Balancing Expected and Worst-Case Utility in Contracting Models with Asymmetric Information and Pooling

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Abstract

We consider a principal-agent contracting problem between a seller and a buyer, where the buyer has single-dimensional private information. The buyer's type is assumed to be continuously distributed on a closed interval. The seller designs a menu of finitely many contracts by pooling the buyer types a priori using a partition scheme. He maximises either his minimum utility, his expected utility, or a combination of both (a multi-objective approach). For each variation, we determine tractable reformulations and the optimal menu of contracts under certain conditions.

These results are applied to a contracting problem with quadratic utilities. We show that the optimal objective value is completely determined by the partition scheme, a single aggregate instance parameter, and a parameter encoding the seller's guaranteed obtained utility. This enables us to derive the optimal partition and exact performance guarantees. Our analysis shows that the seller should always offer at least two contracts in order to have reasonable performance guarantees, resulting in at least 88% of the expected utility compared to offering infinitely many contracts. By also optimising obtained worst-case utility, he can potentially achieve only 64% of the maximum expected utility.

Keywords: mechanism design, asymmetric information, pooling of contracts, multi-objective optimisation

1 Introduction

We consider a principal-agent problem where the principal is a seller of products and where the agent is a potential buyer. The seller has the initiative and market power to make a one-time offer to the buyer in which he presents a menu of contracts. We assume that this is a take-it-or-leave-it offer, i.e., we do not consider repeated offers or renegotiations. Each contract specifies an order quantity $x \in \mathbb{R}_{\geq 0}$ and a side payment $z \in \mathbb{R}$ from the buyer to the seller. The buyer has the market power to accept or reject any contract from the menu. Furthermore, we assume that both the seller and the buyer act individually rationally and want to maximise their own utility. Thus, the buyer will accept an offered contract if this is most beneficial to himself.

The buyer has private information that he does not share with the seller. We consider the case where the buyer's private information can be encoded into a single-dimensional parameter p, referred to as the buyer's *type*. We assume that the buyer's type can take on values in $[\underline{p}, \bar{p}] \subseteq \mathbb{R}$ with $\bar{p} > \underline{p}$ and follows a continuous distribution with strictly positive density function $\omega : [\underline{p}, \bar{p}] \to \mathbb{R}_{>0}$. Although the buyer's type is private, this distribution is known to the seller.

The seller has utility function $\phi_S : \mathbb{R}_{\geq 0} \to \mathbb{R}$ for an order quantity and his net utility also includes the side payment. Thus, if the buyer accepts a contract (x, z), the resulting seller's net utility is $\phi_S(x) + z$. Likewise, a buyer with type p has utility function $\phi_B(\cdot|p) : \mathbb{R}_{\geq 0} \to \mathbb{R}$. His net utility for contract (x, z) is $\phi_B(x|p) - z$. If the buyer rejects all contracts, we assume that his net utility is zero, which is also known as the buyer's *reservation level*.

Due to the buyer's private information, the seller can and will use Mechanism Design to construct a menu of contracts for the buyer to choose from. For a general reference on Mechanism Design for contracting problems, see for example Laffont and Martimort (2002). We consider the case where the seller only offers a limited number of contracts, similar to the Robust Pooling approach in our earlier work Kerkkamp et al. (2017). That is, the seller first decides how many contracts are offered, indicated by $K \in \mathbb{N}_{\geq 1}$. For notational convenience, let $\mathcal{K} = \{1, \ldots, K\}$. Second, the seller partitions $[\underline{p}, \bar{p}]$ into K subintervals $[\underline{p}_k, \bar{p}_k]$ with $\bar{p}_k > \underline{p}_k$ for $k \in \mathcal{K}$. Such a partition is called a *proper K-partition*. Third, the seller constructs a menu of K contracts by solving a specific optimisation model, given below, which depends on the chosen partition. Finally, this menu is offered to the buyer.

From this point onwards, we refer to a contract by (x_k, z_k) with $k \in \mathcal{K}$ and to a menu of contracts by (x, z), where $x = (x_1, \ldots, x_K)$ and $z = (z_1, \ldots, z_K)$. The menu is designed such that for each $k \in \mathcal{K}$ it is for all types in $[\underline{p}_k, \bar{p}_k]$ most beneficial to choose contract (x_k, z_k). We have the following constraints for the menu:

$$\phi_B(x_k|p_k) - z_k \geq 0, \qquad \forall\, p_k \in [\underline{p}_k, \bar{p}_k], k \in \mathcal{K}, \tag{1}$$

$$\phi_B(x_k|p_k) - z_k \geq \phi_B(x_l|p_k) - z_l, \qquad \forall\, p_k \in [\underline{p}_k, \bar{p}_k], k, l \in \mathcal{K}, \tag{2}$$

$$x_k \geq 0, \qquad \forall\, k \in \mathcal{K}. \tag{3}$$

Constraints (3) simply enforce non-negative order quantities. The other constraints (1) and (2) affect which contract the buyer will choose. Constraints (1) ensure individual rationality (IR) for the buyer: for $k \in \mathcal{K}$ and $p_k \in [\underline{p}_k, \bar{p}_k]$ contract (x_k, z_k) must not give a lower net utility than the buyer's reservation level. If (1) does not hold, then type p_k will never accept contract (x_k, z_k). We make the conventional assumption that if the buyer has multiple options which all maximise his net utility, then the seller can convince the buyer to choose from these the most beneficial option to the seller. Consequently, (1) guarantees that all types will choose a contract from the menu. Constraints (2) ensure for $k \in \mathcal{K}$ that contract (x_k, z_k) has the highest net utility for all types in $[\underline{p}_k, \bar{p}_k]$. This is known as incentive compatibility (IC).

Thus, for a menu satisfying constraints (1)-(3) contract (x_k, z_k) is chosen by all types $[\underline{p}_k, \bar{p}_k]$. In other words, contract (x_k, z_k) is chosen by the buyer with probability

$$\omega_k \equiv \int_{\underline{p}_k}^{\bar{p}_k} \omega(p) \mathrm{d}p \qquad \forall\, k \in \mathcal{K}. \tag{4}$$

We consider two objective functions for the seller subject to constraints (1)-(3): he maximises either his expected net utility or his minimum net utility. With the above insight, the seller's expected net utility is given by

$$\sum_{k \in \mathcal{K}} \omega_k \left(\phi_S(x_k) + z_k\right). \tag{5}$$

This leads to the *Maximise Expected net utility* (ME) model: $\max\{(5) : (1)\text{-}(3)\}$. Similarly, the seller's minimum net utility is

$$\min_{k \in \mathcal{K}} \left(\phi_S(x_k) + z_k\right), \tag{6}$$

resulting in the *Maximise Minimum net utility* (MM) model: $\max\{(6) : (1)\text{-}(3)\}$.

It turns out that for a broad class of problems the MM model has multiple optimal solutions, as we will show. The seller can therefore choose from these optimal solutions based on a second criterion. In light of the seller's desire to maximise his utility, we consider the case that the seller selects the optimal MM solution with maximum *expected* net utility. This can be interpreted as a two-stage optimisation approach based on the MM and ME models.

In fact, we generalise this two-stage approach to a multi-objective approach by adding the constraint $\min_{k \in \mathcal{K}}(\phi_S(x_k) + z_k) \geq M$, or equivalently

$$\phi_S(x_k) + z_k \geq M, \qquad \forall\, k \in \mathcal{K}, \tag{7}$$

to the ME model for some parameter $M \in \mathbb{R}$. We note that (7) is also known as the *seller*'s individual rationality constraint, where M is the seller's reservation level. We call the resulting model the *Multi-Objective* (MO) model: $\max\{(5) : (7), (1)\text{-}(3)\}$. Notice that by choosing M sufficiently small/negative (7) is non-restrictive and the MO model becomes the ME model. Likewise, by setting M to the optimal MM objective value, the MO model has the above described two-stage interpretation and finds the optimal MM solution with maximum expected net utility. Hence, the parameter M allows the seller to analyse the trade-off between maximising expected or worst-case net utility in a multi-objective perspective.

We shall refer to the MM, ME, and MO models as pooling models in general. Our goal is to analyse these pooling models, determine the optimal solutions analytically, and apply the results to a concrete contracting problem. In particular, we want to analytically quantify the effect of pooling the buyer types, the chosen partition scheme, and the buyer's reservation level M.

1.1 Literature

In the MM model the seller maximises his minimum net utility, which is often called having a 'maximin' objective or being ambiguity averse in the literature. Here, the minimum is taken over all offered contracts or, equivalently, over all possible realisations of the buyer's type. Only the support $[\underline{p}, \bar{p}]$ of the distribution ω is needed. This is an extreme case of recent Robust Optimisation approaches to Mechanism Design, see for example Bergemann and Schlag (2011) and Pınar and Kızılkale (2016). In these models the minimum is taken over an uncertainty set for ω, e.g., the distribution ω cannot differ too much from a reference distribution. The resulting robust model is typically less conservative than the classical maximim model. For further references on using Robust Optimisation, see Aghassi and Bertsimas (2006), Ben-Tal et al. (2009) and Bergemann and Morris (2005).

Our pooling approach can be viewed as a different application of Robust Optimisation. Consider the classical discrete variant of the contracting problem (see for example Laffont and Martimort (2002)). Here, the buyer's type lies in the set $\{p_1, \ldots, p_K\}$ and follows a discrete distribution. If we associate an uncertainty set $[\underline{p}_k, \bar{p}_k]$ to type p_k, then our pooling model is the Robust Optimisation variant for the discrete model. By considering a continuum of types $[\underline{p}, \bar{p}]$ and using a partition scheme, we are in fact restricting the uncertainty sets $[\underline{p}_k, \bar{p}_k]$ to form a partition of $[\underline{p}, \bar{p}]$.

A property of the pooling approach is to offer finitely many contracts to a continuum of buyer types. There are to our knowledge two papers in the literature that are strongly connected to this approach: Bergemann et al. (2011) and Wong (2014).

Bergemann et al. (2011) consider a linear-quadratic model with limited communication between the seller and the buyer based on Mussa and Rosen (1978). The seller wants to maximise his expected net utility. The limited communication restricts the seller to using a menu with finitely many contracts. In contrast to our pooling approach, they do not partition the types a priori. Instead, their menu maps each buyer type to one of the K contracts without any restrictions. By reformulating the problem into a mean square minimisation problem and applying Quantisation theory, they are able to determine the optimal menu of contracts and the corresponding optimal mapping of buyer types to contracts. In particular, their results show that the restriction to K contracts leads to a loss in performance of the order $\Theta(1/K^2)$ compared to offering infinitely many contracts.

Wong (2014) uses the same modelling approach as Bergemann et al. (2011), but analyses a more general non-linear pricing problem (again maximising the seller's expected net utility). His analysis focusses on the loss in performance when restricting to K contracts instead of infinitely many contracts. In particular, he derives the same $\Theta(1/K^2)$ loss in performance as Bergemann et al. (2011), but under a more general setting.

For the ME model we have shown in Kerkkamp et al. (2017) that the pooling approach and those of Bergemann et al. (2011) and Wong (2014) are equivalent provided that we use the K-partition of $[\underline{p}, \bar{p}]$ that maximises the seller's expected net utility. That is, both approaches lead to partitioning the buyer types. However, as argued in Kerkkamp et al. (2017) and Wong (2014) determining the optimal K-partition is difficult in general. In case the optimal partition cannot be derived, the benefit of our pooling approach is that it allows for heuristic partition schemes in a simple and controlled way. As we will show, the complexity of the pooling models for a given partition is similar to classical discrete contracting models.

The previously discussed papers do not consider multi-objective optimisation. In terms of using a multi-objective approach, Zheng et al. (2015) has to our knowledge the strongest connection to our work. Zheng et al. (2015) consider a continuum of types $[\underline{p}, \bar{p}]$ and the seller offers a menu with infinitely many contracts, i.e., there is no a priori pooling. They suppose that the seller is not confident about the probability distribution ω and model this by a so-called ϵ-contamination: with probability $0 \leq \epsilon \leq 1$ the distribution ω is incorrect and the worst-case outcome of ω occurs. Consequently, the objective function is the weighted sum of the seller's expected net utility and the seller's minimum net utility. The assumed utility functions are $\phi_S(x) = -cx$ and $\phi_B(x|p) = p\chi(x)$, where χ is a strictly increasing, positive, and continuously differentiable concave function. Furthermore, the distribution ω has a non-decreasing hazard rate. They show that for $0 < \epsilon < 1$ the optimal menu effectively pools the types $[\underline{p}, p^*(\epsilon)]$ for some $\underline{p} < p^*(\epsilon) < \bar{p}$ and offers those types the same contract. For the types $(p^*(\epsilon), \bar{p}]$ infinitely many contracts are offered.

To compare their multi-objective approach to ours, we translate the model of Zheng et al. (2015) to our pooling setting. For given $0 \leq \epsilon \leq 1$ the resulting model is to maximize

$$(1 - \epsilon) \sum_{k \in \mathcal{K}} \omega_k \left(\phi_S(x_k) + z_k\right) + \epsilon \min_{k \in \mathcal{K}} \left(\phi_S(x_k) + z_k\right)$$

subject to (1)-(3) with variables x and z. This is equivalent to maximising

$$(1-\epsilon)\sum_{k\in\mathcal{K}}\omega_k\left(\phi_S(x_k)+z_k\right)+\epsilon M$$

subject to (7) and (1)-(3) with variables x, z, and M. We refer to this model as the weighted objective model. Although this model is very similar to our MO model, there are differences. The most obvious difference is that for $\epsilon = 1$ the weighted objective model is our MM model, not our MO model (with correctly corresponding M). In this case, our MO model is a two-stage optimisation model which determines the optimal MM solution that maximises the seller's expected net utility. Typically, the MM model has multiple optimal solutions, whereas the MO model has just one (as we will show later). Furthermore, the parameter M in the MO model has a natural interpretation, namely the seller's reservation level. A similar interpretation of ϵ only follows indirectly after solving the weighted objective model and observing the corresponding optimal M. We will discuss further similarities and differences in more detail during our analysis.

Besides solving the pooling models for given number of contracts K, partition scheme, and seller's reservation level M, we want to quantify the effect of these choices on the corresponding optimal objective values. Due to the complexity we focus on a concrete contracting problem for such an analysis: the Linear-Quadratic-Uniform (LQU) problem adapted from Wong (2014). For the ME model variant of the LQU problem we also refer to Kerkkamp et al. (2017), where we completed the analysis of Wong (2014). Since the MO model generalises the ME model, our results in this paper supersede the mentioned analyses of the LQU problem. Furthermore, we are able to relate results for the LQU problem to Zheng et al. (2015).

1.2 Contribution

We analyse a contracting problem where the seller offers a menu of finitely many contracts to a buyer with a continuum $[\underline{p}, \bar{p}]$ of types. Here, the seller uses a partition scheme to pool the types a priori. Moreover, we consider a multi-objective approach for the seller's objective function which balances expected and worst-case (minimum) net utility. Compared to the literature, we extend related work by either considering a multi-objective approach (Bergemann et al. (2011) and Wong (2014)) or by pooling the buyer types with a partition scheme (Zheng et al. (2015)). Furthermore, there are differences in the modelling approaches as discussed in Section 1.1.

Under commonly used assumptions, we derive tractable reformulations for the pooling models and determine the optimal menu of contracts. The optimal menus all turn out to be the maxima of certain modified joint net utility functions. We apply and extend these results to a concrete contracting problem, namely the LQU problem. In particular, we derive the optimal partition scheme for the LQU problem and the corresponding optimal objective values. Consequently, we can analyse various performance measures that quantify the effect of the number of contracts K offered and of the seller's reservation level M. This leads to performance guarantees that give insight into the trade-off between maximising expected or worst-case net utility. All results are analytical and expressed in closed-form formulas.

The remainder of this paper starts with the general analysis of the pooling models in Section 2, followed by the application to the LQU problem in Section 3. We conclude our findings in Section 4.

2 General analysis

In this section we analyse the three pooling models in detail. First, we present the essential details of the setting and the three models in Section 2.1. In Section 2.2 we derive tractable reformulations for the models under a common assumption on the buyer's utility function. Finally, we determine the optimal solution of the models for a broad class of problems in Section 2.3. All corresponding proofs are given in Appendix A.

2.1 The models

As introduced in Section 1, we consider a seller with utility function $\phi_S : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and a buyer with utility function $\phi_B(\cdot|p) : \mathbb{R}_{\geq 0} \to \mathbb{R}$ for type $p \in [\underline{p}, \bar{p}] \subseteq \mathbb{R}$. The joint utility function is denoted by $\phi_J(\cdot|p) \equiv \phi_S(\cdot) + \phi_B(\cdot|p)$. The buyer's type follows a continuous distribution with strictly positive density function $\omega : [\underline{p}, \bar{p}] \to \mathbb{R}_{>0}$. The seller offers the buyer a menu with a limited number of contracts by pooling the buyer types a priori. In this menu, the k-th contract (x_k, z_k) specifies the order quantity $x_k \in \mathbb{R}_{\geq 0}$ and the side payment $z_k \in \mathbb{R}$ from the buyer to the seller. First, the seller chooses the number of contracts $K \in \mathbb{N}_{\geq 1}$ to offer. Second, the seller partitions $[\underline{p}, \bar{p}]$ into K subintervals $[\underline{p}_k, \bar{p}_k]$ with $\bar{p}_k > \underline{p}_k$ for $k \in \mathcal{K} = \{1, \ldots, K\}$, leading to aggregate probabilities ω_k for $k \in \mathcal{K}$ given by (4). Given this partition/pooling scheme, the seller constructs a menu of K contracts by solving one of our pooling models: the MM model $\max\{(6) : (1)\text{-}(3)\}$, the ME model $\max\{(5) : (1)\text{-}(3)\}$, or the MO model $\max\{(5) : (7), (1)\text{-}(3)\}$.

We focus on the MO model:

$$\max_{x,z} \quad \sum_{k\in\mathcal{K}} \omega_k \left(\phi_S(x_k) + z_k\right),$$

$$\text{s.t.} \quad \phi_S(x_k) + z_k \geq M, \qquad \forall\, k \in \mathcal{K}, \tag{7}$$

$$\phi_B(x_k|p_k) - z_k \geq 0, \qquad \forall\, p_k \in [\underline{p}_k, \bar{p}_k], k \in \mathcal{K}, \tag{1}$$

$$\phi_B(x_k|p_k) - z_k \geq \phi_B(x_l|p_k) - z_l, \qquad \forall\, p_k \in [\underline{p}_k, \bar{p}_k], k, l \in \mathcal{K}, \tag{2}$$

$$x_k \geq 0, \qquad \forall\, k \in \mathcal{K}. \tag{3}$$

The MO model maximises the seller's expected net utility under individual rationality constraints for the seller (7) and the buyer (1), and under incentive compatibility constraints (2). Note that the buyer's reservation level in (1) is assumed to be zero, whereas the seller's reservation level $M \in \mathbb{R}$ in (7) is set by the seller. With the parameter M the seller can balance his expected net utility with his minimum net utility. In particular, by an appropriate choice of M we can solve the ME or MM model with the MO model. That is, if we increase M then the optimal solution transitions from an optimal ME solution to an optimal MM solution. As a final note, if M is too large, then the MO model is infeasible. These insights will be made more concrete in the following analysis.

2.2 Tractable reformulation

In order to obtain tractable reformulations of our models and the results to come, we need to assume additional structure on the buyer's utility function. Assumption 1 states that ϕ_B is non-decreasing in the buyer's type and satisfies the strictly increasing differences property.

Assumption 1. The buyer's utility function ϕ_B satisfies the following properties:

$$\phi_B(x|\lambda) \leq \phi_B(x|\mu) \qquad \forall\, \lambda \leq \mu \in \mathbb{R}, x \geq 0, \tag{8}$$

$$\phi_B(x'|\lambda) - \phi_B(x|\lambda) < \phi_B(x'|\mu) - \phi_B(x|\mu) \qquad \forall\, \lambda < \mu \in \mathbb{R}, 0 \leq x < x'. \tag{9}$$

Note that we implicitly assume that $\phi_B(\cdot|\lambda)$ is defined for all $\lambda \in \mathbb{R}$, not just for $[\underline{p}, \bar{p}]$. However, it follows from the proofs that we only need to consider $2K$ instance-dependent values for λ. Assumption 1, or the stronger Single-Crossing Condition, is common in the Mechanism Design literature and leads to non-decreasingness in the order quantities with respect to the buyer's type (see also Edlin and Shannon (1998), Laffont and Martimort (2002) and Schottmüller (2015)). This is also the case for the pooling models, as shown in Lemma 1.

Lemma 1. *Under Assumption 1, any x satisfies (1)-(3) if and only if $0 \leq x_1 \leq \cdots \leq x_K$.*

Furthermore, for fixed order quantities x, the IR and IC constraints (1) and (2) imply a dual shortest path problem structure on the side payments z. For non-pooling models this has been identified before, see for example Rochet and Stole (2003) and Vohra (2012). For our pooling models, the side payments are even more restricted, leading to the optimal formulas for z given in Lemma 2. In fact, we only need to assume (8) for this result, but we have chosen to merge certain assumptions for readability.

Lemma 2. *Consider the ME, MM, or MO model under Assumption 1. It is necessary and sufficient for optimality to set*

$$z_k = \phi_B(x_k|\underline{p}_k) - \sum_{i=1}^{k-1} \left(\phi_B(x_i|\bar{p}_i) - \phi_B(x_i|\underline{p}_i)\right) \qquad \forall\, k \in \mathcal{K}. \tag{10}$$

With Lemma 2 we can eliminate the side payments from our models. Using Lemma 1 we can then simplify the constraints from infinitely many to K linear constraints. This results in the tractable reformulations as shown in Theorem 3. Recall that $\phi_J(\cdot|p)$ is the joint utility function with respect to type p.

Theorem 3. *Under Assumption 1, the ME model is equivalent to*

$$\max_{0 \le x_1 \le \cdots \le x_K} \sum_{k \in \mathcal{K}} \omega_k \left(\phi_J(x_k|\underline{p}_k) - \left(\phi_B(x_k|\bar{p}_k) - \phi_B(x_k|\underline{p}_k)\right) \sum_{i=k+1}^{K} \frac{\omega_i}{\omega_k} \right), \tag{11}$$

the MM model to

$$\max_{0 \le x_1 \le \cdots \le x_K} \min_{k \in \mathcal{K}} \left(\phi_J(x_k|\underline{p}_k) - \sum_{i=1}^{k-1} \left(\phi_B(x_i|\bar{p}_i) - \phi_B(x_i|\underline{p}_i)\right) \right), \tag{12}$$

and the MO model to

$$\max_{x} \quad \sum_{k \in \mathcal{K}} \omega_k \left(\phi_J(x_k|\underline{p}_k) - \left(\phi_B(x_k|\bar{p}_k) - \phi_B(x_k|\underline{p}_k)\right) \sum_{i=k+1}^{K} \frac{\omega_i}{\omega_k} \right), \tag{13}$$

$$\text{s.t.} \quad \phi_J(x_k|\underline{p}_k) - \sum_{i=1}^{k-1} \left(\phi_B(x_i|\bar{p}_i) - \phi_B(x_i|\underline{p}_i)\right) \ge M, \quad \forall\, k \in \mathcal{K}, \tag{14}$$

$$x_K \ge \cdots \ge x_1 \ge 0. \tag{15}$$

From the reformulations it is clear that the complexity of solving our pooling models depends on the shape of $\phi_J(\cdot|\underline{p}_k)$ and of $\phi_B(\cdot|\bar{p}_k) - \phi_B(\cdot|\underline{p}_k)$ for $k \in \mathcal{K}$. For example, if $\phi_J(\cdot|\underline{p}_k)$ is differentiable and concave, and if $\phi_B(\cdot|\bar{p}_k) - \phi_B(\cdot|\underline{p}_k)$ is linear, then all three models are concave optimisation problems with differential functions, which can be solved numerically in an efficient way. We focus on classifying problems for which the optimal solutions can be described in a unified way.

2.3 Optimal solutions

The next assumption excludes situations where the seller could potentially achieve infinite utility from the menu of contracts, see Assumption 2.

Assumption 2. For any $\lambda \in \mathbb{R}$ the joint utility function $\phi_J(\cdot|\lambda)$ has a maximum on $\mathbb{R}_{\ge 0}$ and on any closed subinterval of $\mathbb{R}_{\ge 0}$.

In Assumption 2, the existence of a maximum on any closed subinterval of $\mathbb{R}_{\ge 0}$ is needed because of a technicality (see the proof of Lemma 4 to come). In particular, Assumption 2 is satisfied if ϕ_S and ϕ_B are continuous functions in the order quantity.

The maximum of $\phi_J(\cdot|\lambda)$ for specific values of λ has a central role in the optimal solutions for our models. Therefore, we have an intermediate result on the maximisers of $\phi_J(\cdot|\lambda)$, see Lemma 4.

Lemma 4. *Under Assumptions 1-2, there exists a non-decreasing function $M^* : \mathbb{R} \to \mathbb{R}$ such that*

$$M^*(\lambda) \equiv \max_{x \ge 0} \{\phi_J(x|\lambda)\}.$$

For $M \le M^*(\underline{p})$, there exists a non-decreasing function $x^*(\cdot|M) : \mathbb{R} \to \mathbb{R}_{\ge 0}$ such that

$$x^*(\lambda|M) \equiv \min \operatorname*{argmax}_{x \ge x^M} \{\phi_J(x|\lambda)\},$$

where $x^M \in \mathbb{R}_{\ge 0}$ is given by $x^M \equiv \min\left\{x \ge 0 : \phi_J(x|\underline{p}) \ge M\right\}$.

Recall that the seller's reservation level M only affects the MO model. For the MM and ME models, we can implicitly use an non-restrictive value for M, namely $M = -\infty$. For $M = -\infty$ we have $x^M = 0$, hence $x^*(\lambda|M)$ optimises over the entire domain $x \ge 0$. In this case we simplify our notation and use $x^*(\lambda)$ instead of $x^*(\lambda|-\infty)$. We can now express the optimal solution for the MM model, see Theorem 5.

Theorem 5. *Under Assumptions 1-2, the optimal objective value for the MM model is $M^*(\underline{p})$, which can be attained by offering a menu with a single contract with order quantity $x^*(\underline{p})$. Note that this does not depend on the partition of $[\underline{p}, \bar{p}]$. Another optimal solution is $x_k = x^*(\underline{p}_k)$ for $k \in \mathcal{K}$, which does depend on the partition.*

A consequence of Theorem 5 is that the complexity of solving the MM model is completely determined by the complexity of maximising $\phi_J(x|\underline{p})$ over $x \geq 0$. For a similar result for the ME and MO models, we need the assumption that each k-th term in the objective function can be written as $\phi_J(x_k|\lambda_k)$ for some λ_k non-decreasing in k. This assumption is formalised in Assumption 3.

Assumption 3. The density function ω and the buyer's utility function ϕ_B are such that there exist parameters $\pi_k \in \mathbb{R}$ for $k \in \mathcal{K}$ satisfying $\pi_1 \leq \cdots \leq \pi_K$ and

$$\phi_B(x|\pi_k) = \phi_B(x|\underline{p}_k) - \left(\phi_B(x|\bar{p}_k) - \phi_B(x|\underline{p}_k)\right) \sum_{i=k+1}^{K} \frac{\omega_i}{\omega_k} \qquad \forall\, x \geq 0, k \in \mathcal{K}. \tag{16}$$

The parameters π_k are strongly related to the virtual valuation of the buyer types (see e.g. Laffont and Martimort (2002)). Under Assumptions 1 and 3 it is trivial to show that $\pi_k < \underline{p}_k$ for all $k \in \mathcal{K} \setminus \{K\}$ and $\pi_K \leq \underline{p}_K$ (see the proof of Theorem 7 to come). In Appendix B we present an example problem class for which we prove that it satisfies Assumption 3 and provide a closed-form expression for π_k. In the example, the buyer's utility function is $\phi_B(x|p) = \psi(x) + p\chi(x)$ for some functions ψ and χ, where χ is strictly increasing and non-negative. Furthermore, ω is a continuous distribution with a non-decreasing hazard rate, e.g., the uniform distribution.

Under the additional assumption, we can derive the optimal solution for the ME model as shown in Theorem 6.

Theorem 6. *Under Assumptions 1-3, an optimal solution for the ME model is $x_k = x^*(\pi_k)$ for $k \in \mathcal{K}$.*

Compared to the MM model, where an optimal solution is given by the maximisers of $\phi_J(\cdot|\underline{p}_k)$ for $k \in \mathcal{K}$, an optimal ME solution is specified by the maximisers of $\phi_J(\cdot|\pi_k)$. In other words, we need to shift the buyer types downwards from $\underline{p}_k$ to π_k.

Last but not least, we have the optimal solution for the MO model. The MO model maximises the seller's expected net utility under the constraint that the seller's minimum net utility is at least his reservation level M. From Theorem 5 we know that the minimum net utility is at most $M^*(\underline{p})$, being the optimal objective value of the MM model. Therefore, any seller's reservation level $M \leq M^*(\underline{p})$ can be satisfied and any $M > M^*(\underline{p})$ is infeasible. Theorem 7 states the optimal MO solution.

Theorem 7. *Under Assumptions 1-3, the MO model is feasible if and only if $M \leq M^*(\underline{p})$, and an optimal solution for the MO model is $x_k = x^*(\pi_k|M)$ for $k \in \mathcal{K}$.*

In particular, if the seller sets $M = M^*(\underline{p})$, then the MO model is a two-stage optimisation which maximises first the seller's minimum net utility and second the seller's expected net utility. Hence, under Assumptions 1-3 we have identified a third optimal MM solution. This leads to the next straightforward corollary.

Corollary 8. *Under Assumptions 1-3, if for each $k \in \mathcal{K}$ the function*

$$\phi_J(x_k|\underline{p}_k) - \sum_{i=1}^{k-1} \left(\phi_B(x_i|\bar{p}_i) - \phi_B(x_i|\underline{p}_i)\right)$$

is concave on $x_K \geq \cdots \geq x_1 \geq 0$, then any convex combination of

$$\begin{aligned} x_k &= x^*(\underline{p}) && \forall\, k \in \mathcal{K}, \\ x_k &= x^*(\underline{p}_k) && \forall\, k \in \mathcal{K}, \\ x_k &= x^*(\pi_k|M^*(\underline{p})) && \forall\, k \in \mathcal{K}, \end{aligned}$$

is an optimal solution for the MM model.

Returning to Theorem 7, if $\phi_J(\cdot|\lambda)$ is concave, then the optimal MO solution can be found in two steps as follows. First, determine the optimal ME solution by using the shifted buyer types π_k, resulting in $x_k = x^*(\pi_k)$ for $k \in \mathcal{K}$. Second, set any $x_k < x^M$ to the threshold order quantity x^M in order to guarantee the seller's reservation level M. Zheng et al. (2015) derive a similar solution structure for their concave setting with infinitely many contracts, where the types $[\underline{p}, p^*]$ for some $\underline{p} < p^* < \bar{p}$ are offered the same contract (like x^M in our case). Returning to our result, the threshold x^M could lead to additional pooling of types as multiple contracts can specify the order quantity x^M. This implies that the original partition of $[\underline{p}, \bar{p}]$ can be improved to increase the seller's expected net utility.

This brings us to one of the decisions the seller has to make: the partition of $[\underline{p}, \bar{p}]$. Based on our results, we have the following strategy for the seller. First, the seller must determine the optimal MM objective value $M^*(\underline{p})$, which is independent of the partition. Second, he must decide on his reservation level $M \leq M^*(\underline{p})$. Third, he chooses the number of contracts K offered. Finally, the seller selects a partition and uses the above results to determine an optimal menu of contracts for the MO model.

Ideally, the seller optimises the partition such that his expected net utility is maximised. Unfortunately, such optimisation appears to be difficult in general. Given the complexity of the analysis, we focus on the so-called Linear-Quadratic-Uniform (LQU) problem adapted from Wong (2014). In Section 3 we derive the optimal partition and analyse performance guarantees for the LQU problem.

3 Application to the LQU problem

In this section, we apply the results of Section 2 to a concrete contracting problem, called the Linear-Quadratic-Uniform (LQU) problem. We formalise the LQU problem in Section 3.1 and translate our general results from Section 2 to this setting in Section 3.2. In Sections 3.3-3.5 we continue the analysis, derive the optimal partition, and determine performance guarantees when using the optimal partition. All corresponding proofs are given in Appendix C.

3.1 The Linear-Quadratic-Uniform problem

In the *Linear-Quadratic-Uniform* (LQU) problem, the seller's utility function is linear in the order quantity: $\phi_S(x) = Px$, where $P \in \mathbb{R}_{>0}$ is the seller's utility per unit of sold product. The buyer's utility function is characterised by a saturation effect: the marginal utility of buying an additional product decreases linearly. That is, for order quantity $x \in \mathbb{R}_{\geq 0}$ the buyer's marginal utility of an additional product is $p - rx$. Here, $p \in [\underline{p}, \bar{p}] \subseteq \mathbb{R}_{>0}$ with $\bar{p} > \underline{p}$ is the buyer's (private) type and $r \in \mathbb{R}_{>0}$ is a saturation rate parameter. Note that p is strictly positive. The buyer's type is assumed to have a uniform distribution, i.e., $\omega(p) = 1/(\bar{p} - \underline{p})$. Consequently, the buyer's utility function is

$$\phi_B(x|p) = \int_0^x (p - ru)\mathrm{d}u = px - \tfrac{1}{2}rx^2.$$

Notice that for large order quantities the buyer's utility is negative, which models for example that excess products must be disposed of at a cost. Furthermore, ordering no products leads to zero utility for the buyer, which is his reservation level.

The pooling of contracts for the LQU problem has been analysed in Wong (2014) and under the name DMU-1 in Kerkkamp et al. (2017), both with the goal to maximise the seller's expected net utility (the ME model). As mentioned in Section 1, we extend the analysis to the MO model, with which we can balance the maximisation of the seller's worst-case and expected net utility.

3.2 Optimal solutions

It is straightforward to verify that the LQU problem satisfies Assumptions 1-3 of Section 2. In particular, since ω is the uniform density function we have

$$\omega_k = \frac{\bar{p}_k - \underline{p}_k}{\bar{p} - \underline{p}},$$

and therefore (16) of Assumption 3 simplifies to

$$\pi_k x - \tfrac{1}{2}rx^2 = \underline{p}_k x - \tfrac{1}{2}rx^2 - (\bar{p}_k - \underline{p}_k)x \sum_{i=k+1}^{K} \frac{\bar{p}_i - \underline{p}_i}{\bar{p}_k - \underline{p}_k} = (\bar{p}_k + \underline{p}_k - \bar{p})x - \tfrac{1}{2}rx^2.$$

Hence, the parameters π_k are

$$\pi_k = \bar{p}_k + \underline{p}_k - \bar{p} \qquad \forall\, k \in \mathcal{K},$$

which satisfy $\pi_1 \leq \cdots \leq \pi_K$ and $\pi_k \leq \underline{p}_k$ for all $k \in \mathcal{K}$. In contrast to the buyer's type p the parameter π_k can be negative for some $k \in \mathcal{K}$, depending on the instance parameters and the partition.

Since $\phi_J(x|\lambda) = (P+\lambda)x - \frac{1}{2}rx^2$, the function M^* stated in Lemma 4 is

$$M^*(\lambda) = \begin{cases} \frac{1}{2r}(P+\lambda)^2 & \text{if } P+\lambda \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

For the MO model, we need to realise that for a non-negative seller's reservation level ($M \leq 0$) the seller's IR constraint (14) is non-restrictive. This follows from Theorem 7 and the definition of $x^*(\cdot|M)$ in Lemma 4. More precisely, since $\phi_J(0|\underline{p}) = 0$ we have $x^M = 0$ for $M \leq 0$ and hence the optimal MO solution is the optimal ME solution. To conclude, the only proper values for M for the MO model are $0 \leq M \leq M^*(\underline{p})$.

Therefore, we only consider $M = \beta M^*(\underline{p})$ for some $\beta \in [0,1]$. For notational convenience, we often switch from M to β. We define $x^\beta \equiv x^{\beta M^*(\underline{p})}$ and $x^*(\cdot|\beta) \equiv x^*(\cdot|\beta M^*(\underline{p}))$, resulting in

$$\begin{aligned} x^\beta &= \min\left\{x \geq 0 : \phi_J(x|\underline{p}) \geq \beta M^*(\underline{p})\right\} \\ &= \min\left\{x \geq 0 : (P+\underline{p})x - \tfrac{1}{2}rx^2 \geq \beta \tfrac{1}{2r}(P+\underline{p})^2\right\} \\ &= \tfrac{1}{r}(1-\sqrt{1-\beta})(P+\underline{p}) \in [0, \tfrac{1}{r}(P+\underline{p})]. \end{aligned}$$

Thus, the function x^* stated in Lemma 4 is

$$x^*(\lambda|\beta) = \tfrac{1}{r}\max\left\{(1-\sqrt{1-\beta})(P+\underline{p}), P+\lambda\right\}. \tag{17}$$

In Corollary 9 we collect and translate the results of Theorems 5, 6, and 7 for the LQU problem.

Corollary 9. *For the MM-LQU model, any convex combination of*

$$\begin{aligned} x_k &= \tfrac{1}{r}(P+\underline{p}) && \forall k \in \mathcal{K}, \\ x_k &= \tfrac{1}{r}(P+\underline{p}_k) && \forall k \in \mathcal{K}, \\ x_k &= \tfrac{1}{r}\max\left\{P+\underline{p}, P-\bar{p}+\bar{p}_k+\underline{p}_k\right\} && \forall k \in \mathcal{K}, \end{aligned}$$

is an optimal solution. For the ME-LQU model, the only optimal solution is

$$x_k = \tfrac{1}{r}\max\left\{0, P-\bar{p}+\bar{p}_k+\underline{p}_k\right\} \qquad \forall k \in \mathcal{K}. \tag{18}$$

For the MO-LQU model and $M = \beta M^*(\underline{p})$ for some $\beta \in [0,1]$, the only optimal solution is

$$x_k = \tfrac{1}{r}\max\left\{(1-\sqrt{1-\beta})(P+\underline{p}), P-\bar{p}+\bar{p}_k+\underline{p}_k\right\} \qquad \forall k \in \mathcal{K}. \tag{19}$$

Note that the optimal solutions for the ME and MO models are unique due to the strict concavity of $\phi_J(\cdot|\lambda)$ (the details are given in the proof of Corollary 9). Furthermore, for certain instances the optimal ME solution (18) results in no trade with a range of buyer types (those for which $P-\bar{p}+\bar{p}_k+\underline{p}_k \leq 0$). Consequently, the seller's worst-case net utility is zero for such instances. It might be preferable to always trade with a potential buyer to at least make some revenue, even if this results in a potentially lower expected net utility. This is exactly what happens with the optimal MO solution (19) for $\beta > 0$: the optimal MO menu always instigates trade with the buyer if $\beta > 0$. A similar result is observed in Zheng et al. (2015) for their concave setting with infinitely many contracts.

We can use Corollary 9 to illustrate a difference between the approach of Zheng et al. (2015) and our MO model. If we translate the results of Zheng et al. (2015) to our pooling setting, their multi-objective approach could lead to a discontinuity at $\beta = 1$ (the equivalent to their $\epsilon = 1$) in x_k as function of β. This is due to the fact that their approach leads to the MM model if $\beta = 1$, which has multiple optimal solutions, as seen in Corollary 9. In contrast, the MO solution is always unique and continuous in β.

Finally, for $\beta = 0$ the optimal MO solution is the optimal ME solution. Similarly, for $\beta = 1$ the optimal MO solution is the (unique) optimal MM solution that also maximises the seller's expected net utility as a two-stage optimisation process. For this reason, we focus completely on the MO model in the results to come.

3.3 Normalising the objective function

To construct an optimal menu of contracts, the seller has to decide on his reservation level M (or equivalently β), the number of contracts K offered, and the partition of the buyer types. To quantify the effect of these decisions, we need to express the optimal objective function value in terms of the stated decisions. We can simply substitute (19) in the objective function (13), but it turns out to be useful to normalise various parameters as follows. First, we redefine the partition as

$$\underline{p}_k = \underline{p} + \delta_{k-1}(\bar{p} - \underline{p}) \qquad \text{and} \qquad \bar{p}_k = \underline{p} + \delta_k(\bar{p} - \underline{p}),$$

where $\delta_0 = 0$, $\delta_k \in [0, 1]$ for $k = 1, \ldots, K - 1$, and $\delta_K = 1$. For a proper K-partition, we have $0 = \delta_0 < \cdots < \delta_K = 1$. Consequently, $\omega_k = \delta_k - \delta_{k-1}$. Second, we introduce the normalisation factor ν and the aggregate instance parameter α:

$$\nu = \frac{2r}{(\bar{p} - \underline{p})^2} > 0 \qquad \text{and} \qquad \alpha = \frac{\bar{p} - \underline{p}}{P + \underline{p}} > 0.$$

As we will show, the (relative) performance measures of interest can be expressed completely in terms of α, β, and δ_k ($k \in \mathcal{K}$). The normalisation factor ν is used to simplify the expressions and cancels out in relative measures.

Let Γ_K be the optimal MO objective value when using a proper K-partition, i.e., using (19). We can express the *normalised* optimal MO objective value $\nu\Gamma_K$ in terms of the introduced normalised/aggregate parameters, see Lemma 10.

Lemma 10. *For any proper K-partition the normalised optimal MO-LQU objective value is given by*

$$\begin{aligned} \nu\Gamma_K = &\sum_{k=1}^{k^\beta} (\delta_k - \delta_{k-1}) \left(\tfrac{\beta}{\alpha^2} + 2(\delta_k + \delta_{k-1} - 1)(1 - \sqrt{1-\beta})\tfrac{1}{\alpha} \right) \\ &+ \sum_{k=k^\beta+1}^{K} (\delta_k - \delta_{k-1}) \left(\tfrac{1}{\alpha} + \delta_k + \delta_{k-1} - 1 \right)^2, \end{aligned} \tag{20}$$

where k^β is the largest index affected by the seller's reservation level:

$$k^\beta = \max\left\{0, \max\left\{k \in \mathcal{K} : \delta_k + \delta_{k-1} < 1 - \tfrac{1}{\alpha}\sqrt{1-\beta}\right\}\right\}.$$

Notice that $k^\beta < K$, since $\delta_K = 1$. Furthermore, for instances with $\alpha \in (0, \sqrt{1-\beta}] \subseteq (0, 1]$ we have $k^\beta = 0$, implying that the seller's reservation level is non-restrictive for all contracts.

Two extreme cases are $\nu\Gamma_1 = \alpha^{-2}$ and the limit of $\nu\Gamma_K$ for $K \to \infty$ using any sensible partition (such as $\delta_k = k/K$):

$$\nu\Gamma_\infty = \int_0^{\delta^\beta} \left(\tfrac{\beta}{\alpha^2} + (4\delta - 2)(1 - \sqrt{1-\beta})\tfrac{1}{\alpha} \right) \mathrm{d}\delta + \int_{\delta^\beta}^1 \left(\tfrac{1}{\alpha} + 2\delta - 1 \right)^2 \mathrm{d}\delta,$$

where δ^β corresponds to the limit of k^β:

$$\delta^\beta = \max\left\{0, \tfrac{1}{2}(1 - \tfrac{1}{\alpha}\sqrt{1-\beta})\right\}.$$

Hence, we get

$$\nu\Gamma_\infty = \begin{cases} \frac{1}{\alpha^2} + \frac{1}{3} & \text{if } \alpha \leq \sqrt{1-\beta} \\ \frac{\beta}{\alpha^2} + \frac{1}{6}(1 + \frac{1}{\alpha}\sqrt{1-\beta})^3 & \text{if } \alpha > \sqrt{1-\beta} \end{cases}. \tag{21}$$

Notice that Γ_∞ is independent of the partition, as should be the case since we are effectively offering infinitely many contracts. Trivially, we have $\Gamma_K \leq \Gamma_\infty$ for any α, β, K, and partition. Hence, we can use Γ_∞ as a benchmark to evaluate the performance of the chosen partition scheme. Recalling the objective of the MO model, a natural choice for the partition is the one which maximises the seller's expected net utility Γ_K. For the LQU problem we are able to determine this optimal partition, as we will show in the next section.

3.4 Optimal partition

The goal of this section is to determine closed-form formulas for the partition that maximises Γ_K (or equivalently $\nu\Gamma_K$) for a given α, β, and K. As mentioned in Kerkkamp et al. (2017) and Wong (2014), it seems to be difficult to determine such closed-form formulas in general, either due to complex system of equations needed to be solved or due to the existence of multiple local optima. However, the structure of the LQU problem allows us to determine closed-form formulas for the optimal partition.

First, we prove that offering the same contract multiple times is suboptimal and that we should use all available contracts. This is intuitively clear, but formalised in Lemma 11.

Lemma 11. *The optimal MO-LQU partition satisfies* $k^\beta \in \{0,1\}$ *and* $0 = \delta_0 < \delta_1 < \cdots < \delta_{K-1} < \delta_K = 1$.

Lemma 11 greatly restricts the value of k^β when determining the optimal partition, making the analysis manageable. We can now derive the optimal MO partition, see Theorem 12.

Theorem 12. *For* $0 < \alpha \le \frac{K}{K-1}\sqrt{1-\beta}$ *the optimal MO-LQU partition satisfies* $k^\beta = 0$ *and is the equidistant partition:*

$$\delta_k^{\text{opt}} = \tfrac{k}{K} \qquad\qquad \forall\, k \in \mathcal{K}.$$

For $\alpha > \frac{K}{K-1}\sqrt{1-\beta}$ *the optimal MO-LQU partition satisfies* $k^\beta = 1$ *and is*

$$\delta_k^{\text{opt}} = 1 - \tfrac{K-k}{2K-1}\left(1 + \tfrac{1}{\alpha}\sqrt{1-\beta}\right) \qquad\qquad \forall\, k \in \mathcal{K}.$$

In particular, for $\beta = 1$ *the equidistant partition is suboptimal for all* $\alpha > 0$.

Sketch of the proof. Since $k^\beta \in \{0,1\}$ by Lemma 11 we only need to consider two variants for the formula of $\nu\Gamma_K$. For each variant we set the gradient to zero, leading to systems of linear equations after simplification, and determine the corresponding maximiser. The maximiser must be in line with the considered value of k^β, resulting in a specification of a valid range of instances: either $0 < \alpha \le \frac{K}{K-1}\sqrt{1-\beta}$ or $\alpha > \frac{K}{K-1}\sqrt{1-\beta}$. These ranges are disjoint and capture all instances $\alpha > 0$, which completes the proof. □

The result in Theorem 12 (and its proof) is a generalisation of the derived optimal partition in Kerkkamp et al. (2017). A remarkable property is that the equidistant partition is optimal for a range of instances, as specified by the relation between α, β, and K. This range of instances increases if β decreases (by lowering the seller's reservation level) or if K decreases (by offering less contracts). Moreover, if the equidistant partition is not optimal, then the optimal partition can be found by increasing δ_1 and partitioning the remaining subinterval $[\delta_1, 1]$ equidistantly for $\delta_2, \ldots, \delta_{K-1}$.

When using the optimal partition the expression for $\nu\Gamma_K$ (20) can be simplified to a similar expression as (21). This is shown in Corollary 13.

Corollary 13. *For the optimal partition the normalised optimal MO-LQU objective value* (20) *is*

$$\nu\Gamma_K^{\text{opt}} = \begin{cases} \frac{1}{\alpha^2} + \frac{1}{3}\left(1 - \frac{1}{K^2}\right) & \text{if } \alpha \le \frac{K}{K-1}\sqrt{1-\beta} \\ \frac{\beta}{\alpha^2} + \frac{2}{3}\frac{K(K-1)}{(2K-1)^2}\left(1 + \frac{1}{\alpha}\sqrt{1-\beta}\right)^3 & \text{if } \alpha > \frac{K}{K-1}\sqrt{1-\beta} \end{cases}. \tag{22}$$

Notice that (22) clearly converges to (21) if $K \to \infty$, as should be the case. Also, on certain intervals $\nu\Gamma_K^{\text{opt}}$ and $\nu\Gamma_\infty$ either differ by a constant $-1/(3K^2)$ or by a factor $4K(K-1)/(2K-1)^2$ in the cubic term. In the next section, we analyse the relative difference between Γ_K^{opt} and Γ_∞ in more detail to obtain performance guarantees.

3.5 Performance guarantees

In this section we analyse the performance of the optimal menu of contracts when using the optimal partition. We consider two performance measures: the pooling performance (Section 3.5.1) and the reservation level performance (Section 3.5.2). Both measure the effectiveness of pooling the buyer types compared to offering infinitely many contracts in their own way. In particular, the first measure is useful when the seller's reservation level cannot be adjusted, whereas the second is insightful when the seller's reservation level is a decision variable.

3.5.1 Pooling performance

The *pooling performance* $\Gamma_K^{\text{opt}}/\Gamma_\infty$ is the fraction of the seller's expected net utility attained by offering K contracts instead of infinitely many contracts. Here, the seller uses the optimal menu of contracts and the optimal partition, as derived in Sections 3.2 and 3.4. In other words, it measures how much is lost due to the pooling of the buyer types by offering a limited number of contracts. Note that the seller's reservation level M (or β) must always be satisfied by both menus corresponding to Γ_K^{opt} and Γ_∞.

In Figure 1 we illustrate the attained pooling performance for two example instances in terms of β and K. Here, we use $\alpha = 2$ and $\alpha = \frac{9}{4} + \frac{3}{4}\sqrt{5} \approx 3.927$. We have chosen for $\alpha = 2$ because this is the threshold value in (22) for $K = 2$ and $\beta = 0$. The reason for the other chosen instance will be given in the next section. We observe in Figure 1 that for $\beta = 0$ the pooling performances are 88% ($K = 2$) and 96% ($K = 3$) for both instances. As β increases the pooling performance increases. However, the rate of increase differs significantly between $K = 2$ and $K = 3$, and between the two instances. Finally, notice that for fixed $K \in \{2, 3\}$ and for any $0 < \beta \leq 1$ the pooling performance is higher for $\alpha = 2$ than for the other instance.

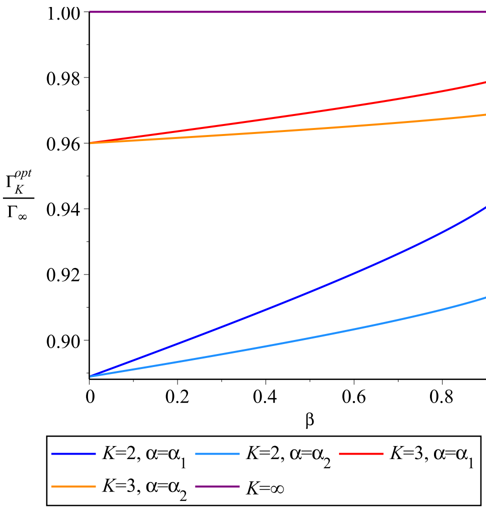

Figure 1: Pooling performance for the MO-LQU model with the optimal partition, where α is fixed to $\alpha_1 = 2$ or $\alpha_2 = \frac{9}{4} + \frac{3}{4}\sqrt{5}$.

By analysing expressions (21) and (22) for Γ_∞ and Γ_K^{opt}, respectively, we can generalise the above observations. Furthermore, we are able to derive guarantees for the pooling performance, see Theorem 14.

Theorem 14. *For the optimal partition the MO-LQU pooling performance $\Gamma_K^{\text{opt}}/\Gamma_\infty$ is continuous and non-increasing in α, and continuous and non-decreasing in β. In particular, we have the tight pooling performance guarantee*

$$\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty} \geq \frac{4K(K-1)}{(2K-1)^2} \qquad \forall\, \alpha > 0, 0 \leq \beta \leq 1. \tag{23}$$

Sketch of the proof. The idea is to consider all cases that occur based on (21) and (22). Analysing the closed-form and manageable formula for each case leads to the following insights.

- For $K = 1$ we have $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_1}{\Gamma_\infty} < 0$ for all $\alpha > 0$.
- For $K > 1$ we have
  - if $\beta = 0$: $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty} < 0$ for $0 < \alpha < \frac{K}{K-1}$ and $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty} = 0$ for $\alpha \geq \frac{K}{K-1}$,
  - if $0 < \beta \leq 1$: $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty} < 0$ for all $\alpha > 0$.
- We have $\frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty} = 0$ for $0 < \alpha \leq \sqrt{1-\beta}$ and $\frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty} > 0$ for $\alpha > \sqrt{1-\beta}$.

Hence, for any $0 \leq \beta \leq 1$ we obtain a tight pooling performance guarantee by taking the limit $\alpha \to \infty$, resulting in (23). The full proof is given in Appendix C. □

In Figure 2 we show the attained pooling performance for various choices of α, β, and K. The aggregate instance parameter α increases when the seller's utility per unit of sold product P decreases or when the buyer's type interval $[\underline{p}, \bar{p}]$ widens (under certain conditions). We can interpret the first case (decreasing P) as a higher investment risk in products, since a product provides less utility. The second case (widening $[\underline{p}, \bar{p}]$) can be interpreted as an increase in uncertainty on the buyer's identity. Hence, Theorem 14 implies that an increase in investment risk or in the uncertainty on the buyer's identity has a negative effect on the pooling performance.

In contrast, increasing the seller's reservation level (encoded in β) has a positive effect on the pooling performance, provided that the seller's reservation level is restrictive ($\alpha > \sqrt{1-\beta}$). However, if we want to give a guarantee for the pooling performance that holds for any instance, then this positive effect has no influence. In fact, the seller's reservation level does not affect the pooling performance guarantee. This follows from the proof of Theorem 14, where we show that the guarantee (23) is tight for any $0 \leq \beta \leq 1$. Table 1 shows the values of this guarantee for $K = 1, \ldots, 5$.

From Figure 2 and Table 1 we can conclude that the seller should not offer a single contract due to poor pooling performance in general. In contrast, offering two contracts already leads to a pooling performance guarantee of 88% and offering three contracts results in 96%. Recall that for $\beta = 0$ the MO model is equivalent to the ME model. Therefore, the bounds for $\beta = 0$ correspond to the results in Kerkkamp et al. (2017) and Wong (2014). Our analysis shows that the same bounds hold for the MO model for any $0 \leq \beta \leq 1$. Thus, the pooling of buyer types leads to a simpler menu of contracts and can be done with a controllable loss in the seller's expected net utility.

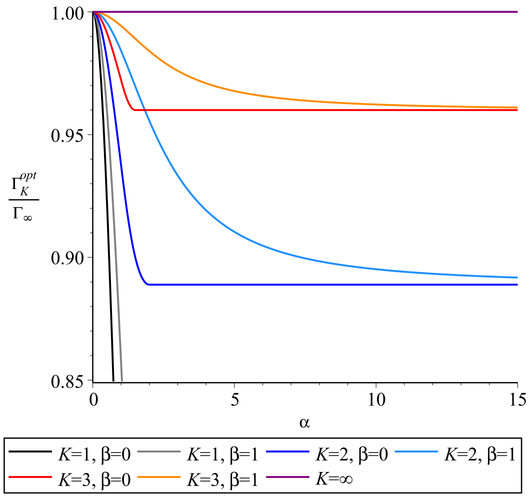

Figure 2: Attained pooling performance $\Gamma_K^{\text{opt}}/\Gamma_\infty$ for the MO-LQU model with the optimal partition.

K	Tight lower bound for $\Gamma_K^{\text{opt}}/\Gamma_\infty$
1	0
2	0.888
3	0.960
4	0.979
5	0.987
∞	1

Table 1: Pooling performance guarantees for the MO-LQU model with the optimal partition.

3.5.2 Reservation level performance

The *reservation level performance* $\Gamma_K^{\text{opt}}/\Gamma_\infty^{\beta=0}$ is similar to the pooling performance, except that the used benchmark $\Gamma_\infty^{\beta=0}$ disregards the seller's reservation level. That is, the numerator Γ_K^{opt} depends on β as before, but the denominator $\Gamma_\infty^{\beta=0}$ always uses $\beta = 0$. In particular, $\Gamma_\infty^{\beta=0}$ is the seller's maximum attainable expected net utility over all β and K. We can use the reservation level performance to quantify how much expected net utility is lost by the seller's reservation level, again in terms of the number of contracts offered.

We first consider the attained reservation level performance for two example instances, see Figure 3. As before, we use $\alpha = 2$ and $\alpha = \frac{9}{4} + \frac{3}{4}\sqrt{5} \approx 3.927$. Realise that for $\beta = 0$ the reservation level performance and the pooling performance are the same. In contrast to the pooling performance, the reservation level performance decreases when β increases, as seen in Figure 3. We observe that for $\alpha = 2$ the performance is less sensitive to changes in β for low values of β compared to the other instance. For both instances there is a steep decrease in performance when β approaches 1, i.e., when the seller fully considers his worst-case utility.

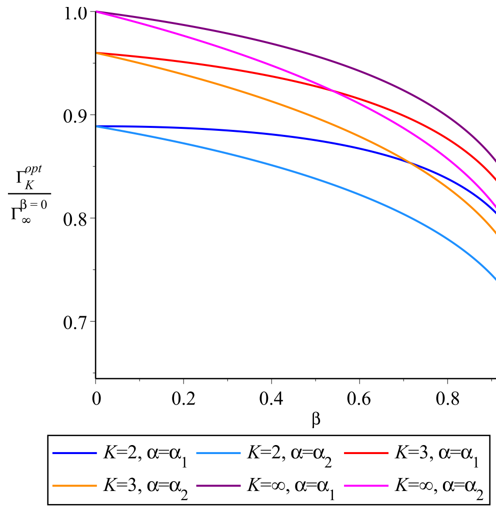

Figure 3: Reservation level performance for the MO-LQU model with the optimal partition, where α is fixed to $\alpha_1 = 2$ or $\alpha_2 = \frac{9}{4} + \frac{3}{4}\sqrt{5}$.

Similar to Theorem 14, we are able to generalise the above observations and determine guarantees for the reservation level performance. These results are shown in Theorem 15, where the term 'unimodal' refers to being non-increasing at first and then non-decreasing.

Theorem 15. *For the optimal partition the MO-LQU reservation level performance* $\Gamma_K^{\text{opt}}/\Gamma_\infty^{\beta=0}$ *is continuous and unimodal in* α*, and continuous and non-increasing in* β*. In particular, we have the tight reservation level performance guarantee*

$$\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} \geq \frac{8K(K-1)\left(4K(K-1)+(2K-1)\sqrt{2K^2-2K+1}+1\right)}{\left(6K(K-1)+(2K-1)\sqrt{2K^2-2K+1}+1\right)^2} \qquad \forall\, \alpha>0, 0\leq\beta\leq 1. \tag{24}$$

Sketch of the proof. We need to consider all cases that occur based on (21) and (22). By analysing each case, we obtain the following results.

- For $K=1$ we have $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_1}{\Gamma_\infty^{\beta=0}} < 0$ and $\frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_1}{\Gamma_\infty^{\beta=0}} = 0$ for all $\alpha > 0$.
- For $K > 1$ we have
  - if $\beta = 0$: $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} < 0$ for $0 < \alpha < \frac{K}{K-1}$ and $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} = 0$ for $\alpha \geq \frac{K}{K-1}$,
  - if $0 < \beta \leq 1$: $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} < 0$ for $0 < \alpha < \alpha^*$, $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} = 0$ for $\alpha = \alpha^*$, and $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} > 0$ for $\alpha > \alpha^*$,
  - $\frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} = 0$ for $0 < \alpha \leq \frac{K}{K-1}\sqrt{1-\beta}$ and $\frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} < 0$ for $\alpha > \frac{K}{K-1}\sqrt{1-\beta}$.

Here, the minimiser α^* is defined for $K > 1$ and $0 < \beta \leq 1$ by

$$\alpha^* = 1 + \frac{(2K(K-1)+1)\beta + (2K-1)\sqrt{\beta\left(2K(K-1)(1-\sqrt{1-\beta})+\beta\right)}}{2K(K-1)(1-\sqrt{1-\beta})}, \tag{25}$$

which satisfies $\alpha^* > 1$ and $\alpha^* > \frac{K}{K-1}\sqrt{1-\beta}$ if it exists.

Therefore, the reservation level performance guarantee is zero for $K=1$ and for $K>1$ it follows by taking $\beta = 1$ and evaluating the performance for $\alpha = \alpha^*$. The full proof is given in Appendix C. □

Note that in Figures 1 and 3 the example instance with $\alpha = \frac{9}{4} + \frac{3}{4}\sqrt{5}$ corresponds to the minimiser (25) for $K = 2$ and $\beta = 1$. The attained reservation level performance for various choices of α, β, and K is depicted in Figure 4. For $\beta = 0$ the reservation level performance and the pooling performance are equivalent (see also Figure 2). For $0 < \beta \leq 1$ there is a unique minimiser for the reservation level performance, namely α^* stated in (25). Hence, for any given β we can easily determine the minimum reservation level performance. We omit the verbose exact expressions, simply use (21), (22), and (25).

Instead, we depict the resulting minima in Figure 5, which are tight reservation level performance guarantees for each $0 \le \beta \le 1$.

In Figure 5 the values for $\beta = 0$ correspond to (23) and the values for $\beta = 1$ to (24). As seen in Figure 5, the reservation level performance guarantee decreases more rapidly for larger values of β. These guarantees are also given in Table 2 for certain choices of K and β. For example, increasing β from 0 to $\frac{3}{4}$ leads to approximately the same percentage point decrease in the guarantee as increasing β from $\frac{9}{10}$ to 1. Furthermore, notice that even with infinitely many contracts ($K = \infty$) it is not always possible to obtain full reservation level performance. If we let $K \to \infty$, then the bound in (24) is

$$\tfrac{8+4\sqrt{2}}{11+6\sqrt{2}} \approx 0.7009.$$

In other words, with infinitely many contracts the seller obtains at least 70% of the maximum expected net utility if he first maximises his worst-case net utility ($\beta = 1$) and this bound can be attained depending on the instance. Similarly, when using two (three) contracts, the seller achieves at least 64% (68%) when first maximising his worst-case net utility, and these bounds can be attained (see Table 2). Lowering the seller's reservation level raises these guarantees. For example, for $\beta = \frac{1}{2}$ these are 83%, 89%, and 92% for K equal to 2, 3, and ∞, respectively.

Overall, we conclude that the seller's reservation level has a significant impact on the seller's expected net utility, irrespective of the number of contracts offered. In terms of pooling performance, increasing the seller's reservation level has a positive effect, whereas it has a negative effect in terms of the reservation level performance. In any case, the seller should always offer at least two contracts in order to have a reasonable reservation level performance guarantee.

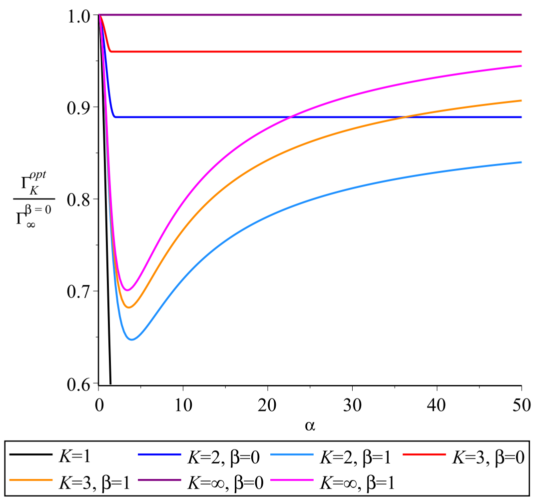

Figure 4: Attained reservation level performance $\Gamma_K^{\text{opt}}/\Gamma_\infty^{\beta=0}$ for the MO-LQU model with the optimal partition.

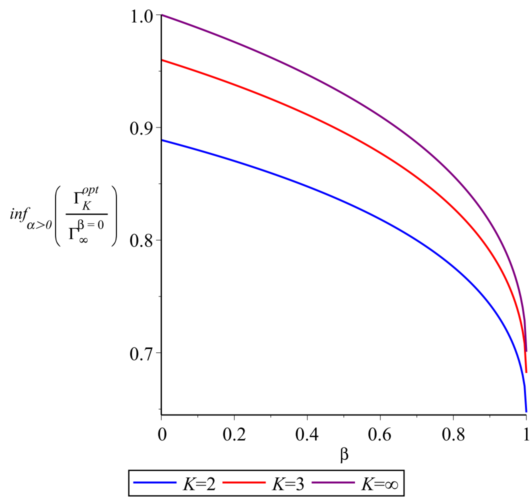

Figure 5: Tight reservation level performance guarantees in terms of β for the MO-LQU model with the optimal partition.

K	Tight lower bound for $\Gamma_K^{\text{opt}}/\Gamma_\infty^{\beta=0}$				
	$\beta = 0$	$\beta = \frac{1}{2}$	$\beta = \frac{3}{4}$	$\beta = \frac{9}{10}$	$\beta = 1$
1	0	0	0	0	0
2	0.888	0.834	0.788	0.742	0.647
3	0.960	0.895	0.842	0.790	0.682
4	0.979	0.912	0.857	0.802	0.691
5	0.987	0.919	0.863	0.808	0.695
∞	1	0.929	0.872	0.816	0.700

Table 2: Reservation level performance guarantees for the MO-LQU model with the optimal partition.

4 Concluding remarks

When faced with a continuum $[\underline{p}, \bar{p}]$ of buyer types, the seller can pool the buyer types to obtain a simpler menu of finitely many contracts. We analysed a pooling approach where the seller partitions the set of types $[\underline{p}, \bar{p}]$ a priori into $K \in \mathbb{N}_{\geq 1}$ subintervals $[\underline{p}_k, \bar{p}_k]$ for $k \in \{1, \ldots, K\}$. The resulting menu consists of K contracts and is designed such that the types in the k-th subinterval $[\underline{p}_k, \bar{p}_k]$ choose the k-th contract in the menu. In addition to pooling, we considered multiple objective functions for the seller: he maximises either his minimum net utility, his expected net utility, or a combination of both (resulting in a multi-objective approach). We modelled the multi-objective approach by maximising the seller's expected net utility under the additional constraint that his minimum net utility must be at least his reservation level. Here, the seller's reservation level is an additional model parameter decided by the seller.

Our analysis shows that under commonly used assumptions the three considered pooling models have tractable reformulations and that the optimal menus are maxima of certain modified joint net utility functions. In particular, the maximum obtainable minimum net utility is the maximum joint net utility with respect to the lowest buyer type $\underline{p}$. Using this property, the seller can fine-tune his reservation level in the multi-objective pooling model to balance his expected and worst-case net utility. Effectively, the multi-objective model encompasses the other models. With this model the seller can, for example, first maximise his minimum net utility, followed by his expected net utility (as a two-stage approach).

The considered pooling models depend on the chosen partition scheme and the seller's reservation level. We considered a contracting problem with quadratic utilities to quantify the effect of these decisions made by the seller. For this problem we first derived the optimal partition scheme, which then led to manageable formulas for the corresponding optimal objective value. In turn, these formulas can be used to determine various performance measures. We note that these results are analytical/exact and hold for any number of contracts.

We focussed on two performance measures. The first is the pooling performance, which is the obtained expected net utility by offering K menus compared to infinitely many contracts. It quantifies purely the effect of pooling the buyer types. The second is the reservation level performance, which is the obtained expected net utility compared to ignoring the seller's reservation level and using infinitely many contracts. Here, the benchmark is the highest attainable expected net utility over all seller's reservation levels and all number of contracts. Both measures have been fully analysed, which resulted in performance guarantees (lower bounds) in terms of the number of contracts K offered. For example, offering a single contract has poor performance (in both measures) and is ill-advised. In contrast, offering two, three, or infinitely many contracts leads to a pooling performance guarantee of 88%, 96%, and 100%, respectively. The corresponding reservation level performance guarantees are 64%, 68%, and 70%, respectively. Note that the latter guarantees are overall bounds and can be made more specific for a fixed seller's reservation level. In particular, a reservation level near the maximum feasible value is costly in performance. All mentioned bounds can be attained for certain instances and are therefore tight.

From our analysis, we conclude that pooling of buyer types results in a simpler menu of contracts and any loss in performance can be controlled by the number of contracts offered. High performance can already be achieved with up to five contracts. Furthermore, a multi-objective optimisation approach can be performed by including the seller's reservation level as a decision parameter. The seller's reservation level has a significant impact on the seller's expected net utility, irrespective of the number of contracts offered. Increasing the reservation level has a positive effect on the pooling performance, but a negative effect on the reservation level performance. Therefore, the seller has to balance his expected and worst-case net utility and can use the stated performance measures to justify his choices.

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A Proofs of Section 2

This appendix contains the proofs of the results in Section 2. We note that certain proofs are similar to or generalisations of those found in Kerkkamp et al. (2017).

Proof of Lemma 1. First, we show the necessity of $x_1 \leq \cdots \leq x_K$. Suppose $x_k > x_{k+1}$ for some $k, k+1 \in \mathcal{K}$ and consider (2) between contracts k with $\underline{p}_k$ and $k+1$ with $\underline{p}_{k+1}$. Adding both IC constraints leads to

$$\phi_B(x_k|\underline{p}_k) - \phi_B(x_{k+1}|\underline{p}_k) \geq \phi_B(x_k|\underline{p}_{k+1}) - \phi_B(x_{k+1}|\underline{p}_{k+1}).$$

Since $\underline{p}_k < \underline{p}_{k+1}$, this contradicts (9) of Assumption 1. Hence, $x_1 \leq \cdots \leq x_K$ must hold.

Second, we show sufficiency of $0 \leq x_1 \leq \cdots \leq x_K$. Let $x \in \mathbb{R}^K_{\geq 0}$ be non-decreasing and set $z \in \mathbb{R}^K$ to

$$z_k = \phi_B(x_k|\underline{p}_k) - \sum_{i=1}^{k-1} \left(\phi_B(x_i|\bar{p}_i) - \phi_B(x_i|\underline{p}_i)\right) \qquad \forall\, k \in \mathcal{K}. \tag{26}$$

It remains to check feasibility of (x, z). Fix $k \in \mathcal{K}$ and $p_k \in [\underline{p}_k, \bar{p}_k]$. For $l \in \mathcal{K}$ with $k < l$ we have

$$\begin{aligned} z_k - z_l &\overset{(26)}{=} \phi_B(x_k|\underline{p}_k) - \phi_B(x_l|\underline{p}_l) + \sum_{i=k}^{l-1} \left(\phi_B(x_i|\bar{p}_i) - \phi_B(x_i|\underline{p}_i)\right) = \sum_{i=k+1}^{l} \left(\phi_B(x_{i-1}|\underline{p}_i) - \phi_B(x_i|\underline{p}_i)\right) \\ &\overset{(9)}{\leq} \sum_{i=k+1}^{l} \left(\phi_B(x_{i-1}|p_k) - \phi_B(x_i|p_k)\right) = \phi_B(x_k|p_k) - \phi_B(x_l|p_k). \end{aligned}$$

Likewise, let $l \in \mathcal{K}$ with $l < k$, then

$$\begin{aligned} z_k - z_l &\overset{(26)}{=} \phi_B(x_k|\underline{p}_k) - \phi_B(x_l|\underline{p}_l) - \sum_{i=l}^{k-1} \left(\phi_B(x_i|\bar{p}_i) - \phi_B(x_i|\underline{p}_i)\right) = \sum_{i=l+1}^{k} \left(\phi_B(x_i|\underline{p}_i) - \phi_B(x_{i-1}|\underline{p}_i)\right) \\ &\overset{(9)}{\leq} \sum_{i=l+1}^{k} \left(\phi_B(x_i|p_k) - \phi_B(x_{i-1}|p_k)\right) = \phi_B(x_k|p_k) - \phi_B(x_l|p_k). \end{aligned}$$

Hence, all IC constraints (2) hold. Furthermore, we have

$$\phi_B(x_k|p_k) - z_k \overset{(2)}{\geq} \phi_B(x_1|p_k) - z_1 \overset{(8)}{\geq} \phi_B(x_1|\underline{p}_1) - z_1 \overset{(26)}{=} 0.$$

Thus, all IR constraints (1) are satisfied and the solution is feasible. □

Proof of Lemma 2. Let $x \in \mathbb{R}^K_{\geq 0}$ be feasible, i.e., there exists a $z \in \mathbb{R}^K$ such that (x, z) satisfies (1)-(3) and for the MO model also (7). The proof consists of two parts: first we show that (10) holds for contract $k = 1$ and then for the other contracts in the menu ($k > 1$).

First, realise that for an optimal z at least one IR constraint (1) must hold with equality. If this is not the case, we can increase all z_k by adding some $\epsilon > 0$ until at least one IR constraint is tight. This new solution is still feasible, as (2) only considers the difference $z_k - z_l$, which is unaffected. For the MO model (7) would trivially still hold. Moreover, the objective value of the new solution is strictly larger for the ME, MM, and MO models. Hence, if no IR constraint is tight we have a contradiction.

Now, suppose that $z_1 < \phi_B(x_1|\underline{p}_1)$, then for $k \in \mathcal{K}$ we have for all $p_k \in [\underline{p}_k, \bar{p}_k]$ that

$$\phi_B(x_k|p_k) - z_k \overset{(2)}{\geq} \phi_B(x_1|p_k) - z_1 \overset{(8)}{\geq} \phi_B(x_1|\underline{p}_1) - z_1 > 0.$$

The result implies that no IR constraint is tight, which is suboptimal as argued above. Hence, for an optimal z it must hold that $z_1 = \phi_B(x_1|\underline{p}_1)$.

Second, fix $k \in \mathcal{K}$ with $k > 1$ and consider the following IC constraints (2) between contracts k and $k-1$:

$$\phi_B(x_k|\bar{p}_{k-1}) - \phi_B(x_{k-1}|\bar{p}_{k-1}) \leq z_k - z_{k-1} \leq \phi_B(x_k|\underline{p}_k) - \phi_B(x_{k-1}|\underline{p}_k).$$

Since $\bar{p}_{k-1} = \underline{p}_k$, this implies that

$$z_k - z_{k-1} = \phi_B(x_k|\underline{p}_k) - \phi_B(x_{k-1}|\underline{p}_k).$$

Using our earlier result that $z_1 = \phi_B(x_1|\underline{p}_1)$, we obtain the following formula:

$$z_k = \sum_{i=2}^{k} \left(\phi_B(x_i|\underline{p}_i) - \phi_B(x_{i-1}|\underline{p}_i)\right) + \phi_B(x_1|\underline{p}_1),$$

which can be rewritten into (10). □

Proof of Theorem 3. We use Lemmas 1 and 2 to eliminate the variable z. The equivalent MM model follows immediately. The ME model becomes

$$\max_{0 \le x_1 \le \dots \le x_K} \sum_{k \in \mathcal{K}} \omega_k \Big(\phi_J(x_k|\underline{p}_k) - \sum_{i=1}^{k-1} \left(\phi_B(x_i|\bar{p}_i) - \phi_B(x_i|\underline{p}_i)\right) \Big).$$

Collecting all x_k terms results in the stated formulation, where we use that $\omega_k > 0$ for all $k \in \mathcal{K}$. Finally, the MO model follows by combining these insights. □

Proof of Lemma 4. By Assumption 2 the maximum of $x \mapsto \phi_J(x|\lambda)$ is attainable for any $\lambda \in \mathbb{R}$. Thus, M^* is well-defined and non-decreasing since (8) holds by Assumption 1. Next, for $M \le M^*(\underline{p})$ the threshold x^M is well-defined. By Assumptions 1 and 2, we can construct the stated function $x^*(\cdot|M)$ by selecting the smallest maximiser (if there are multiple). There is a technicality in this argument, which we discuss at the end of this proof. We continue with the proof of the non-decreasingness of $x^*(\cdot|M)$. Suppose the constructed $x^*(\cdot|M)$ is not non-decreasing, then there exist $\lambda < \mu$ with $x^*(\lambda|M) > x^*(\mu|M)$. By definition of the smallest maximiser, we have

$$\phi_J(x^*(\lambda|M)|\lambda) > \phi_J(x^*(\mu|M)|\lambda),$$
$$\phi_J(x^*(\mu|M)|\mu) \ge \phi_J(x^*(\lambda|M)|\mu).$$

Adding both inequalities and cancelling common terms leads to

$$\phi_B(x^*(\lambda|M)|\lambda) - \phi_B(x^*(\mu|M)|\lambda) > \phi_B(x^*(\lambda|M)|\mu) - \phi_B(x^*(\mu|M)|\mu),$$

which contradicts (9) of Assumption 1.

The technicality regarding the existence of $x^*(\cdot|M)$ is as follows. We have to show that $\phi_J(\cdot|\lambda)$ always has a maximum on $[x^M, \infty)$ for any $M \le M^*(\underline{p})$ and any $\lambda \in \mathbb{R}$. First, for $M = -\infty$ we have $x^M = 0$ and Assumption 2 guarantees the existence of the maximum. Hence, the non-decreasing function $x^*(\cdot| -\infty)$ exists. For notational convenience, let $x^*(\lambda) = x^*(\lambda| -\infty)$. Second, $x^M \le x^*(\underline{p})$ for any $M \le M^*(\underline{p})$ by definition. This implies that $x^*(\lambda|M) = x^*(\lambda)$ for $\lambda \ge \underline{p}$, i.e., the restriction to $x \ge x^M$ has no effect for $\lambda \ge \underline{p}$. For $\lambda < \underline{p}$ we have that for $x \ge x^*(\underline{p})$

$$\phi_J(x|\lambda) - \phi_J(x^*(\underline{p})|\lambda) \overset{(9)}{\le} \phi_J(x|\underline{p}) - \phi_J(x^*(\underline{p})|\underline{p}) \le 0.$$

Here, the last inequality follows from the fact that $x^*(\underline{p})$ maximises $\phi_J(\cdot|\underline{p})$ by definition. Thus, the maximum of $\phi_J(\cdot|\lambda)$ on $[x^M, \infty)$ (if it exists) must be attained in the closed interval $[x^M, x^*(\underline{p})]$. By Assumption 2 this maximum exists. □

Proof of Theorem 5. First, we use induction to prove that the solution $x_k = x^*(\underline{p}_k)$ for $k \in \mathcal{K}$ is optimal. Then, we show that the resulting optimal objective value can also be attained using a menu with only a single contract.

In order to do so, we need the following insight. Suppose $x_{k+1} = x^*(\underline{p}_{k+1})$ and compare the k-th and $(k+1)$-th terms of (12). We claim that these two terms satisfy

$$\phi_J(x_k|\underline{p}_k) - \sum_{i=1}^{k-1} \left(\phi_B(x_i|\bar{p}_i) - \phi_B(x_i|\underline{p}_i)\right) \le M^*(\underline{p}_{k+1}) - \sum_{i=1}^{k} \left(\phi_B(x_i|\bar{p}_i) - \phi_B(x_i|\underline{p}_i)\right),$$

where we have substituted $x_{k+1} = x^*(\underline{p}_{k+1})$, resulting in the term $M^*(\underline{p}_{k+1})$. The common terms cancel out in this expression, leading to

$$\phi_J(x_k|\bar{p}_k) \leq M^*(\underline{p}_{k+1}).$$

Since $\bar{p}_k = \underline{p}_{k+1}$, this inequality holds by definition of $M^*(\underline{p}_{k+1})$, which proves our claim. This implies that if $x_{k+1} = x^*(\underline{p}_{k+1})$ the $(k+1)$-th term does not affect the objective value and can be omitted in (12). Hence, if $x_l = x^*(\underline{p}_l)$ for all $l > k$ for some $k \in \mathcal{K}$, we only need to consider the first k terms of (12) for the remaining optimisation problem.

We continue with the induction proof that $x^*(\underline{p}_k)$ is optimal. First, we relax the feasibility constraint $x_1 \leq \cdots \leq x_K$. Second, notice that x_K only appears in the K-th term in (12). Therefore, we can optimise x_K independently for this term and optimally set $x_K = x^*(\underline{p}_K)$. Third, suppose that for some $k \in \mathcal{K}$ we have $x_l = x^*(\underline{p}_l)$ for all $l > k$. The remaining optimisation problem has decision variables $x_1, \dots, x_k$. By the above mentioned insight, we only need to consider the first k terms of (12). As such, x_k only appears in the k-th term of (12) and we can optimise x_k independently as seen before. This results in $x_k = x^*(\underline{p}_k)$. By induction, we end up with $x_k = x^*(\underline{p}_k)$ for all $k \in \mathcal{K}$, which is optimal for the relaxed problem as the induction proof shows. Since x^* is non-decreasing by definition, $x_k = x^*(\underline{p}_k)$ is also feasible and optimal for the MM model.

Finally, by using the above mentioned insight it follows that for this optimum only the first term $(k = 1)$ of (12) affects the objective value. Hence, the resulting optimal objective value is $M^*(\underline{p})$. The same objective value is attained by offering a single contract with order quantity $x = x^*(\underline{p})$, which does not depend on K or the partition of $[\underline{p}, \bar{p}]$. □

Proof of Theorem 6. Relax the feasibility constraint $x_1 \leq \cdots \leq x_K$ in (11) to obtain a separable optimisation problem for each $k \in \mathcal{K}$. By Assumption 3, the solution $x_k = x^*(\pi_k)$ for $k \in \mathcal{K}$ is optimal for this relaxed problem. Since $\pi_k \leq \pi_{k+1}$ and x^* is non-decreasing by definition, we have that $x_k \leq x_{k+1}$ for all $k \in \mathcal{K}$. Therefore, the relaxed optimum is feasible for the ME model and thus optimal. □

Proof of Theorem 7. Since $\underline{p}_1 = \underline{p}$ and x^M is the smallest value such that $\phi_J(x|\underline{p}) \geq M$, constraint (14) for $k = 1$ implies $x_1 \geq x^M$. Now relax all feasibility constraints (14)-(15), but add the implied constraints $x_k \geq x^M$ for $k \in \mathcal{K}$. By definition, $x_k = x^*(\pi_k|M)$ for $k \in \mathcal{K}$ is an optimal solution for the resulting separable optimisation problem.

It remains to verify that the proposed solution is also feasible for the MO model. Notice that $0 \leq x^M \leq x^*(\pi_1|M) \leq \cdots \leq x^*(\pi_K|M)$ by definition of $x^*(\cdot|M)$ and since $\pi_k \leq \pi_{k+1}$ for all $k \in \mathcal{K}$. Thus, we need to check if (14) holds for all $k \in \mathcal{K}$.

First, notice that $\pi_k \leq \underline{p}_k$ must hold for all $k \in \mathcal{K}$ by the assumptions. The proof is as follows. By (8) and (16) we have $\phi_B(\cdot|\pi_k) \leq \phi_B(\cdot|\underline{p}_k)$. If $\pi_k > \underline{p}_k$, then the previous result and (8) imply $\phi_B(\cdot|\pi_k) = \phi_B(\cdot|\underline{p}_k)$, which trivially violates (9). In fact, $\pi_k < \underline{p}_k$ must hold for all $k \in \mathcal{K} \setminus \{K\}$: if $\pi_k = \underline{p}_k$ for some $k < K$ then (16) implies $\phi_B(\cdot|\bar{p}_k) = \phi_B(\cdot|\underline{p}_k)$, which again trivially violates (9).

Second, we show a useful implication of the definition of $x^*(\cdot|M)$. For some $k \in \mathcal{K}$, consider any order quantity $\bar{x}$ with $x^M \leq \bar{x} \leq x_k = x^*(\pi_k|M)$. By definition of $x_k = x^*(\pi_k|M)$, we have

$$\phi_J(x_k|\pi_k) \geq \phi_J(\bar{x}|\pi_k). \tag{27}$$

Since $\pi_k \leq \underline{p}_k$ and $\bar{x} \leq x_k$, we have

$$\begin{aligned}
\phi_J(x_k|\underline{p}_k) &= \phi_S(x_k) + \phi_B(x_k|\pi_k) + \big(\phi_B(x_k|\underline{p}_k) - \phi_B(x_k|\pi_k)\big) \\
&\overset{(27)}{\geq} \phi_S(\bar{x}) + \phi_B(\bar{x}|\pi_k) + \big(\phi_B(x_k|\underline{p}_k) - \phi_B(x_k|\pi_k)\big) \\
&\overset{(9)}{\geq} \phi_S(\bar{x}) + \phi_B(\bar{x}|\pi_k) + \big(\phi_B(\bar{x}|\underline{p}_k) - \phi_B(\bar{x}|\pi_k)\big) \\
&= \phi_J(\bar{x}|\underline{p}_k).
\end{aligned}$$

Finally, the above result with $k = 1$ and $\bar{x} = x^M$ shows that (14) holds for $k = 1$:

$$\phi_J(x_1|\underline{p}_1) \geq \phi_J(x^M|\underline{p}_1) \geq M.$$

Likewise, using the above result for $k \in \mathcal{K}$ and $\bar{x} = x_{k-1}$ gives

$$\phi_J(x_k|\underline{p}_k) \geq \phi_J(x_{k-1}|\underline{p}_k).$$

Subtracting $(\phi_B(x_i|\bar{p}_i) - \phi_B(x_i|\underline{p}_i))$ for $i = 1, \dots, k-1$ from both sides leads to

$$\phi_J(x_k|\underline{p}_k) - \sum_{i=1}^{k-1} \left(\phi_B(x_i|\bar{p}_i) - \phi_B(x_i|\underline{p}_i)\right) \geq \phi_J(x_{k-1}|\underline{p}_{k-1}) - \sum_{i=1}^{k-2} \left(\phi_B(x_i|\bar{p}_i) - \phi_B(x_i|\underline{p}_i)\right).$$

These are the left-hand sides of (14) for k and $k-1$. Repeatedly applying this result for $k, k-1, \dots, 1$ gives

$$\phi_J(x_k|\underline{p}_k) - \sum_{i=1}^{k-1} \left(\phi_B(x_i|\bar{p}_i) - \phi_B(x_i|\underline{p}_i)\right) \geq \phi_J(x_1|\underline{p}_1) \geq M,$$

where we have derived the last inequality earlier. Hence, (14) holds for all $k \in \mathcal{K}$. We conclude that that $x_k = x^*(\pi_k|M)$ is feasible for the MO model and therefore optimal. □

Proof of Corollary 8. We can rewrite the MM model into

$$\begin{aligned} \max_{x,u} \quad & u, \\ \text{s.t.} \quad & \phi_J(x_k|\underline{p}_k) - \sum_{i=1}^{k-1} \left(\phi_B(x_i|\bar{p}_i) - \phi_B(x_i|\underline{p}_i)\right) \geq u, && \forall\, k \in \mathcal{K}, \\ & x_K \geq \dots \geq x_1 \geq 0. \end{aligned}$$

This is a concave optimisation problem by the additional assumption of this corollary, hence any convex combination of optimal solutions is also optimal. It remains to verify that all stated solutions are optimal for the MM model. The first two stated menus are optimal for the MM model as shown in Theorem 5. The third stated menu is also feasible and optimal for the MM model by construction, due to Assumption 3 and the choice of $M = M^*(\underline{p})$. See the proof of Theorem 7 for additional details regarding feasibility. □

B Examples that satisfy Assumption 3

In this appendix, we give an example problem class that satisfies Assumptions 1 and 3. Consider a buyer with utility function given by $\phi_B(x|p) \equiv \psi(x) + p\chi(x)$, where the functions $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and $\chi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ do not depend on the type p. Furthermore, χ is strictly increasing and non-negative. Finally, ω is a (strictly positive) continuous distribution with a non-decreasing hazard rate. We show that the stated assumptions hold for this problem class.

We first verify that Assumption 1 holds, i.e., (8) and (9). For $\lambda \leq \mu \in \mathbb{R}$ and $x \geq 0$ we have

$$\phi_B(x|\lambda) - \phi_B(x|\mu) = (\lambda - \mu)\chi(x) \leq 0,$$

since χ is non-negative. For $\lambda < \mu \in \mathbb{R}$ and $0 \leq x < x'$ we get

$$\phi_B(x'|\lambda) - \phi_B(x|\lambda) - \phi_B(x'|\mu) + \phi_B(x|\mu) = (\lambda - \mu)(\chi(x') - \chi(x)) < 0,$$

as χ is strictly increasing. Thus, Assumptions 1 is satisfied.

Next, we show that Assumption 3 holds. In order to define π_k, we need to consider (16):

$$\phi_B(x|\pi_k) = \phi_B(x|\underline{p}_k) - \left(\phi_B(x|\bar{p}_k) - \phi_B(x|\underline{p}_k)\right) \sum_{i=k+1}^{K} \frac{\omega_i}{\omega_k} = \psi(x) + \Big(\underline{p}_k - (\bar{p}_k - \underline{p}_k) \sum_{i=k+1}^{K} \frac{\omega_i}{\omega_k}\Big)\chi(x).$$

Hence, π_k is the coefficient of $\chi(x)$ in the above expression:

$$\pi_k \equiv \underline{p}_k - (\bar{p}_k - \underline{p}_k) \sum_{i=k+1}^{K} \frac{\omega_i}{\omega_k} \qquad \forall\, k \in \mathcal{K}.$$

Notice that $\pi_k < \underline{p}_k$ for $k \in \mathcal{K} \setminus \{K\}$ and $\pi_K = \underline{p}_K$. In order to have $\pi_1 \leq \dots \leq \pi_K$ and thus Assumption 3 to hold, we need conditions on the probability distribution ω. As stated, we assume that ω has a non-decreasing hazard rate, which implies that

$$\frac{\omega(v)}{1 - \int_{\underline{p}}^{v} \omega(p)\mathrm{d}p} \geq \frac{\omega(u)}{1 - \int_{\underline{p}}^{u} \omega(p)\mathrm{d}p} \qquad \forall\, u, v \in [\underline{p}, \bar{p}], u \leq v,$$

or equivalently

$$\frac{1}{\omega(u)}\int_u^{\bar{p}}\omega(p)\mathrm{d}p \geq \frac{1}{\omega(v)}\int_v^{\bar{p}}\omega(p)\mathrm{d}p \qquad \forall\, u,v \in [\underline{p},\bar{p}], u \leq v.$$

Since ω is assumed to be continuous, by the Mean Value Theorem there exist $\hat{p}_k \in (\underline{p}_k, \bar{p}_k)$ for $k \in \mathcal{K}$ such that

$$\omega(\hat{p}_k) = \frac{1}{\bar{p}_k - \underline{p}_k}\int_{\underline{p}_k}^{\bar{p}_k}\omega(p)\mathrm{d}p = \frac{\omega_k}{\bar{p}_k - \underline{p}_k}.$$

We now have for $k \in \mathcal{K}$ that

$$\begin{aligned}
\pi_k - \pi_{k+1} &= \underline{p}_k - (\bar{p}_k - \underline{p}_k)\sum_{i=k+1}^{K}\frac{\omega_i}{\omega_k} - \underline{p}_{k+1} + (\bar{p}_{k+1} - \underline{p}_{k+1})\sum_{i=k+2}^{K}\frac{\omega_i}{\omega_{k+1}} \\
&= \frac{\bar{p}_{k+1} - \underline{p}_{k+1}}{\omega_{k+1}}\sum_{i=k+2}^{K}\omega_i - \frac{\bar{p}_k - \underline{p}_k}{\omega_k}\sum_{i=k}^{K}\omega_i \\
&= \frac{1}{\omega(\hat{p}_{k+1})}\int_{\bar{p}_{k+1}}^{\bar{p}}\omega(p)\mathrm{d}p - \frac{1}{\omega(\hat{p}_k)}\int_{\underline{p}_k}^{\bar{p}}\omega(p)\mathrm{d}p \\
&< \frac{1}{\omega(\hat{p}_{k+1})}\int_{\hat{p}_{k+1}}^{\bar{p}}\omega(p)\mathrm{d}p - \frac{1}{\omega(\hat{p}_k)}\int_{\hat{p}_k}^{\bar{p}}\omega(p)\mathrm{d}p \leq 0.
\end{aligned}$$

Here, the first inequality follows from $\omega(p) > 0$ for all $p \in [\underline{p}, \bar{p}]$ and the last inequality from the non-decreasing hazard rate. Hence, $\pi_k < \pi_{k+1}$ for all $k \in \mathcal{K}$ and Assumption 3 is satisfied.

C Proofs of Section 3

In this appendix we give all proofs of the results in Section 3. We note that certain proofs are similar to or generalisations of those found in Kerkkamp et al. (2017).

Proof of Corollary 9. First, the optimal ME and MO solutions follow from Theorems 6 and 7, and the optimal MM solutions from Corollary 8. Second, since the function $x \mapsto \phi_J(x|\lambda)$ for the LQU problem is strictly concave and differentiable for any $\lambda \in \mathbb{R}$, it has a unique maximiser. From the relaxations used in the proofs of Theorems 6 and 7 it follows that the stated optima are the only optima. □

Proof of Lemma 10. Consider the optimal MO solution (19). The first term in the maximisation corresponds to the case where the seller's reservation level is restrictive for contract $k \in \mathcal{K}$. This is the case if

$$\begin{aligned}
& (1 - \sqrt{1-\beta})(P + \underline{p}) > P - \bar{p} + \bar{p}_k + \underline{p}_k \\
\iff\quad & (1 - \sqrt{1-\beta})(P + \underline{p}) > P + \underline{p} + (\delta_k + \delta_{k-1} - 1)(\bar{p} - \underline{p}) \\
\iff\quad & (1 - \sqrt{1-\beta})\tfrac{1}{\alpha} > \tfrac{1}{\alpha} + (\delta_k + \delta_{k-1} - 1) \\
\iff\quad & 1 - \tfrac{1}{\alpha}\sqrt{1-\beta} > \delta_k + \delta_{k-1}.
\end{aligned}$$

Since $\delta_k + \delta_{k-1} < \delta_{k+1} + \delta_k$ for all $k \in \mathcal{K}$, we can determine the largest index affected by the seller's reservation level:

$$k^\beta = \max\left\{0, \max\left\{k \in \mathcal{K} : \delta_k + \delta_{k-1} < 1 - \tfrac{1}{\alpha}\sqrt{1-\beta}\right\}\right\}.$$

Combining our results, the optimal MO objective value is (by recalling Assumption 3)

$$\begin{aligned}
\Gamma_K &= \sum_{k\in\mathcal{K}}\omega_k\phi_J(x_k|\pi_k) = \sum_{k\in\mathcal{K}}\omega_k\left((P+\pi_k)x_k - \tfrac{1}{2}rx_k^2\right) \\
&= \tfrac{1}{r}\sum_{k=1}^{k^\beta}\omega_k\left((P+\pi_k)(1-\sqrt{1-\beta})(P+\underline{p}) - \tfrac{1}{2}(1-\sqrt{1-\beta})^2(P+\underline{p})^2\right) \\
&\quad + \tfrac{1}{2r}\sum_{k=k^\beta+1}^{K}\omega_k(P+\pi_k)^2.
\end{aligned}$$

Since $(1-\sqrt{1-\beta})^2 = -\beta + 2(1-\sqrt{1-\beta})$, $(\pi_k - \underline{p})/(\bar{p}-\underline{p}) = \delta_k + \delta_{k-1} - 1$, and

$$\frac{P+\pi_k}{\bar{p}-\underline{p}} = \frac{P+\bar{p}_k+\underline{p}_k-\bar{p}}{\bar{p}-\underline{p}} = \frac{P+2\underline{p}-\bar{p}}{\bar{p}-\underline{p}} + \delta_k + \delta_{k-1} = \frac{1}{\alpha} + \delta_k + \delta_{k-1} - 1,$$

the normalised optimal MO objective value is

$$\begin{aligned} \nu\Gamma_K = &\sum_{k=1}^{k^\beta}(\delta_k - \delta_{k-1})\left(\tfrac{\beta}{\alpha^2} + 2(\delta_k+\delta_{k-1}-1)(1-\sqrt{1-\beta})\tfrac{1}{\alpha}\right) \\ &+ \sum_{k=k^\beta+1}^{K}(\delta_k-\delta_{k-1})\left(\tfrac{1}{\alpha}+\delta_k+\delta_{k-1}-1\right)^2. \end{aligned}$$

This completes the proof. □

Proof of Lemma 11. Let Δ be an optimal partition. First, suppose that $k^\beta(\Delta) \geq 2$. Construct a new partition $\hat{\Delta}$ with $\hat{\delta}_1 = \delta_{k^\beta}$, $\hat{\delta}_k = \delta_{k^\beta+1}$ for $1 < k \leq k^\beta$, and $\hat{\delta}_k = \delta_k$ otherwise. By construction, we have $k^\beta(\hat{\Delta}) = 1$, since

$$\begin{aligned} \hat{\delta}_1 + \hat{\delta}_0 = \delta_{k^\beta} \leq \delta_{k^\beta} + \delta_{k^\beta-1} &< 1 - \tfrac{1}{\alpha}\sqrt{1-\beta}, \\ \hat{\delta}_k + \hat{\delta}_{k-1} \geq \delta_{k^\beta+1} + \delta_{k^\beta} &\geq 1 - \tfrac{1}{\alpha}\sqrt{1-\beta}, \qquad \text{for } k = 2,\ldots,K. \end{aligned}$$

Here we use the definition of k^β. Since we have

$$\sum_{k=1}^{k^\beta}(\delta_k-\delta_{k-1})(\delta_k+\delta_{k-1}) - (\delta_{k^\beta}-\delta_0)(\delta_{k^\beta}+\delta_0) = \sum_{k=1}^{k^\beta}(\delta_k^2 - \delta_{k-1}^2) - \delta_{k^\beta}^2 = 0,$$

it is straightforward to verify that

$$\begin{aligned} \nu\Gamma_K(\Delta) - \nu\Gamma_K(\hat{\Delta}) = &\sum_{k=1}^{k^\beta}(\delta_k-\delta_{k-1})\left(\tfrac{\beta}{\alpha^2} + 2(\delta_k+\delta_{k-1}-1)(1-\sqrt{1-\beta})\tfrac{1}{\alpha}\right) \\ &- (\delta_{k^\beta}-\delta_0)\left(\tfrac{\beta}{\alpha^2} + 2(\delta_{k^\beta}+\delta_0-1)(1-\sqrt{1-\beta})\tfrac{1}{\alpha}\right) = 0. \end{aligned}$$

Hence, the new partition $\hat{\Delta}$ is also optimal and we can assume without loss of generality that $k^\beta(\Delta) \in \{0,1\}$.

Second, suppose that $\delta_{i-1} = \delta_i < \delta_{i+1}$ for some $i \in \{1,\ldots,K-1\}$. By the first part of this proof, we know that $i+1 \geq 2 > k^\beta(\Delta) \in \{0,1\}$. Case I: if $k^\beta < i$, then there exists an $0 < \epsilon < 1$ such that

$$(1-\epsilon)\delta_{i+1} + (1+\epsilon)\delta_i \geq 1 - \tfrac{1}{\alpha}\sqrt{1-\beta}.$$

We construct a new partition $\hat{\Delta}$ with $\hat{\delta}_i = (1-\epsilon)\delta_{i+1} + \epsilon\delta_i$ and $\hat{\delta}_k = \delta_k$ otherwise. By construction, we have $\hat{\delta}_{i-1} < \hat{\delta}_i < \hat{\delta}_{i+1}$ and $k^\beta(\hat{\Delta}) \leq k^\beta(\Delta) < i$. The difference in the resulting normalised optimal objective value is

$$\begin{aligned} \nu\Gamma_K(\Delta) - \nu\Gamma_K(\hat{\Delta}) &= 0 + (\delta_{i+1}-\delta_i)(\tfrac{1}{\alpha}+\delta_{i+1}+\delta_i-1)^2 \\ &\quad - (\hat{\delta}_i - \hat{\delta}_{i-1})(\tfrac{1}{\alpha}+\hat{\delta}_i+\hat{\delta}_{i-1}-1)^2 - (\hat{\delta}_{i+1}-\hat{\delta}_i)(\tfrac{1}{\alpha}+\hat{\delta}_{i+1}+\hat{\delta}_i-1)^2 \\ &= (\delta_{i+1}-\delta_i)(\tfrac{1}{\alpha}+\delta_{i+1}+\delta_i-1)^2 \\ &\quad - (1-\epsilon)(\delta_{i+1}-\delta_i)(\tfrac{1}{\alpha}+(1-\epsilon)\delta_{i+1}+(1+\epsilon)\delta_i-1)^2 \\ &\quad - \epsilon(\delta_{i+1}-\delta_i)(\tfrac{1}{\alpha}+(2-\epsilon)\delta_{i+1}+\epsilon\delta_i-1)^2 < 0. \end{aligned}$$

Here, the inequality follows from the strict convexity of the quadratic function and contradicts the optimality of Δ. Case II: if $k^\beta = 1 = i$, then $\delta_1 = \delta_0 = 0$. Construct a new partition $\hat{\Delta}$ with $\hat{\delta}_1 = \delta_2$ and $\hat{\delta}_k = \delta_k$ otherwise. This leads to $k^\beta(\hat{\Delta}) = 0 < k^\beta(\Delta)$ and the same (optimal) objective value. We can now apply either the previous case or the following cases.

Finally, suppose $0 = \delta_0 < \cdots < \delta_{i-1} < \delta_i = \cdots = \delta_K = 1$ for some $i \in \{1, \ldots, K-1\}$. Notice that $k^\beta(\Delta) < i$. Case I: if $i = 1$ and $\beta = 1$, then $k^\beta(\Delta) = 0$ and $\nu\Gamma_K(\Delta) = 1/\alpha^2$. Construct a new partition with $0 < \hat{\delta}_1 < 1$ and $\hat{\delta}_k = \delta_k = 1$ otherwise. This leads to $k^\beta(\hat{\Delta}) = 1$ and the following contradiction:

$$\nu\Gamma_K(\hat{\Delta}) = \hat{\delta}_1(\tfrac{1}{\alpha^2} + 2(\hat{\delta}_1 - 1)\tfrac{1}{\alpha}) + (1 - \hat{\delta}_1)(\tfrac{1}{\alpha} + \hat{\delta}_1)^2 = \tfrac{1}{\alpha^2} + (1 - \hat{\delta}_1)\hat{\delta}_1^2 > \tfrac{1}{\alpha^2} = \nu\Gamma_K(\Delta).$$

Case II: if $i > 1$ or $\beta < 1$, then there exists an $0 < \epsilon < 1$ such that

$$(1 - \epsilon)\delta_i + (1 + \epsilon)\delta_{i-1} \geq 1 - \tfrac{1}{\alpha}\sqrt{1 - \beta}.$$

Construct a new partition with $\hat{\delta}_i = (1 - \epsilon)\delta_i + \epsilon\delta_{i-1}$ and $\hat{\delta}_k = \delta_k$ otherwise. We have $\hat{\delta}_{i-1} < \hat{\delta}_i < \hat{\delta}_{i+1}$ and $k^\beta(\hat{\Delta}) = k^\beta(\Delta)$. Consequently, we get the contradiction

$$\begin{aligned}
\nu\Gamma_K(\Delta) - \nu\Gamma_K(\hat{\Delta}) &= (\delta_i - \delta_{i-1})(\tfrac{1}{\alpha} + \delta_i + \delta_{i-1} - 1)^2 + 0 \\
&\quad - (\hat{\delta}_i - \hat{\delta}_{i-1})(\tfrac{1}{\alpha} + \hat{\delta}_i + \hat{\delta}_{i-1} - 1)^2 - (\hat{\delta}_{i+1} - \hat{\delta}_i)(\tfrac{1}{\alpha} + \hat{\delta}_{i+1} + \hat{\delta}_i - 1)^2 \\
&= (\delta_i - \delta_{i-1})(\tfrac{1}{\alpha} + \delta_i + \delta_{i-1} - 1)^2 \\
&\quad - (1 - \epsilon)(\delta_i - \delta_{i-1})(\tfrac{1}{\alpha} + (1 - \epsilon)\delta_i + (1 + \epsilon)\delta_{i-1} - 1)^2 \\
&\quad - \epsilon(\delta_i - \delta_{i-1})(\tfrac{1}{\alpha} + (2 - \epsilon)\delta_i + \epsilon\delta_{i-1} - 1)^2 < 0.
\end{aligned}$$

To conclude, an optimal partition Δ must satisfy $k^\beta(\Delta) \in \{0, 1\}$ and $0 < \delta_1 < \cdots < \delta_{K-1} < 1$. □

Proof of Theorem 12. By Lemma 11 we only need to consider the cases $k^\beta = 0$ and $k^\beta = 1$. Case I: suppose $k^\beta = 0$, then

$$\nu\Gamma_K = \sum_{k=1}^{K}(\delta_k - \delta_{k-1})(\tfrac{1}{\alpha} + \delta_k + \delta_{k-1} - 1)^2.$$

This expression is quadratic in δ_k for $k \in \{1, \ldots, K-1\}$, since the cubic terms cancel out. Setting the gradient to zero, leads to

$$(\delta_{k+1} - \delta_{k-1})(\delta_{k+1} + \delta_{k-1} - 2\delta_k) = 0, \qquad \forall\, k \in \{1, \ldots, K-1\}.$$

Since $\delta_{k+1} > \delta_{k-1}$ by Lemma 11, $\delta_k = \frac{1}{2}(\delta_{k+1} + \delta_{k-1})$ must hold. Solving this system of linear equalities, results in the equidistant partition:

$$\delta_k = \tfrac{k}{K}.$$

Since $k^\beta = 0$, it must hold that

$$\tfrac{1}{K} = \delta_1 = \delta_1 + \delta_0 \geq 1 - \tfrac{1}{\alpha}\sqrt{1 - \beta} \qquad \Longleftrightarrow \qquad \alpha \leq \tfrac{K}{K-1}\sqrt{1 - \beta}.$$

Case II: suppose $k^\beta = 1$, then

$$\nu\Gamma_K = \delta_1\left(\beta\tfrac{1}{\alpha^2} + 2(\delta_1 - 1)(1 - \sqrt{1 - \beta})\tfrac{1}{\alpha}\right) + \sum_{k=2}^{K}(\delta_k - \delta_{k-1})(\tfrac{1}{\alpha} + \delta_k + \delta_{k-1} - 1)^2. \tag{28}$$

This expression is cubic in δ_1 and quadratic in δ_k for $k \in \{2, \ldots, K-1\}$. Setting the gradient to zero leads to

$$\begin{aligned}
-3\delta_1^2 - 2(\delta_2 + 2\tfrac{1}{\alpha}\sqrt{1 - \beta} - 2)\delta_1 + \delta_2^2 - 1 + 2\tfrac{1}{\alpha}\sqrt{1 - \beta} - \tfrac{1-\beta}{\alpha^2} &= 0, \\
(\delta_{k+1} - \delta_{k-1})(\delta_{k+1} + \delta_{k-1} - 2\delta_k) &= 0, \qquad \forall\, k \in \{2, \ldots, K-1\}.
\end{aligned}$$

The roots for the first equation are $\delta_1 = \frac{1}{3}(\delta_2 + 1 - \frac{1}{\alpha}\sqrt{1 - \beta})$ and $\delta_1 = 1 - \delta_2 - \frac{1}{\alpha}\sqrt{1 - \beta}$, where the first root is the largest. The second set of equations are as before, implying $\delta_k = \frac{1}{2}(\delta_{k+1} + \delta_{k-1})$ for $k > 1$. By substituting $\delta_K = 1$ we can express δ_k as an affine function of δ_{k-1} for $k > 1$. Consequently, δ_k is an affine function of δ_1 for $k > 1$. Hence, after substitution of δ_k, (28) remains a cubic function in δ_1, whose leading term is $-\delta_1^3$. We conclude that the optimal value for δ_1 is the larger root of the corresponding derivative.

The resulting system of linear equations, $\delta_1 = \frac{1}{3}(\delta_2 + 1 - \frac{1}{\alpha}\sqrt{1-\beta})$ and $\delta_k = \frac{1}{2}(\delta_{k+1} + \delta_{k-1})$ for $k \in \{2, \dots, K-1\}$, has the following solution:

$$\delta_k = 1 - \tfrac{K-k}{2K-1}(1 + \tfrac{1}{\alpha}\sqrt{1-\beta}), \qquad \forall\, k \in \{1, \dots, K-1\}.$$

For this partition to be valid with $k^\beta = 1$, we must have

$$1 - \tfrac{K-1}{2K-1}(1 + \tfrac{1}{\alpha}\sqrt{1-\beta}) = \delta_1 = \delta_1 + \delta_0 < 1 - \tfrac{1}{\alpha}\sqrt{1-\beta} \qquad \iff \qquad \alpha > \tfrac{K}{K-1}\sqrt{1-\beta}.$$

Likewise, we have for $k \in \{2, \dots, K-1\}$ that

$$\alpha > 0 \qquad \implies \qquad 2 - \tfrac{2K-2k+1}{2K-1}(1 + \tfrac{1}{\alpha}\sqrt{1-\beta}) = \delta_k + \delta_{k-1} \geq 1 - \tfrac{1}{\alpha}\sqrt{1-\beta}.$$

This implies that for the partition indeed $k^\beta = 1$. Finally, clearly $\delta_{K-1} < 1$, so we only need to verify that $\delta_1 > 0$:

$$1 - \tfrac{K-1}{2K-1}(1 + \tfrac{1}{\alpha}\sqrt{1-\beta}) > 0 \qquad \iff \qquad \alpha > \tfrac{K-1}{K}\sqrt{1-\beta}.$$

Thus, the partition is valid for $\alpha > \frac{K}{K-1}\sqrt{1-\beta}$.

Notice that both cases for k^β are disjoint and cover all possible values of $\alpha > 0$. This completes the proof. □

Proof of Corollary 13. By Theorem 12, for $\alpha \leq \frac{K}{K-1}\sqrt{1-\beta}$ we have $k^\beta = 0$ and $\delta_k = k/K$. The expressions (20) simplifies to

$$\nu\Gamma_K^{\text{opt}} = \sum_{k=1}^{K} \tfrac{1}{K}\left(\tfrac{1}{\alpha} + \tfrac{2k-1}{K} - 1\right)^2 = \tfrac{1}{\alpha^2} + \tfrac{1}{3}\left(1 - \tfrac{1}{K^2}\right).$$

For $\alpha > \frac{K}{K-1}\sqrt{1-\beta}$ we have $k^\beta = 1$ and (20) becomes

$$\begin{aligned}
\nu\Gamma_K^{\text{opt}} &= \left(1 - \tfrac{K-1}{2K-1}(1 + \tfrac{1}{\alpha}\sqrt{1-\beta})\right)\left(\tfrac{\beta}{\alpha^2} - \tfrac{2K-2}{2K-1}(1 + \tfrac{1}{\alpha}\sqrt{1-\beta})(1 - \sqrt{1-\beta})\tfrac{1}{\alpha}\right) \\
&\quad + \sum_{k=2}^{K} \tfrac{1}{2K-1}(1 + \tfrac{1}{\alpha}\sqrt{1-\beta})\left(\tfrac{1}{\alpha} + 1 - \tfrac{2K-2k+1}{2K-1}(1 + \tfrac{1}{\alpha}\sqrt{1-\beta})\right)^2 \\
&= \tfrac{\beta}{\alpha^2} + \tfrac{2}{3}\tfrac{K(K-1)}{(2K-1)^2}(1 + \tfrac{1}{\alpha}\sqrt{1-\beta})^3.
\end{aligned}$$

In particular, these expressions converge to (21) as $K \to \infty$, as should be the case. □

Proof of Theorem 14. Continuity of $\Gamma_K^{\text{opt}}/\Gamma_\infty$ is trivially verified by (21) and (22). For readability, we state the properties that will be proved in the end:

- For $K = 1$ we have $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_1}{\Gamma_\infty} < 0$ for all $\alpha > 0$.
- For $K > 1$ we have
  - if $\beta = 0$: $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty} < 0$ for $0 < \alpha < \frac{K}{K-1}$ and $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty} = 0$ for $\alpha \geq \frac{K}{K-1}$,
  - if $0 < \beta \leq 1$: $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty} < 0$ for all $\alpha > 0$.
- We have $\frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty} = 0$ for $0 < \alpha \leq \sqrt{1-\beta}$ and $\frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty} > 0$ for $\alpha > \sqrt{1-\beta}$.

We start with the proof for $K = 1$, which is considered separately to prevent issues with division by zero. Note that $\nu\Gamma_1 = \alpha^{-2}$ and that there is no partition to optimise in this case. Therefore, for $0 < \alpha \leq \sqrt{1-\beta}$ it is trivial to show that

$$\frac{\Gamma_1}{\Gamma_\infty} = \frac{3}{\alpha^2 + 3}, \qquad \frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_1}{\Gamma_\infty} = -\frac{6\alpha}{(\alpha^2+3)^2} < 0, \qquad \frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_1}{\Gamma_\infty} = 0.$$

For $\alpha > \sqrt{1-\beta}$, we have

$$\frac{\Gamma_1}{\Gamma_\infty} = \frac{6\alpha}{6\alpha\beta + (\alpha + \sqrt{1-\beta})^3}.$$

The corresponding derivatives are

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_1}{\Gamma_\infty} = -\frac{6(2\alpha-\sqrt{1-\beta})(\alpha+\sqrt{1-\beta})^2}{\left(6\alpha\beta+(\alpha+\sqrt{1-\beta})^3\right)^2} < 0,$$
$$\frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_1}{\Gamma_\infty} = \frac{9\alpha(\alpha-\sqrt{1-\beta})^2}{\sqrt{1-\beta}\left(6\alpha\beta+(\alpha+\sqrt{1-\beta})^3\right)^2} > 0,$$

since $\alpha > \sqrt{1-\beta}$. Hence, for any $0 \le \beta \le 1$ the infimum for the pooling performance is

$$\inf_{\alpha>0}\frac{\Gamma_1}{\Gamma_\infty} = \lim_{\alpha\to\infty}\frac{\Gamma_1}{\Gamma_\infty} = 0.$$

We continue with the proof for $K > 1$. Based on (21) and (22) we need to differentiate three cases. Case I: for $0 < \alpha \le \sqrt{1-\beta}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\mathrm{opt}}}{\Gamma_\infty} = -\frac{6\alpha}{K^2(\alpha^2+3)^2} < 0, \qquad \frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_K^{\mathrm{opt}}}{\Gamma_\infty} = 0.$$

Case II: for $\sqrt{1-\beta} < \alpha \le \frac{K}{K-1}\sqrt{1-\beta}$, which can only occur for $0 \le \beta < 1$, we get

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\mathrm{opt}}}{\Gamma_\infty} = 6\frac{(K^2-1)\sqrt{1-\beta}\alpha^4 + 2\left((K^2-1)\beta-1\right)\alpha^3}{K^2\left(6\alpha\beta+(\alpha+\sqrt{1-\beta})^3\right)^2} + 6\frac{-\left(K^2(\beta+2)+1-\beta\right)\sqrt{1-\beta}\alpha^2 + K^2(1-\beta)^{3/2}}{K^2\left(6\alpha\beta+(\alpha+\sqrt{1-\beta})^3\right)^2}. \tag{29}$$

We claim that (29) is strictly negative on $\sqrt{1-\beta} < \alpha < \frac{K}{K-1}\sqrt{1-\beta}$ and that (29) at $\alpha = \frac{K}{K-1}\sqrt{1-\beta}$ is either zero (if $\beta = 0$) or strictly negative (if $0 < \beta < 1$). Clearly, the denominator is always strictly positive. Hence, it is sufficient to focus on the numerator. Let $f(\alpha) = c_4\alpha^4 + c_3\alpha^3 + c_2\alpha^2 + c_0$ denote the numerator. Recall that $K > 1$. Since this case cannot occur for $\beta = 1$, we have $c_4 > 0$, $c_3 \in \mathbb{R}$, $c_2 < 0$, and $c_0 > 0$.

First, by Descartes' Sign Rule the number of positive real roots of f is bounded by 2, namely by the number of sign changes in the sequence c_4, c_3, c_2, and c_0.

Second, we evaluate the numerator f for certain values for α:

$$\lim_{\alpha\downarrow 0} f(\alpha) = 6K^2(1-\beta)^{3/2} > 0, \qquad \lim_{\alpha\to\infty} f(\alpha) = \infty > 0,$$
$$f(\sqrt{1-\beta}) = -24(1-\beta)^{3/2} < 0, \qquad f\left(\tfrac{K}{K-1}\sqrt{1-\beta}\right) = -6\frac{K^2(K+1)\beta(1-\beta)^{3/2}}{(K-1)^3} \le 0,$$

where we use that $K > 1$ and that this case cannot occur for $\beta = 1$. By continuity of f we conclude that there is a positive real root on $(0, \sqrt{1-\beta})$ and on $[\frac{K}{K-1}\sqrt{1-\beta}, \infty)$.

Thus, f has exactly two positive real roots. If $0 < \beta < 1$ both fall outside of $(\sqrt{1-\beta}, \frac{K}{K-1}\sqrt{1-\beta}]$. Furthermore, since f is strictly negative on the borders of this interval, it is strictly negative on the entire interval. If $\beta = 0$, one of the roots is the border point $\frac{K}{K-1}$. The same conclusions hold for (29).

Furthermore, using $\alpha > \sqrt{1-\beta}$ we have that the derivative to β is

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_K^{\mathrm{opt}}}{\Gamma_\infty} = \frac{3\alpha(\alpha-\sqrt{1-\beta})^2((K^2-1)\alpha^2+3K^2)}{K^2\sqrt{1-\beta}\left(6\alpha\beta+(\alpha+\sqrt{1-\beta})^3\right)^2} > 0.$$

Case III: for $\alpha > \frac{K}{K-1}\sqrt{1-\beta}$ it holds that

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\mathrm{opt}}}{\Gamma_\infty} = -\frac{6\beta\left(2\alpha^3+3\sqrt{1-\beta}\alpha^2-(1-\beta)^{3/2}\right)}{(2K-1)^2\left(6\alpha\beta+(\alpha+\sqrt{1-\beta})^3\right)^2} \le 0, \tag{30}$$
$$\frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_K^{\mathrm{opt}}}{\Gamma_\infty} = \frac{3\alpha(2\sqrt{1-\beta}\alpha^3+3(2-\beta)\alpha^2+6\sqrt{1-\beta}\alpha+2-\beta^2-\beta)}{(2K-1)^2\sqrt{1-\beta}\left(6\alpha\beta+(\alpha+\sqrt{1-\beta})^3\right)^2} > 0,$$

where we use that $\alpha > \frac{K}{K-1}\sqrt{1-\beta} > \sqrt{1-\beta}$. Note that (30) is zero if $\beta = 0$ and strictly negative if $0 < \beta \le 1$.

We conclude that the derivative of the pooling performance is non-negative (if $\beta = 0$) or strictly negative (if $0 < \beta \leq 1$) with respect to α in all cases. Hence, the infimum is reached for $\alpha \to \infty$:

$$\inf_{\alpha>0} \frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty} = \lim_{\alpha\to\infty} \frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty} = \frac{4K(K-1)}{(2K-1)^2}.$$

Here, the limit trivially follows from (21) and (22). Furthermore, notice that this infimum holds for any $0 \leq \beta \leq 1$, implying that this bound is tight for any $0 \leq \beta \leq 1$. □

Proof of Theorem 15. Continuity of $\Gamma_K^{\text{opt}}/\Gamma_\infty^{\beta=0}$ is trivially verified by (21) and (22). For readability, we make the following claims, which are all proved in the end:

- For $K = 1$ we have $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_1}{\Gamma_\infty^{\beta=0}} < 0$ and $\frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_1}{\Gamma_\infty^{\beta=0}} = 0$ for all $\alpha > 0$.
- For $K > 1$ we have
  - if $\beta = 0$: $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} < 0$ for $0 < \alpha < \frac{K}{K-1}$ and $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} = 0$ for $\alpha \geq \frac{K}{K-1}$,
  - if $0 < \beta \leq 1$: $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} < 0$ for $0 < \alpha < \alpha^*$, $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} = 0$ for $\alpha = \alpha^*$, and $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} > 0$ for $\alpha > \alpha^*$,
  - $\frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} = 0$ for $0 < \alpha \leq \frac{K}{K-1}\sqrt{1-\beta}$ and $\frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} < 0$ for $\alpha > \frac{K}{K-1}\sqrt{1-\beta}$.

Here, the minimiser α^* is defined for $K > 1$ and $0 < \beta \leq 1$ by

$$\alpha^* = 1 + \frac{(2K(K-1)+1)\beta + (2K-1)\sqrt{\beta\left(2K(K-1)(1-\sqrt{1-\beta})+\beta\right)}}{2K(K-1)(1-\sqrt{1-\beta})}$$

and we claim that $\alpha^* > 1$ and $\alpha^* > \frac{K}{K-1}\sqrt{1-\beta}$ if it exists.

We first focus on the tight reservation level performance guarantee. Using the above claims, in particular on the derivative to β, we conclude that it is sufficient to consider $\beta = 1$ to derive the tight guarantee for the reservation level performance. Therefore, we have to consider two cases: $K = 1$ and $K > 1$ with $\beta = 1$.

First, consider the case $K = 1$. We have for $0 < \alpha \leq 1$ that

$$\frac{\Gamma_1}{\Gamma_\infty^{\beta=0}} = \frac{3}{\alpha^2+3}, \qquad \frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_1}{\Gamma_\infty^{\beta=0}} = -\frac{6\alpha}{(\alpha^2+3)^2} < 0, \qquad \frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_1}{\Gamma_\infty^{\beta=0}} = 0.$$

Likewise, for $\alpha > 1$

$$\frac{\Gamma_1}{\Gamma_\infty^{\beta=0}} = \frac{6\alpha}{(\alpha+1)^3}, \qquad \frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_1}{\Gamma_\infty^{\beta=0}} = -\frac{6(2\alpha-1)}{(\alpha+1)^4} < 0, \qquad \frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_1}{\Gamma_\infty^{\beta=0}} = 0.$$

Thus, for any $0 \leq \beta \leq 1$ the reservation level performance guarantee is

$$\inf_{\alpha>0}\frac{\Gamma_1}{\Gamma_\infty^{\beta=0}} = \lim_{\alpha\to\infty}\frac{\Gamma_1}{\Gamma_\infty^{\beta=0}} = 0. \tag{31}$$

Second, consider $K > 1$ and $\beta = 1$. We need to discern two cases based on α.

Case I: for $0 < \alpha \leq 1$ we have

$$\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} = \frac{2K(K-1)\alpha^2 + 3(2K-1)^2}{(2K-1)^2(\alpha^2+3)}, \qquad \frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} = -\frac{6(2K^2-2K+1)\alpha}{(2K-1)^2\left(\alpha^2+3\right)^2} < 0.$$

Case II: for $\alpha > 1$ we get

$$\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} = \frac{4K(K-1)\alpha^3 + 6(2K-1)^2\alpha}{(2K-1)^2(\alpha+1)^3},$$

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} = 6\frac{2K(K-1)\alpha^2 - (8K^2-8K+2)\alpha + 4K^2 - 4K + 1}{(2K-1)^2(\alpha+1)^4}. \tag{32}$$

The derivative (32) has roots

$$\alpha^{\pm} = 2 + \frac{1 \pm (2K-1)\sqrt{2K^2-2K+1}}{2K(K-1)}.$$

Note that α^+ corresponds to α^* for this case. Evaluating the formula in (32) for $\alpha = 1$ gives

$$-\frac{3}{8}\frac{2K(K-1)+1}{(2K-1)^2} < 0.$$

Since (32) is a parabola that opens upward, we have $\alpha^- < 1 < \alpha^+$. Hence, the reservation level performance has a minimum at α^+. By combining Case I and Case II, we conclude that α^+ is the global minimum for $K > 1$ and $\beta = 1$.

As argued above, for $K > 1$ the tight reservation level performance guarantee follows from evaluating $\Gamma_K^{\text{opt}}/\Gamma_\infty^{\beta=0}$ at $\alpha = \alpha^+$ and $\beta = 1$, resulting in

$$\inf_{0\le\beta\le1}\inf_{\alpha>0}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} = \frac{8K(K-1)\left(4K(K-1)+(2K-1)\sqrt{2K^2-2K+1}+1\right)}{\left(6K(K-1)+(2K-1)\sqrt{2K^2-2K+1}+1\right)^2}.$$

This formula also works for $K = 1$ (resulting in a value of 0, see also (31)).

It remains to prove all our claims. The proofs for $K = 1$ have already been given. Therefore, consider the case $K > 1$. Unfortunately, the proofs are somewhat tedious work. We have to distinguish four cases.

Case I: for $0 < \alpha \le 1$ and $\alpha \le \frac{K}{K-1}\sqrt{1-\beta}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} = -\frac{6\alpha}{K^2\left(\alpha^2+3\right)^2} < 0, \qquad \frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} = 0.$$

Case II: for $1 < \alpha \le \frac{K}{K-1}\sqrt{1-\beta}$ the derivatives are

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} = 6\frac{(K^2-1)\alpha^2 - K^2(2\alpha-1)}{K^2(\alpha+1)^4}, \qquad \frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} = 0.$$

The roots of the derivative to α are $\frac{K}{K+1} < 1$ and $\frac{K}{K-1} \ge \frac{K}{K-1}\sqrt{1-\beta}$. Notice that the numerator is a parabola that opens upward. For $\beta = 0$ the derivative to α is strictly negative on $1 < \alpha < \frac{K}{K-1}$ and zero at $\alpha = \frac{K}{K-1}$. For $0 < \beta \le 1$ it is strictly negative on the entire interval $1 < \alpha \le \frac{K}{K-1}\sqrt{1-\beta}$.

Case III: for $\frac{K}{K-1}\sqrt{1-\beta} < \alpha \le 1$ we get

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} = -6\frac{K(K-1)\sqrt{1-\beta}\left(\alpha^4-(\beta+2)\alpha^2+(1-\beta)\right)+(2K^2-2K+1)\beta\alpha^3}{(2K-1)^2\alpha^2\left(\alpha^2+3\right)^2}. \tag{33}$$

Let f be the numerator of (33). For $\beta = 0$ this case cannot occur. For $\beta = 1$ the function f simplifies to a cubic function with roots $\alpha = 0$. Hence, it follows trivially that (33) is strictly negative for $\alpha > 0$. For $0 < \beta < 1$ it holds that $f(\alpha) < 0$ on $\alpha > 0$ if and only if

$$g(\alpha) = -6\alpha^4 - 6\frac{2K^2-2K+1}{K(K-1)}\frac{\beta}{\sqrt{1-\beta}}\alpha^3 + 6(\beta+2)\alpha^2 - 6(1-\beta) < 0 \qquad \forall\,\alpha > 0,$$

where the quartic function g differs from f by a positive factor. Let the quartic function h be defined by

$$h(\alpha) = -6\alpha^4 - 12\tfrac{\beta}{\sqrt{1-\beta}}\alpha^3 + 6(\beta+2)\alpha^2 - 6(1-\beta).$$

Since $(2K^2-2K+1)/(K(K-1)) > 2$ for $K > 1$, we have $g(\alpha) < h(\alpha)$ for all $\alpha > 0$. The discriminant of h is zero. By using well-known properties of quartic formulas we conclude that h has two distinct real roots and one double real root. The shape of h now follows from evaluating it for certain points:

$$\lim_{\alpha\to-\infty} h(\alpha) = -\infty < 0, \qquad h(-1) = 12\beta\left(1+\tfrac{1}{\sqrt{1-\beta}}\right) > 0, \qquad h(0) = -6(1-\beta) < 0.$$

This trivially implies that $h(\alpha) \le 0$ for all $\alpha > 0$ by evaluating all possible shapes of h. Thus, $g(\alpha) < 0$, $f(\alpha) < 0$, and (33) is strictly negative for all $\alpha > 0$.

The derivative to β is given by

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_K^{\mathrm{opt}}}{\Gamma_\infty^{\beta=0}} = -3\frac{K(K-1)\alpha^2 - (2K^2-2K+1)\sqrt{1-\beta}\alpha + K(K-1)(1-\beta)}{(2K-1)^2\sqrt{1-\beta}\alpha(\alpha^2+3)}, \tag{34}$$

which has roots $\frac{K-1}{K}\sqrt{1-\beta}$ and $\frac{K}{K-1}\sqrt{1-\beta}$ (both smaller than the considered α). Note that the numerator of (34) is a parabola that opens downward. Hence, (34) is strictly negative.

Case IV: for $\alpha > \frac{K}{K-1}\sqrt{1-\beta}$ and $\alpha > 1$ it holds that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\mathrm{opt}}}{\Gamma_\infty^{\beta=0}} &= 6\frac{2K(K-1)(1-\sqrt{1-\beta})\alpha^2 - \left((4K^2-4K+2)\beta + 4K(K-1)(1-\sqrt{1-\beta})\right)\alpha}{(2K-1)^2(\alpha+1)^4} \\ &\quad + 6\frac{-2K(K-1)(1-\beta)^{3/2} + 2K(K-1)(1+\beta) + \beta}{(2K-1)^2(\alpha+1)^4}.\end{aligned} \tag{35}$$

If $\beta = 0$, then (35) is always equal to zero. For $\beta > 0$ the numerator of (35) is a parabola that opens upward with roots

$$\alpha^{\pm} = 1 + \frac{\big(2K(K-1)+1\big)\beta \pm (2K-1)\sqrt{\beta\big(2K(K-1)(1-\sqrt{1-\beta})+\beta\big)}}{2K(K-1)(1-\sqrt{1-\beta})}.$$

We claim that α^+ is a local minimiser for the reservation level performance, which turns out to be the global minimiser by checking all other cases. This claim is proved by showing that $\alpha^- < 1 < \alpha^+$ and, if needed, that $\alpha^- < \frac{K}{K-1}\sqrt{1-\beta} < \alpha^+$. This implies that (35), with a parabola that opens upward as numerator, is strictly negative for $\alpha < \alpha^+$, zero at $\alpha = \alpha^+$, and strictly positive for $\alpha > \alpha^+$. Hence, α^+ is a local minimiser.

We continue to prove our claim. Evaluating (35) for $\alpha = 1$ results in the value

$$-\frac{3}{8}\beta\frac{2K(K-1)(1-\sqrt{1-\beta})+1}{(2K-1)^2} < 0.$$

This implies that $\alpha^- < 1 < \alpha^+$. If $\frac{K}{K-1}\sqrt{1-\beta} \leq 1$ the proof for this case if complete. Otherwise, $\frac{K}{K-1}\sqrt{1-\beta} > 1$ or equivalently

$$\beta < \frac{2K-1}{K^2}.$$

Evaluating (35) for $\alpha = \frac{K}{K-1}\sqrt{1-\beta}$ gives

$$-6\frac{(K-1)^3((K+1)\beta - 2K(1-\sqrt{1-\beta}))}{(K(1+\sqrt{1-\beta})-1)^4}, \tag{36}$$

which is zero only if $\beta = 0$ or if $\beta = \frac{4K}{(K+1)^2}$ and strictly negative in between these values. Since $\frac{2K-1}{K^2} = \frac{4K}{(K+1)^2}$ only if $K = 1$ (excluding negative values), we conclude that we are considering β satisfying

$$0 < \beta < \frac{2K-1}{K^2} < \frac{4K}{(K+1)^2}.$$

For such β the value (36) is strictly negative. This implies that $\alpha^- < \frac{K}{K-1}\sqrt{1-\beta} < \alpha^+$, completing the proof for this case.

The derivative to β is

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_K^{\mathrm{opt}}}{\Gamma_\infty^{\beta=0}} = -6\frac{K(K-1)\alpha^2 - (2K^2-2K+1)\sqrt{1-\beta}\alpha + K(K-1)(1-\beta)}{(2K-1)^2\sqrt{1-\beta}(\alpha+1)^3}, \tag{37}$$

with roots $\frac{K-1}{K}\sqrt{1-\beta}$ and $\frac{K}{K-1}\sqrt{1-\beta}$ (both smaller than the considered α). As seen before, the numerator is a parabola that opens downward, which implies that (37) is strictly negative. □