4. GOAL PROGRAMMING

Goal Programming, an approach designed and further developed mainly by Charnes and Cooper, was one of the earliest practical techniques in multiple criteria decision making. In our opinion, goal programming is one of the stronger methods available, especially for capital budgeting and financial planning with multiple goals. Its use of aspiration levels and pre-emptive priorities closely corresponds to decision making in practice. Furthermore, goal programming possesses a number of attractive technical properties.

Several empirical findings from decision making practice are indeed rather convincing in demonstrating the practical usefulness of goal programming. As mentioned by several authors, the method corresponds quite well to the results of the behavioral theory of the firm. In practice, decision makers are aiming at various goals, formulated as aspiration levels. The intensity with which the goals are strived for may vary for each goal, in other words, different 'weights' may be assigned to different goals. The use of aspiration levels in decision making is also reported by scientists from other fields, for instance, psychology (see for a short overview Monarchi et al. [1976]. In the same way, pre-emptive priorities are also known in real life problems. Support for this essentially lexicographic viewpoint is provided a.o. by Fishburn [1974] and Monarchi et al. [1976]. A more concrete example of the correspondence of goal programming and practice is provided by Ijiri [1965], who regards goal programming as an extension of break even analysis, which is widely used in business practice.

The above plea for goal programming is of a somewhat theoretical nature. However, the operational usefulness of goal programming has also been recognized in practice, as shown by the many applications which have been reported in literature (see Nijkamp and Spronk [1979] for an overview).

One of the technical advantages of goal programming is that it always provides a solution, even if none of the goals are realizable, provided the feasible region is non-empty. This is due to the inclusion of deviational variables, which show whether the goals are attained or not. In the latter case they measure the distance between the realized and aspired goal levels. Another advantage of goal programming is that it does not require very sophisticated solution procedures. Especially the linear goal programming problems can be solved by easily available linear programming routines.

4.1. General Formulation

In problems involving multiple goals a number (say m) of goal variables, depending on a number of instrumental variables $x_1, x_2, \ldots, x_n$ (in vector notation $\underline{x}$), can be identified. From the set of admissable values of the instrumental variables $\underline{x}$, an element (or set of elements) must be selected, resulting in certain aspired levels of the goal variables. If such instrument vectors do exist, they are solutions of

$$\begin{aligned} &\underline{b} = \underline{g}(\underline{x}) \\ &\underline{x} \in R, \text{ where } R = \{\underline{x} | \underline{h}(\underline{x}) \leq \underline{h}\}, \end{aligned} \tag{4.1}$$

and $\underline{b}$ is the m x 1 vector of aspired levels of the goal variables $g_j(\underline{x})$, $j = 1, \ldots, m$; and R denotes the feasible region, being the set of admissable values of the instrumental variables $\underline{x}$, given the k x 1 vector of available resources, $\underline{h}$. In this section we assume both $\underline{g}(\underline{x})$ and $\underline{h}(\underline{x})$ to be linear in $\underline{x}$. Then, the problem can be formalized as finding the set of vectors $\underline{x}$, for which

$$\begin{aligned} &\underline{b} = \underline{g}(\underline{x}) = A.\underline{x} \\ &\underline{x} \in R, \text{ where } R = \{\underline{x} | \underline{h}(\underline{x}) = B.\underline{x} \leq \underline{h}\}, \end{aligned} \tag{4.2}$$

A is a matrix of order m x n, and B is a matrix of order k x n.

Due to the possible incompatibility of goals - at least within the feasible region R - we have to account for deviations of the goal variables from the aspired levels $b_1, b_2, \ldots, b_m$. This can be done by introducing the m x 1 vectors $\underline{y}^+$ en $\underline{y}^-$ which measure respectively the over- and underachievements of the stated goals. Because over- and underachievement of a given goal cannot take place at the same time, we impose the conditions $y_i^+ . y_i^- = 0$ for $i = 1,2, \ldots, m$; $\underline{y}^+ \geq 0$ and $\underline{y}^- \geq 0$. Assuming that a <u>dispreference function</u> $f(\underline{y}^+,\underline{y}^-)$ can be specified which reflects the undesirability of the deviations properly, the problem becomes

$$
(4.3) \qquad \begin{aligned}
&\text{Min} \quad f(\underline{y}^+,\underline{y}^-) \ , \text{ s.t.} \\
&A.\underline{x} - \underline{y}^+ + \underline{y}^- = \underline{b} \ , \\
&y_i^+ . y_i^- = 0 \text{ for } i = 1, \ldots, m; \\
&\underline{x} \in R, \text{ and } \underline{y}^+, \underline{y}^- \geq \underline{0}
\end{aligned}
$$

In goal programming, this formulation is often extended as follows. First, divide the m goals in p ($p < m$) classes $C_t, t = 1, \ldots, p$; in such a way that the goals in the first class have a preemptive priority above the goals in the second class and so forth. In this essentially lexicographic approach (see also Fishburn [1974]), the goals in a certain priority class must be satisfied 'as well as possible' before paying attention to the goals of subsequent classes. Defining b_{ts} as the s'th goal of the t'th priority class and

m_t ($\sum_{t=1}^{p} m_t = m$) as the number of goals in class C_t, we can formulate this sequential problem as

First:

$\text{Min } f_1(\underline{y}^+,\underline{y}^-) = f(y^+_{11},y^+_{12},\ldots,y^+_{1m_1}, y^-_{11},y^-_{12},\ldots,y^-_{1m_1})$, s.t.

$A.\underline{x} - \underline{y}^+ + \underline{y}^- = \underline{b}$,

$y^+_i.y^-_i = 0$ for $i = 1, \ldots, m$;

$\underline{x} \in R$, and $\underline{y}^+,\underline{y}^- \geq \underline{0}$

If this problem has only one optimal $\underline{x}$ vector, stop.

Otherwise:

$\text{Min } f_2(\underline{y}^+,\underline{y}^-) = f(y^+_{21},y^+_{22},\ldots,y^+_{2m_2},y^-_{21},y^-_{22},\ldots,y^-_{2m_2})$, s.t.

$f_1(\underline{y}^+,\underline{y}^-) = \text{Min}\{f_1(\underline{y}^+,\underline{y}^-)\}$,

$A.\underline{x} - \underline{y}^+ + \underline{y}^- = \underline{b}$

$y^+_i.y^-_i = 0$ for $i = 1, \ldots, m$;

$\underline{x} \in R$, and $\underline{y}^+,\underline{y}^- \geq \underline{0}$

(4.4) If this problem has only one optimal $\underline{x}$ vector, stop.

Otherwise:

. . .

. . .

. . .

If this problem has only one optimal $\underline{x}$ vector, stop.

Otherwise:

$\text{Min } f_p(\underline{y}^+,\underline{y}^-) = f(y^+_{p1},y^+_{p2},\ldots,y^+_{pm_p},y^-_{p1},y^-_{p2},\ldots,y^-_{pm_p})$, s.t.

$f_i(\underline{y}^+,\underline{y}^-) = \text{Min}\{f_i(\underline{y}^+,\underline{y}^-)\}$ for $i = 1,2, \ldots, p-1$.

$A.\underline{x} - \underline{y}^+ + \underline{y}^- = \underline{b}$,

$y^+_i.y^-_i = 0$ for $i = 1, \ldots, m$;

$\underline{x} \in R$, and $\underline{y}^+,\underline{y}^- \geq \underline{0}$

In general, each step of the minimization in (4.4) reduces the feasible region remaining for the fulfilment of the lower priority goals. Let C_t be the first class in the priority ordering where not all goals can be fulfilled simultaneously (thus for

$t' < t$ all goals in C_t are attainable). Then we need to specify the dispreference function $f_t(\underline{y}^+,\underline{y}^-)$ in order to choose among the goals within C_t with goal levels $b_{t1}, b_{t2}, \ldots, b_{tm_t}$. By definition, this dispreference function is a function of the deviations from the aspired goal levels, and can thus be regarded as a distance function requiring some distance measure.

A number of alternative specifications of such distance functions can be derived from the general Minkovski metric, used among others in welfare economics (cf. Bartels and Nijkamp [1976]). As a welfare criterion, the Minkovski metric can be specified as follows

$$\text{(4.5)} \qquad \text{Min } f = \left\{ \sum_{i=1}^{m} \left| \frac{z_i - \bar{z}}{\bar{z}} \right|^p \right\}^{1/p}$$

where $\bar{z}_i$ is the average income (or some other appropriate welfare indicator) in class $i (i=1,\ldots,m)$ and $\bar{z}$ is the overall average income. More generally, the term in the inner brackets can be interpreted as a standardized distance from a predetermined norm. Realizing there is not a single norm (as $\bar{z}$ in (4.5)) in (4.3) and (4.4), we can adapt (4.5) in terms of the multiple goal problem (4.3) to

$$\text{(4.6)} \qquad \text{Min } f = \left\{ \sum_{i=1}^{m} \left| \frac{\underline{a(i)}.\underline{x} - b_i}{b_i} \right|^p \right\}^{1/p}$$

where $\underline{a(i)}$ is the i'th row of A. Unfortunately, this form does not discriminate between positive and negative deviations from the stated goal levels $b_i, i, = 1, \ldots, m$. In order to bridge this gap note that (cf. Charnes and Cooper [1977 ,p.9])

$$\text{(4.7a)} \qquad y_i^+ = \frac{|\underline{a(i)}.\underline{x} - b_i| + (\underline{a(i)}.\underline{x} - b_i)}{2} \quad \text{for } i = 1, \ldots, m;$$

represents the positive deviations, and

$$(4.7b) \qquad \bar{y}_i = \frac{|\underline{a(i)}.\underline{x}-b_i| - (\underline{a(i)}.\underline{x}-b_i)}{2} \qquad \text{for } i = 1, \ldots, m;$$

represents the negative deviations. Substracting (4.7b) from (4.7a) gives

$$(4.8) \qquad y_i^+ - y_i^- = \underline{a(i)}.\underline{x} - b_i \qquad \text{for } i = 1, \ldots, m;$$

which exactly equals the goal restrictions in (4.3). Adding (4.7a) and (4.7b) gives

$$(4.9) \qquad y_i^+ + y_i^- = |\underline{a(i)}.\underline{x} - b_i|$$

which can be substituted in (4.6), to get $f(y^+,y^-)$ as a function of the standardized deviations from the aspired goal levels:

$$(4.10) \qquad \text{Min } f = f(\underline{y}^+,\underline{y}^-) = \{ \sum_{i=1}^{m} \left(\frac{y_i^+ + y_i^-}{|b_i|}\right)^p \}^{1/p}$$

Then of course, to exclude the possibility of y_i^+ and y_i^- being both positive at the same time, $y_i^+.y_i^- = 0$ must hold for $i = 1, \ldots, m$. Assuming $b_i > 0$, this condition permits us[1)] to write (4.10) as

$$(4.11) \qquad \text{Min } f(\underline{y}^+,\underline{y}^-) = \{ \sum_{i=1}^{m} \left(\frac{y_i^+}{b_i}\right)^p + \sum_{i=1}^{m} \left(\frac{y_i^-}{b_i}\right)^p \}^{1/p}$$

This form is preferable to (4.10) if not all deviations are valued

1) Since $y_i^+.y_i^- = 0$, either $y_i^+ = 0$ or $y_i^- = 0$ or $y_i^+ = y_i^- = 0$. Assume $y_i^+ = 0$. Then:

$$\left(\frac{y_i^+ + y_i^-}{b_i}\right)^p = \left(\frac{y_i^-}{b_i}\right)^p = \left(\frac{y_i^-}{b_i}\right)^p + \left(\frac{0}{b_i}\right)^p = \left(\frac{y_i^-}{b_i}\right)^p + \left(\frac{y_i^+}{b_i}\right)^p. \qquad \text{q.e.d.}$$

For either $y_i^- = 0$ or $y_i^+ = y_i^- = 0$ the proof is similar.

equally by the decision maker. In that case (4.11) permits different weights for the positive and negative deviations from the same aspiration level whereas (4.10) does not. The specification of the weights in (4.11) may be given by

$$\text{(4.12)} \qquad \text{Min } f(\underline{y}^+,\underline{y}^-) = \{ \sum_{i=1}^{m} \alpha_i^+ \cdot \left(\frac{y_i^+}{b_i}\right)^p + \sum_{i=1}^{m} \alpha_i^- \cdot \left(\frac{y_i^-}{b_i}\right)^p \}^{1/p}$$

Obviously both (4.10) and (4.11) or their weighted forms, may serve as the (generally non-linear) dispreference function $f(\underline{y}^+,\underline{y}^-)$ in (4.3).

It is easily seen that both (4.10) and (4.11) include some very popular metrics. That is, for $p = 1$ we get an absolute value metric and for $p = 2$ we get a Euclidean metric, while for $p \to \infty$ a Chebychev metric results which in turn shows close resemblance to the minimax criterion (cf. also Charnes and Cooper [1961, app. A] and Lane [1970]). Of course different metrics will usually generate different results. As was pointed out by Ijiri [1965, p.56], this means, that apart from the determination of the weights α_i, the prescription of the metric constitutes a problem of managerial choice. One may wish to simulate this choice by extrapolating from managerial actions in the past. In that case an *ex post* estimation of the p value in (4.6) or (4.12) must be made. For a further discussion on the sensitivity of the results for the p value used, we refer to Nijkamp and Rietveld [1977].

Under special circumstances, (4.12) can be used in another way. That is, if $p = 1$ it can be used as an index of goal attainment. For the sake of simplicity let us assume that only the y^- variables are opportune, and furthermore that

(4.13) $\sum_{i=1}^{n} \alpha_i^- = 1$, $A\underline{x} \geq \underline{0}$, and $b_i > 0$ for all i

Instead of (4.11) we then get

(4.14) $$f^*(\underline{y}^-) = \sum_{i=1}^{n} \alpha_i^- \cdot \frac{y_i^-}{b_i}$$

where all terms $\frac{y_i}{b_i}$ must lie in the closed interval [0,1]. If the aspiration level b_i is completely reached the term is zero and if the underattainment is at maximum we have $y_i^- \leq b_i$ so that $\frac{y_i^-}{b_i} \leq 1$. Obviously (4.14) is at its maximum value (which is 1 by virtue of the condition that the weights α_i add up to 1) when all aspiration levels are maximally underattained and is at its minimum level (which is 0) when all goals are simultaneously attained. Such an index might be used in the evaluation and comparison of algorithms designed for the multiple goal problem.

4.2. The Objective Function

Goal programming is often used to model some kind of satisficing behaviour. Nevertheless, the class of objective functions which can be incorporated within the goal programming framework, provides the possibility to represent other kinds of behaviour. In this section, we shall give an outline of this class of objective functions for the case that p = 1, i.e. for linear goal programming.

In general, the objective function is a function of the deviations from certain aspired goal levels. Goal programs may include aspired goal levels of one goal variable only or, alternatively, they may include goal levels of several distinct goal variables. Consequently, the objective functions may relate either to one or to several goal variables. These two classes of objective functions will be discussed below.

We will pay special attention to the '*one-sided goal programming*' formulation, in which the objective function includes only one deviation variable for each aspired goal level. It appears that many objective functions can be effectively translated in this 'one-sided' way. One of the advantages is an easier access to the techniques developed for solving the linear goal program and its dual. Furthermore, such a formulation has some advantages within the interactive approach to goal programming which will be described in Chapter 6.

Objective Functions Relating to One Goal Variable Only

If only one goal variable has been defined, the decision maker may have formulated either one or several aspired goal levels. We will first consider the case in which <u>only one aspiration level</u> (say b) has been defined. Then, the objective function becomes

(4.15) $$\text{Min}\{\alpha^+.y^+/b+ \bar{\alpha}.y^-/b\}$$

For the sake of simplicity we assume $b > 0$ to be incorporated in α^+ and α^-, by which we can write

(4.16) $$\text{Min}\{\alpha^+.y^+ + \bar{\alpha}.y^-\}$$

The factors α^+ and α^- are not 'weights' in the usual sense, simply because $y^+.y^- = 0$ by which y^+ and y^- cannot be 'traded-off' against each other. However, both α^+ and α^- can be given values -1,1 or zero in order to indicate whether the deviation in question must be maximized, minimized, or is not relevant for the decision maker's preferences. In Table 4.1 we combine these possibilities (cf. Ijiri [1965]). Furthermore, we show the effect of these functionals being minimized without explicitly taking account of the condition $y^+.y^- = 0$.

Table 4.1. The effect of different functionals $\alpha^+.y^+ + \alpha^-.y^-$ being minimized

		value of α^+		
		-1	0	1
value of α^-	-1	infinite solution	infinite solution	minimum goal variable
	0	infinite solution		minimum over-attainment
	1	maximum goal variable	minimal under-attainment	exact attainment

The infinite solutions occur because both y^+ and y^- can be made arbitrarily large without violating the goal constraint

$$A.\underline{x} - y^+ + y^- = b, \tag{4.17}$$

where A in this case is a (n x 1) vector. The functionals in the lower left and upper right corners of Table 4.1. bring about maximization and minimization of the goal variable concerned. In general, these functionals do not produce unique values for y^+ and y^-.[1] Of course, other methods can be used to minimize or maximize a single goal variable. One possibility is to directly

1) See also Ijiri [1965, p.40 ff.]. If the simplex procedure is used, a unique solution can be guaranteed (see p.83).

minimize or maximize the goal variable without using the deviational variables. Otherwise one can keep in line with the goal programming formulation by defining an aspiration level that is known to be unattainable and then minimizing the relevant deviational variable. In this way the only three functionals of importance are those within the marked lines of Table 4.1. This means that introduction of two deviational variables in the objective function is strictly necessary only in the razor - edge case of exact attainment of the aspiration level. The other finite cases are one-sided goal programming problems.

Even in the case when only one goal variable, say $g(\underline{x})$, has been defined, it may be desirable to use several aspiration levels, say $\underline{b} = b_1, \ldots, b_v$; for this single goal variable. In this case, the most general form for the objective function becomes

$$(4.18) \qquad \text{Min}\{ \sum_{u=1}^{v} (\alpha_u^+ . y_u^+ + \alpha_u^- . y_u^-) \},$$

which may be specified in several ways. First, a goal variable may be a piecewise linear function of the instrumental variables. Such a functional form may be defined in its own right (see Ijiri [1965]) or as an approximation of a continuous function (see e.g. Goodman [1974] and Laurent [1976]). We will deal with the 'kinks' in the piecewise linear function as if they are aspiration levels. In Figure 4.1 we give an example of a goal variable which is a piecewise linear function of one instrumental variable. In this example, the goal variable represents a one-product firm's total revenue, being a function of the firm's volume of sales. Here the β_u, $u = 1, \ldots, v$; are activity levels, for which the marginal contribution of the activity level x to the value of the goal variable g(x) changes. Furthermore, if $\beta_1 < x < \beta_v$ where $\beta_1 = 0$ and c_u is defined as the marginal contribution to the goal variable for activity levels between β_u and β_{u+1}, then according to Ijiri (Ibid) the function g(x) can be written in a linear form

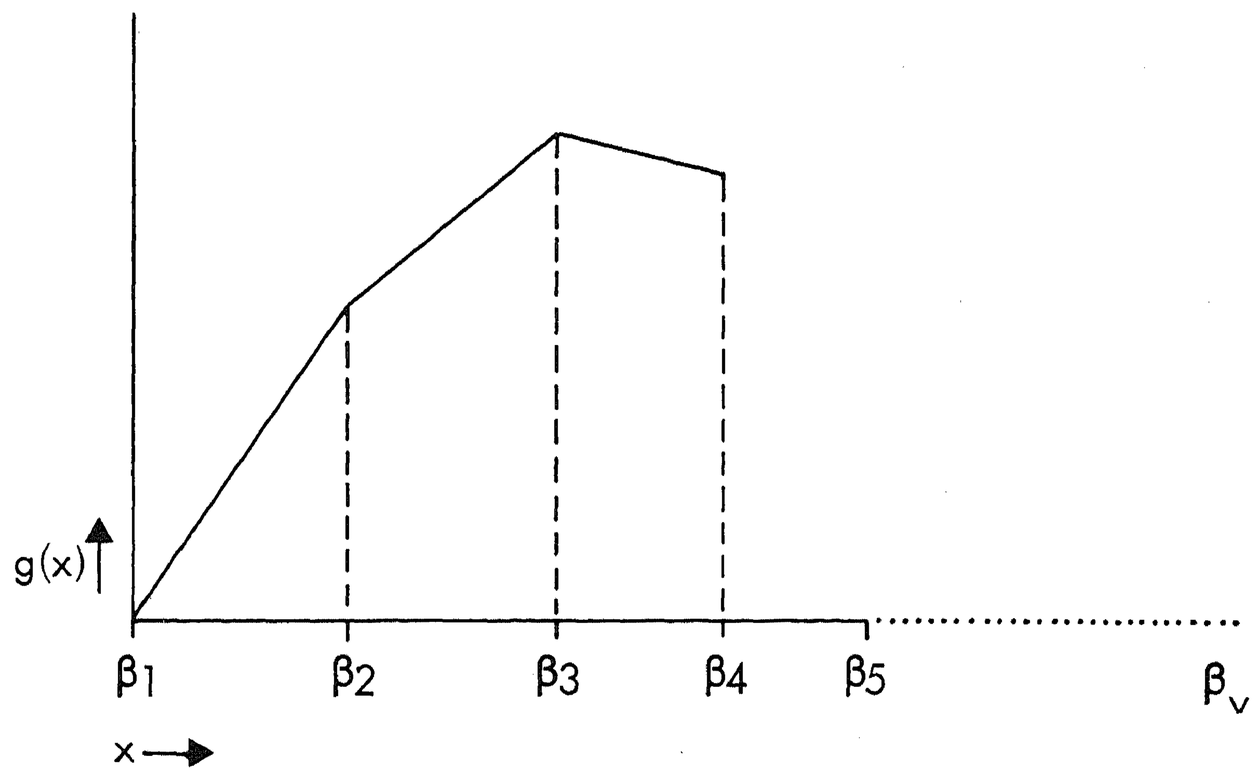

Figure 4.1. A goal variable being a piecewise linear function of one instrumental variable

as

$$g(x) = \sum_{u=1}^{v} (c_u - c_{u-1}) . z_u^+,$$

(4.19) $$x - z_u^+ + z_u^- = \beta_u, \qquad \text{for } u = 1, \ldots, v.$$

$$z_u^+ . z_u^- = 0 \text{ and } x, z_u^+, z_u^- \geq 0$$

Assume next that the decision maker has defined the aspiration level g^* of $g(x)$. Then, introducing two additional variables to measure the deviation from this aspired goal level, a goal restriction can be formulated, as well as an objective function (see Table 4.1) in terms of the previously added deviational variables.
If, in contrast with the situation in Figure 4.1. $g(x)$ is monotone non-decreasing or monotone non-increasing in x, it is not always

necessary to include the two additional deviational variables. Let $g(x)$ be monotone non-decreasing. Its value in the points β_u are then given by

$$g(b_1) = 0 \text{ and}$$

(4.20)

$$g(b_u) = \sum_{w=1}^{u-1} c_w.(\beta_{w+1}-\beta_w) \qquad \text{for } u = 2, \ldots, v.$$

Given these values it is easy to trace the interval $[\beta_k,\beta_{k+1}]$ containing the activity level x^* for which $g(x^*) = g^*$, the aspired level of $g(x)$. This is done by determining the first $k, k = 1, \ldots, v-1$ for which

(4.21) $$g(\beta_k) \leq g^* \leq g(\beta_{k+1})$$

holds. Next define $g_u(x)$, $u = 1, \ldots, v-1$ as the linear functions which for $\beta_u \leq x \leq \beta_{u+1}$ coincide with $g(x)$. Clearly $g_u(x)$ can be defined as:

(4.22) $$g_1(x) = c_1.x \text{ for } \beta_1 \leq x \leq \beta_2, \text{ and}$$

$$g_u(x) = \sum_{w=1}^{u-1} c_w.(\beta_{w+1}-\beta_w) + c_u(x-\beta_u) \qquad u = 2, \ldots, v-1.$$

$$\text{for } \beta_u \leq x \leq \beta_{u+1}$$

We then consider $g_u(x)$, $u = 1, \ldots, k$ (k being the index of the interval containing x^*) as goal variables and $g_u(\beta_{u+1}) = g(\beta_{u+1})$ as

their respective aspiration levels. The goal restrictions become

$$(4.23) \qquad g_u(x) - y_u^+ + y_u^- = g(\beta_{u+1}) \qquad \text{for } u = 1, \ldots, k,$$

which after substitution of (4.20) and (4.22) can be reduced to

$$(4.24) \qquad c_u.x - y_u^+ + y_u^- = c_u.\beta_{u+1} \qquad \text{for } u = 1, \ldots, k.$$

The distance to the aspired level g* can then be minimized by

$$(4.25) \qquad \text{Min } \{ \sum_{u=1}^{k} M_u.y_u \},$$

where $M_u > M_{u+1}$ are factors used to represent the pre-emptive priority of the minimization of y_u over the minimization of y_{u+1}.[1)] Thus this problem is also written in the one-sided way.
In recent years, several authors (see e.g. Collomb [1971], Charnes and Cooper [1977] and Charnes, Cooper, Klingman and Niehaus [1979]) have extended the capacity of goal programming to deal with piecewise linear functions. This has resulted in a new technique known as 'goal interval programming'.

A second reason for more than one aspiration level being defined for the same goal variable occurs when the aspiration level g* of g(x) has been replaced by a range of aspiration levels, which we shall describe by $[g_L^*, g_U^*]$. In this case the decision maker prefers values of g(x) within this interval to values $g(x) < g_L^*$ or values $g(x) > g_U^*$. Examples in practice can easily be found. Enterprises may wish to hold their inventories of goods and cash within certain limits. In financial planning it may be desirable that growth should not be too high or too low. (It will be clear from these examples that the arguments for the determination of the lower bound aspiration level g_L^* may differ from those for the upper bound

1) See also (4.4) and p.84.

aspiration level g_U^*). In this case the goal programming problem is

$$\text{Min } \{f = \alpha_L^- \cdot y_L^- + \alpha_U^+ \cdot y_U^+\}, \text{ s.t.}$$

$$g(x) - y_L^+ + y_L^- = g_L^*,$$

$$(4.26) \qquad g(x) - y_U^+ + y_L^- = g_U^*,$$

$$y_L^+ \cdot y_L^- = y_U^+ \cdot y_U^- = 0$$

$x \in R$, and all variables non-negative.

The weights α_L^- and α_U^+ can be used to represent the relative preference of y_L^- and y_U^+ with respect to other goal variables, if introduced. Note that no trade-off can occur between y_L^- and y_U^+ ($y_U^+ > 0$ implies $y_L^+ > 0$ and $y_L^- = 0$ by $y_L^+ \cdot y_L^- = 0$). Problems may arise when $g(x)$ itself is a piecewise linear function. As shown in Figure 4.2a, there may be two mutual exclusive subsets of the domain (here $[x_1, x_2]$ and $[x_3, x_4]$) for which $g(x) \in [g_L^*, g_U^*]$. This means that the function f to be minimized in (4.26) is not convex in x for all functional forms of $g(x)$. To illustrate this point we have depicted f in (4.26) as a function of the $g(x)$ in Figure 4.2a (whereby $\alpha_L^- = \alpha_U^+ = 2$). Consequently, $f(x)$ can be expressed as a function of x, as shown in Figure 4.2b.

Objective Functions Relating to Several Goal Variables

When the objective function refers to two or more goal variables, the relative and pre-emptive priorities between the goal variables must be elucidated and expressed in mathematical form. To illustrate the flexibility of goal programming with respect to

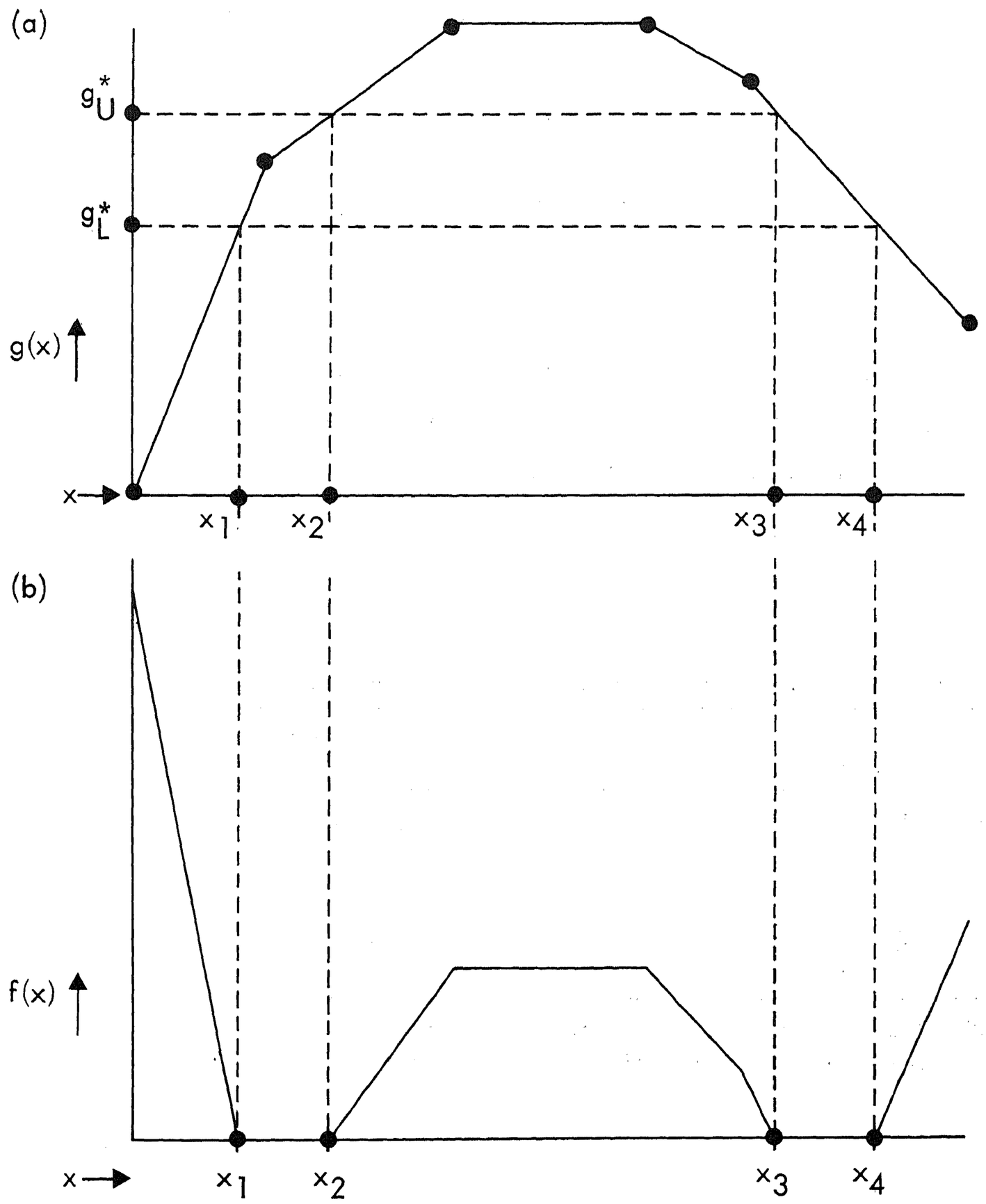

Figure 4.2. Example of a dispreference function which is not convex in x

this kind of objective functions, we give the following example. Assume an enterprise's management has two goal variables, viz. sales represented by $g_S(\underline{x})$ and profit represented by $g_p(\underline{x})$. Next consider the following linear goal programming formulation.

$$
\begin{aligned}
&\text{Min } \{f = 3.y_p^- - 2.y_p^+ + 1.y_S^- - 0.75y_S^+\}, \text{ s.t.} \\
&g_p(\underline{x}) - y_p^+ + y_p^- = g_p^*, \\
&g_S(\underline{x}) - y_S^+ + y_S^- = g_S^*, \\
&\underline{x} \in R
\end{aligned}
\tag{4.27}
$$

By this formulation the feasible region R has been subdivided into four different regions. Each subregion has its own trade-off between profit and sales, as shown in Table 4.2.

Table 4.2. Example of possible trade-offs between profit and sales

	$g_p(\underline{x}) \leq g_p^*$	$g_p(\underline{x}) > g_p^*$
$g_S(\underline{x}) \leq g_S^*$	3/1 = 3	2/1 = 2
$g_S(\underline{x}) > g_S^*$	3/.75 = 4	2/.75 = 2.66

Remark: the trade-offs are given in terms of goal variables and not in terms of the deviations.

Of course, the trade-off between two goal variables may change at more goal levels than in the above example. This situation can be handled by translating each of these 'critical' goal levels as a

goal constraint (which has been defined in Section 3.1 as a constraint including deviational variables).

Furthermore, note that the objective function f can be written as the sum of 2 piecewise linear functions f_i, one for each of the two goal variables.

$$
\begin{aligned}
& f = f_1 + f_2, \text{ with} \\
(4.28) \qquad & f_1 = 3.y_p^- - 2.y_p^+, \\
& f_2 = 1.y_S^- - 0.75y_S^+
\end{aligned}
$$

Because f_i is a piecewise linear function of the goal variable g_i, which itself is a linear or piecewise linear function of the instrumental variables $\underline{x}$, f_i also becomes a piecewise linear function in $\underline{x}$, which can be handled by the methods described earlier in this section (see (4.18) - (4.25)). This not only holds for the example, but also in the more general case in which there are several goal variables and a more complex trade-off pattern.

In general, if all f_i are monotone non-increasing[1)] (or all are monotone non-decreasing) in $\underline{x}$ and thus f is of the same type, the problem can be formulated as a one-sided goal programming problem. Consider $g_i(\underline{x})$ with 'kinks' in its dispreference function f_i for $g_i(\underline{x}) = g^*_{ij}$, $j = 1,2,\ldots, m-1$ and let $g^*_{im} = \max\{g_i(\underline{x}) \mid \underline{x} \in R\}$. The problem can then be written as

1) Note that functions that do not meet this condition (as f in (4.26)) can sometimes be written as the sum of two linear or piecewise linear functions.

$$\text{Min } \{ \sum_{j=1}^{m} \alpha_{ij}^{-} . y_{ij}^{-} \}, \text{ s.t.}$$

$$(4.29) \qquad g_i(\underline{x}) - y_{ij}^{+} + y_{ij}^{-} = g_{ij}^{*} \qquad \text{for } j = 1, \ldots, m;$$

$$\text{and } \underline{x} \in R$$

In this formulation the weights α_{ij}^{-} do not represent the slope of the dispreference function f_i because $y_{ij}^{-} > 0$ implies $y_{ik}^{-} > 0$ for $k = j+1, \ldots, m$. This means that the dispreference for some point $g^* \in [g_{ij-1}, g_{ij}]$ is given by $\sum_{k=j}^{m} \alpha_{ik}^{-} . y_{ik}^{-}$.

It has been shown in this section (for $p = 1$) that different kinds of objective functions can be incorporated within goal programming. Furthermore, it appeared that many linear goal problems can be written in the 'one-sided' way. Such a formulation gives an easier access to the interactive approach to goal programming which is described in Chapter 6.

We next turn to a number of solution procedures for goal programming.

4.3. Solution Procedures

In minimizing the distance function (4.11), good approximations of the minimum function value may be obtained by means of mathematical programming techniques, as shown for example by Barrodale and Roberts [1970]. For certain values of the parameter p in the distance function, even an exact solution can be obtained. In this section we will discuss the cases in which $p \to \infty$, $p = 2$ and $p = 1$ respectively. In the next section the latter case will be elaborated further.

The Case in which $p \to \infty$

If p should approach infinity, it can be shown (cf. Beckenbach and Bellman [1961]) that (4.11) is equivalent to

$$\text{(4.30)} \qquad \underset{i}{\text{Min}}\ \underset{i}{\text{Max}}\ \{\alpha_i^+.(y_i^+/b_i),\alpha_i^-.(y_i^-/b_i)\}.$$

According to Charnes and Cooper [1977], Lane [1970] and Zukhovitsky and Avdeyeva [1966] this can be put into a linear programming format as

$$\text{(4.31)} \qquad \begin{array}{ll} \text{Min } \{\lambda\}, \text{ s.t.} & \\ (\alpha_i^+/b_i).(-b_i+\underline{a(i)}.\underline{x}) \leq \lambda & \text{for } i = 1, \ldots, m; \\ (\alpha_i^-/b_i).(b_i-\underline{a(i)}.\underline{x}) \leq \lambda & \text{for } i = 1, \ldots, m; \\ \underline{x} \in R, & \end{array}$$

where $\underline{a(i)}$ is the i'th row of matrix A. This in fact minimaxing metric has been employed a.o. in the STEP-method of Benayoun et al. [1974] and in an interactive approach by Nijkamp and Rietveld [1976].

The Case in which $p = 2$

In discussing this case, we at first do not discriminate between positive and negative deviations from the goal levels[1)], so

1) With the help of (4.7a) and (4.7b) it is straightforward to discriminate between positive and negative deviations from the goal levels

that by refering to (4.3) and (4.6) the problem can be stated as

$$(4.32) \qquad \text{Min } Q = \left\{ \sum_{i=1}^{m} \left| \frac{\underline{a}(i).\underline{x}-b_i}{b_i} \right|^2 \right\}^{\frac{1}{2}}$$

$$\text{s.t. } \underline{x} \in R,$$

which closely resembles a quadratic programming format[1]. In fact, a unique solution to (4.32) can be found by solving

$$(4.33) \qquad \text{Min } W = \underline{x}'.B'.B.\underline{x} - 2b'.B.\underline{x}$$

$$\text{s.t. } \underline{x} \in R$$

where B is an (m x n) matrix which results when A is premultiplied by the (m x m) diagonal matrix whose elements are equal to $1/b_i$ for i = 1, ..., m, provided B'B and therefore A'A are positive definite. In order to meet this condition, A must have full column rank, which is of course not always the case. For instance the number of instruments n may exceed the number of goals m. In that case other methods can be used to solve the problem. For instance, Charnes and Cooper [1961] show that a positive semidefinite matrix A'.A can be reduced to a positive definite one by perturbing it to A'.A + ε.I for any $\varepsilon > 0$. Ijiri [1970] suggested another solution procedure which will next be described.

Ijiri promotes the idea of using the <u>generalized inverse</u> of a matrix for solving the multiple goal problem (4.3), no matter whether m < n, m = n or m > n. Here we summarize his results in

1) See again (4.7a) and (4.7b).

broad lines only. Every matrix A - not necessarily being a non-singular, square or non-zero matrix - has its own unique generalized inverse, A^+ , which is defined as

$$A^+ = (A'.A)^{-1}.A', \tag{4.32}$$

where A' is the transpose of A. Important features of the generalized inverse are 'its least squares property' (see below) and the fact that the generalized inverse of a non-singular square matrix is identical to its ordinary inverse.

The use of the generalized inverse can be shown when we take from (4.2) the set of goals

$$A.\underline{x} = \underline{b} \tag{4.33}$$

Let us assume the existence of at least one $\underline{x}$ satisfying all the goals b_i, i = 1, ..., m. Then the complete set of solutions is given by

$$\underline{x} = A^+.\underline{b} + (I-A^+.A).\underline{z} \tag{4.34}$$

where I is the (nxn) identity matrix and $\underline{z}$ is an arbitrary (nx1) vector. The matrix $(I-A^+A)$ maps $\underline{z}$ into the null space of A. If this null space contains only $\underline{0}$, $\underline{x}$ is unique. Otherwise, there is a set of solutions (given by (4.34)) all satisfying (4.33) which contains more than one element (in general it has an infinite number of elements because $\underline{z}$ may be freely chosen). Within this set of solutions, goals not included in $\underline{b}$ may be strived for.

As stated in (4.2) the choice of the instrumental variables $\underline{x}$ may be subject to constraints. These can be translated into constraints on the choice of the vectors $\underline{z}$. For instance, we can substitute (4.34) in the set of restrictions

(4.35) $$B.\underline{x} \leq \underline{h},$$

which results in

(4.36) $$B.(I-A^{+}.A).\underline{z} \leq \underline{h} - B.A.\underline{b},$$

by which $\underline{x}$ is indirectly restrained to (4.35).

Before formulating (4.34) we assumed the existence of at least one $\underline{x}$ satisfying all goals b_i. In the absence of such a solution it can be shown (Ibid, p.59 and p.161) that the least squares property of A^{+} is equivalent to the fact that $\underline{x} = A^{+}.b$ minimizes the ℓ_2 metric

(4.37) $$\left\{ \sum_{i=1}^{m} (\underline{a(i)x} - b_i)^2 \right\}^{\frac{1}{2}},$$

where $\underline{a(i)}$ is the i'th row of A. Of course this metric can be modified in order to incorporate the subjective weights α_i, $i = 1, \ldots, m$ to be assigned to the deviations. This yields

(4.38) $$\left\{ \sum_{i=1}^{m} \alpha_i.(\underline{a(i)x}-b_i)^2 \right\}^{\frac{1}{2}},$$

which is minimized by $\underline{x} = (I_\alpha.A)^{+}.\ I_\alpha.\underline{b}$, where I_α is the diagonal matrix of order m with diagonal elements $\sqrt{\alpha_i}$.

As was indicated in Section 4.1, goals are not always commensurable. The suggested approach there was to divide the set of m goals into the pre-emptive priority classes C_t, $t = 1, \ldots, p$. Then the goals of class C_t had to be fulfilled before those of $C_{t'}$ if and only if $t < t'$. The problem of incommensurability is thus restrained to the first priority class in which not all goals can be satisfied simultaneously. The same approach is applicable when

generalized inverses are used (Ibid.,pp.57-62). In that case (4.33) has to be partitioned into

$$(4.39) \qquad A_t.\underline{x} = \underline{b}_t \quad \text{for } t = 1, \ldots, p;$$

where A_t is the matrix of the coefficients of the goal variables that are in the t'th priority class C_t and $\underline{b}_t$ is the vector of aspiration levels for these goal variables. The set of solutions (if not empty) which satisfy the goals of the first priority class can now be found by

$$(4.40) \qquad \underline{x} = A_1^+.\underline{b}_1 + (I-A_1^+.\underline{a(1)}).\underline{z}.$$

Within this set of solutions - denoted by R_1 - the goals of the second priority class must be met. The problem is then to find the solutions of

$$(4.41) \qquad A_2.\underline{x} = \underline{b}_2 \quad \text{s.t.}$$

$$\underline{x} \in R_1$$

If the solution set R_2 of (4.41) is not empty one can proceed by determining the set of solutions that satisfy the goals of the third priority class. This procedure continues until a priority class C_k is found of which not all goals can be reached simultaneously. Accordingly we have to find the solution in R_{k-1} which has minimum distance to the goals of class C_k, being $b_{k1}, \ldots, b_{km_k}$. If this distance is measured by the ℓ_2 metric, then by virtue of (4.38) and (4.41) the problem is to find the solution of

$$(4.42) \qquad I_{\alpha m_k}.A_k.\underline{x} = I_{\alpha m_k}.\underline{b}_k,$$

$$\underline{x} \in R_{k-1},$$

where $I_{\alpha m_k}$ is a diagonal matrix of order $(m_k x m_k)$, with diagonal elements $\sqrt{\alpha_{ks}}$, $s = 1, \ldots, m_k$ - being the square roots of the weights assigned to the m_k goals of the k'th priority class C_i. In practice a unique least squares solution to (4.42) can be found by the procedure described in (4.33) through (4.36). However, one serious drawback of this procedure should be stressed. That is, first the set of unconstrained least squares solutions is calculated. Then this set is intersected with the set of solutions being feasible (in this case R_{k-1}) in order to form the final solution set (here R_k). Thus, it is tacitly assumed there <u>is</u> a non-empty intersection. However, this assumption may not be satisfied. In that case the problem remains to find a least squares solution <u>within</u> the region R_{k-1}.

To summarize, the solution of the stated problem by means of generalized inverses is not a simple task. It becomes even worse when the instrumental variables $\underline{x}$ are subject to exogenous constraints as stated in (4.35). Furthermore, under- and overachievement of the stated goals b_i, $i = 1, \ldots, m$; were assumed to be equally undesirable. If this assumption is relaxed so as to include goals where under- and overachievement are being valued differently, the approach using generalized inverses becomes unclear. Fortunately, much research has been recently carried out and will surely be carried out in the near future, not at least because of its applicability in constrained regression (see e.g. Ben-Israel and Greville [1974] and Theil [1971]). Therefore, efficient procedures for this kind of multiple goal problems may be developed soon.

The Case in which p = 1

Next, we will discuss a number of solution procedures for the

ℓ_1 specification of the objective function. Moreover we conjecture that m goals are to be ordered in $p(1 \leq p \leq m)$ pre-emptive priority classes. Charnes and Cooper [1961] and Ijiri [1965] paid a lot of attention to the ℓ_1 specification of the multiple goal problem; in fact they developed the linear programming formulation of this problem, which is now known as linear goal programming. By reference to (4.3) and (4.11), their formulation can be written in the standardized form by

$$\text{(4.43)} \qquad \text{Min } \{ \sum_{i=1}^{m} \alpha_i^+ . y_i^+ / b_i + \sum_{i=1}^{m} \alpha_i^- . y_i^- / b_i \}$$

$$\text{s.t. } A.\underline{x} - I.\underline{y}^+ + I.\underline{y}^- = \underline{b}$$

$$\text{and } \underline{x} \in R,$$

where I is the m x m identity matrix and all other symbols have been defined as before. In this linear programming format there is of course no place for the non-linear constraints $y_i^+ . y_i^- = 0$, $(i=1,\ldots,m)$. However, the latter are automatically fulfilled when the simplex method of linear programming is used. Then the y_i^+ and y_i^- vectors are linearly dependent, and hence they cannot appear together in the same basic solution (cf. Charnes et al. [1955]). Strictly speaking, the non-linear conditions need only be met at the optimum (cf. Charnes and Cooper [1977]). If the weights α_i^+ and α_i^- are non-negative these conditions are fulfilled as a consequence of the minimization process. This means that in this case methods other than the simplex routine can also be used. For the sake of illustration consider

$$\text{(4.44)} \qquad \text{Min}\{\alpha^+ . y^+ + \alpha^- . y^-\}, \text{ with } \alpha^+, \alpha^- \geq 0,$$

$$\text{s.t: } \underline{a}' . \underline{x} - y^+ + y^- = b$$

$$\text{and } \underline{x} \in R$$

where $\underline{a}$ is a (nx1) vector and b is a scalar. Assume a solution exists with $y^+ = t > 0$, $y^- = u > 0$ and $t > u$. This solution is not optimal because the value of the objective function declines when both y^+ and y^- are reduced by the same amount. Thus define $y^+ = t - \delta$ and $y^- = u - \delta$, which gives

$$
\begin{aligned}
f &= \alpha^+.y^+ + \alpha^-.y^- = \\
&= \alpha^+.(t-\delta) + \alpha^-.(u-\delta) = \qquad (4.45)\\
&= (\alpha^+.t+\alpha^-.u) - (\alpha^+ + \alpha^-).\delta,
\end{aligned}
$$

where the first term represents the value of the first solution and the second term its possible reduction. This reduction is an increasing function of δ, which , however, is restricted by the non-negativity restrictions on y^+ and y^-. The latter can be stated as $t-\delta \geq 0$ and $u-\delta \geq 0$. Since we assumed $t > u$, the maximal value of δ (and consequently the minimal value of f) is reached for $\delta = u$. Finally, the goal restriction in (4.44) remains unaffected because both y^+ and y^- are reduced by the same amount.

In order to handle the <u>pre-emptive priorities</u> discussed in Section 4.1, the deviations must be weighted by the non-Archimedean values M_t. For instance, if goal t has pre-emptive priority above goal t', the weights M_t and $M_{t'}$ are chosen such that

$$M_t >> M_{t'}, \qquad (4.46)$$

which means that no real number γ - however large - can give

$$\gamma M_{t'} = M_t \qquad (4.47)$$

By this procedure a lexicographic ordering is obtained because no trade-offs between goals of different priority classes are possible. Within standard linear programming these preemptive priorities raise special problems to which we shall return below.

Another possibility is to give one goal absolute priority above another. This can be done by constraints on the deviational variables. For instance

$$y_i^+ + y_j^+ \leq \delta \tag{4.48}$$

to insure that the sum of two positive deviations under no circumstances exceeds the given value δ. These restrictions may only be used if they do not violate the non-linear constraints $y_i^+ . y_i^- = 0$.

In principle, the goal problem (4.43) can be solved by the usual linear programming techniques. However, introduction of the non-Archimedean factors M_t calls for a different approach. This stems from the fact there is no foolproof method which translates the non-Archimedean factors M_t *a priori* in real-valued weights μ_t, which can then be used within a linear programming format. One might consider replacing the M_t by real numbers μ_t having the property that μ_1, belonging to the most important priority class, is 'much bigger' than μ_2, the next important priority class and so on, where 'much bigger' is defined in an arbitrary way. Such procedures are of course *ad hoc* and lack proof of correctness.

Kornbluth [1973] and Salkin and Kornbluth [1973] use a straightforward procedure to handle pre-emptive priorities. Analogous to the approach using generalized inverses ((4.40) - (4.42)), they first determine the part of the feasible region, R_1, in which the goals of the first priority class are met. Given R_1, they try to fulfil the goals of the second priority class, and so.

By reformulating (4.43) including pre-emptive priorities, we get

$$\text{Min } D = \sum_{t=1}^{p} M_t \left(\sum_{s=1}^{m_t} \alpha_{ts}^{+} . y_{ts}^{+} / b_{ts} + \alpha_{ts}^{-} . y_{ts}^{-} / b_{ts} \right),$$

$$\text{(4.49)} \qquad \text{s.t. } A.\underline{x} - \underline{y}^{+} + \underline{y}^{-} = \underline{b},$$

$$\text{and } \underline{x} \in R,$$

where $M_1 >> M_2 >> \ldots >> M_p$ and $\sum_{t=1}^{p} m_t = m$, while the other elements are defined as before. This problem is solved through the following sequential optimization process (Salkin and Kornbluth [1973]):

(i) Set $t = 1$;

(ii) Minimize $D_t = \sum_{s=1}^{m_t} \alpha_{ts}^{+} . y_{ts}^{+} / b_{ts} + \alpha_{ts}^{-} . y_{ts}^{-} / b_{ts}$;

subject to the constraints in (4.49). Let the minimum be D_t^*;

(iii) If the solution is unique (non-degenerate), stop; otherwise

(iv) Flag out variables which are not part of the alternative bases of (ii) and add the constraint

$$D_t^* = \sum_{s=1}^{m_t} \alpha_{ts}^{+} . y_{ts}^{+} / b_{ts} + \alpha_{ts}^{-} . y_{ts}^{-} / b_{ts};$$

(v) If all goals have been considered, stop; otherwise put $t = t + 1$ and go to (ii).

It will be clear that this procedure guarantees an optimal solution within a number of $p' \leq p$ runs of the standard linear programming problem (ii). Nevertheless it is possible to solve the problem in one run by using an adapted version of the simplex procedure, as proposed by Mao [1969] and Lee [1972]. This will be dealt with in the next section.

4.4. An Adapted Simplex Procedure

In this section we describe an adapted simplex procedure presented by Lee [1972]. The notation used may be clarified with the help of the initial standard simplex tableau in Table 4.3.

Table 4.3. The initial standard simplex tableau

c_j			c_1 . . . c_n		
	V	B	x_1 . . . x_n	s_n . . . s_m	
	s_1	b_1	a_{11} . . . a_{1n}	I_{mxm} =	
	.	.	. .		
	.	.	. .	Identity Matrix	
	.	.	. .		
	s_m	b_m	a_{m1} . . . a_{mn}		
$z_j - c_j$		0	$-c_1$ $-c_n$		

In this tableau the instrumental variables are $x_1 \ldots x_n$ and the slack variables $s_1 \ldots s_m$. The column headed V represents the vector of variables chosen in the basis. In this initial tableau the basis consists of the slack variables only. The values of the basic variables are given in the column headed B. The top row shows the variables' contribution to the value of the objective function. These values are duplicated in the lefthand column for the variables that are chosen into the basis. The bottom row gives the value of the objective function for the basis at hand. It also gives the negatives of the net changes in the value of the objective function (z_j-c_j) induced by the introduction of the respective non-basic variables into the basis (where the changes are measured per unit of the entering variable).

Procedure

The following elaboration of the adapted simplex method departs from the general formulation (4.49). For the sake of simplicity, we assume all constraints to have been formulated as goal constraints. The initial tableau for this minimization problem contains no slack variables since their role can be played by the deviational variables y_{ts}^-. Because we now have to deal with p preemptive priority classes, one single $z_j - c_j$ row is not enough. Instead, p rows are introduced (one for each priority class) which are labeled $(z_j-c_j)^1$ up to and including $(z_j-c_j)^p$. The format of the initial tableau is shown in Table 4.4. The value of the objective function to be minimized is equal to $\sum_{t=1}^{p} z_t$, where z_t has been defined as the weighted sum of the deviations from the goals is in the tth pre-emptive priority class, so that in the initial solution

$$z_t = \sum_{s=1}^{m_t} M_t.\alpha_{ts}.y_{ts}^- \tag{4.50}$$

In the adapted simplex tableau all z_j's are shown separately. Accordingly the (z_j-c_j) row has been replaced by a matrix, consisting of the rows $(z_j-c_j)^p$ through $(z_j-c_j)^1$. The tth row of this matrix, $(z_j-c_j)^t$, gives the net influences on the value of z_t by entering the respective non-basic variables into the basis. The net influence on the objective function as a whole by introducing some non-basic variable is then found by summing up the elements of its column in the (Z-C) matrix.

With the construction of the initial tableau we have established the first step of the adapted simplex algorithm. The <u>second step</u> consists of the identification of the new entering variable. To that end we find the highest pre-emptive priority class that has not been completely attained. This is the first class k whose corresponding

Table 4.4. The initial adapted simplex tableau

c_j				$M_1\alpha^-_{11} \cdots M_1\alpha^-_{m_1 1} \cdots\cdots M_p\alpha^-_{1p} \cdots M_p\alpha^-_{m_p p}$	$M_1\alpha^+_{11} \cdots M_1\alpha^+_{m_1 1} \cdots\cdots M_p\alpha^+_{1p} \cdots M_p\alpha^+_{m_p p}$
	V	B	$x_1 \cdots x_n$	$y^-_{11} \cdots y^-_{m_1 1} \cdots\cdots y^-_{1p} \cdots y^-_{m_p p}$	$y^+_{11} \cdots y^+_{m_1 1} \cdots\cdots y^+_{1p} \cdots y^+_{m_p p}$
$M_1 \cdot \alpha^-_{11}$	y^-_{11}	b_1	$a_{11} \cdots a_{1n}$		
.	.	.	. .		
.	.	.	. .		
.	.	.	. .		
$M_1 \cdot \alpha^-_{m_1 1}$	$y^-_{m_1 1}$	.	. .	In the initial tableau this is the (m x m) identity matrix	In the initial tableau this is the (m x m) identity matrix
.	.	.	. .		
.	.	.	. .		
.	.	.	. .		
.	.	.	. .		
.	.	.	. .		
$M_p \cdot \alpha^-_{1p}$	y^-_{1p}	.	. .		
.	.	.	. .		
.	.	.	. .		
.	.	.	. .		
$M_p \alpha^-_{m_p p}$	$y^-_{m_p p}$	b_m	$a_{m1} \cdots a_{mn}$		
$(z_j - c_j)^p$		z_p	In the initial tableau $(z_1 - c_1) = M_1 (\sum_{u=1}^{m_1} \alpha_{u1} \cdot a_{u1})$	In the initial tableau this is the (p x m) null matrix	In the initial tableau the southwest-northeast diagonal of this matrix consists of the elements $M_1(\alpha^-_{11} - \alpha^+_{11})$ through $M_p(\alpha^-_{m_p p} - \alpha^+_{m_p p})$. All other elements are zero.
.		.			
.		.			
$(z_j - c_j)^1$		z_1			

z_k value is positive. We then proceed to identifying the variable with the largest positive $(z_j-c_j)^k$ value. This will enter the basis in the next iteration. If two non-basic variables both have the maximum $(z_j-c_j)^k$ value, the tie may be broken by inspecting their respective $(z_j-c_j)^{k+1}$ values. The variable with the greater $(z_j-c_j)^{k+1}$ is then chosen.

In the <u>third step</u> the pivot row is found. This is the row which has minimum positive or zero value of the right hand constant divided by the coefficients in the pivot column. The associated variable leaves the basis in the next iteration. If there are two variables with minimum value remove the variable with the higher priority factor.

The new basic feasible solution is determined in the <u>fourth step</u>. First, the pivot row is divided by the pivot element. Secondly the new basic variable is eliminated from all other rows in the tableau.

The <u>fifth step</u> treats the optimality check. If $z_t = 0$ for $t = 1, \ldots, p$; all goals have been attained. If there exists a positive value of z_t, say z_k, examine the positive elements of $(z_j-c_j)^k$. In the case that such a positive element does not exist without its column containing a negative element in the higher priority rows $(z_j-c_j)^t$, $t = 1, \ldots, k-1$, the optimal solution has been reached. In other cases return to the second step.

The adapted simplex procedure is not very different from the 'standard' simplex procedure. The main difference is the pivot selection, which in the adapted procedure has to deal with non-Archimedean pivot elements. For a more detailed discussion of the adapted simplex procedure we refer to Lee [1972].

Sensitivity Analysis and Duality

The adapted simplex procedure has been developed in order to handle the pre-emptive priority factors M_t within a single simplex program. Both sensitivity analysis and duality are influenced by the character of the priority structure. Below, we shall discuss the two major differences as compared with 'standard' procedures (cf. also Ignizio [1976]). First, old questions receive new answers. Notably, changes in weighting factors, right-hand side constants and technological coefficients may affect more than one priority level of the final solution. Consequently, these influences must be shown and analyzed separately. Secondly, the nature of the priority structure raises new questions. For instance, the influence of a reordening of priority levels or of the addition of a new goal may be investigated.

Within an unchanged pre-emptive priority structure we can study the influence of changes in the weighting factors α_{ts}. When the variable weighted by α_{ts} is non-basic in the final solution, a change in α_{ts} only results in a direct change of its corresponding element in the t'th row of the (Z-C) matrix. Whether the new tableau remains optimal is checked by the fifth step of the adapted simplex procedure. A change in the weighting factor α_{ts} of a basic variable can affect all elements of the t'th row of the (Z-C) matrix as well as the achievement value z_t of the t'th priority level. Again a new optimality check is necessary.

Changes in the right-hand side constants vector $\underline{b}$ result in a modified column vector $\underline{b}$ in the final tableau and in changes of the achievement levels z_t, $t = 1, \ldots, p$. If the new solution remains feasible, it too is optimal because the elements of the (Z-C) matrix are unaffected by changes in $\underline{b}$. An infeasible solution occurs if

at least one of the elements of the vector $\underline{b}$ in the final solution is negative. In that case the optimal solution can be found by using the 'dual adapted simplex method' developed by Ignizio (Ibid).

A modification of the technological coefficients α_{ts} yields no problems as long as non-basic variables are concerned. An initial change in the column vector of technological coefficients of the non-basic variable x_j affects no other elements of the final tableau but the elements of x_j itself and its accompanying column in the (Z-C) matrix (the latter being the multidimensional opportunity loss associated with the introduction of x_j). Again an optimality check becomes necessary. The introduction of a new activity is easily handled by considering it as a set of changes in an old activity of which all technological coefficients are equal to zero. The analysis of changes in technological coefficients of basic variables is not straightforward and involves a simultaneous consideration of both the primal and the dual problem.

The addition of a new goal involves the introduction in the final tableau of both a goal restriction and at least one weighting factor (associated with the positive or negative deviational variable of this new goal). After eliminating the basic variables from the new row, the achievement levels Z_t, $t = 1, \ldots, p$; and the matrix (Z-C) can be recalculated. Again, an optimality check is necessary. Furthermore, the right-hand side constant of the new goal constraint may have become negative. This can be handled by the dual adapted simplex method mentioned above. To study the effect of reordening pre-emptive priority levels the same method is available. This implies a recalculation of the achievement levels Z_j and the (Z-C) matrix, followed by the optimality check.

Effects of Changing the Goal Levels

We next consider changes in the goal levels b_i for the case that all goals have the same pre-emptive priority. Consider the primal problem written as

$$\text{Max}\{-(\underline{\alpha}^+)'.\underline{y}^+-(\underline{\alpha}^-)'.\underline{y}^-\}, \text{ s.t.}$$

$$A.\underline{x} - \underline{y}^+ + \underline{y}^- = \underline{b}$$

(4.51)

$$B.\underline{x} \leq \underline{h}, \text{ and}$$

$$\underline{x}, \underline{y}^+, \underline{y}^- \geq \underline{0}, \text{ where}$$

$\underline{\alpha}^+$ and $\underline{\alpha}^-$ are m dimensional vectors of weighting factors. The dual problem can then be formulated as

$$\text{Min } \{\underline{b}'.\underline{u} + \underline{h}'.\underline{v}\}$$

$$A'.\underline{u} + B'.\underline{v} \geq \underline{0}$$

(4.52)

$$\underline{\alpha}^+ \geq \underline{u} \geq -\underline{\alpha}^-, \text{ and}$$

$$\underline{v} \geq \underline{0}, \underline{u} \text{ unrestricted in sign}$$

The Kuhn-Tucker conditions connecting problems (4.51) and (4.52) are given by

$$\underline{u}'.(\underline{y}^+-\underline{y}^-) = 0, \text{ and}$$

(4.53)

$$\underline{v}'.(\underline{h}-B.\underline{x}) = 0$$

By defining $\underline{u} = \underline{u}^- - \underline{u}^+$, where $\underline{u}^-, \underline{u}^+ \geq 0$, we can reformulate (4.52) as

$$\text{Min}\{b'.(-\underline{u}^+ + \underline{u}^-) + h'.\underline{v}\}, \text{ s.t.}$$

$$A'.(-\underline{u}^+ + \underline{u}^-) + B'.\underline{v} \geq 0,$$

$$(4.53) \qquad \underline{u}^- \leq \underline{\alpha}^+,$$

$$\underline{u}^+ \leq \underline{\alpha}^-, \text{ and}$$

$$\underline{u}^+, \underline{u}^-, v \geq 0$$

The variables u_i^+, u_i^- and $(u_i^+ - u_i^-)$ can be interpreted as the decrease in the value of the objective function when the right-hand side constant b_i is augmented by one unit (provided the solution remains feasible and optimal, which we shall assume hereafter). Such a shift in b_i causes one out of two possible effects. Either y_i^+ grows by one unit or y_i^- declines by one unit (a positive shift of b_i causes the opposite effects). This is shown in Figure 4.3 where the effects on y_i^+ and y_i^- are shown for a negative shift in the aspiration level b_i of a given goal variable $g_i(\underline{x})$ and a given solution $g_i(\underline{x}) = S$.
The increase of the objective function caused by a negative shift of b_i is given by $u_i = -u_i^+ + u_i^-$. Obviously $-u_i^+$ represents the effect of a decrease in y_i^-, whereas u_i^- represents the effect of a possible increase of y_i^-. The effect on the objective function of a unit increase of y_i^+ and of a unit decrease of y_i^- must be equal to the weights of these variables in the objective function of the primal problem. These effects are thus α_i^+ and $-\alpha_i^-$ respectively. Because these effects are not exactly the same when b_i is shifted upwards, we list all possibilities in (4.54).

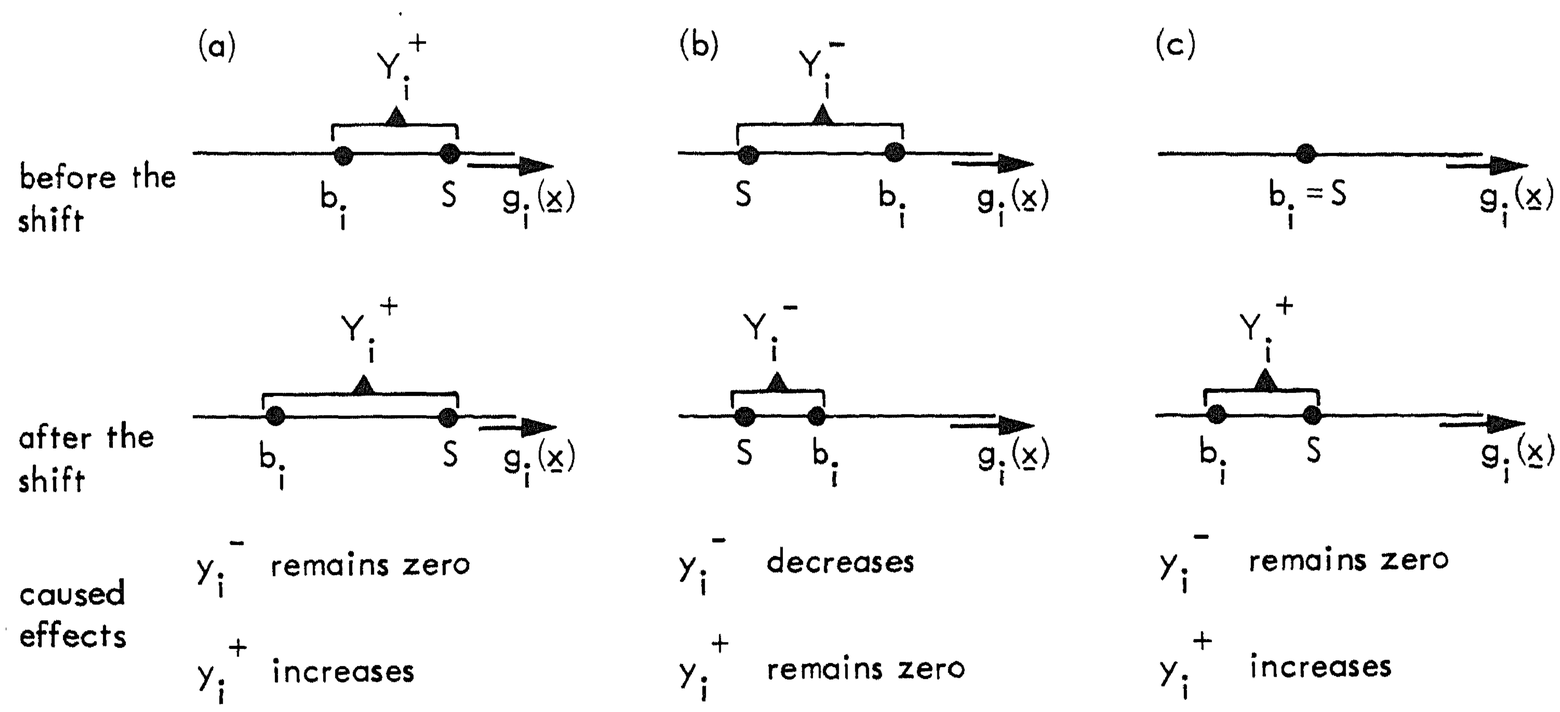

Figure 4.3. The effects of a negative shift of the aspiration level b_i on the deviational variables y_i^+ and y_i^-

$$
\begin{aligned}
u_i^+ &= \alpha_i^- \text{ for } y_i^+ = 0 \text{ and } y_i^- > 0 \\
&= \alpha_i^- \text{ for } y_i^+ = y^- = 0 \text{ and when } b_i \text{ is raised} \\
&= 0 \text{ in all other cases} \qquad (4.54) \\
u_i^- &= \alpha_i^+ \text{ for } y_i^- = 0 \text{ and } y_i^+ > 0 \\
&= \alpha_i^+ \text{ for } y_i^+ = y_i^- = 0 \text{ and when } b_i \text{ is lowered.} \\
&= 0 \text{ in all other cases.}
\end{aligned}
$$

4.5. Concluding Remarks

Goal programming has been extended in various ways. From several possible extensions we only discussed the inclusion of piecewise linear functions in the goal programming model. Kornbluth [1973] paid special attention to ratio form goals as 'return on assets', 'profit to sales', etc. (see also Chapter 8). In some cases these

goals may induce computational obstacles, which according to Charnes and Cooper [1977] can be circumvented if the Chebychev metric (see p.64) is used. Ignizio [1976] proposes a method to include integer conditions within the goal programming framework. Discussions about the extension towards stochastic goal programming can be found in Charnes and Cooper [1963] and Contini [1968].

In addition to the use of goal programming for multiple criteria decision making, this technique is suitable for other areas of application. According to Charnes and Cooper [1977] goal programming was developed (in 1952) to obtain constrained regression estimates for an executive compensation formula which would conform to management policies (cf. Charnes et al. [1955]). The use of goal programming in econometrics has been studied further (see Charnes and Cooper [1975]) and has been advocated for other application areas. For instance, Spivey and Tamura [1970] present goal programming as an alternative to simultaneous equation models for use in macro economic decision making. For them, important advantages of goal programming are its access to sensitivity analysis, its flexibility in formulating the preference function, and its ability to deal with political and institutional variables explicitly.

An important drawback of goal programming is its need for fairly detailed *a priori* information on the decision maker's preferences. Goal programming requires definition of aspiration levels, partitioning of the set of goals into pre-emptive priority classes and assessment of weights within these classes. A problem not to be overlooked is the critical role of the aspiration levels and the pre-emptive priorities defined by the decision maker. Once a goal obtains pre-emptive priority above another, no trade-off between these goals is admitted anymore. In other words first order priority goals determine the solution space for goals of less important priority classes. This means that the definition of the goal levels must be completely reliable or that some kind of bargaining about

these levels must remain possible.

Clearly, the elucidation of the decision maker's preferences in this 'direct' way is not a sinecure. We do agree with the scholars advocating interactive methods, which neither require an explicit representation or specification of the decision maker's preference function nor an explicit quantitative representation of trade-offs among conflicting goal variables.

Unfortunately, most of the usual interactive approaches lack some of the advantages of 'traditional' goal programming, such as for instance the possibility to include pre-emptive priorities. Furthermore, in contrast with most existing interactive methods, goal programming is particularly suitable in situations of satisficing behaviour. Together, these advantages of goal programming justify the effort to seek an interactive variant of the traditional approach. One such effort is described in the following chapters.

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