Multiplicative decomposition and index number theory: an empirical application of the Sato-Vartia decomposition

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Abstract In De Boer (2006) the *additive* decomposition of the aggregate change in a variable into its factors was considered. I proposed to use the "ideal" Montgomery decomposition, developed in index number theory, rather than the commonly used methods in structural decomposition analysis and applied it to the example analyzed by Dietzenbacher and Los (1998) (D&L). In this paper I consider the *multiplicative* decomposition and argue that from a theoretical point of view the "ideal" Sato-Vartia decomposition is to be preferred to the geometric average of the polar decompositions and that from a computational point of view it is to be preferred to the geometric average of all elementary decompositions. Application to the example of D&L reveals that the three methods yield results that are very close to each other.

Keywords: multiplicative decomposition, index number theory, structural decomposition analysis.

1. Introduction

In a previous paper, De Boer (2006), I considered the *additive* decomposition of the aggregate change in a variable V, $V(1) - V(0)$, the comparison period being denoted by 1 and the base period by 0.

In the framework of structural decomposition analysis (SDA), Dietzenbacher and Los (1998),(D&L), proved, that in case of n factors the number of possible elementary additive decompositions, which from a theoretical point of view are equivalent, is equal to n! There are (n!)/2 "mirror pairs" (base and comparison period reversed, see De Haan, 2001), among which the pair of the *polar decompositions*. Each of the decompositions is *complete*, in the sense that there is no residual term, and is *zero value robust*, i.e. zero values can be dealt with. But none of them satisfies the requirement of *time reversal* which states that if we reverse base period and comparison period the decomposition should yield the reverse result, i.e. $V(0) - V(1) = -[V(1) - V(0)]$. Arithmetic averages of the mirror pairs can be shown to satisfy time reversal. In their empirical application D&L decomposed the additive change in sectoral labor costs (V) into the effects of changes in labor costs per unit, technical changes, and changes in final demand mix and in final demand levels. In their example there are 4!=24 elementary decompositions. They considered the *arithmetic* average of the two polar decompositions and the arithmetic average of all 24 decompositions and found out that both averages were quite close to each other. The disadvantage of the arithmetic average of the two polar decompositions is that it does not obey the requirement of *factor reversal*: if we reverse two factors we do not get the same result. This means that the order of appearance of the factors in the decomposition matters. The arithmetic average of all elementary decompositions meets the requirement of factor reversal, but the computation of n! decompositions are needed.

In index number theory a decomposition that obeys factor reversal is called "*ideal*". In the previous paper I proposed to use alternatively the "ideal" Montgomery decomposition. For the example of D&L I showed that the results of the Montgomery decomposition were very close to those either obtained by means of the arithmetic average of the two polar decompositions or by the arithmetic average of all elementary decompositions.

In this paper I consider the *multiplicative* decomposition of the aggregate change in a variable V, i.e. $DV[1,0] = V(1)/V(0)$, in the framework of the D&L example. In section 2 I link for the case of n=2 factors (price and quantity) SDA to index number theory. We have 2!=2 elementary decompositions: the "polar ones", none of them satisfying the requirement of *time reversal* which, in the multiplicative case, states that if we reverse base period and comparison period the decomposition should yield the reciprocal result; i.e. $DV[1,0] \times DV[0,1] = 1$. The commonly used solution is to take the *geometric* average of the two polar decompositions which satisfies time reversal. We show that the geometric average of the two polar decompositions, in case n=2 the only elementary decompositions, is nothing but the product of the price and quantity index of Fisher. If we reverse the two factors we obtain the same result: the Fisher index is *ideal*. But there are also other indices that are ideal. For the problem at hand the Sato-Vartia index is particularly suited and I present it in section 2 as well. Section 3 is devoted to the case of more than two factors. I take as example the multiplicative decomposition of the example of D&L and I give the formula for the elementary decompositions and of the Sato-Vartia decomposition. Section 4 contains the results of the Sato-Vartia decomposition and of the geometric average of the two polar decompositions and the geometric average of all decompositions. It turns out that they are remarkably close to each other. Section 5, finally, contains our conclusions.

2. Structural decomposition analysis and ideal indices in case of two factors

2.1. Notation

Let $p_i(1)$ and $p_i(0)$ denote the prices of commodity i (= 1,…,n) in comparison and base period, and let $q_i(1)$ and $q_i(0)$ be the corresponding quantities. Then,

$$v_i(1) = p_i(1)q_i(1) \quad \text{and} \quad v_i(0) = p_i(0)q_i(0) \tag{1}$$

are the expenditure on commodity i (= 1,…,n) in comparison and base period, whereas

$$V(1) = \sum_{i=1}^{n} v_i(1) = \sum_{i=1}^{n} p_i(1)q_i(1) \quad \text{and} \quad V(0) = \sum_{i=1}^{n} v_i(0) = \sum_{i=1}^{n} p_i(0)q_i(0) \tag{2}$$

are total expenditure in comparison and in base period, and

$$s_i(1) = \frac{v_i(1)}{V(1)} \quad \text{and} \quad s_i(0) = \frac{v_i(0)}{V(0)} \tag{3}$$

are the shares of commodity i (=1,…,n) in comparison and base period.

Finally, we define the ratio of total expenditure in comparison and in base period:

$$DV[1,0] = \frac{V(1)}{V(0)} = \frac{\sum_{i=1}^{n} p_i(1)q_i(1)}{\sum_{i=1}^{n} p_i(0)q_i(0)} \tag{4}$$

2.2. Structural decomposition analysis and the "ideal" indices of Fisher

In the terminology of structural decomposition analysis we have to decompose (4) into its factors "price" and "quantity". One possible solution, the so-called *first polar decomposition*, is:

$$DV[1,0] = \frac{\sum_{i=1}^{n} p_i(1)q_i(1)}{\sum_{i=1}^{n} p_i(0)q_i(1)} \times \frac{\sum_{i=1}^{n} p_i(0)q_i(1)}{\sum_{i=1}^{n} p_i(0)q_i(0)} \tag{5}$$

It is easily seen that if we reverse **base and comparison period** (0 to 1 and 1 to 0) that for the first polar decomposition (5) generally $DV[1,0] \times DV[0,1] \neq 1$ holds true. In the terminology of index number theory, the first polar decomposition does not meet the requirement of **time reversal**:

$$DV[1,0] \times DV[0,1] = 1 \tag{6}$$

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However, this is not the only possibility. By reversing the time periods in the **weights** (0 to 1, and 1 to 0) we obtain the *second polar decomposition*:

$$DV[1,0] = \frac{\sum_{i=1}^{n} p_i(1)q_i(0)}{\sum_{i=1}^{n} p_i(0)q_i(0)} \times \frac{\sum_{i=1}^{n} p_i(1)q_i(1)}{\sum_{i=1}^{n} p_i(1)q_i(0)} \tag{7}$$

As is easily seen, the second polar decomposition, (7), does not meet the requirement of time reversal (6), either. The solution that is commonly adopted in structural decomposition analysis is to take the *geometric mean of the two polar decompositions*:

$$\left[\frac{\sum_{i=1}^{n} p_i(1)q_i(1)}{\sum_{i=1}^{n} p_i(0)q_i(1)} \times \frac{\sum_{i=1}^{n} p_i(0)q_i(1)}{\sum_{i=1}^{n} p_i(0)q_i(0)} \right]^{1/2} \times \left[\frac{\sum_{i=1}^{n} p_i(1)q_i(0)}{\sum_{i=1}^{n} p_i(0)q_i(0)} \times \frac{\sum_{i=1}^{n} p_i(1)q_i(1)}{\sum_{i=1}^{n} p_i(1)q_i(0)} \right]^{1/2} \tag{8}$$

which can easily be shown to meet the requirement of time reversal.

In order to link the structural decomposition approach to the index number approach we rewrite (8) to:

$$\left[\frac{\sum_{i=1}^{n} p_i(1)q_i(1)}{\sum_{i=1}^{n} p_i(0)q_i(1)} \times \frac{\sum_{i=1}^{n} p_i(1)q_i(0)}{\sum_{i=1}^{n} p_i(0)q_i(0)} \right]^{1/2} \times \left[\frac{\sum_{i=1}^{n} p_i(1)q_i(1)}{\sum_{i=1}^{n} p_i(1)q_i(0)} \times \frac{\sum_{i=1}^{n} p_i(0)q_i(1)}{\sum_{i=1}^{n} p_i(0)q_i(0)} \right]^{1/2} \tag{9}$$

In index number theory (omitting the, commonly used, factor 100) the first term is the definition of the *Fisher* price index (P^F), defined as the geometric mean of the price indices of *Paasche* (P^P) and of Laspeyres (P^L):

$$P^P = \frac{\sum_{i=1}^{n} p_i(1)q_i(1)}{\sum_{i=1}^{n} p_i(0)q_i(1)} \quad \text{and} \quad P^L = \frac{\sum_{i=1}^{n} p_i(1)q_i(0)}{\sum_{i=1}^{n} p_i(0)q_i(0)}$$

The second term of (8) is the *Fisher* quantity index (Q^F) defined as the geometric mean of the quantity indices of *Paasche*(Q^P) and *Laspeyres* (Q^L):

$$Q^P = \frac{\sum_{i=1}^{n} p_i(1)q_i(1)}{\sum_{i=1}^{n} p_i(1)q_i(0)} \quad \text{and} \quad Q^L = \frac{\sum_{i=1}^{n} p_i(0)q_i(1)}{\sum_{i=1}^{n} p_i(0)q_i(0)}$$

Consequently, the geometric mean of the two polar decompositions yields:

$$DV[1,0] = P^F \times Q^F \tag{10}$$

It can easily seen that if in the formula of the Fisher price index, i.e. the first term of (9), we reverse the factors (p to q and q to p) that we obtain the formula given in the second term, the Fisher quantity index. Indices that exhibit this property of **factor reversal** are called **"ideal".**

2.3. Another ideal index: Sato-Vartia[1]

Sato (1976) and, independently, Vartia (1974,1976) have discovered another pair of ideal price and quantity indices. In the derivation use will be made of the logarithmic mean[2] that for two *positive* numbers a and b is defined as:

$$L(a,b) = \frac{a-b}{\ln(a/b)} \text{ and } L(a,a) = a \tag{11}$$

Balk (2003) supplies a simple derivation from the identity:

$$\sum_{i=1}^{n} [s_i(1) - s_i(0)] = 0 \tag{12}$$

that easily follows from the adding-up of shares to 1 (see (3)).

Consider the logarithmic mean of $s_i(1)$ and $s_i(0)$:

$$L[s_i(1), s_i(0)] = \frac{s_i(1) - s_i(0)}{\ln[s_i(1)/s_i(0)]} \tag{13}$$

Using (12), it follows from (11) that:

$$\sum_{i=1}^{n} L[s_i(1), s_i(0)] \ln[s_i(1)/s_i(0)] = 0 \tag{14}$$

Using (3) and (4), expression (14) can be rewritten to:

$$\sum_{i=1}^{n} L[s_i(1), s_i(0)] \left[\ln\{v_i(1)/v_i(0)\} - \ln\{DV(1,0)\} \right] = 0$$

from which it follows that:

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$$\ln\{DV(1,0)\} = \frac{\sum_{i=1}^{n} L[s_i(1),s_i(0)]\left[\ln\{v_i(1)/v_i(0)\}\right]}{\sum_{j=1}^{n} L[s_j(1),s_j(0)]}$$

where we changed the index of summation in the denominator from i to j.

Using definition (1) we obtain:

$$\ln\{DV(1,0)\} = \frac{\sum_{i=1}^{n} L[s_i(1),s_i(0)]\left[\ln\{p_i(1)/p_i(0)\}\right]}{\sum_{j=1}^{n} L[s_j(1),s_j(0)]} + \frac{\sum_{i=1}^{n} L[s_i(1),s_i(0)]\left[\ln\{q_i(1)/q_i(0)\}\right]}{\sum_{j=1}^{n} L[s_j(1),s_j(0)]} \qquad (15)$$

Taking the exponent, it follows from (15) that the Sato-Vartia decomposition reads:

$$DV(1,0) = \prod_{i=1}^{n}\left[\frac{p_i(1)}{p_i(0)}\right]^{L[s_i(1),s_i(0)]/\sum_{j=1}^{n} L[s_j(1),s_j(0)]} \prod_{i=1}^{n}\left[\frac{q_i(1)}{q_i(0)}\right]^{L[s_i(1),s_i(0)]/\sum_{j=1}^{n} L[s_j(1),s_j(0)]} \qquad (16)$$

The first term in (16) is the definition of the price index of Sato-Vartia (P^{SV}) and the second one the quantity index of Sato-Vartia (Q^{SV}), so that (16) can, alternatively, be written as:

$$DV[1,0] = P^{SV} \times Q^{SV} \qquad (17)$$

As is seen from (13), the logarithmic mean is *symmetric* in $s_i(1)$ and $s_i(0)$, i.e. $L[s_i(1),s_i(0)] = L[s_i(0),s_i(1)]$. Consequently, it easily follows from (16) that the Sato-Vartia decomposition satisfies *time reversal*. Because the exponents are the same, it is easily seen as well, that (16) satisfies factor reversal, so that the index numbers are *ideal*. Finally, Ang *et al.* (1998) proved that in an empirical application zeros can be replaced by epsilon small positive numbers. Consequently, the Sato-Vartia decomposition satisfies the requirement of *zero-robustness* as well.

3. The case of many factors: the example of Dietzenbacher and Los (D&L)

3.1. The D&L model

In their application D&L used the input-output tables at basic prices for the Netherlands of 1986 and 1992. Defining the following vectors and matrices:

w: the 214x1 vector of sectoral labor costs;

u: the 214x1 vector of sectoral labor costs per unit of this sector's output (in money terms);

$\hat{u}$:   the 214x214 diagonal matrix with u on the main diagonal;

q: the 214x1 vector of sectoral outputs;

A: the 214x214 matrix of technical coefficients a_{ij}, measuring the input from sector i in sector j, per unit of sector j's output;

B: the 214x5 matrix of bridge coefficients b_{jk}, measuring the fraction of the final demand in category k that is spent on products from sector i, describing the final demand mix;

f: the 5x1 vector with total final demands[3] in each of the five categories, i.e. private consumption, government consumption, exports, investments, and other items (imputed bank services, VAT, trade and transport margins),

they consider the model:

$$w = \hat{u}q$$

$$q = Aq + Bf$$

of which the solution is:

$$w = \hat{u}LBf \qquad (18)$$

where: $L = (I - A)^{-1}$ is the Leontief inverse.

In sum notation (18) reads:

$$w_i = u_i \sum_{j=1}^{n} \sum_{k=1}^{m} l_{ij} b_{jk} f_k \qquad i = 1,...,n \qquad (19)$$

where w_i, u_i and f_k are the typical elements of the vectors w, u and f, respectively; and l_{ij} and b_{jk} the typical elements of the matrices L and B, respectively, whereas n=214 and m=5.

2.2. Elementary decompositions

The change in labor cost of sector i is defined to be (see (19)):

$$DW_i = \frac{w_i(1)}{w_i(0)} = DU_i \times \frac{\sum_{j=1}^{n} \sum_{k=1}^{m} l_{ij}(1)b_{jk}(1)f_k(1)}{\sum_{j=1}^{n} \sum_{k=1}^{m} l_{ij}(0)b_{jk}(0)f_k(0)}, \qquad (20)$$

where $DU_i = \dfrac{u_i(1)}{u_i(0)}$ is the change in the labor cost per unit.

We want to decompose the second term at the right hand side of (20) into technical changes, $\dfrac{l_{ij}(1)}{l_{ij}(0)}$; changes in the final demand mix, $\dfrac{b_{jk}(1)}{b_{jk}(0)}$, and changes in the final demand levels, $\dfrac{f_k(1)}{f_k(0)}$.

A possible solution is the *first polar decomposition*. In order to alleviate the notation, we define:

(i) the effects in technical changes:

$$DL_{i1} = \frac{\displaystyle\sum_{j=1}^{n}\sum_{k=1}^{m} l_{ij}(1) \times b_{jk}(1) \times f_k(1)}{\displaystyle\sum_{j=1}^{n}\sum_{k=1}^{m} l_{ij}(0) \times b_{jk}(1) \times f_k(1)}$$

(ii) the effects of changes in the final demand mix:

$$DB_{i1} = \frac{\displaystyle\sum_{j=1}^{n}\sum_{k=1}^{m} l_{ij}(0) \times b_{jk}(1) \times f_k(1)}{\displaystyle\sum_{j=1}^{n}\sum_{k=1}^{m} l_{ij}(0) \times b_{jk}(0) \times f_k(1)}$$

(iii) the effects of the changes in the final demand levels:

$$DF_{i1} = \frac{\displaystyle\sum_{j=1}^{n}\sum_{k=1}^{m} l_{ij}(0) \times b_{jk}(0) \times f_k(1)}{\displaystyle\sum_{j=1}^{n}\sum_{k=1}^{m} l_{ij}(0) \times b_{jk}(0) \times f_k(0)}$$

Then, we have the multiplicative decomposition:

$$DW_i = DU_i \times DL_{i1} \times DB_{i1} \times DF_{i1} \tag{21}$$

However, this is only one of the 3! = 6 elementary decompositions that are distinguished by Dietzenbacher and Los (1998).

By inverting the time periods in the **weights** (0 to 1 and 1 to 0) we obtain the second polar decomposition that we denote by the subscript 6 (we give in the appendix the four other elementary decompositions that, in our software, are denoted by the subscripts 2,…,5):

$$DL_{i6} = \frac{\sum_{j=1}^{n}\sum_{k=1}^{m}l_{ij}(1) \times b_{jk}(0) \times f_{k}(0)}{\sum_{j=1}^{n}\sum_{k=1}^{m}l_{ij}(0) \times b_{jk}(0) \times f_{k}(0)}$$

$$DB_{i6} = \frac{\sum_{j=1}^{n}\sum_{k=1}^{m}l_{ij}(1) \times b_{jk}(1) \times f_{k}(0)}{\sum_{j=1}^{n}\sum_{k=1}^{m}l_{ij}(1) \times b_{jk}(0) \times f_{k}(0)}$$

$$DF_{i6} = \frac{\sum_{j=1}^{n}\sum_{k=1}^{m}l_{ij}(1) \times b_{jk}(1) \times f_{k}(1)}{\sum_{j=1}^{n}\sum_{k=1}^{m}l_{ij}(1) \times b_{jk}(1) \times f_{k}(0)}$$

Then, we have the multiplicative decomposition:

$$DW_i = DU_i \times DL_{i6} \times DB_{i6} \times DF_{i6} \tag{22}$$

Neither (21), nor (22) satisfies the requirement of time reversal. As before the geometric mean:

$$DW_i = DU_i \times (DL_{i1} \times DL_{i6})^{1/2} \times (DB_{i1} \times DB_{i6})^{1/2} \times (DF_{i1} \times DF_{i6})^{1/2} \tag{23}$$

satisfies time reversal, but it does not any longer satisfy factor reversal! The geometric mean of the two polar decompositions is not any longer **"ideal"** when the number of factors is larger than 2. This means that the order of appearance of the factors matters: if we would take for instance $b_{jk}f_{k}l_{ij}$ we will get another result for the geometric mean (23).

From the 6 elementary decompositions we can derive many combinations that satisfy time reversal, but only of them is "ideal": the geometric mean of *all* 6 elementary decompositions:

$$DW_i = DU_i \times \left[\prod_{j=1}^{6}DL_{ij}\right]^{1/6} \times \left[\prod_{j=1}^{6}DB_{ij}\right]^{1/6} \times \left[\prod_{j=1}^{6}DF_{ij}\right]^{1/6} \tag{24}$$

but it requires the computation of six decompositions.

3.4. Application of the Sato-Vartia decomposition to the D&L model

We define the shares:

$$s_{ijk}(1) = \frac{u_i(1)l_{ij}(1)b_{jk}(1)f_k(1)}{w_i(1)}$$

and

$$s_{ijk}(0) = \frac{u_i(0)l_{ij}(0)b_{jk}(0)f_k(0)}{w_i(0)}$$

and the logarithmic mean:

$$L[s_{ijk}(1), s_{ijk}(1)] = \frac{s_{ijk}(1) - s_{ijk}(0)}{\ln[s_{ijk}(1)/s_{ijk}(0)]}$$

Next, we define (compare expression (16)):

$$DL_{iSV} = \prod_{j=1}^{n}\prod_{k=1}^{m}\left[\frac{l_{ij}(1)}{l_{ij}(0)}\right]^{L[s_{ijk}(1),s_{ijk}(0)]/\sum_{j=1}^{n}\sum_{k=1}^{m}L[s_{ijk}(1),s_{ijk}(0)]}$$

$$DB_{iSV} = \prod_{j=1}^{n}\prod_{k=1}^{m}\left[\frac{b_{jk}(1)}{b_{jk}(0)}\right]^{L[s_{ijk}(1),s_{ijk}(0)]/\sum_{j=1}^{n}\sum_{k=1}^{m}L[s_{ijk}(1),s_{ijk}(0)]}$$

and

$$DF_{iSV} = \prod_{j=1}^{n}\prod_{k=1}^{m}\left[\frac{f_k(1)}{f_k(0)}\right]^{L[s_{ijk}(1),s_{ijk}(0)]/\sum_{j=1}^{n}\sum_{k=1}^{m}L[s_{ijk}(1),s_{ijk}(0)]}$$

Then, the Sato-Vartia decomposition of the ratio of sectoral labor cost of sector i reads:

$$DW_i = DU_i \times DL_{iSV} \times DB_{iSV} \times DF_{iSV} \tag{25}$$

This decomposition obviously satisfies time reversal, but also factor reversal, i.e. the Sato-Vartia decomposition is "ideal". It is clear that generalization of this decomposition to the case of more than three factors is straightforward.

To summarize this section, I prefer the use of the "ideal" Sato-Vartia approach that only requires one decomposition for any number of factors, rather than the use of the geometric mean of the two polar decompositions which is not ideal so that the order of appearance of the factors matters, or the use of the geometric mean of all decompositions that is "ideal", but requires the computation of an ever increasing number of decompositions when the number of factors increases.

4. Results

In table 1 I give the results for nine sectors: four with a large percentage growth and five with large absolute growth (the results of the other sectors show the same pattern).

Table 1. Results

Sector	Geometric mean of all decompositions	standard deviation of all decompositions	geometric mean of polar decompositions	Sato-Vartia decomposition
Sectors with largest percentage growth				
156 economic advising agents	$w(86) = 539$	$w(92) = 1436$	$DW = 2.664$	$DU = 1.015$
DL	1.640	0.0817	1.640	1.641
DB	1.265	0.0640	1.264	1.263
DF	1.266	0.0017	1.266	1.266
153 computer services	$w(86) = 1245$	$w(92) = 2842$	$DW = 2.283$	$DU = 1.096$
DL	1.416	0.0380	1.417	1.416
DB	1.151	0.0310	1.150	1.151
DF	1.278	0.0014	1.279	1.279
127 beverage serving services (no lodging)	$w(86) = 189$	$w(92) = 416$	$DW = 2.201$	$DU = 1.085$
DL	1.084	0.0140	1.084	1.082
DB	1.433	0.0179	1.433	1.436
DF	1.306	0.0010	1.306	1.306
157 other business services	$w(86) = 1032$	$w(92) = 2249$	$DW = 2.179$	$DU = 1.005$
DL	1.384	0.0460	1.385	1.391
DB	1.237	0.0426	1.236	1.230
DF	1.267	0.0035	1.267	1.267
Sectors with the largest absolute growth				
121 wholesale trade	$w(86) = 13212$	$w(92) = 21712$	$DW = 1.568$	$DU = 1.144$
DL	1.015	0.0006	1.015	1.015
DB	1.032	0.0008	1.032	1.032
DF	1.309	0.0007	1.309	1.309
123 retail trade	$w(86) = 7726$	$w(92) = 12225$	$DW = 1.582$	$DU = 1.160$
DL	0.999	0.0000	0.999	0.999
DB	1.035	0.0000	1.035	1.035
DF	1.320	0.0000	1.320	1.320
146 railways, communication services, taxi and coach enterprises	$w(86) = 5385$	$w(92) = 8232$	$DW = 1.529$	$DU = 1.049$
DL	1.059	0.0036	1.059	1.059
DB	1.064	0.0036	1.064	1.065
Df	1.294	0.0006	1.294	1.294

171 special (primary) education (for handicapped children)	$w(86) = 8221$	$w(92) = 10863$	$DW = 1.321$	$DU = 1.152$
DL	0.999	0.0000	0.999	0.999
DB	0.939	0.0000	0.939	0.939
DF	1.223	0.0000	1.223	1.223
162 local government	$w(86) = 6933$	$w(92) = 9417$	$DW = 1.358$	$DU = 1.097$
DL	0.973	0.0013	0.973	0.973
DB	1.030	0.0011	1.030	1.030
DF	1.236	0.0007	1.236	1.236

In the heading for each sector I give (omitting the index i):

$w(86)$: sectoral labor costs in 1986;

$w(92)$: sectoral labor costs in 1992;

$DW = \dfrac{w(92)}{w(86)}$: ratio of labor costs in 1992 and1986;

$DU = \dfrac{u(92)}{u(86)}$: ratio of labor costs per unit of output (in money terms) in 1992 and1986,

which are the same for all decompositions.
 In the second column I give the geometric average of all decompositions; in column three the standard deviation are presented from which it can be inferred that the elementary decompositions did not vary much. Therefore, the geometric mean of the two polar decompositions, given in column four is very close to the geometric average of all decompositions. Column five, finally, contains the results for the Sato-Vartia decomposition which are very close to the results obtained in columns two and four.
 I conclude that from an empirical point of view the three methods yield (almost) the same results.

5. Concluding remarks

 In this paper I have tried to argue that from a theoretical point of view the use of the "ideal" Sato-Vartia approach that only requires one decomposition for any number of factors, is to be preferred to the use of the geometric mean of the two polar decompositions which is not ideal so that the order of appearance of the factors matters, and to the use of the geometric mean of all decompositions that is "ideal", but requires the computation of an ever increasing number of decompositions when the number of factors increases. In the example at hand it turned out that from an empirical point of view the three methods yielded almost the same results.

[1] Siegel(1945) derives another ideal index. Both Balk (2003) and Ang *et al.* (2004) argue that its formula is relatively complex, see the appendix of *Ang et al.* in which they give the formula for n=4. The Montgomery-Vartia index, the multiplicative variant of the Montgomery index that De Boer (2006) used for the additive decomposition does not meet the requirement that it is not linear homogeneous in prices (quantities), see Balk (2003).

[2] The properties (Balk, 2003) are: $\min(a,b) \le L(a,b) \le \max(a,b)$; $L(a,b)$ is continuous;

$L(\lambda a, \lambda b) = \lambda L(a,b)$; $L(a,b) = L(b,a)$; and $\sqrt{ab} \le L(a,b) \le \dfrac{a+b}{2}$. We give his footnote 1:

"The logarithmic mean was introduced in the economics literature by Törnqvist in 1935 in an unpublished memo of the Bank of Finland; see Törnqvist *et al.*(1985) …..A proof of the fact that $(ab)^{1/2} \le L(a,b) \le (a+b)/2$ was provided by Lorenzen (1990)".

[3] In De Boer (2006) it is argued that a change in stocks is not an appropriate final demand category and he split a stock change over all other items of the row according to the pertinent shares in total output.

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Appendix The four non-polar elementary decompositions (e=2,…,5)

E	DL_{ie}	DB_{ie}	DF_{ie}
2	$$\frac{\sum_{j=1}^{n}\sum_{k=1}^{m}l_{ij}(1)b_{jk}(1)f_{k}(1)}{\sum_{j=1}^{n}\sum_{k=1}^{m}l_{ij}(0)b_{jk}(1)f_{k}(1)}$$	$$\frac{\sum_{j=1}^{n}\sum_{k=1}^{m}l_{ij}(0)b_{jk}(1)f_{k}(0)}{\sum_{j=1}^{n}\sum_{k=1}^{m}l_{ij}(0)b_{jk}(0)f_{k}(0)}$$	$$\frac{\sum_{j=1}^{n}\sum_{k=1}^{m}l_{ij}(0)b_{jk}(1)f_{k}(1)}{\sum_{j=1}^{n}\sum_{k=1}^{m}l_{ij}(0)b_{jk}(1)f_{k}(0)}$$
3	$$\frac{\sum_{j=1}^{n}\sum_{k=1}^{m}l_{ij}(1)b_{jk}(0)f_{k}(1)}{\sum_{j=1}^{n}\sum_{k=1}^{m}l_{ij}(0)b_{jk}(0)f_{k}(1)}$$	$$\frac{\sum_{j=1}^{n}\sum_{k=1}^{m}l_{ij}(1)b_{jk}(1)f_{k}(1)}{\sum_{j=1}^{n}\sum_{k=1}^{m}l_{ij}(1)b_{jk}(0)f_{k}(1)}$$	$$\frac{\sum_{j=1}^{n}\sum_{k=1}^{m}l_{ij}(0)b_{jk}(0)f_{k}(1)}{\sum_{j=1}^{n}\sum_{k=1}^{m}l_{ij}(0)b_{jk}(0)f_{k}(0)}$$
4	$$\frac{\sum_{j=1}^{n}\sum_{k=1}^{m}l_{ij}(1)b_{jk}(0)f_{k}(0)}{\sum_{j=1}^{n}\sum_{k=1}^{m}l_{ij}(0)b_{jk}(0)f_{k}(0)}$$	$$\frac{\sum_{j=1}^{n}\sum_{k=1}^{m}l_{ij}(1)b_{jk}(1)f_{k}(1)}{\sum_{j=1}^{n}\sum_{k=1}^{m}l_{ij}(1)b_{jk}(0)f_{k}(1)}$$	$$\frac{\sum_{j=1}^{n}\sum_{k=1}^{m}l_{ij}(1)b_{jk}(0)f_{k}(1)}{\sum_{i=1}^{n}\sum_{j=1}^{m}l_{ij}(1)b_{jk}(0)f_{k}(0)}$$
5	$$\frac{\sum_{j=1}^{n}\sum_{k=1}^{m}l_{ij}(1)b_{jk}(1)f_{k}(0)}{\sum_{j=1}^{n}\sum_{k=1}^{m}l_{ij}(0)b_{jk}(1)f_{k}(0)}$$	$$\frac{\sum_{j=1}^{n}\sum_{k=1}^{m}l_{ij}(0)b_{jk}(1)f_{k}(0)}{\sum_{j=1}^{n}\sum_{k=1}^{m}l_{ij}(0)b_{jk}(0)f_{k}(0)}$$	$$\frac{\sum_{j=1}^{n}\sum_{k=1}^{m}l_{ij}(1)b_{jk}(1)f_{k}(1)}{\sum_{j=1}^{n}\sum_{k=1}^{m}l_{ij}(1)b_{jk}(1)f_{k}(0)}$$