the properties of the optimal partition for DMU-1 are indeed exceptional.

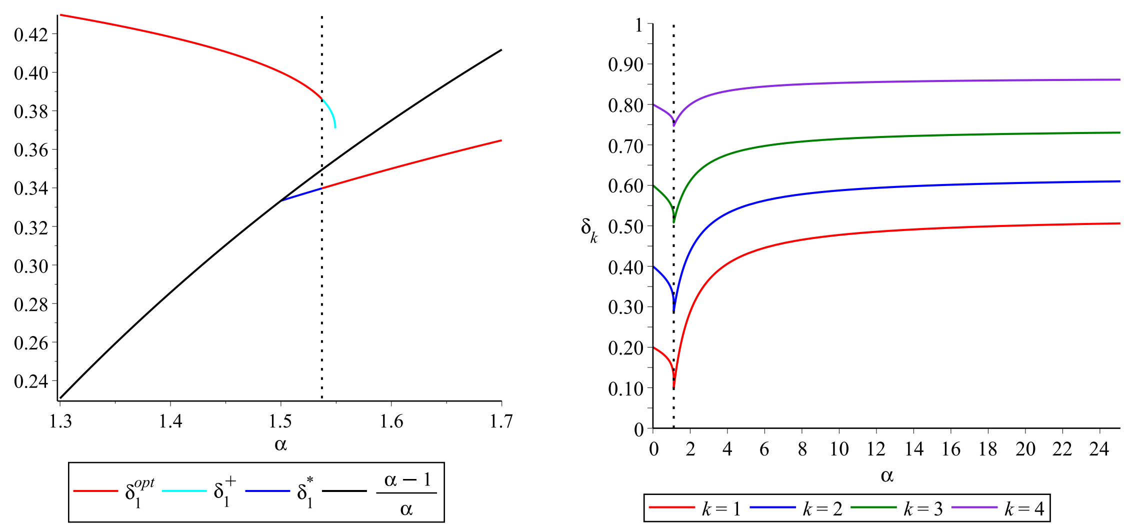
Lemma 10 shows that the equidistant partition is never optimal for the DMU-2 problem (except in the limit $\alpha \to 0$). Furthermore, as α increases, the optimal partition point first decreases, then jumps to a lower value, and finally increases. Thus, if the uncertainty in the buyer's efficiency is large enough the optimal menu refuses trade with the most inefficient types. Moreover, as this uncertainty increases, trade is refused for more types, as is the case for the DMU-1 problem.

For a general number of contracts K, we can attempt to imitate the proof of Lemma 10.

However, this requires to solve a complicated non-linear system of equalities, for which a general solution seems impossible. Instead, we optimise the partition numerically using the gradient-based methodology described in Appendix A.5. Although the used solver can only guarantee local optimality, its performance is stable and the results correspond to our theoretical results when available. Therefore, all results indicate that the method finds the global optimum.

We see a similar structure in the optimised partition for $K \geq 2$ as observed for the optimal partition for $K = 2$: decreasing in α at first, then a discontinuous jump to a lower value, and finally increasing in α. See Figure 3b for the optimised partition for $K = 5$. Notice that the optimised partition points are not bounded by the equidistant partition points, as is the case for DMU-1.

To conclude, this analysis for the DMU-2 shows the special structure of the DMU-1, for which the equidistant partition can be optimal and general formulas can be determined. For the DMU-2 problem, we can numerically optimise the partition for any number of contracts. In the next section, we compare the performance of the equidistant and optimised partitions.



(a) Optimal partition δ_1^{opt} for $K = 2$ contracts. (b) Optimised partition for $K = 5$ contracts.

Figure 3: DMU-2: optimised partition in terms of α.

2.6.2 Performance of partition schemes

As in Section 2.5.2, we compare the pooling performance Γ_K/Γ_∞ for the equidistant and optimised partitions. Recall that the related formulas for Γ_∞ and Γ_K^{equi} are (2.13), (2.14), and (2.15). As explained in the previous section, we only have numerical results for the optimised partition.

Figure 4 shows the pooling performance for $K \in \{1, 2, 3\}$ for the equidistant and optimised partitions. First of all, Γ_1/Γ_∞ is not shown completely, because it goes to zero for $\alpha \to \infty$ as seen for the DMU-1 problem. In contrast to DMU-1, the performance of the equidistant partition has local minima and maxima, and the infimum is typically attained at some finite value for α (so not for $\alpha \to \infty$). Furthermore, for a fixed instance, an equidistant $(K+1)$-partition does not always perform better than an equidistant K-partition. For example, for $\alpha = 20$ the equidistant 4-partition outperforms the equidistant 5-partition. The lower bounds on the pooling performance are given in Table 2. Note that the lower bounds for the equidistant partition with 4 and 5 contracts are effectively the same.

For α such that $k^* = 2$ for the optimised partition, we see that the pooling performance is constant and minimal. For $K = 2$ this can be verified with Lemma 10. This property also holds for DMU-1. Table 2 also includes the lower bounds for the optimised partition.

To conclude, as for the DMU-1 problem, offering a single robust contract is not recommended. However, by offering only a few contracts, high pooling performance can be achieved of at least 88% (equidistant partition) or 92% (optimised partition). The partition can be optimised using numerical methods, which is in particular beneficial for up to five contracts.

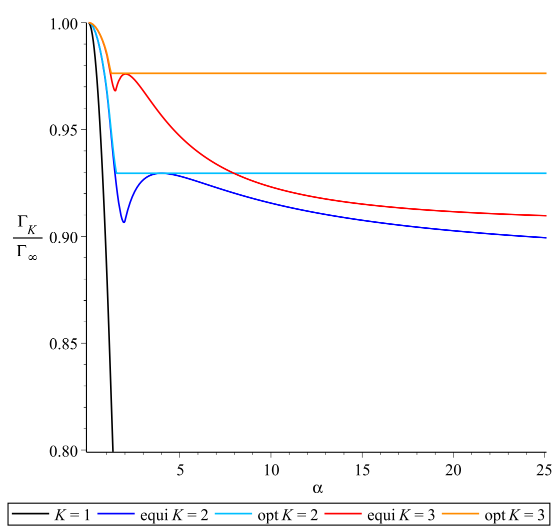


Figure 4: DMU-2: the pooling performance Γ_K/Γ_∞ for the equidistant and optimised partitions as functions of the instance parameter α.

K	Equidistant LB	Optimised LB
1	0	0
2	0.8838	0.9295
3	0.9065	0.9763
4	0.9681	0.9882
5	0.9681	0.9929
6	0.9842	0.9953
∞	1	1

Table 2: DMU-2: lower bounds for the pooling performance Γ_K/Γ_∞ for the equidistant and optimised partitions.

3 Contracting for minimising costs

In this section we analyse the robust pooling model in the setting of cost minimisation using the same approach as in Section 2. We formalise the model in Section 3.1. Although the models for maximising utility and for minimising cost have a similar structure, they are not equivalent under the considered assumptions. Nevertheless, the general analysis is similar and will be provided in Appendix B.1. When needed, we will highlight the differences between cost minimisation and utility maximisation. We apply the robust pooling model to a classical cost minimisation model based on the Economic Order Quantity setting in Section 3.2.

3.1 The model

As in Section 2, the principal is a seller of products and the agent a buyer. The seller offers contracts to the buyer, which specify the order quantity $x \in \mathbb{R}_{\geq 0}$ and a side payment $z \in \mathbb{R}$. In contrast to utility maximisation, here we define z to be the side payment from the seller to the buyer to be consistent with the literature related to the model considered in Section 3.2.

As before, we assume that the buyer's private information can be captured by a parameter $p \in [\underline{p}, \bar{p}] \subseteq \mathbb{R}_{\geq 0}$ with $\bar{p} > \underline{p}$. The buyer's cost for order quantity x is $\phi_B(x|p)$ and the corresponding seller's cost is $\phi_S(x)$. The seller applies the same robust pooling approach as before. First, the seller decides how many contracts are offered, denoted by $K \in \mathbb{N}_{\geq 1}$. Second, he divides the interval $[\underline{p}, \bar{p}]$ into K subintervals, using a proper K-partition. Third, the seller designs a menu of K contracts by solving the following optimisation model:

$$
\begin{aligned}
\min_{x,z} \quad & \sum_{k\in\mathcal{K}} \omega_k \left(\phi_S(x_k) + z_k\right), && \\
\text{s.t.} \quad & \phi_B(x_k|p_k) - z_k \le \Theta, && \forall\, p_k \in [\underline{p}_k, \bar{p}_k], k \in \mathcal{K}, \qquad (3.1)\\
& \phi_B(x_k|p_k) - z_k \le \phi_B(x_l|p_k) - z_l, && \forall\, p_k \in [\underline{p}_k, \bar{p}_k], k, l \in \mathcal{K}, \qquad (3.2)\\
& x_k \ge 0, && \forall\, k \in \mathcal{K}.
\end{aligned}
$$

Except for (3.1), the model is essentially the same as in Section 2.4. Constraints (3.1) ensure individual rationality for the buyer, which need further clarification. The parameter $\Theta \in \mathbb{R}_{\ge 0}$ is the buyer's reservation level: if the buyer's net cost for a contract would exceed Θ he will not accept it. In the literature, Θ is often the cost for ordering at an outside option. Hence, Θ is also called the *outside option* or *default option*.

For utility maximisation problems, such as the DMU-n problem in Section 2.4, it is common that the default option is to have no trade and thus zero utility. This implies $\Theta = 0$. Therefore, we did not include Θ in the model description in Section 2. For cost minimisation problems this is not the case, as there is no common natural default option. For example, if the default option is to have no trade, which (virtual) cost should be assigned? If the default is to use an outside option, does the corresponding cost depend on $[\underline{p}, \bar{p}]$ or not? For the problem analysed in Section 3.2, and its default option, it is useful to mention Θ explicitly in the model as it effects the results.

To prepare for Section 3.2, we make the following assumption on the buyer's cost function to derive a simpler reformulation of the robust pooling model.

Assumption 5. The buyer's cost function is $\phi_B(x|p) = \psi(x) + p\chi(x)$, where the functions $\psi : \mathbb{R}_{\ge 0} \to \mathbb{R}$ and $\chi : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ do not depend on the type p. Moreover, χ is non-decreasing and non-negative.

Although we can rewrite the cost minimisation problem into a utility maximisation problem, Assumption 5 does not fit into the framework of Section 2, because of the resulting negative term $-p\chi(x)$ in the buyer's utility function.

Under Assumption 5, we can derive results equivalent to those in Sections 2.2 and 2.3, which are given in Appendix B.1. The proofs and results are essentially identical, with the following highlighted exceptions. First, the change of variables by redefining the side payment includes the outside option Θ:

$$z_k = \psi(x_k) + y_k - \Theta.$$

Consequently, Θ appears as a constant in the objective function of the reformulated models.

Second, the structure of the optimal side payments and the feasible region is 'reversed' in terms of the contract indices k. For example, the feasible region is $x_1 \geq \cdots \geq x_K \geq 0$. However, in terms of buyer type's efficiency the result is not reversed. Here, a buyer with a lower parameter p is more efficient, since he has lower costs for an order quantity.

Finally, under Assumption 5 the robust pooling model is equivalent to the limited variety and discrete models. For the discrete model, the highest type $\bar{p}_k$ must be chosen as representative for $[\underline{p}_k, \bar{p}_k]$, i.e., again the most inefficient type. This can be shown using a similar argument as in Section 2.3.

We continue to apply robust pooling and these results to the Economic Order Quantity model in the next section.

3.2 Economic order quantity problem

We consider a contracting model to which we can apply the robust pooling setting of Section 3.1. The considered cost functions are those of the classical Economic Order Quantity (EOQ) model, which model the average cost of a trade agreement over an infinite horizon. The context of the problem is as follows.

The buyer has external demand with constant rate $d \in \mathbb{R}_{>0}$ on an infinite time horizon, which must be satisfied without backlogging. He can order products at the seller, which has an ordering cost of $f \in \mathbb{R}_{>0}$ for the buyer. Furthermore, the buyer has an inventory holding cost of $h \in \mathbb{R}_{>0}$ per product and time unit. The buyer's holding cost h is the private parameter. To minimise his own costs, the buyer orders if and only if his inventory is depleted (the zero-inventory property). Therefore, an order quantity of $x \in \mathbb{R}_{>0}$ products leads to a total cost per time unit of $\phi_B(x) = df\frac{1}{x} + \frac{1}{2}hx$.

The seller has a similar cost structure: setup cost $F \in \mathbb{R}_{>0}$ and inventory holding cost $H \in \mathbb{R}_{>0}$. Production takes place with constant rate $p \in \mathbb{R}_{>d}$ and according to a just-in-time lot-for-lot policy.

This leads to a total costs per time unit for the seller of $\phi_S(x) = dF\frac{1}{x} + \frac{1}{2}H\frac{d}{p}x$.

To simplify notation, we define $R = dF$, $P = \frac{1}{2}H\frac{d}{p}$, $r = df$, and $p = \frac{1}{2}h$. Hence, the buyer's cost function is

$$\phi_B(x|p) = r\tfrac{1}{x} + px,$$

where $r \in \mathbb{R}_{>0}$ is a fixed parameter and $p \in [\underline{p}, \bar{p}] \subseteq \mathbb{R}_{\geq 0}$ the buyer's private parameter. We assume that the distribution of p is uniform. Likewise, the seller's costs are given by $\phi_S(x) = R\frac{1}{x} + Px$ for fixed parameters $R, P \in \mathbb{R}_{>0}$.

Given this setting, the seller constructs a menu of contracts using the robust pooling approach of Section 3.1. We refer to this problem as the *Economic Order Quantity* (EOQ) problem, which is analysed in detail in the following sections. In Section 3.2.1, we determine the optimal solution and corresponding optimal objective value. Section 3.2.2 focuses on performance measures. We show that performance measures can be expressed in terms of an instance parameter α, similar to the DMU-n problem. In Section 3.2.3, we analyse the optimal partition for the EOQ problem. Finally, the derived results are used in Section 3.2.4 to determine the performance of the equidistant partition and the optimised partition.

3.2.1 Optimal solution and objective value

Since the EOQ setting satisfies the assumptions of Appendix B.1, we can apply the results of this appendix. In particular, we obtain the optimal solution of the EOQ problem as a corollary, see Corollary 11.

Corollary 11. *For given $K \in \mathbb{N}_{\geq 1}$ and proper K-partition of $[\underline{p}, \bar{p}]$, the optimal solution for the EOQ problem is given by*

$$x_k = \sqrt{\frac{R + r}{P - \underline{p} + \bar{p}_k + \underline{p}_k}} \qquad \text{and} \qquad z_k = r\tfrac{1}{x_k} + \bar{p}_k x_k + \sum_{i=k+1}^{K} (\bar{p}_i - \underline{p}_i) x_i - \Theta.$$

Hence, $x_1 > \cdots > x_K > 0$ and trade always occurs.

The corresponding optimal objective value Γ_K is

$$\Gamma_K = 2\sqrt{R + r} \sum_{k=1}^{K} \frac{\bar{p}_k - \underline{p}_k}{\bar{p} - \underline{p}} \sqrt{P - \underline{p} + \bar{p}_k + \underline{p}_k} - \Theta. \tag{3.3}$$

Proof. We apply Theorem 17 stated in Appendix B.1 to the EOQ problem. Since $\phi_S(x) + \psi(x) = (R+r)\frac{1}{x} + Px$ it is straightforward to determine that $k^* = K$ and $\hat{k} = 1$ (note that the definitions of k^* and $\hat{k}$ differ from Section 2). The results now follow directly from the theorem after simplifying the expressions. □

Recall that for the DMU-n problem the optimal menu could include contracts that instigated no trade ($x_k = 0$ for some $k \in \mathcal{K}$). By Corollary 11 trade always occurs for the EOQ problem ($x_k > 0$ for all $k \in \mathcal{K}$).

As in Section 2.4 there are two extreme choices for K, namely $K = 1$ and $K = \infty$. The optimal objective value Γ_1 is the highest expected cost for the seller when using robust pooling:

$$\Gamma_1 = 2\sqrt{R+r}\sqrt{P+\bar{p}} - \Theta.$$

In contrast, Γ_∞ is the lowest expected cost for the seller:

$$\begin{aligned}\Gamma_\infty &= 2\frac{\sqrt{R+r}}{\bar{p}-\underline{p}} \int_{\underline{p}}^{\bar{p}} \sqrt{P - \underline{p} + 2p}\, \mathrm{d}p - \Theta \\ &= \frac{2}{3}\frac{\sqrt{R+r}}{\bar{p}-\underline{p}} \left(\left(P + 2\bar{p} - \underline{p}\right)^{\frac{3}{2}} - \left(P + \underline{p}\right)^{\frac{3}{2}} \right) - \Theta.\end{aligned}$$

Again, we recognise that Γ_K is the composite midpoint rule for numerical integration applied to the integrand of Γ_∞.

3.2.2 Performance measures

We redefine the partition into terms of δ as in Section 2.4:

$$\underline{p}_k = \underline{p} + \delta_{k-1}(\bar{p} - \underline{p}) \qquad \text{and} \qquad \bar{p}_k = \underline{p} + \delta_k(\bar{p} - \underline{p}),$$

where $\delta_0 = 0$, $\delta_k \in [0,1]$ for $k = 1, \ldots, K-1$, and $\delta_K = 1$. Thus, (3.3) becomes

$$\Gamma_K = 2\sqrt{R+r}\sum_{k=1}^{K}(\delta_k - \delta_{k-1})\sqrt{P + \underline{p} + (\delta_k + \delta_{k-1})(\bar{p} - \underline{p})} - \Theta. \tag{3.4}$$

We introduce the same instance parameter $\alpha \in \mathbb{R}_{>0}$ as for the DMU-n problem:

$$\alpha = \frac{\bar{p} - \underline{p}}{P + \underline{p}},$$

but a different normalisation factor ν:

$$\nu = \left(2\sqrt{R+r}\sqrt{\bar{p}-\underline{p}}\right)^{-1}.$$

Consequently, the normalised optimal objective values are given by

$$\nu\Gamma_K = \sum_{k=1}^{K}(\delta_k - \delta_{k-1})\sqrt{\tfrac{1}{\alpha} + \delta_k + \delta_{k-1}} - \tfrac{\Theta}{\nu}, \tag{3.5}$$

$$\nu\Gamma_\infty = \tfrac{1}{3}\left(\left(\tfrac{1}{\alpha}+2\right)^{\frac{3}{2}} - \left(\tfrac{1}{\alpha}\right)^{\frac{3}{2}}\right) - \tfrac{\Theta}{\nu}. \tag{3.6}$$

For performance measures that use differences, such as $(\Gamma_1 - \Gamma_K)/(\Gamma_1 - \Gamma_\infty)$, the outside option Θ cancels out. Therefore, these performance indicators are 1-dimensional functions in terms of α. However, the relative improvement $(\Gamma_1 - \Gamma_K)/\Gamma_1$, for example, is more difficult to analyse, since Θ/ν is in the denominator.

For other EOQ contracting problems in the literature it is common to assume that by default the buyer places orders using his own individually optimal order quantity. Hence, Θ is the corresponding minimal cost for the buyer. This worst-case assumption fits the conservative approach of robust pooling. In terms of the robust pooling model, this assumption leads to

$$\Theta^* = \inf_{p\in[\underline{p},\bar{p}]}\inf_{x\geq 0}\phi_B(x|p) = 2\sqrt{r\underline{p}}, \tag{3.7}$$

which implies that

$$\tfrac{\Theta^*}{\nu} = \sqrt{\frac{r}{R+r}}\sqrt{\frac{\underline{p}}{\bar{p}-\underline{p}}}.$$

From this point onwards, we assume that the outside option Θ is set according to (3.7).

When determining performance bounds, we take supremum or infimum of the performance measure with respect to all possible instances. This often means that Θ^*/ν must be as large or as small as possible. For example, consider $(\Gamma_1 - \Gamma_K)/\Gamma_1$. For fixed $\alpha > 0$, we want that Θ^*/ν is as small (large) as possible for the infimum (supremum). Now notice that any fixed α can be attained for any $R > 0$ and $P > 0$ by using the parameters $\underline{p}$ and $\bar{p}$. Thus, the infimum can be reached for $R \to \infty$, for which

$$\lim_{R\to\infty}\tfrac{\Theta^*}{\nu} = 0.$$

Likewise, the supremum can be reached for $R \to 0$ and $P \to 0$, which implies that

$$\lim_{P \to 0} \lim_{R \to 0} \frac{\Theta^*}{\nu} = \lim_{P \to 0} \sqrt{\frac{1}{\alpha} - \frac{P}{\bar{p} - \underline{p}}} = \frac{1}{\sqrt{\alpha}}.$$

To conclude, when assuming (3.7) the bounds for $(\Gamma_1 - \Gamma_K)/\Gamma_1$ and similar performance measures can still be determined by a 1-dimensional function of α.

3.2.3 Optimal partition

As is the case for the DMU-2 problem, a general formula for the optimal partition for the EOQ problem seems impossible. We do note that the optimal partition only depends on α and in particular not on Θ. Furthermore, the optimal partition must be a proper K-partition, see Lemma 12.

Lemma 12. *For any $K \in \mathbb{N}_{\geq 1}$ an optimal partition Δ must satisfy $0 = \delta_0 < \delta_1 < \cdots < \delta_{K-1} < \delta_K = 1$.*

Proof. The proof mimics that of Lemma 6, but is simpler due to the lack of k^*. We can simply take $\epsilon = 1/2$ in both cases in the proof and use the strict concavity of the square root function on $\mathbb{R}_{\geq 0}$. See Appendix B.2 for the details. □

We show the difficulty of finding formulas for the optimal partition by deriving the optimal partition for $K = 2$, see Lemma 13.

Lemma 13. *For the EOQ problem and $K = 2$, the optimal partition is*

$$\delta_1^{\text{opt}} = \frac{1}{6\alpha}(\sqrt{\alpha^2 + 8\alpha + 4} + \alpha - 2),$$

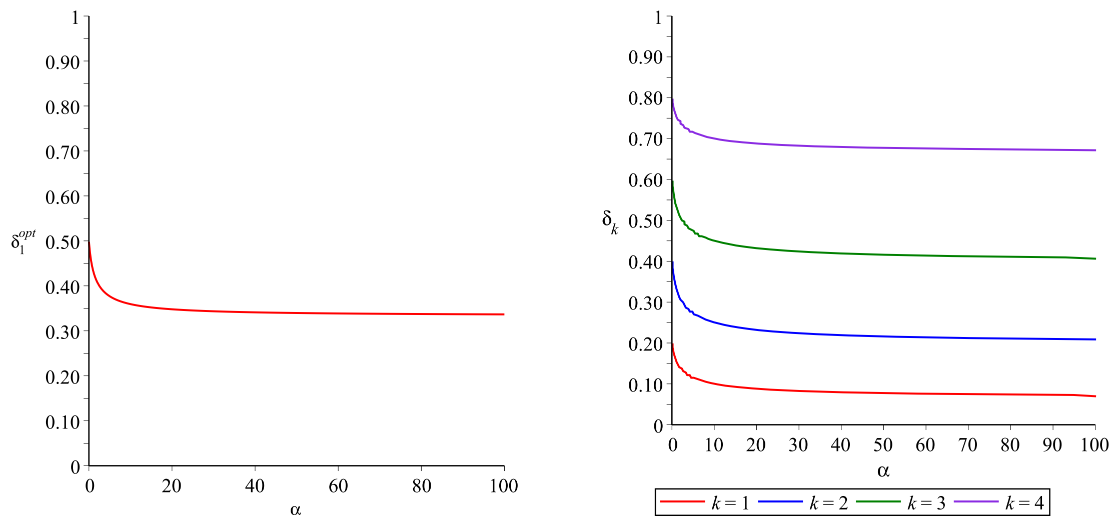
which satisfies the tight bounds $\frac{1}{3} < \delta_1 < \frac{1}{2}$.

Proof. We set the derivative of Γ_2 to zero and verify that the resulting partition is the global minimum. The details are in Appendix B.2. □

For a general number of contracts, we need to solve a complicated non-linear system of equalities. However, a numerical approach is viable. We apply a similar methodology as in Section 2.6.1. Notice that we do not need to account for k^*, which simplifies the procedure. See Figure 5a for the optimal partition δ_1^{opt} for $K = 2$ and Figure 5b for the optimised partition for $K = 5$.

Both Lemma 13 for $K = 2$ and the numerical results for $K \geq 2$ show that $\delta_k^{\text{opt}} \leq \delta_k^{\text{equi}}$. This is also the case for DMU-1, but not DMU-2. Thus, whether δ_k^{opt} is bounded by δ_k^{equi} seems to be a problem-specific property.

To conclude, by using a numerical solution approach, we can determine optimised partitions for the EOQ problem. We continue by determining the pooling performance of the equidistant and optimised partitions.



(a) Optimal partition δ_1^{opt} for $K = 2$ contracts. (b) Optimised partition for $K = 5$ contracts.

Figure 5: EOQ: optimised partition in terms of α.

3.2.4 Performance of partition schemes

In Section 3.2.4 we have shown that the infimum and supremum of relative performance measures can still be expressed as 1-dimensional functions of α. In particular, the pooling performance is calculated by rewriting Γ_K/Γ_∞ into $1 + (\Gamma_K - \Gamma_\infty)/\Gamma_\infty$. Thus, for upper bounds on the pooling performance, we use formulas (3.5) and (3.6) with $\Theta^*/\nu = \alpha^{-1/2}$.

The results are shown in Figure 6 and Table 3. The performances for the equidistant and optimised partitions have roughly the same shape as Γ_1/Γ_∞, i.e., there is a global maximum for a finite α and a (lower) asymptote for $\alpha \to \infty$.

Compared to the results of Section 2.4, the dominant difference is that a single robust contract performs reasonably well with a pooling performance of 107%. It is not arbitrarily bad as is the case for the DMU-n problem. We believe the reason is twofold. First, when minimising costs, there is a natural lowest cost possible. In contrast, when maximising utility without budgets, there is no natural limitation. Second, the EOQ cost functions are known for being relatively insensitive to small perturbations in the order quantity or the cost parameters.

From the results, we see that the optimised partition performs only marginally better than the equidistant partition. For example, for $K = 2$ the absolute difference in pooling performance is about 0.4%.

We conclude that robust pooling obtains exceptionally good performances for the EOQ problem. Offering a single robust contract is viable, but it is recommended to offer a few more contracts for a better performance guarantee.

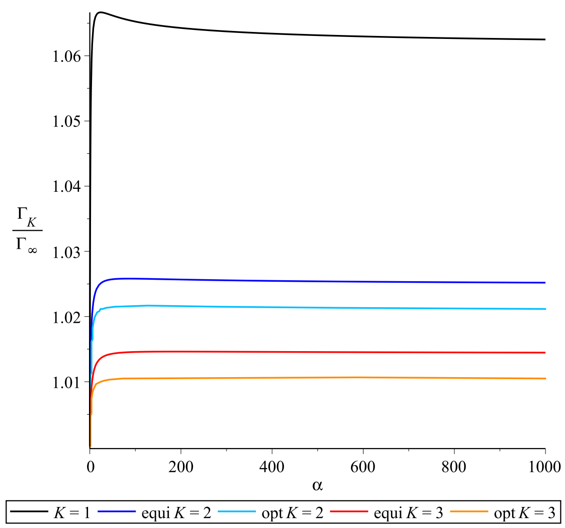


Figure 6: EOQ: pooling performance Γ_K/Γ_∞ for the equidistant and optimised partitions as functions of the instance parameter α.

K	Equidistant UB	Optimised UB
1	1.0667	1.0667
2	1.0259	1.0218
3	1.0147	1.0108
4	1.0098	1.0065
5	1.0071	1.0043
6	1.0055	1.0031
∞	1	1

Table 3: EOQ: upper bounds for the pooling performance Γ_K/Γ_∞ for the equidistant and optimised partitions.

4 Conclusion

We have presented and analysed a new modelling approach for principal-agent contracting models, called robust pooling. This approach considers a buyer whose type follows a continuous distribution on the interval $[\underline{p}, \bar{p}] \subseteq \mathbb{R}$. The seller wants to offer a menu with a finite number of contracts $K \in \mathbb{N}_{\geq 1}$. In our approach, the seller partitions $[\underline{p}, \bar{p}]$ into K subintervals and designs a menu with a single contract for each subinterval. The menu is constructed such that each type will choose its intended contract, making the menu robust to the buyer's private information.

With the robust pooling modelling approach we can compare offering different number of contracts in a natural and consistent way. Furthermore, we can determine performance guarantees in terms of the number of contracts offered, provided that the problem can be analysed (analytically or numerically) in sufficient detail. The existing classical continuous and discrete approaches are not suitable for this analysis. The continuous approach does not handle offering finitely many contracts. For the discrete approach such analysis requires changing the distribution of the buyer's type. This makes any comparison inconsistent, already from a modelling point of view. An exception is when the discrete model turns out to be robust, i.e., when the representative of each subinterval is its most inefficient type and the representatives have suitable probabilities. However, this robustness is not apparent from the discrete model, but follows from our robust pooling analysis.

Compared to the limited variety approach from the literature, we restrict the pooling of types to use a partition of $[\underline{p}, \bar{p}]$. We make this restriction to structure the buyer's choice, to obtain simple and intuitive mechanisms, to guarantee the accuracy of the extracted information on the buyer's type, and to promote the experimentation with partition schemes. For example, the seller can use the equidistant partition as a simple heuristic. After observing the buyer's chosen contract, the seller can narrow down the buyer's type to the corresponding subinterval. Thus, the accuracy of the extracted information is related to the width of the subintervals and is straightforwardly controlled by the seller by varying the number of contracts.

In Section 2 we have applied robust pooling to utility maximisation problems and in Section 3 to cost minimisation problems. The robust pooling model can be reformulated and simplified under certain assumptions on the buyer's utility/cost function, which are not uncommon in the literature. In particular, we have analysed two problems in detail: the DMU-n problem, based on a decreasing marginal utility, and the EOQ problem, based on the economic order quantity setting.

Our application of robust pooling to these two problems leads to new insights into the performances of partition schemes of $[\underline{p}, \bar{p}]$. A natural choice is to partition $[\underline{p}, \bar{p}]$ equidistantly. For the DMU-1 problem, the equidistant partition is optimal for a fully specified family of instances, but is suboptimal for other instances. The optimality seems to be a special property of DMU-1, since it is suboptimal for the DMU-2 and EOQ problems.

It is difficult to say whether the equidistant partition performs good enough for a given number of contracts. This depends on what performance is acceptable. Naturally, the performance of the equidistant partition reaches that of the optimal partition as the number of contracts increases. However, the idea for robust pooling is to offer only a few contracts. For the DMU-1 problem, it is definitely worthwhile to optimise the partition when using up to five contracts. For example, offering 3 contracts with the optimal partition achieves at least 96% of the best possible expected utility (corresponding to infinitely many contracts). For the equidistant partition this is 88%. For the DMU-2 and EOQ problems the difference in performance is smaller. In fact, for the EOQ problem the equidistant partition performs exceptionally well.

Overall, we conclude that robust pooling with only a few contracts, say 3 to 5, leads to high performances and is a viable approach. Offering only a single contract is not advised, since being able to distinguish between inefficient and efficient types is needed for good performances. For example, the optimal menu for the DMU-n problem refuses trade with the most inefficient types in certain cases. Offering a single contract can lead to arbitrarily bad performance for the DMU-n problem.

A possible extension to our analysis is to consider partition heuristics. Based on the results of DMU-1 and DMU-2, we can design the following partition heuristic: optimise only the first partition point and partition the remaining subinterval equidistantly. This is in fact optimal for DMU-1. However, numerically optimising the entire partition is relatively straightforward. Thus, such heuristics should have a clear benefit to (numerically) optimising the entire partition. For example, a heuristic should follow a rule of thumb and not require numerical optimisation.

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A Addendum to Section 2

A.1 Proofs for Section 2.2

Proof of Lemma 1. Let x be feasible, i.e., there exists an y such that (x, y) is a feasible solution. The proof consists of two parts: first we show that (2.7) holds for contract $k = 1$, then we focus on the other contracts in the menu ($k > 1$).

First, realise that for an optimal y at least one IR constraint (2.5) must hold with equality. If this is not the case, we can increase all y_k by adding some $\epsilon > 0$ until at least one IR constraint is tight. This new solution is still feasible, as (2.6) only considers the difference $y_k - y_l$, which is unaffected. Moreover, the objective value of the new solution is strictly larger.

Now, suppose that $y_1 < \underline{p}_1\chi(x_1)$, then for $k \in \mathcal{K}$ we have for all $p_k \in [\underline{p}_k, \bar{p}_k]$ that

$$p_k\chi(x_k) - y_k \geq \underline{p}_k\chi(x_k) - y_k \overset{(2.6)}{\geq} \underline{p}_k\chi(x_1) - y_1 \geq \underline{p}_1\chi(x_1) - y_1 > 0.$$

Here, we use that χ is non-negative. The result implies that no IR constraint is tight, which is suboptimal as argued above. Hence, for an optimal y it must hold that $y_1 = \underline{p}_1\chi(x_1)$.

Second, fix $k \in \mathcal{K}$ with $k > 1$ and consider the following IC constraints between contracts k and $k - 1$:

$$\bar{p}_{k-1}(\chi(x_k) - \chi(x_{k-1})) \leq y_k - y_{k-1} \leq \underline{p}_k(\chi(x_k) - \chi(x_{k-1})).$$

Since $\bar{p}_{k-1} = \underline{p}_k$, this implies that

$$y_k - y_{k-1} = \underline{p}_k(\chi(x_k) - \chi(x_{k-1})).$$

Using our earlier result that $y_1 = \underline{p}_1\chi(x_1)$, we obtain the following formula:

$$y_k = \sum_{i=2}^{k} \underline{p}_i(\chi(x_i) - \chi(x_{i-1})) + \underline{p}_1\chi(x_1),$$

which can be rewritten into (2.7). $\square$

Proof of Lemma 2. First, we show the necessity of $x_1 \leq \cdots \leq x_K$. Let $k, k+1 \in \mathcal{K}$ and consider

(2.6) for $\underline{p}_k$ and $\underline{p}_{k+1}$:

$$\underline{p}_k\chi(x_k) - y_k \geq \underline{p}_k\chi(x_{k+1}) - y_{k+1} \qquad \text{and} \qquad \underline{p}_{k+1}\chi(x_{k+1}) - y_{k+1} \geq \underline{p}_{k+1}\chi(x_k) - y_k.$$

Adding both IC constraints leads to $(\underline{p}_k - \underline{p}_{k+1})(\chi(x_k) - \chi(x_{k+1})) \geq 0$. Since $\underline{p}_k < \underline{p}_{k+1}$, this implies that $\chi(x_k) \leq \chi(x_{k+1})$ and $x_k \leq x_{k+1}$ since χ is non-decreasing.

Second, we show sufficiency of $0 \leq x_1 \leq \cdots \leq x_K$. Let $x \geq 0$ be non-decreasing and set y according to (2.7). Since χ is non-decreasing, we have $\chi(x_k) \leq \chi(x_{k+1})$ for $k \in \mathcal{K}$. It remains to check feasibility of (x, y). Fix $k \in \mathcal{K}$ and $p_k \in [\underline{p}_k, \bar{p}_k]$. For $l \in \mathcal{K}$ with $k < l$ we have

$$\begin{aligned} y_k - y_l &= \sum_{i=k}^{l-1}(y_i - y_{i+1}) \overset{(2.7)}{=} \sum_{i=k}^{l-1} \underline{p}_{i+1}(\chi(x_i) - \chi(x_{i+1})) \\ &\leq \sum_{i=k}^{l-1} \bar{p}_k(\chi(x_i) - \chi(x_{i+1})) = \bar{p}_k(\chi(x_k) - \chi(x_l)) \leq p_k(\chi(x_k) - \chi(x_l)). \end{aligned}$$

Likewise, let $l \in \mathcal{K}$ with $l < k$, then

$$\begin{aligned} y_k - y_l &= \sum_{i=l+1}^{k} (y_i - y_{i-1}) \overset{(2.7)}{=} \sum_{i=l+1}^{k} \underline{p}_i(\chi(x_i) - \chi(x_{i-1})) \\ &\leq \sum_{i=l+1}^{k} \underline{p}_k(\chi(x_i) - \chi(x_{i-1})) = \underline{p}_k(\chi(x_k) - \chi(x_l)) \leq p_k(\chi(x_k) - \chi(x_l)). \end{aligned}$$

Hence, all IC constraints (2.6) hold. Furthermore,

$$p_k\chi(x_k) - y_k \geq \underline{p}_k\chi(x_k) - y_k \overset{(2.6)}{\geq} \underline{p}_k\chi(x_1) - y_1 \geq \underline{p}_1\chi(x_1) - y_1 = 0.$$

Thus, all IR constraints (2.5) are satisfied and the solution is feasible. □

Proof of Theorem 4. The uniform distribution (Assumption 4) implies that $\omega_k = (\bar{p}_k - \underline{p}_k)/(\bar{p} - \underline{p})$. Therefore, we can simplify the summation in (2.8) as follows:

$$(\bar{p}_k - \underline{p}_k) \sum_{i=k+1}^{K} \frac{\omega_i}{\omega_k} = \sum_{i=k+1}^{K} (\bar{p}_i - \underline{p}_i) = \bar{p} - \bar{p}_k.$$

Substituting these expressions in the result of Theorem 3 gives the desired formulation of the optimisation problem.

For the optimal solution, we first relax the constraint $x_1 \leq \cdots \leq x_K$ to obtain a separable optimisation problem. Since the objective of the relaxed problem is strictly concave and differentiable (Assumption 3), its optimal solution can easily be determined. Consider contract $k \in \mathcal{K}$. We distinguish three cases.

Case I: if

$$\frac{\mathrm{d}}{\mathrm{d}x_k}(\phi_S + \psi)(0) + \bar{p}_k + \underline{p}_k - \bar{p} \leq 0,$$

then it is optimal for the relaxed problem to set $x_k = 0$. Otherwise, the optimal x_k for the relaxed problem satisfies $x_k > 0$.

Case II: if

$$\lim_{x_k \to \infty} \left(\frac{\mathrm{d}}{\mathrm{d}x_k}(\phi_S + \psi)(x_k) + \bar{p}_k + \underline{p}_k - \bar{p} \right) \geq 0,$$

then it is optimal to set $x_k = \infty$. Otherwise, a finite x is optimal. Here, we use that the above limit is zero only if it is an asymptote. This holds since $\phi_S + \psi$ is strictly concave, implying that its derivative is strictly decreasing. Furthermore, this shows that Cases I and II are indeed mutually exclusive for (fixed) k.

Case III: if the above cases do not hold the optimal x_k is found by setting the derivative to zero:

$$\frac{\mathrm{d}}{\mathrm{d}x_k}\Big(\phi_S(x_k) + \psi(x_k) + (\bar{p}_k + \underline{p}_k - \bar{p})x_k\Big) = 0 \qquad \Longleftrightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}x_k}(\phi_S + \psi)(x_k) = -(\bar{p}_k + \underline{p}_k - \bar{p}).$$

Since $\phi_S + \psi$ is strictly concave and differentiable, its derivative is continuous and invertible. Furthermore, Cases I and II are excluded, so the following value is well-defined and strictly positive:

$$\hat{x}_k = \left(\frac{\mathrm{d}}{\mathrm{d}x_k}(\phi_S + \psi) \right)^{-1} (\bar{p} - \bar{p}_k - \underline{p}_k),$$

which is the optimum for the relaxed problem. Notice that the definitions of k^* and $\hat{k}$ imply that Case III corresponds to $k \in \mathcal{K}$ such that $k^* \leq k \leq \hat{k}$. By strict concavity of $\phi_S + \psi$ we know that its derivative is strictly decreasing. Furthermore, realise that $\bar{p}_k + \underline{p}_k - \bar{p} < \bar{p}_{k+1} + \underline{p}_{k+1} - \bar{p}$ for all $k \in \mathcal{K}$. Therefore, we have $0 < \hat{x}_{k^*} < \cdots < \hat{x}_{\hat{k}}$.

Combining all cases leads to a solution satisfying $0 \leq x_1 \leq \cdots \leq x_K$, which is feasible for the non-relaxed problem. Hence, this is the optimal solution to our original problem. $\square$

A.2 Proofs for Section 2.4

Proof of Lemma 6. Consider an optimal partition Δ that does not satisfy the properties stated in the lemma. First, recall the definition of $k^* = \min\left\{k \in \mathcal{K} : \delta_k + \delta_{k-1} > \frac{\alpha-1}{\alpha}\right\}$. Suppose $k^*(\Delta) > 2$, which requires $K > 2$. Construct a new partition $\hat{\Delta}$ with $\hat{\delta}_1 = \delta_{k^*-1}$, $\hat{\delta}_k = \delta_{k^*}$ for $1 < k < k^*$ and $\hat{\delta}_k = \delta_k$ otherwise. By construction, the partition $\hat{\Delta}$ leads to the same objective value as Δ and is therefore an optimal partition. Furthermore, we have

$$\begin{aligned} \hat{\delta}_1 + \hat{\delta}_0 = \delta_{k^*-1} + \delta_0 \leq \delta_{k^*-1} + \delta_{k^*-2} \leq \tfrac{\alpha-1}{\alpha} & \\ \hat{\delta}_k + \hat{\delta}_{k-1} \geq \delta_{k^*} + \delta_{k^*-1} > \tfrac{\alpha-1}{\alpha}, & \qquad \text{for } k = 2, \ldots, K. \end{aligned}$$

Thus, $k^*(\hat{\Delta}) = 2$. Therefore, by applying this transformation, we can assume without loss of generality that the optimal partition Δ satisfies $k^*(\Delta) \in \{1, 2\}$.

Second, we modify the partition Δ into a strictly better one, which is a contradiction. The details require two cases to be analysed.

Case I: there exists an index $i \in \{1, \ldots, K-1\}$ such that $\delta_{i-1} = \delta_i < \delta_{i+1}$. Notice that $i + 1 \geq 2 \geq k^*(\Delta) \in \{1, 2\}$. Therefore, $\delta_{i+1} + \delta_i > \frac{\alpha-1}{\alpha}$ and there exists an $0 < \epsilon < 1$ such that

$$(1-\epsilon)\delta_{i+1} + (1+\epsilon)\delta_i > \tfrac{\alpha-1}{\alpha}.$$

Construct a new partition $\hat{\Delta}$ by setting $\hat{\delta}_i = (1-\epsilon)\delta_{i+1} + \epsilon\delta_i$ and $\hat{\delta}_k = \delta_k$ otherwise. By construction, we have $\hat{\delta}_{i-1} < \hat{\delta}_i < \hat{\delta}_{i+1}$, $i \geq k^*(\hat{\Delta})$, and $k^*(\hat{\Delta}) \leq k^*(\Delta)$. The normalised objective value corresponding to $\hat{\Delta}$ differs from that of Δ as follows: the terms

$$\sum_{k=\max\{i,k^*(\Delta)\}}^{i+1} (\delta_k - \delta_{k-1}) \left(\tfrac{1}{\alpha} + \delta_k + \delta_{k-1} - 1\right)^{\frac{n+1}{n}} = (\delta_{i+1} - \delta_i) \left(\tfrac{1}{\alpha} + \delta_{i+1} + \delta_i - 1\right)^{\frac{n+1}{n}}$$

are replaced by

$$\begin{aligned} &(\hat{\delta}_i - \hat{\delta}_{i-1}) \left(\tfrac{1}{\alpha} + \hat{\delta}_i + \hat{\delta}_{i-1} - 1\right)^{\frac{n+1}{n}} + (\hat{\delta}_{i+1} - \hat{\delta}_i) \left(\tfrac{1}{\alpha} + \hat{\delta}_{i+1} + \hat{\delta}_i - 1\right)^{\frac{n+1}{n}} \\ &\quad = (1-\epsilon)(\delta_{i+1} - \delta_i) \left(\tfrac{1}{\alpha} + (1-\epsilon)\delta_{i+1} + (1+\epsilon)\delta_i - 1\right)^{\frac{n+1}{n}} \\ &\qquad + \epsilon(\delta_{i+1} - \delta_i) \left(\tfrac{1}{\alpha} + (2-\epsilon)\delta_{i+1} + \epsilon\delta_i - 1\right)^{\frac{n+1}{n}} \\ &\quad > (\delta_{i+1} - \delta_i) \left(\tfrac{1}{\alpha} + \delta_{i+1} + \delta_i - 1\right)^{\frac{n+1}{n}}. \end{aligned}$$

The inequality follows from strict convexity of the function $(\cdot)^{\frac{n+1}{n}}$ on $\mathbb{R}_{\geq 0}$. This implies that $\hat{\Delta}$ is strictly better than Δ, which contradicts the optimality of Δ.

Case II: $\delta_{K-1} = \delta_K = 1$. Define $i = \min\{k \in \{1, \dots, K\} : \delta_k = \delta_K\}$ to be the first partition point that coincides with δ_K. Notice that $\delta_0 < \delta_i$ and $\delta_i + \delta_{i-1} = 1 + \delta_{i-1} \geq 1 > \frac{\alpha-1}{\alpha}$, so $i \geq k^*(\Delta)$. Therefore, there exists an $0 < \epsilon < 1$ such that $(1-\epsilon)\delta_i + (1+\epsilon)\delta_{i-1} > \frac{\alpha-1}{\alpha}$.

Construct a new partition $\hat{\Delta}$ by setting $\hat{\delta}_i = (1-\epsilon)\delta_i + \epsilon\delta_{i-1}$ and $\hat{\delta}_k = \delta_k$ otherwise. By construction, we have $\hat{\delta}_{i-1} < \hat{\delta}_i < \hat{\delta}_{i+1}$ and $k^*(\hat{\Delta}) = k^*(\Delta)$. The normalised objective value corresponding to $\hat{\Delta}$ differs from that of Δ as follows: the terms

$$\sum_{k=\max\{i,k^*(\Delta)\}}^{i+1} (\delta_k - \delta_{k-1}) \left(\tfrac{1}{\alpha} + \delta_k + \delta_{k-1} - 1\right)^{\frac{n+1}{n}} = (\delta_i - \delta_{i-1}) \left(\tfrac{1}{\alpha} + \delta_i + \delta_{i-1} - 1\right)^{\frac{n+1}{n}}$$

are replaced by

$$\begin{aligned}
&(\hat{\delta}_i - \hat{\delta}_{i-1}) \left(\tfrac{1}{\alpha} + \hat{\delta}_i + \hat{\delta}_{i-1} - 1\right)^{\frac{n+1}{n}} + (\hat{\delta}_{i+1} - \hat{\delta}_i) \left(\tfrac{1}{\alpha} + \hat{\delta}_{i+1} + \hat{\delta}_i - 1\right)^{\frac{n+1}{n}} \\
&\quad = (1-\epsilon)(\delta_i - \delta_{i-1}) \left(\tfrac{1}{\alpha} + (1-\epsilon)\delta_i + (1+\epsilon)\delta_{i-1} - 1\right)^{\frac{n+1}{n}} \\
&\qquad + \epsilon(\delta_i - \delta_{i-1}) \left(\tfrac{1}{\alpha} + (2-\epsilon)\delta_i + \epsilon\delta_{i-1} - 1\right)^{\frac{n+1}{n}} \\
&\quad > (\delta_i - \delta_{i-1}) \left(\tfrac{1}{\alpha} + \delta_i + \delta_{i-1} - 1\right)^{\frac{n+1}{n}}.
\end{aligned}$$

Hence, $\hat{\Delta}$ is strictly better than Δ, which contradicts the optimality of Δ. This concludes the proof. □

A.3 Proofs for Section 2.5

Proof of Lemma 8. By Lemma 6 we know that $k^* \in \{1, 2\}$ for the optimal partition. First, we analyse the objective function Γ_K when we consider k^* as a parameter independent of the chosen partition (which it is not). To simplify notation, we use the normalised $\nu\Gamma_K$, which does not affect the optimality of a partition. Suppose $k^* = 1$, then the normalised objective function is

$$\begin{aligned}
\nu\Gamma_K|_{k^*=1} = \Big(&(\delta_1 - \delta_0)(\tfrac{1}{\alpha} + \delta_1 + \delta_0 - 1)^2 + (\delta_2 - \delta_1)(\tfrac{1}{\alpha} + \delta_2 + \delta_1 - 1)^2 \\
&+ (\delta_3 - \delta_2)(\tfrac{1}{\alpha} + \delta_3 + \delta_2 - 1)^2 + (\delta_4 - \delta_3)(\tfrac{1}{\alpha} + \delta_4 + \delta_3 - 1)^2 + \cdots \\
&+ (\delta_{K-1} - \delta_{K-2})(\tfrac{1}{\alpha} + \delta_{K-1} + \delta_{K-2} - 1)^2 \\
&+ (\delta_K - \delta_{K-1})(\tfrac{1}{\alpha} + \delta_K + \delta_{K-1} - 1)^2\Big).
\end{aligned}$$

Since terms cancel out, this is a quadratic function for each δ_k, $k = 1, \ldots, K-1$. Note that $\delta_0 = 0$ and $\delta_K = 1$ are fixed for any partition. Setting the gradient to zero gives $(\delta_{k+1} - \delta_{k-1})(\delta_{k+1} + \delta_{k-1} - 2\delta_k) = 0$ for all $k \in \{1, \ldots, K-1\}$. By Lemma 6 we know that $\delta_{k+1} > \delta_{k-1}$ must hold for the optimal partition, so the only possibility is $\delta_k = \frac{1}{2}(\delta_{k+1} + \delta_{k-1})$ for all $k \in \{1, \ldots, K-1\}$. The solution to this linear system of equalities is

$$\delta_k = \frac{k}{K} \equiv \delta_k^{\text{equi}},$$

which is the equidistant partition Δ^{equi}.

Likewise, suppose $k^* = 2$, then we have

$$\begin{aligned}\nu\Gamma_K|_{k^*=2} = \Big(&(\delta_2 - \delta_1)(\tfrac{1}{\alpha} + \delta_2 + \delta_1 - 1)^2 \\ &+ (\delta_3 - \delta_2)(\tfrac{1}{\alpha} + \delta_3 + \delta_2 - 1)^2 + (\delta_4 - \delta_3)(\tfrac{1}{\alpha} + \delta_4 + \delta_3 - 1)^2 + \cdots \\ &+ (\delta_{K-1} - \delta_{K-2})(\tfrac{1}{\alpha} + \delta_{K-1} + \delta_{K-2} - 1)^2 \\ &+ (\delta p - \delta_{K-1})(\tfrac{1}{\alpha} + \delta_K + \delta_{K-1} - 1)^2\Big).\end{aligned}$$

This is cubic in δ_1 and quadratic in the other δ_k ($k \in \{2, \ldots, K-1\}$). Setting the gradient to zero gives

$$(1 - \tfrac{1}{\alpha} + \delta_2 - 3\delta_1)(\tfrac{1}{\alpha} - 1 + \delta_2 + \delta_1) = 0$$

and $(\delta_{k+1} - \delta_{k-1})(\delta_{k+1} + \delta_{k-1} - 2\delta_k) = 0$ for $k \in \{2, \ldots, K-1\}$. The roots of the derivative of the cubic function are

$$\delta_1 = \tfrac{1}{3}(1 - \tfrac{1}{\alpha} + \delta_2) \qquad \text{and} \qquad \delta_1 = 1 - \tfrac{1}{\alpha} - \delta_2.$$

By closer investigation of the shape of this cubic function, the larger value of these two corresponds to the maximum. Solving the system of linear equations for both cases results in:

$$\begin{aligned}\delta_1 &= \tfrac{1}{3}(1 - \tfrac{1}{\alpha} + \delta_2) \qquad &\Longrightarrow \qquad \delta_k &= \frac{K+k-1}{2K-1} - \frac{K-k}{2K-1}\frac{1}{\alpha}, \\ \delta_1 &= 1 - \tfrac{1}{\alpha} - \delta_2 \qquad &\Longrightarrow \qquad \delta_k &= \frac{K+k-3}{2K-3} - \frac{K-k}{2K-3}\frac{1}{\alpha}.\end{aligned}$$

Since δ_1 is larger in the first case, this is the correct solution. Hence,

$$\delta_k = \frac{K+k-1}{2K-1} - \frac{K-k}{2K-1}\frac{1}{\alpha} = 1 - \frac{K-k}{2K-1}\left(\frac{1}{\alpha}+1\right) \equiv \delta_k^{\text{cubic}}.$$

We refer to this partition as Δ^{cubic}.

Finally, we check which of these partitions is feasible. First, we check correctness with k^*. We have ($\Delta^{\text{equi}} \Rightarrow k^* = 1$) if and only if $\delta_1^{\text{equi}} > \frac{\alpha-1}{\alpha}$, which is $\frac{1}{K} > \frac{\alpha-1}{\alpha}$ or equivalently $\alpha < \frac{K}{K-1}$. Likewise, ($\Delta^{\text{cubic}} \Rightarrow k^* = 2$) if and only if $\delta_1^{\text{cubic}} \leq \frac{\alpha-1}{\alpha}$ and $\delta_2^{\text{cubic}} + \delta_1^{\text{cubic}} > \frac{\alpha-1}{\alpha}$. These conditions require the following:

$$\begin{aligned}
\delta_1^{\text{cubic}} \leq \tfrac{\alpha-1}{\alpha} \quad &\iff \quad 1 - \tfrac{K-1}{2K-1}\left(\tfrac{1}{\alpha}+1\right) \leq \tfrac{\alpha-1}{\alpha} \quad &\iff \quad \alpha \geq \tfrac{K}{K-1}, \\
\delta_2^{\text{cubic}} + \delta_1^{\text{cubic}} > \tfrac{\alpha-1}{\alpha} \quad &\iff \quad 2 - \tfrac{2K-3}{2K-1}\left(\tfrac{1}{\alpha}+1\right) > \tfrac{\alpha-1}{\alpha} \quad &\iff \quad \alpha > -1.
\end{aligned}$$

Moreover, we need to check $\delta_1^{\text{cubic}} > \delta_0 = 0$:

$$\delta_1^{\text{cubic}} > 0 \quad \iff \quad 1 - \tfrac{K-1}{2K-1}\left(\tfrac{1}{\alpha}+1\right) > 0 \quad \iff \quad \alpha > \tfrac{K-1}{K}.$$

This condition is trivially satisfied for the range $\alpha \geq \frac{K}{K-1}$ corresponding to Δ^{cubic}.

To conclude, for $\alpha < K/(K-1)$ the optimal partition is Δ^{equi}, whereas for $\alpha \geq K/(K-1)$ the optimal partition is Δ^{cubic}. $\square$

A.4 Proofs for Section 2.6

Proof of Lemma 10. First, we consider $k^* = 1$, for which

$$\nu\Gamma_2 = \delta_1\left(\tfrac{1}{\alpha}+\delta_1-1\right)^{\frac{3}{2}} + (1-\delta_1)\left(\tfrac{1}{\alpha}+\delta_1\right)^{\frac{3}{2}}.$$

Setting the derivative with respect to δ_1 to zero and solving the equation for δ_1 results in two solutions:

$$\delta_1^+ = \tfrac{1}{30}\left(\sqrt{36\tfrac{1}{\alpha^2}-15}+15-6\tfrac{1}{\alpha}\right) \qquad \text{and} \qquad \delta_1^- = \tfrac{1}{30}\left(-\sqrt{36\tfrac{1}{\alpha^2}-15}+15-6\tfrac{1}{\alpha}\right).$$

The partition point δ_1^+ is a local maximum, whereas δ_1^- does not maximise the objective. Furthermore, δ_1^+ is valid for $\alpha \in (0, \frac{2}{5}\sqrt{15}]$, i.e., it is feasible and corresponds to $k^* = 1$.

Second, consider $k^* = 2$, where the normalised optimal objective is

$$\nu\Gamma_2 = (1-\delta_1)\left(\tfrac{1}{\alpha}+\delta_1\right)^{\frac{3}{2}}.$$

Again, setting its derivative to δ_1 to zero, leads to a single feasible solution

$$\delta_1^* = 1 - \tfrac{2}{5}(\tfrac{1}{\alpha}+1).$$

The partition point δ_1^* is valid for $\alpha \in [\frac{3}{2}, \infty)$.

In contrast to the DMU-1 problem, the valid intervals overlap: $\alpha \in (0, 1.5491]$ for $k^* = 1$ and $\alpha \in [1.5, \infty)$ for $k^* = 2$. It turns out that the optimal δ_1 switches from δ_1^+ to δ_1^* (a discontinuous jump) as α increases. The partitions δ_1^+ and δ_1^* are both optimal for $\alpha^{\text{trans}} \approx 1.5371$ (the exact expression for α^{trans} is too verbose). This is illustrated in Figure 7, where the maximum on the left corresponds to δ_1^* and that on the right to δ_1^+.

Finally, the lower bound on δ_1^{opt} is attained at the switch to δ_1^* (at α^{trans}) and the upper bound is reached for $\alpha \to \infty$. □

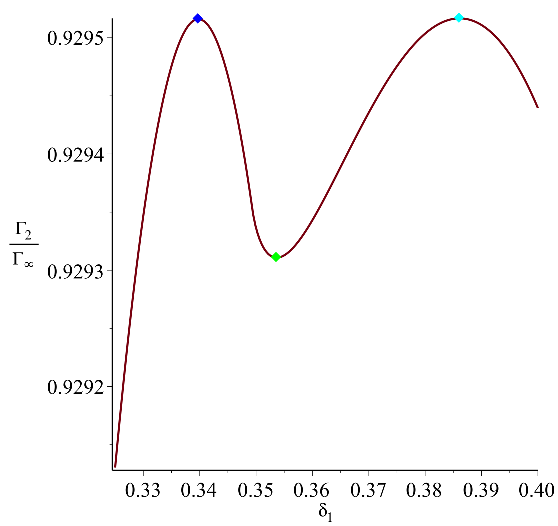

Figure 7: DMU-2: pooling performance Γ_2/Γ_∞ at α^{trans} in terms of the partition δ_1. The shown points are δ_1^* (left), δ_1^- (middle), and δ_1^+ (right).

A.5 Numerical solver

We describe the used methodology to numerically optimise the partition for the DMU-2 problem of Section 2.6. For each α, we have to maximise Γ_K or equivalently Γ_K/Γ_∞, so we can use the formulas of the normalised objective values $\nu\Gamma_K$ and $\nu\Gamma_\infty$. However, the formula for Γ_K contains index k^*, which depends on the partition. From Lemma 6 we know that $k^* \in \{1, 2\}$ for the optimal partition. Therefore, we simply optimise twice: for $k^* = 1$ in the formula with the restriction $\delta_1 > \frac{\alpha-1}{\alpha}$, and for $k^* = 2$ with the restrictions $\delta_1 \leq \frac{\alpha-1}{\alpha}$ and $\delta_1 + \delta_2 > \frac{\alpha-1}{\alpha}$. The optimal partition is the best of the resulting partitions. Note that $k^* = 1$ is always optimal for $0 < \alpha \leq 1$, since $k^* = 2$ is infeasible for this range.

For DMU-1 and DMU-2, we have inspected the shape of Γ_K as a function of the used partition for $K = 2$ and $K = 3$. From our observations, Γ_K with k^* fixed to 1 or 2 is a smooth function with a clear maximum on the respective domain that can be found using a gradient-based search. Thus, we apply a gradient-based search to maximise Γ_K with k^* fixed to either 1 or 2. To be precise, we use Maple's built-in solver 'NLPSolve' for non-linear programs with only bounds on the variables, which uses the Modified-Newton method, and verify the feasibility of the obtained partition. That is, we check if $0 = \delta_1 < \delta_2 < \cdots < \delta_{K-1} < \delta_K = 1$ and if the partition indeed results in the used value for k^*.

We have also used Maple's built-in solver 'NLPSolve' for constrained non-linear programs, which uses Sequential Quadratic Programming. With this solver we can enforce the required constraints on δ_k directly. Note that we still separately solve for k^* fixed to either 1 or 2. Both solvers give the same results, also when specifying different starting solutions.

The used methodology only guarantees to find a local maximum. However, the numerical solver always finds the same local optimum, i.e., it is stable. Furthermore, the results are consistent with our available theoretical results, such as for DMU-1 with any K and for DMU-2 with $K = 2$. Therefore, all results indicate that the numerical solver is able to find the global maximum.

B Addendum to Section 3

B.1 Reformulation and analysis

We follow the approach of Section 2 to reformulate the robust pooling model for cost minimisation, given in Section 3.1. First, we make Assumption 5, so the buyer's cost function is given by $\phi_B(x|p) = \psi(x) + p\chi(x)$. Under this assumption, we perform a change of variables by redefining the side payment as

$$z_k = \psi(x_k) + y_k - \Theta.$$

Notice that compared to Section 2 we include the outside option Θ in this change of variables. Substitution leads to an equivalent model with simpler constraints:

$$\begin{aligned}
\min_{x,y} \quad & \sum_{k\in\mathcal{K}} \omega_k \left(\phi_S(x_k) + \psi(x_k) + y_k\right) - \Theta, && \\
\text{s.t.} \quad & p_k\chi(x_k) - y_k \leq 0, && \forall\, p_k \in [\underline{p}_k, \bar{p}_k], k \in \mathcal{K}, \quad \text{(B.1)}\\
& p_k\chi(x_k) - y_k \leq p_k\chi(x_l) - y_l, && \forall\, p_k \in [\underline{p}_k, \bar{p}_k], k, l \in \mathcal{K}, \quad \text{(B.2)}\\
& x_k \geq 0, && \forall\, k \in \mathcal{K}.
\end{aligned}$$

This formulation clearly shows that Θ has no effect on the optimal order quantities and is simply a constant to be included in the side payment. We continue with the structure of the optimal solution.

Lemma 14. *Under Assumption 5, for any feasible x it is optimal to set*

$$y_k = \bar{p}_k\chi(x_k) + \sum_{i=k+1}^{K} (\bar{p}_i - \underline{p}_i)\chi(x_i) \qquad \forall\, k \in \mathcal{K}. \tag{B.3}$$

Proof. The proof is essentially the same as that of Lemma 1. The only difference is that the optimal y must satisfy $y_K = \bar{p}_K\chi(x_K)$. In other words, we have a tight IR constraint for $k = K$ instead of $k = 1$.

First, realise that for an optimal y at least one IR constraint (B.1) must hold with equality. If this is not the case, we can decrease all y_k by subtracting some $\epsilon > 0$ until at least one IR constraint is tight. This new solution is still feasible, as (B.2) only considers the difference $y_k - y_l$, which is unaffected. Moreover, the objective value of the new solution is strictly smaller.

Suppose that $y_K > \bar{p}_K\chi(x_K)$, then for $k \in \mathcal{K}$ we have for all $p_k \in [\underline{p}_k, \bar{p}_k]$ that

$$p_k\chi(x_k) - y_k \leq \bar{p}_k\chi(x_k) - y_k \overset{\text{(B.2)}}{\leq} \bar{p}_k\chi(x_K) - y_K \leq \bar{p}_K\chi(x_K) - y_K < 0.$$

That is, no IR constraint is tight, which is a contradiction. Hence, for an optimal y we must have $y_K = \bar{p}_K\chi(x_K)$.

Second, fix $k \in \mathcal{K}$ with $k < K$ and consider the IC constraints between contracts k and $k+1$:

$$\bar{p}_k(\chi(x_k) - \chi(x_{k+1})) \leq y_k - y_{k+1} \leq \underline{p}_{k+1}(\chi(x_k) - \chi(x_{k+1})).$$

Since $\bar{p}_k = \underline{p}_{k+1}$, this implies that

$$y_k - y_{k+1} = \bar{p}_k(\chi(x_k) - \chi(x_{k+1})).$$

Using our earlier result that $y_K = \bar{p}_K \chi(x_K)$, we obtain the following formula:

$$y_k = \sum_{i=k}^{K-1} \bar{p}_i(\chi(x_i) - \chi(x_{i+1})) + \bar{p}_K \chi(x_K),$$

which can be rewritten into (B.3). $\square$

Lemma 14 shows that the side payment z_k for contract $k \in \mathcal{K}$ only depends on the order quantities of contracts with a higher index $(k+1, \ldots, K)$. In terms of indices, this dependency is reversed in Lemma 1. However, in terms of efficiency the result is not reversed. Thus, both lemmas state that the side payment depends on the order quantities corresponding to less efficient buyers. We observe this phenomenon also in the feasible region, see Lemma 15.

Lemma 15. *Under Assumption 5, any x is feasible if and only if $x_1 \geq \cdots \geq x_K \geq 0$.*

Proof. The proof is the same as that of Lemma 2, except that all inequalities related to the constraints are reversed.

First, we show the necessity of $x_1 \geq \cdots \geq x_K$. Let $k, k+1 \in \mathcal{K}$ and consider (B.2) for $\bar{p}_k$ and $\bar{p}_{k+1}$:

$$\bar{p}_k \chi(x_k) - y_k \leq \bar{p}_k \chi(x_{k+1}) - y_{k+1} \qquad \text{and} \qquad \bar{p}_{k+1} \chi(x_{k+1}) - y_{k+1} \leq \bar{p}_{k+1} \chi(x_k) - y_k.$$

Adding both IC constraints leads to $(\bar{p}_k - \bar{p}_{k+1})(\chi(x_k) - \chi(x_{k+1})) \leq 0$. Since we have chosen $\bar{p}_k < \bar{p}_{k+1}$, this implies that $\chi(x_k) \geq \chi(x_{k+1})$ and thus $x_k \geq x_{k+1}$.

Second, we show sufficiency of $x_1 \geq \cdots \geq x_K \geq 0$. Let $x \geq 0$ be non-increasing and set y according to (B.3). It remains to check feasibility of (x, y). Fix $k \in \mathcal{K}$ and $p_k \in [\underline{p}_k, \bar{p}_k]$. For $l \in \mathcal{K}$

with $k < l$ we have

$$
\begin{aligned}
y_k - y_l &= \sum_{i=k}^{l-1} (y_i - y_{i+1}) \overset{(B.3)}{=} \sum_{i=k}^{l-1} \bar{p}_i(\chi(x_i) - \chi(x_{i+1})) \\
&\geq \sum_{i=k}^{l-1} \bar{p}_k(\chi(x_i) - \chi(x_{i+1})) = \bar{p}_k(\chi(x_k) - \chi(x_l)) \geq p_k(\chi(x_k) - \chi(x_l)).
\end{aligned}
$$

Likewise, let $l \in \mathcal{K}$ with $l < k$, then

$$
\begin{aligned}
y_k - y_l &= \sum_{i=l+1}^{k} (y_i - y_{i-1}) \overset{(B.3)}{=} \sum_{i=l+1}^{k} \bar{p}_{i-1}(\chi(x_i) - \chi(x_{i-1})) \\
&\geq \sum_{i=l+1}^{k} \underline{p}_k(\chi(x_i) - \chi(x_{i-1})) = \underline{p}_k(\chi(x_k) - \chi(x_l)) \geq p_k(\chi(x_k) - \chi(x_l)).
\end{aligned}
$$

Hence, all IC constraints (B.2) hold. Furthermore,

$$
p_k\chi(x_k) - y_k \leq \bar{p}_k\chi(x_k) - y_k \overset{(B.2)}{\leq} \bar{p}_k\chi(x_K) - y_K \leq \bar{p}_K\chi(x_K) - y_K = 0.
$$

Thus, all IR constraints (B.1) are satisfied and the solution is feasible. □

With these Lemmas we reformulate the robust pooling problem in terms of only the order quantities x, see Theorem 16. Again, notice the slight changes compared to Theorem 3.

Theorem 16. *Under Assumption 5, the robust pooling model with infinitely many constraints is equivalent to the following problem with finitely many and linear constraints:*

$$
\min_{x_1 \geq \cdots \geq x_K \geq 0} \sum_{k \in \mathcal{K}} \omega_k \left(\phi_S(x_k) + \psi(x_k) + \Big(\bar{p}_k + (\bar{p}_k - \underline{p}_k) \sum_{i=1}^{k-1} \frac{\omega_i}{\omega_k} \Big) \chi(x_k) \right) - \Theta.
$$

Proof. By Lemma 14 we can substitute the optimal formula (B.3) for y into the optimisation model. By Lemma 15 we conclude that the IR and IC constraints hold if and only if $x_1 \geq \cdots \geq x_K \geq 0$. This leads to the equivalent optimisation problem

$$
\min_{x_1 \geq \cdots \geq x_K \geq 0} \sum_{k \in \mathcal{K}} \omega_k \Big(\phi_S(x_k) + \psi(x_k) + \bar{p}_k\chi(x_k) + \sum_{i=k+1}^{K} (\bar{p}_i - \underline{p}_i)\chi(x_i) \Big) - \Theta,
$$

which can be rewritten into to formulation of the theorem by collecting the terms of x_k. □

We make additional assumptions to find a closed form solution, see Assumptions 6 and 7.

Assumption 6. The function $\phi_S + \psi$ is strictly convex and differentiable on $\mathbb{R}_{\geq 0}$. The function χ is given by $\chi(x) = x$.

Assumption 7. The distribution on the private parameter p is uniform: $\omega(p) = 1/(\bar{p} - \underline{p})$, so $\omega_k = (\bar{p}_k - \underline{p}_k)/(\bar{p} - \underline{p})$ for all $k \in \mathcal{K}$.

These assumptions allow us to derive Theorem 17, which corresponds to Theorem 4 for utility maximisation.

Theorem 17. *Under Assumptions 5-7, the robust pooling model is equivalent to the following convex problem:*

$$\min_{x_1 \geq \cdots \geq x_K \geq 0} \sum_{k \in \mathcal{K}} \frac{\bar{p}_k - \underline{p}_k}{\bar{p} - \underline{p}} \Big(\phi_S(x_k) + \psi(x_k) + (\bar{p}_k + \underline{p}_k - \underline{p}) x_k \Big) - \Theta.$$

The optimal solution is given by

$$x_k = \begin{cases} \infty & \text{if } k < \hat{k} \\ \left(\frac{\mathrm{d}}{\mathrm{d}x} (\phi_S + \psi) \right)^{-1} (\underline{p} - \bar{p}_k - \underline{p}_k) & \text{if } \hat{k} \leq k \leq k^* \\ 0 & \text{if } k > k^* \end{cases} \cdot$$

and satisfy $\infty > x_{\hat{k}} > \cdots > x_{k^*} > 0$. *Here, the index of the last non-zero order quantity is*

$$k^* = \max \left\{ 0, \max \left\{ k \in \mathcal{K} : \bar{p}_k + \underline{p}_k - \underline{p} < -\frac{\mathrm{d}}{\mathrm{d}x} (\phi_S + \psi)(0) \right\} \right\},$$

and the index of the first finite order quantity is

$$\hat{k} = \min \left\{ K + 1, \min \left\{ k \in \mathcal{K} : \bar{p}_k + \underline{p}_k - \underline{p} > - \lim_{x \to \infty} \frac{\mathrm{d}}{\mathrm{d}x} (\phi_S + \psi)(x) \right\} \right\}.$$

Proof. The proof is similar to that of Theorem 3, except that we use the strict convexity instead of strict concavity of $\phi_S + \psi$.

The uniform distribution implies that $\omega_k = (\bar{p}_k - \underline{p}_k)/(\bar{p} - \underline{p})$. Therefore, we can simplify the summation as follows:

$$(\bar{p}_k - \underline{p}_k) \sum_{i=1}^{k-1} \frac{\omega_i}{\omega_k} = \sum_{i=1}^{k-1} (\bar{p}_i - \underline{p}_i) = \underline{p}_k - \underline{p}.$$

Substituting these expressions in the result of Theorem 16 gives the desired formulation.

For the structure of the optimal solution, we first relax the constraint $x_1 \geq \cdots \geq x_K$ to obtain a separable optimisation problem. Since the objective of the relaxed problem is strictly convex and differentiable (Assumption 6), its optimal solution can easily be determined. Consider contract $k \in \mathcal{K}$. We distinguish three cases.

Case I: if

$$\frac{\mathrm{d}}{\mathrm{d}x_k}(\phi_S + \psi)(0) + \bar{p}_k + \underline{p}_k - \underline{p} \geq 0,$$

then it is optimal for the relaxed problem to set $x_k = 0$. Otherwise, the optimal x_k for the relaxed problem satisfies $x_k > 0$.

Case II: if

$$\lim_{x_k \to \infty} \left(\frac{\mathrm{d}}{\mathrm{d}x_k}(\phi_S + \psi)(x_k) + \bar{p}_k + \underline{p}_k - \underline{p} \right) \leq 0,$$

then it is optimal to set $x_k = \infty$. Otherwise, a finite x is optimal. Here, we use that the above limit is zero only if it is an asymptote. This holds since $\phi_S + \psi$ is strictly convex, implying that its derivative is strictly increasing. Furthermore, this shows that Cases I and II are indeed mutually exclusive for (fixed) k.

Case III: if the above cases do not hold the optimal x_k is found by setting the derivative to zero:

$$\frac{\mathrm{d}}{\mathrm{d}x_k}\Big(\phi_S(x_k) + \psi(x_k) + (\bar{p}_k + \underline{p}_k - \underline{p})x_k\Big) = 0 \qquad \Longleftrightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}x_k}(\phi_S + \psi)(x_k) = -(\bar{p}_k + \underline{p}_k - \underline{p}).$$

Since $\phi_S + \psi$ is strictly convex and differentiable, its derivative is continuous and invertible. Furthermore, Cases I and II are excluded, so the following value is well-defined and strictly positive:

$$\hat{x}_k = \left(\frac{\mathrm{d}}{\mathrm{d}x_k}(\phi_S + \psi) \right)^{-1} (\underline{p} - \bar{p}_k - \underline{p}_k),$$

which is the optimum for the relaxed problem. Notice that the definitions of k^* and $\hat{k}$ imply that Case III corresponds to $k \in \mathcal{K}$ such that $\hat{k} \leq k \leq k^*$. By strict convexity of $\phi_S + \psi$ we know that its derivative is strictly increasing. Furthermore, realise that $\bar{p}_k + \underline{p}_k - \underline{p} < \bar{p}_{k+1} + \underline{p}_{k+1} - \underline{p}$ for all $k \in \mathcal{K}$. Therefore, we have $\hat{x}_{\hat{k}} > \cdots > \hat{x}_{k^*} > 0$.

Combining all cases leads to a solution satisfying $x_1 \geq \cdots \geq x_K \geq 0$, which is feasible for the non-relaxed problem. Hence, this is the optimal solution to our original problem. □

B.2 Proofs for Section 3.2

Proof of Lemma 12. Consider an optimal partition that does not satisfy the properties stated in the lemma. We modify this partition into a strictly better one, which is a contradiction. The details require two cases to be analysed.

Case I: there exists a partition point $i \in \{1, \ldots, K-1\}$ such that $\delta_{i-1} = \delta_i < \delta_{i+1}$. Construct a new partition $\hat{\Delta}$ by setting $\hat{\delta}_i = (1/2)\delta_{i+1} + (1/2)\delta_i$ and $\hat{\delta}_k = \delta_k$ otherwise. By construction, we have $\hat{\delta}_{i-1} < \hat{\delta}_i < \hat{\delta}_{i+1}$. The normalised objective value corresponding to $\hat{\Delta}$ differs from that of Δ as follows: the terms

$$\sum_{k=i}^{i+1} (\delta_k - \delta_{k-1}) \left(\tfrac{1}{\alpha} + \delta_k + \delta_{k-1}\right)^{\frac{n}{n+1}} = (\delta_{i+1} - \delta_i) \left(\tfrac{1}{\alpha} + \delta_{i+1} + \delta_i\right)^{\frac{n}{n+1}}$$

are replaced by

$$\begin{aligned}
&(\hat{\delta}_i - \hat{\delta}_{i-1}) \left(\tfrac{1}{\alpha} + \hat{\delta}_i + \hat{\delta}_{i-1}\right)^{\frac{n}{n+1}} + (\hat{\delta}_{i+1} - \hat{\delta}_i) \left(\tfrac{1}{\alpha} + \hat{\delta}_{i+1} + \hat{\delta}_i\right)^{\frac{n}{n+1}} \\
&\quad = (1/2)(\delta_{i+1} - \delta_i) \left(\tfrac{1}{\alpha} + (1/2)\delta_{i+1} + (3/2)\delta_i\right)^{\frac{n}{n+1}} \\
&\qquad + (1/2)(\delta_{i+1} - \delta_i) \left(\tfrac{1}{\alpha} + (3/2)\delta_{i+1} + (1/2)\delta_i\right)^{\frac{n}{n+1}} \\
&\quad < (\delta_{i+1} - \delta_i) \left(\tfrac{1}{\alpha} + \delta_{i+1} + \delta_i\right)^{\frac{n}{n+1}} .
\end{aligned}$$

The inequality follows from strict concavity of the function $(\cdot)^{\frac{n}{n+1}}$ on $\mathbb{R}_{\geq 0}$. This implies that $\hat{\Delta}$ is strictly better than Δ, which contradicts the optimality of Δ.

Case II: $\delta_{K-1} = \delta_K = 1$. Define $i = \min\{k \in \{1, \ldots, K\} : \delta_k = \delta_K\}$ to be the first partition point that coincides with δ_K. Notice that $\delta_0 < \delta_i$. Construct a new partition $\hat{\Delta}$ by setting $\hat{\delta}_i = (1/2)\delta_i + (1/2)\delta_{i-1}$ and $\hat{\delta}_k = \delta_k$ otherwise. By construction, we have $\hat{\delta}_{i-1} < \hat{\delta}_i < \hat{\delta}_{i+1}$. The normalised objective value corresponding to $\hat{\Delta}$ differs from that of Δ as follows: the terms

$$\sum_{k=i}^{i+1} (\delta_k - \delta_{k-1}) \left(\tfrac{1}{\alpha} + \delta_k + \delta_{k-1}\right)^{\frac{n}{n+1}} = (\delta_i - \delta_{i-1}) \left(\tfrac{1}{\alpha} + \delta_i + \delta_{i-1}\right)^{\frac{n}{n+1}}$$

are replaced by

$$
\begin{aligned}
(\hat{\delta}_i - \hat{\delta}_{i-1}) & \left(\tfrac{1}{\alpha} + \hat{\delta}_i + \hat{\delta}_{i-1}\right)^{\frac{n}{n+1}} + (\hat{\delta}_{i+1} - \hat{\delta}_i)\left(\tfrac{1}{\alpha} + \hat{\delta}_{i+1} + \hat{\delta}_i\right)^{\frac{n}{n+1}} \\
&= (1/2)(\delta_i - \delta_{i-1})\left(\tfrac{1}{\alpha} + (1/2)\delta_i + (3/2)\delta_{i-1}\right)^{\frac{n}{n+1}} \\
&\quad + (1/2)(\delta_i - \delta_{i-1})\left(\tfrac{1}{\alpha} + (3/2)\delta_i + (1/2)\delta_{i-1}\right)^{\frac{n}{n+1}} \\
&< (\delta_i - \delta_{i-1})\left(\tfrac{1}{\alpha} + \delta_i + \delta_{i-1}\right)^{\frac{n}{n+1}}.
\end{aligned}
$$

Hence, $\hat{\Delta}$ is strictly better than Δ, which contradicts the optimality of Δ. This concludes the proof. □

Proof of Lemma 13. For $K = 2$ the normalised optimal objective value is given by

$$\nu\Gamma_2 = \delta_1\sqrt{\tfrac{1}{\alpha} + \delta_1} + (1-\delta_1)\sqrt{\tfrac{1}{\alpha} + 1 + \delta_1} - \tfrac{\Theta^*}{\nu}.$$

The derivative with respect to δ_1 is

$$\frac{1-\delta_1}{2\sqrt{\frac{1}{\alpha} + \delta_1 + 1}} + \sqrt{\tfrac{1}{\alpha} + \delta_1} - \sqrt{\tfrac{1}{\alpha} + \delta_1 + 1} + \frac{\delta}{2\sqrt{\frac{1}{\alpha} + \delta_1}}$$

and has a single root for $\alpha > 0$, namely $\delta_1^{\text{opt}} = \frac{1}{6\alpha}(\sqrt{\alpha^2 + 8\alpha + 4} + \alpha - 2)$. Furthermore, for $\delta_1 = 0$ the derivative is

$$-\sqrt{\tfrac{1}{\alpha} + 1} + \sqrt{\tfrac{1}{\alpha}} + \tfrac{1}{2}\left(\tfrac{1}{\alpha} + 1\right)^{-1/2} < 0.$$

The inequality follows from the strict concavity of the square-root function. Likewise, for $\delta_1 = 1$ the derivative is

$$-\sqrt{\tfrac{1}{\alpha} + 2} + \sqrt{\tfrac{1}{\alpha} + 1} + \tfrac{1}{2}\left(\tfrac{1}{\alpha} + 1\right)^{-1/2} > 0.$$

Hence, the root of the derivative is indeed the minimiser of the optimal objective value. The bounds follow from the following estimates:

$$
\begin{aligned}
\delta_1^{\text{opt}} &< \frac{1}{6\alpha}(\sqrt{(2\alpha+2)^2} + \alpha - 2) = \tfrac{1}{2}, \\
\delta_1^{\text{opt}} &> \frac{1}{6\alpha}(\sqrt{(\alpha+2)^2} + \alpha - 2) = \tfrac{1}{3}.
\end{aligned}
$$

The bounds correspond to the limits $\lim_{\alpha\to 0}\delta_1(\alpha) = \frac{1}{2}$ and $\lim_{\alpha\to\infty}\delta_1(\alpha) = \frac{1}{3}$. □