6. INTERACTIVE MULTIPLE GOAL PROGRAMMING

6.1. Definitions and Assumptions

Interactive Multiple Goal Programming (IMGP) starts from the assumption that the decision maker has defined a number of goal variables $g_1(\underline{x}), \ldots, g_m(\underline{x})$, these being concave functions of the instrumental variables $x_1, \ldots, x_n$ ($\underline{x}$ in vector notation).

The decision maker's preferences with respect to the possible solutions can be modelled, at least in principle, by means of a *preference relation* which is *reflexive, transitive* and *complete*. These properties can be defined as follows.[1] Consider α, α' and α'' as elements of the set A of feasible solutions. Furthermore, let us introduce the preference relation $\alpha \geq \alpha'$ for any pair of elements α and α' in A, having the linguistic interpretation that α' is not preferred to α. Reflexivity means that for all elements α in A, $\alpha \geq \alpha$. The relation is transitive, if $\alpha \geq \alpha'$ and $\alpha' \geq \alpha''$ imply $\alpha \geq \alpha''$. The completeness property holds if for any pair α, α' in A with $\alpha \neq \alpha'$, either $\alpha \geq \alpha'$ or $\alpha' \geq \alpha$. If a preference relation possesses these three properties (i.e. reflexivity, transitivity and completeness), it is called a *total quasi ordering* (cf. Rietveld [1980] and Takayama [1974]). IMGP is not intended to model the total quasi ordering for all elements of A. Instead, it aims at finding the most preferred element (or subset of most preferred elements) of A. A crucial assumption is that such a subset exists and can be specified in terms of the goal variables.

1) We refer to Rietveld [1980] for a more detailed discussion on preference relations in the framework of multiple objective programming.

Obviously, A corresponds to the image in goal value space of the set R of feasible instrumental values in instrumental value space. The feasible region R is assumed to be convex, bounded and closed.

IMGP is less restrictive than most other, comparable interactive methods (we will return to this point later on). We will discuss some of the kinds of preference relations that may give rise to successful application of IMGP.

First, the preference relation may be a *lexicographic ordering* (see Chapter 4). This ordering may either be based on a hierarchical ranking of the goal variables or on a hierarchical ranking of the deviational variables going with a series of aspired goal levels.[1)] Assume that for a goal variable (or deviational variable) of given priority rank, alternative α yields a better value than does alternative α'. Assume furthermore that for the more important goal variables (deviational variables), alternative α does not yield worse values than does alternative α'. In this case, alternative α is preferred to alternative α', without regard to the performance of alternative α' for lower priority goals.

A total quasi ordering can also be tackled by means of IMGP if the ordering is *weakly convex*. A preference ordering is weakly convex if for any pair of alternatives α, α' in A, $\alpha \geq \alpha'$ implies $t\alpha + (1-t)\alpha' \geq \alpha'$, $0 < t < 1$, where $\alpha \neq \alpha'$. If a preference ordering is representable by a real-valued preference function (see below),

1) As explained in Chapter 4, the maximization (minimization) of a goal variable can also be represented as the minimization of the deviation from an attainable aspiration level of this goal variable.

then the *quasi-concavity* of the preference function[1] corresponds to the weak convexity of the preference ordering. The condition of weakly convex preference orderings ensures that the subset of most preferred alternatives in the convex set A is connected (see Takayama [1974]).

As illustrated below for preference relations represented by a real-valued preference function, the assumption of weakly convexity allows for a great variety of different preference structures.

As shown by Debreu [1959], a total quasi ordering for all elements of A can be represented by means of a real-valued continuous preference function, if the set A is connected and the total quasi ordering is *continuous*. In general, a preference relation is continuous if given an alternative α' which is preferred to an alternative α, all alternatives α'' which are very close to alternative α' are also preferred to α (see Rietveld [1980] for further details).

In the remainder of this and in the following section it is convenient to assume preference relations that can be represented, in principle, by means of a concave preference function. From the above exposition it should be clear that we consider this assumption as a sufficient but not as a necessary condition for the intended use of IMGP (see also Section 6.4).

Given the concavity of the preference function f, quite a few different preference patterns can be incorporated. A number of examples are given in Figure 6.1, where f is a function of one goal variable $g(\underline{x})$ only. Because f is generally not a known function of $g(\underline{x})$, it is very helpful if we know that f is either monotone increasing (Figure 6.1a) or monotone decreasing (Figure 6.1b). In these

1) A real-valued function defined over a convex set $A \subset R^n$ is quasi-concave if $f(\alpha) \geq f(\alpha')$ implies $f[t\alpha+(1-t)\alpha'] \geq f(\alpha')$ for all pairs α,α' in A and $0 \leq t \leq 1$.

cases, the maximization of f can be accomplished by means of the maximization (respectively, minimization) of $g(\underline{x})$ subject to $\underline{x} \in R$. In the cases sketched in Figures 6.1c - 6.1f, the solution procedure

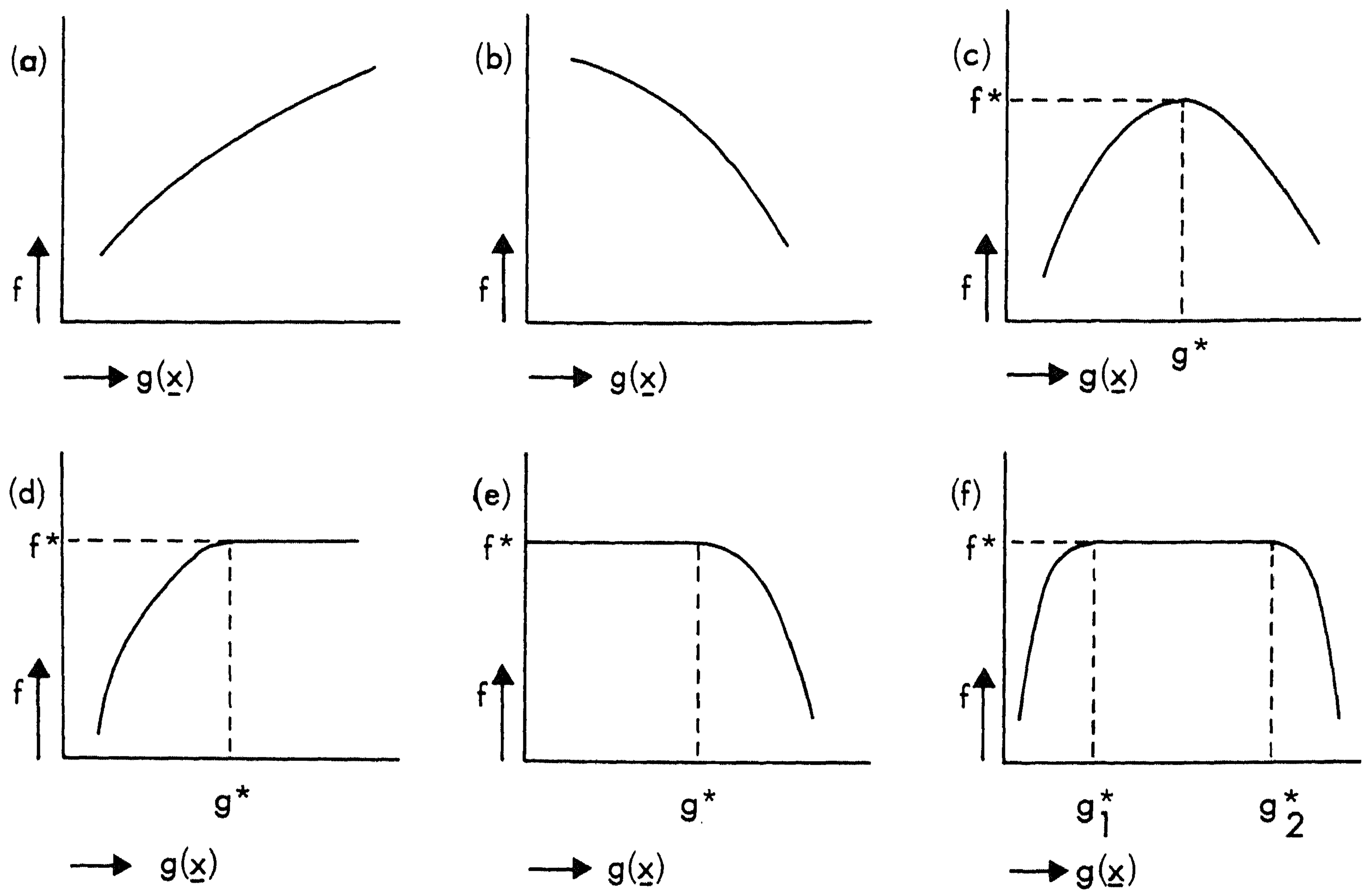

Figure 6.1. The preference function f as a function of one goal variable only

is less straightforward, since f is neither monotone increasing nor monotone decreasing in $g(\underline{x})$. A natural idea is to split the goal variable $g(\underline{x})$ into two other goal variables, one to be maximized ($g_1(\underline{x})$), the other to be minimized ($g_2(\underline{x})$). For the example in Figure 6.1c, this would give (assuming that g^* is known)

$$\text{(6.1)} \qquad \begin{array}{l} \text{Max}\{g_1(\underline{x}) = g(\underline{x})\} \quad \text{s.t.} \quad \underline{x} \in R \text{ and } g(\underline{x}) \leq g^* \\ \text{and} \\ \text{Min}\{g_2(x) = g(\underline{x})\} \quad \text{s.t.} \quad \underline{x} \in R \text{ and } g(x) \geq g^* \end{array}$$

In this way, the problem is divided into two problems. However, as shown in Chapter 4, this formulation can be simplified by using the two-sided goal programming formulation. In this case, we would get

$$(6.2) \qquad \begin{aligned} &\text{Min}\{y^+ + y^-\}, \quad \text{s.t.} \\ &\underline{x} \in R, \\ &g(\underline{x}) - y^+ + y^- = g^* \text{ , and} \\ &y^+ . y^- = 0 \end{aligned}$$

In the same way, the problem in Figure 6.1d can be solved by minimizing y^-, whereas the problem in Figure 6.1e can be solved by minimizing y^+. The problem in Figure 6.1f is a little more complicated because there two 'threshold values', i.e. g_1^* and g_2^*, occur. In this case, two goal restrictions should be formulated. Then the objective function to be minimized includes the deviational variable y_1^- associated with g_1^*, and the variable y_2^+ associated with g_2^*.

The above analysis assumes that the threshold values g^*, g_1^* and g_2^* are known. If this is not the case, some complications arise (cf. Nijkamp and Spronk [1978] for a discussion).

Interactive multiple goal programming needs no more *a priori* information on the decision maker's preferences than other interactive multiple objective programming methods. However, although accurate *a priori* information about the decision maker's preferences may be difficult to obtain, there is usually some information contained in most decision situations. It would be a pity to let this information be unused. On the other hand, it must be realized that the *a priori* information is not always fool-proof. Furthermore, the decision maker may change his mind while dealing with the problem. Interactive multiple goal programming tries to use the *a priori* information in a fruitful manner, by offering the decision maker the opportunity during the interactive process to reconsider his *a priori* information. The *a priori* information used in this method mainly consists of aspiration levels, but relative and pre-emptive priority factors can also be incorporated within the procedure.

In IMGP, the decision maker has to provide information about his preferences on the basis of a solution and a potency matrix presented to him. A solution is a vector of (minimum) values for the respective goal variables. The potency matrix consists of two vectors, representing the ideal and the pessimistic solution, respectively. The ideal solution shows the maximum value for each of the goal variables separately, given the goal values of the pessimistic solution concerned. The pessimistic solution lists a lower value for each of the goal variables separately, either defined directly by the decision maker or, in some cases in which this is possible and useful, derived mathematically from known properties of his preference structure and the set of alternatives. The decision maker merely has to indicate whether or not a solution is satisfactory, and if not, which of the pessimistic goal values should be raised. He does not have to specify how much these goal values should be raised. Nor is there any need to specify weighting factors. (However, if he is able to specify this kind of information, it can be used within the procedure). Then a new solution is presented to him together with a new potency matrix. The decision maker has to indicate whether the shifts in the solution are outweighed by the shifts in the potency matrix. If not, a new solution is calculated and so forth.

At this point it may be useful to pay more attention to the definition of 'solution', as introduced above. Note that a 'solution' has been defined as a vector of goal values, thus being a vector in the goal value space (cf. Chapter 3). Obviously, a solution is not necessarily feasible, i.e. there may be no vector $\underline{x}$ in the feasible region of the space of instruments, which corresponds with this solution (for instance, the ideal solution as defined above). In IMGP, the goal values of a given solution are formulated as restrictions in the space of instruments, which are thus added to the set of restrictions R, describing the region of feasible instrument values.

In the example shown in Figure 6.2a, $g_1(\underline{x}) = x_1$ and $g_2(\underline{x}) = x_2$ (which are both to be maximized), so that the instrument value space and the goal value space are equivalent. It is easily seen that the solution S is infeasible, i.e. there is no $\underline{x}$ in ABCDE for which S can be obtained. However, in this case, if the goal values of S are formulated as inequality constaints in the instrument value space (ES and SC respectively), there is an infinite number of $\underline{x}$-vectors yielding a better solution than S.

In Figure 6.2b, the two spaces are not equivalent. We have defined $g_1(\underline{x}) = x_1$ and $g_2(\underline{x}) = -x_1$, which are both to be maximized. The worst value for $g_1(\underline{x})$ is reached in point G, whereas the worst value for $g_2(\underline{x})$ is reached in point I. If we want to combine these two worst values in a solution S, this can be represented in the goal value space, although the solution as such is infeasible. However, in the space of instruments, no $\underline{x}$-vector exists - whether

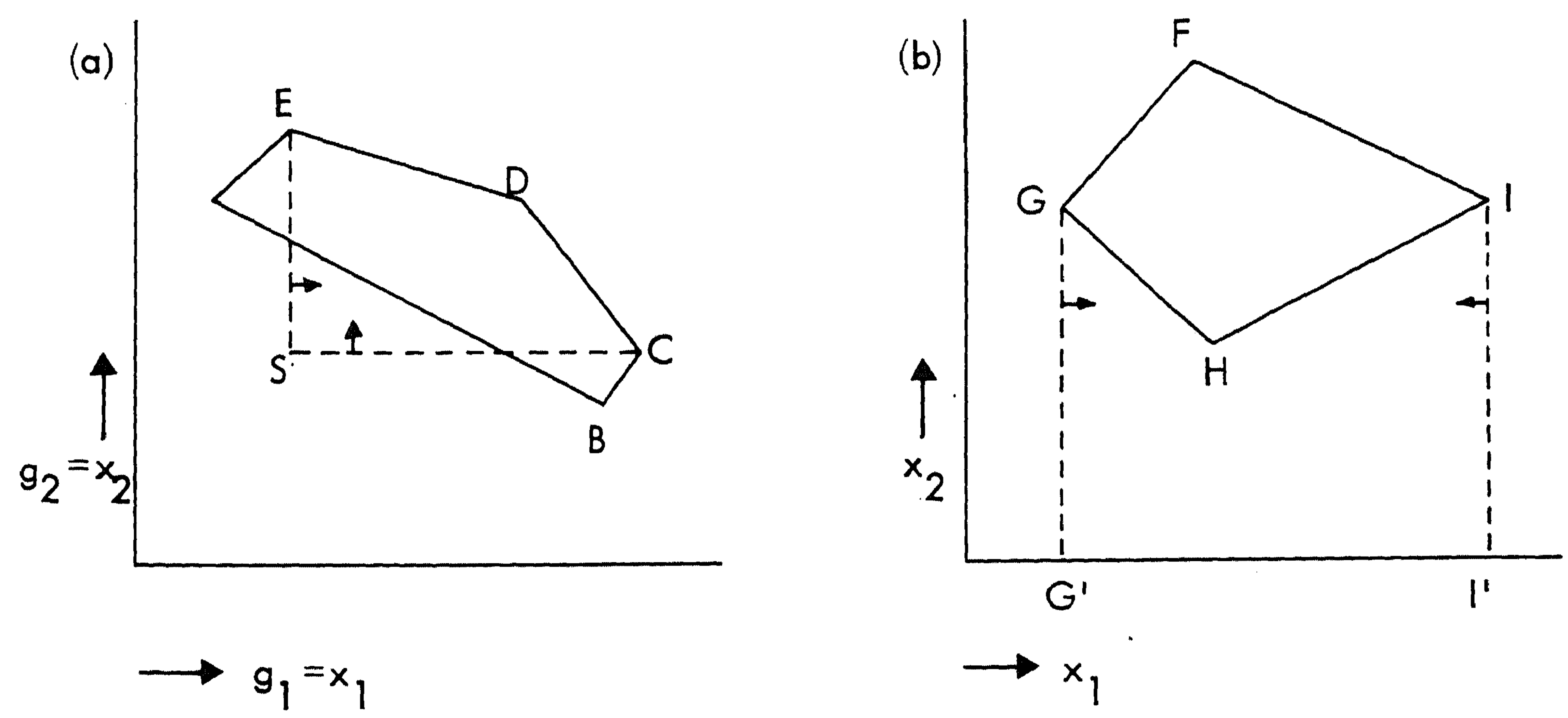

Figure 6.2. The relationship between solutions in the goal value space and in the instrument value space

feasible or not - which yields this solution. Again, formulating these goal values as constraints added to the feasible region leaves an infinite number of $\underline{x}$-vectors which offer better solutions than S (in this case, the set of these $\underline{x}$-vectors is obviously equal to the feasible region FGHI).

6.2. Description of the Procedure

At each iteration of Interactive Multiple Goal Programming (IMGP), a new solution (as defined in Section 6.1) is proposed by changing one or more elements of a former solution. The consequences of these changes are then calculated. To simplify the explanation, we first describe the method while assuming that at each iteration, one and only one element of the solution will undergo a change. Next, we will discuss a number of possible modifications of the method, including the case in which more elements can change during the same iteration.

Step 0 - Given the requirements described in the preceding section, identify the goal variables $g_i(\underline{x}), i = 1, \ldots, m$; as functions of $\underline{x}$, the vector of instrumental variables $x_1, x_2, \ldots, x_n$. Then specify the feasible set R within which an optimal solution must be found. If the decision maker's preferences could be described by a real-valued preference function f (note, however, that we do not make any attempt in this direction), this function should be a concave function of both $g_i(\underline{x}), i = 1, \ldots, m$; and $x_i, i = 1, \ldots, n$. An optimal solution is then defined by

$$\text{Max } f = f(g_1(\underline{x}), \ldots, g_m(\underline{x})), \text{ s.t.} \qquad (6.3)$$
$$\underline{x} \in R$$

To simplify our exposition, we assume further

(6.4) $\frac{\partial f}{\partial g_i} > 0$ for $i = 1, \ldots, m$;

so that we presuppose a higher value of each of the goal variables is preferred to a lower value of (the same) goal variable.[1)]

Step 1 - Successively maximize each of the m goal variables $g_i(\underline{x})$ separately and denote the maxima by g_i^* and the m corresponding combinations of the instrumental variables by $\underline{x}^{i*}$, $i = 1, \ldots, m$. It is usually not necessary to accept a value of $g_i(\underline{x})$ which is lower than g_i^{min}, defined as

$$(6.5) \qquad g_i^{min} = \underset{j=1,\ldots,m}{\mathrm{Min}} \{g_i(\underline{x}^{j*})\},$$

the lowest value of $g_i(\underline{x})$ resulting from the successive maximizations of the goal variables.[2)] Then, the final solution $\underline{S}^*$ must be found between the 'ideal' (but usually infeasible) solution $\underline{I}$, and the 'pessimistic' solution $\underline{Q}$, which are defined respectively as

$$(6.6) \qquad \begin{aligned} \underline{I} &= [g_1^*, g_2^*, \ldots, g_m^*] \text{ and} \\ \underline{Q} &= [g_1^{min}, g_2^{min}, \ldots, g_m^{min}] \end{aligned}$$

1) Note, that this assumption is only made to simplify the exposition. All preference relations satisfying the conditions discussed in Section 6.1, can be included without any difficulties.

2) An obvious case, for which (6.5) is a valid limit on the choice of $\underline{S}^*$, is for $m = 2$, i.e. bicriterion optimization. Other cases are discussed in Appendix 6.a. If (6.5) is not applicable, the g_i^{min} might be assessed directly by the decision maker.

To facilitate the notation, we include the ideal solution $\underline{I}$ and the pessimistic solution $\underline{Q}$ in the (2xm) 'potency matrix' P.

Step 2 - For each goal variable $g_i(\underline{x})$, the decision maker may have defined aspiration levels g_{ij} $(j=2,\ldots,k_i-1)$ with the following property

$$g_i^{min} < g_{i2} < g_{i3} < \ldots < g_{ik_i-1} < g_i^* \tag{6.7}$$

Furthermore we define

$$\begin{aligned} g_{i1} &= g_i^{min} \quad \text{and} \\ g_{ik_i} &= g_i^* \end{aligned} \tag{6.8}$$

In the following steps these goal values are used in constructing trial solutions $\hat{S}_i$ which have to be evaluated by the decision maker. Because proposed goal levels are sometimes regarded as too high, we need the auxiliary vector δ with elements δ_j $(j=1,\ldots,m)$ corresponding to the m goal variables. We define δ_j as the difference of the lowest level of $g_j(\underline{x})$ being <u>rejected</u> by the decision maker and the highest level of $g_j(\underline{x})$ being <u>accepted</u> thus far. At the first stage of the procedure, no proposals have been made and consequently, no goal level has been rejected. Therefore we put $\delta_j = 0$ for $j = 1, \ldots, m$ during the first step. However, if for a certain goal variable j' no aspiration level has been specified, we define $\delta_{j'} = g_{j'}^* - g_{j'}^{min}$ for reasons which will become clear in step 6.

Step 3 - Define the initial solution as

$$\underline{S}_1 = [g_{11}, g_{21}, \ldots, g_{m1}], \tag{6.9}$$

which is thus equal to the pessimistic solution defined in (6.6). Present this solution, together with the potency matrix P_1, to the decision maker.

Step 4 - If the proposed solution is satisfactory for the decision maker, he may accept it; if not, continue with step 5. Define R_i as the subset of R defined by the goal levels in $\underline{S}_i$.

Step 5 - The decision maker then has to answer the following question: 'Given the provisional solution $\underline{S}_i$, which goal variable should be improved first'?[1)]

Step 6 - Let us assume that the decision maker wants to augment the j-th goal variable. Then construct a trial pessimistic solution $\hat{\underline{S}}_{i+1}$, which differs with respect to $\underline{S}_i$ only as far as the value of the j-th goal variable is concerned (denoted by $g_j(\underline{x})_{\hat{\underline{S}}_{i+1}}$ and $g_j(\underline{x})_{\underline{S}_i}$ respectively).

<u>If</u> $\delta_j = 0$ no proposed value of $g_j(\underline{x})$ has been rejected thus far, by which we can propose the next higher aspiration level listed in step 2. <u>If</u> $\delta_j > 0$, a value of $g_j(\underline{x})$ which exceeds the current solution by an amount δ_j has been rejected by the decision maker. In this case, define[2)]

$$g_j(\underline{x})_{\hat{\underline{S}}_{i+1}} = g_j(\underline{x})_{\underline{S}_i} + \tfrac{1}{2}.\delta_j \tag{6.10}$$

1) Later, we will discuss the case in which the decision maker wants to raise more than one goal variable at one time.

2) Here, the decision maker may wish to define a new aspiration level. In our opinion, it is wise to give him explicitly the opportunity to do so.

When a provisional value for $g_j(\underline{x})$ has been calculated in one of both above-mentioned ways, we introduce the restriction:

(6.11) $$g_j(\underline{x}) \geq g_j(\underline{x})_{\hat{\underline{S}}_{i+1}}$$

and proceed to step 7.

Step 7 - Combine the restriction formulated in step 6 or in step 9 with the set of restrictions describing the feasible region R_i. Next calculate a new potency matrix, as in step 2, but subject to the new set of restrictions. Label this potency matrix $\hat{P}_{i+1}$.

Step 8 - Confront the decision maker with $\underline{S}_i$ and $\hat{\underline{S}}_{i+1}$ on the one hand and with P_i and $\hat{P}_{i+1}$ on the other hand. The shifts in the potency matrix can be viewed as a 'sacrifice' for reaching the proposed solution. If (8b) the decision maker considers this sacrifice to be justified, accept the proposed solution by putting $\underline{S}_{i+1} = \hat{\underline{S}}_{i+1}$ and $P_{i+1} = \hat{P}_{i+1}$. Furthermore, in the computer algorithm (see Figure 6.3), put $\delta_j = \frac{1}{2}.\delta_j$ (which is only relevant for $\delta_j > 0$) and return to step 4. If (8a) the decision maker considers the sacrifice unjustified, the proposed value of $g_j(\underline{x})$ is obviously too high. Therefore, drop the constraint added in step 7 and proceed to step 9.

Step 9 - We now know that, in the decision maker's view $g_j(\underline{x})_{\underline{S}_i}$ is too low and that $g_j(\underline{x})_{\hat{\underline{S}}_{i+1}}$ is too high. By definition, we thus may set δ_j equal to the difference between these two values. A new proposal value $\hat{\underline{S}}_{i+1}$ is then calculated by defining

(6.12) $$g_j(\underline{x})_{\hat{\underline{S}}_{i+1}} = g_j(\underline{x})_{\underline{S}_i} + \tfrac{1}{2}\cdot\delta_j$$

As in step 6, we add the restriction that $g_j(\underline{x})$ must equal or exceed the new proposal value and go to step 7 in order to calculate a new potency matrix $\hat{P}_{i+1}$.

If the decision maker is unwilling to indicate a single goal variable which should be improved in value, one could present him the option to define a set of goal variables which should be augmented in value at the same time. Then, the procedure must be modified slightly. These modifications have been included in Figure 6.3, and are discussed below.

Changing more than one Goal Value Simultaneously.

Step 5*- Instead of one goal variable, more than one goal variable to be augmented is chosen.

Step 6*- Find proposal values for all goal variables selected in 5* in the way a new value was calculated for the single goal variable in 6.

Step 7*- Before calculating the potency matrix $\hat{P}_{i+1}$ the set of restrictions is extended with the restrictions formulated in 6*.

Step 8*- If the decision maker considers the sacrifices too heavy to approve the solution, he should indicate which of the goal variables, having a higher value in $\hat{\underline{S}}_{i+1}$ than in $\underline{S}_i$, should be reduced in 9*.

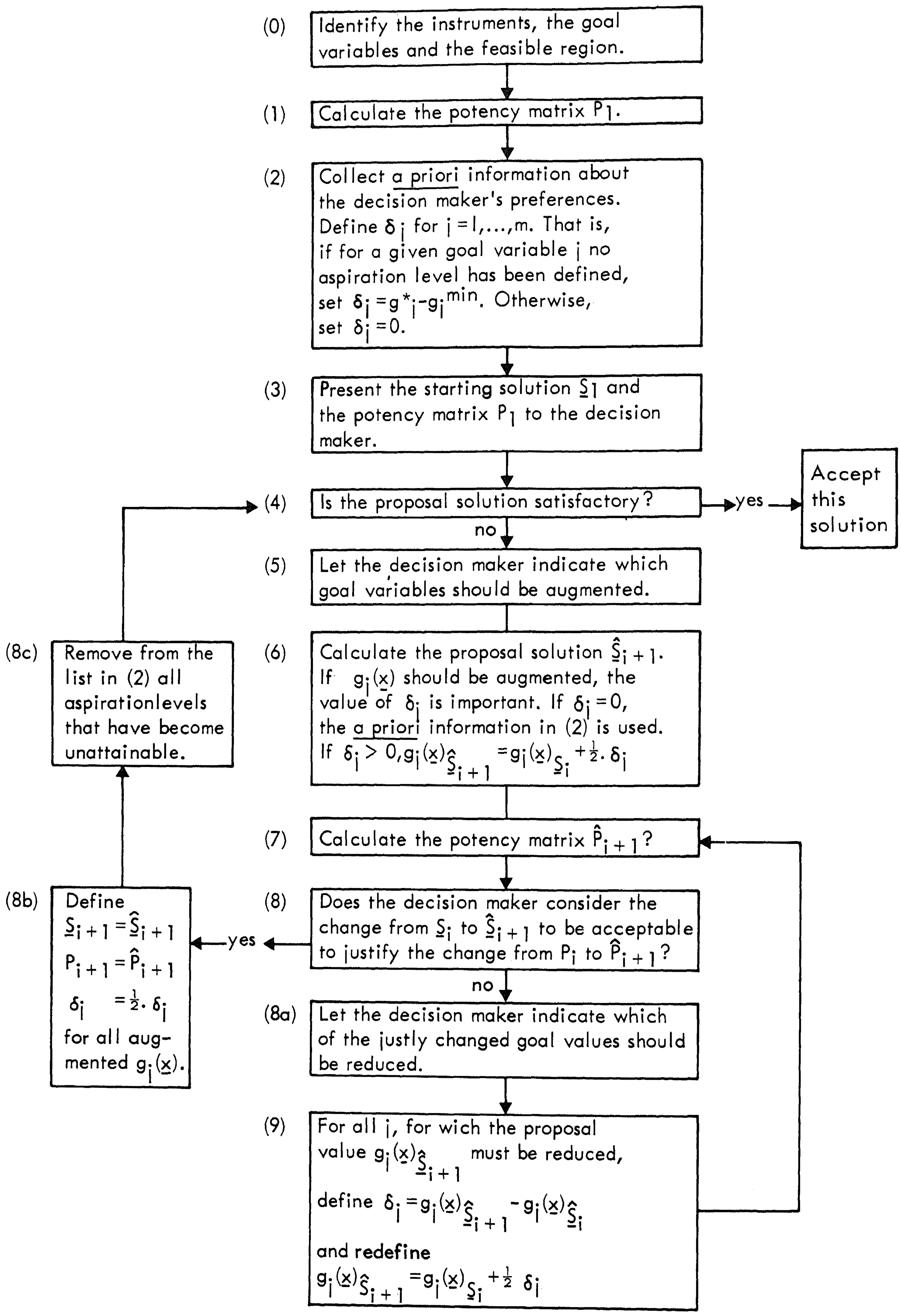

Figure 6.3. A flow chart of the extended interactive multiple goal programming procedure

Step 9*- Calculate a new proposal solution by reducing all goal variables indicated in 8* in the same way that the single goal variable was reduced in 9.

Other Options Available

In IMGP, there are more options available to the decision maker. One option considers the fact that, in general, interactive procedures induce learning effects (cf. Chapter 5). This implies that the decision maker may change his mind during the interactive process or may feel that he has made some errors. This is the main reason why the decision maker must have the opportunity to return to an earlier iteration or even to restart the whole process. As in most interactive procedures, these options can be given in IMGP without any difficulties.

An example of a learning effect occurs, for instance, when the decision maker recognizes that a proposed shift in a single goal variable is outweighted by a simultaneous shift in two or more other goal variables. He may then wish to return to the preceding solution to ask for such a simultaneous shift. He may also require additional information regarding the state of the problem. IMGP offers several possibilities. First, the total 'pay-off' matrix[1)] underlying the potency matrix may be presented to the decision maker. Second, depending on the method used, the optimizations may produce a considerable amount of dual information on the state of the problem (cf. also Chapter 4). Third, to measure the above-mentioned simultaneous shifts of goal variables, one may introduce a proxy goal variable, such as, for instance, the sum of all other goal variables. Of course, many other devices to provide information regarding the state of the problem can be proposed.

1) The pay-off matrix is the matrix of goal values $g_i(\underline{x}^{j*})$, with $i = 1, \ldots, m$; and $j = 1, \ldots, m$.

6.3. IMGP in Linear Terms

Given the fairly modest requirements described in Section 6.1, many methods are available for use within the IMGP procedure[1]. For instance, many mathematical programming techniques may be useful. As an illustration, we describe IMGP in linear terms (with respect to the instruments $\underline{x}$), by which it becomes accessible for linear (goal) programming routines. The additional advantages of such a linear format of IMGP are discussed at the end of this section.

If the feasible region R can be described by means of linear restrictions, <u>and</u> the goal variables $g_i(\underline{x})$, $i = 1, \ldots, m$; are linear or piecewise linear and concave in the instruments $\underline{x}$, IMGP can use standard linear programming routines.

The calculatory steps in IMGP consist of the computation of the potency matrices. The first potency matrix, P_1, is calculated in step 1. Whenever the lower bounds on the values of one or more of the $g_i(\underline{x})$ are augmented (in step 6) or when some of these bounds are decreased (in step 9), the accompanying potency matrix is calculated in step 7. In all three cases the structure of the problem is identical. Each of the goal variables must successively be maximized (or minimized) within the feasible region R and conditioned by a set of lower bounds (or upper bounds) on the values of the goal variables. The problem can thus be written as

$$
\begin{array}{llr}
 & \mathrm{Max}(\mathrm{c.q.Min}) \quad \{g_i(\underline{x})\}, \quad \text{s.t.} & \\
(6.13) & \underline{x} \in R, \text{ and} & \text{for } i = 1, \ldots, m; \\
 & g_j(\underline{x}) \geq (\leq) \hat{g}_j \quad \text{for } j = 1, \ldots, m; &
\end{array}
$$

1) Even if all requirements are not fulfilled, IMGP can sometimes be used, although with a few modifications. For example, in Section 7.4, we demonstrate the use of IMGP in decision problems with a finite number of alternatives (a situation which is in conflict with the convexity condition on the feasible region).

where $\hat{g}_j$ denotes the proposed value of $g_j(\underline{x})$ for the problem at hand.

It is possible to formulate the set of problems in (6.13) in another, but also uniform way, i.e. as minimization problems differing only in the coefficients of the objective function. In this approach, which is closely related to the goal programming formulations in Chapter 4, we formulate for each of the goal variables two restrictions

$$
(6.14) \qquad
\begin{array}{ll}
g_j(\underline{x}) - \hat{y}_j^{+} + \hat{y}_j^{-} = \hat{g}_j & \text{for } j = 1, \ldots, m; \text{ and} \\
g_j(\underline{x}) - \overset{*}{y}{}_j^{+} + \overset{*}{y}{}_j^{-} = g_j^{*} & \text{for } j = 1, \ldots, m;
\end{array}
$$

where $\hat{g}_j$ denotes the proposed value of $g_j(\underline{x})$ in the problem at hand, and g_j^* its maximum value (or minimum value) in the first solution S_1 (thereby only constrained by $\underline{x} \in R$). The y^+ and y^- values measure the overattainment and underattainment with respect to the aspired levels $\hat{g}$ and $\overset{*}{g}$. The problem can then be formulated as a goal programming problem. Let us assume that $g_i(\underline{x})$ should be maximized, given a set of proposed values for the goal variables. This maximization problem can be translated as

$$
(6.15) \qquad
\begin{array}{l}
\text{Min}\{M_1 . \sum_{j=1}^{m} (\alpha_j^+ . \hat{y}_j^+ + \alpha_j^- . \hat{y}_j^-) + M_2 . \overset{*}{y}{}_i^-\} \text{ s.t.} \\
\underline{x} \in R \text{ and s.t. } (6.14) \\
\alpha_j^+ = 1 \text{ and } \alpha_j^- = 0 \text{ if } f \text{ is a decreasing function of } g_j(\underline{x}) \\
\alpha_j^+ = 0 \text{ and } \alpha_j^- = 1 \text{ if } f \text{ is an increasing function of } g_j(\underline{x})
\end{array}
$$

The non-Archimedean (cf. Charnes and Cooper [1977] and Chapter 4) weighing factors M_1 and M_2 have the property $M_1 >>> M_2$ by which pre-emptive priority is given to attain the proposal values $\hat{g}_j$, $j = 1, \ldots, m$, before $g_i(\underline{x})$ can be maximized by means of the minimization of $\overset{*}{y}{}_j^-$. We assumed that the variables $g_j(\underline{x})$ could be formulated in such a way that f was monotone non-decreasing or monotone non-increasing in $g_j(\underline{x})$. In the first case, the proposal value $\hat{g}_j$ must be considered as a lower bound (which means $\hat{y}_j^-$ must be zero) and in the second case, $\hat{g}_j$ must be considered as an upper bound (by which $\hat{y}_j^+$ must be zero). In (6.15) we assumed $g_i(\underline{x})$ was to be maximized. Minimization of $g_i(\underline{x})$ can easily be achieved by replacing $\overset{*}{y}_i^-$ in (6.15) by $\overset{*}{y}_i^+$.

Advantages Related to the Linear Format

As suggested in (6.14) and (6.15), IMGP can make straightforward use of goal programming routines. That is, for each proposal solution, a set of goal programs can be formulated. These differ mutually only with respect to one element in the objective function, being the y_i^{*-}, i = 1, ..., m to be minimized. By means of these goal programs, a potency matrix based on the proposal solution can be constructed. This has to be carried out for each new proposal solution. However, the goal programs belonging to different proposal solutions only differ with respect to some of the right-hand side constants, being the goal levels which have been changed. Clearly, this formulation gives access to specific goal programming routines as proposed for example by Lee [1972], (see also Chapter 4). However, standard linear programming packages can also be used. PL/I programs using IBM's MPSX-package are given by Ouwerkerk and Spronk [1978] and Hartog et al. [1979].

A main advantage of the linear format of the problem is that each solution of a goal program contains useful information about the effects of a shift of the right-hand side constants (see Chapter 4). In an extensive overview, Isermann [1977] argues that duality in multiple objective linear programming is even more relevant than in standard linear programming. Besides the economic implications of duality, he illustrates its decision-oriented relevance. He shows how information from the dual may be employed in the decision maker's search for a compromise solution. In the same sense Kornbluth [1978] proposes a method in which information from the (fuzzy) dual is systematically used in an interactive way.

Furthermore, the linear format of IMGP has all the advantages of goal programming, as discussed in Chapter 4. In its linear format, IMGP may also benefit from the widespread attention paid to linear programming, both in theory and practice. Special procedures developed for linear programming may also be useful in linear IMGP. As an example, procedures to identify redundant constraints in a linear programming problem may be used to identify 'redundant goal constraints' and redundant 'goal variables' (cf. Gal and Leberling [1977] and Spronk and Telgen [1979]).

6.4. Existence, Feasibility, Uniqueness and Convergency

A 'solution' is identified by a vector of minimum (or maximum) values imposed on the respective goal variables. It is easily seen, that given the ideal and the pessimistic solution, there is always at least one combination of the goal variables which is bounded by the ideal and the pessimistic solution, for which a feasible combination of the instrumental variables exists. For instance, consider the vector of goal values which is determined by $\underline{x}^{i*}$, the combination of instrumental variables which maximizes the i-th goal variable, $g_i(\underline{x})$. By definition, this vector is bounded both by the ideal and the pessimistic solution. By the convexity of R, also the convex combinations of the $\underline{x}_i^*$ are feasible.

During the successive iterations of IMGP, the goal values in the successive solutions are repeatedly shifted upwards by the decision maker, thus adding new constraints to the existing set of constraints. Because R is convex in $\underline{x}$ and because the newly added constraints are linear in $\underline{x}$, the part of the feasible region R which remains feasible after adding the constraints (denoted by R_i,

$i=1,2,\ldots;$) remains convex in $\underline{x}$. This means that at each iteration of IMGP there exists a vector of goal values, which is bounded by the ideal and the pessimistic solution of the reduced feasible region, for which a feasible combination of the instruments exists.

Given the assumptions underlying IMGP, a most preferred solution is not necessarily unique. Notably, the assumption that the decision maker's preferences can be modelled by means of weakly convex preference relations is not a sufficient condition to guarantee a most preferred solution. For instance, even satisficing behaviour can be represented by weakly convex preference relations. However, if the preference relations have the *dominance* property[1)], it is easily seen that IMGP yields efficient solutions.

The next question is whether IMGP converges to a most preferred solution (either unique or not). Of course, the convergency properties partly depend on the abilities of the decision maker. We therefore first assume that the decision maker is able to answer the questions posed by IMGP, that his answers are consistent, correct and finally that his preferences do not change during the interactive process of IMGP. Given these assumptions it can be shown that IMGP terminates in a finite number of iterations within an ε-neighbourhood of a most preferred solution.

First, starting from an accepted solution $\underline{S}_i$, the next solution will be accepted after a finite number of steps.
Let us define A_i as the subset of A (see Section 6.1) of which the elements meet the minimally required goal levels defined by $\underline{S}_i$. We

1) A preference relation has the dominance property if for all pairs α,α' in A holds that α is preferred to α' if all goal values of α equal or exceed the corresponding values in α'.

know that $\underline{S}_i$ is preferred to all preceding solutions and that S_i can be improved. Thus the following condition must have been met.

$$(6.16) \qquad \exists\alpha' \in A_i[\alpha' \geq \alpha, \quad \forall\alpha \in \{A-A_i\}]$$

Let us next assume that the decision maker wants to improve $\underline{S}_i$ by raising the value of $g_k(\underline{x})$ thus obtaining a new solution $\underline{S}_{i+1}$. If the preferences of the decision maker can be modelled by means of a lexicographic ordering (see Section 6.1), the elements of A_{i+1} (the subset of A defined by $\underline{S}_{i+1}$) will satisfy condition

$$(6.17) \quad \forall\alpha' \in A_{i+1}[\alpha' \geq \alpha, \ \forall\alpha \in \{A_i-A_{i+1}\}]$$

If the preferences of the decision maker can be modelled by means of a weakly convex ordering as defined in Section 6.1, the elements of A_{i+1} will satisfy

$$(6.18) \quad \forall\alpha \in \{A_i-A_{i+1}\}[\exists\alpha' \in A_{i+1}[\alpha' \geq \alpha]]$$

Let us assume, that given $\underline{\delta}_i$ and given either (6.17) or (6.18), $g_k(\underline{x})$ can be increased by an amount of $\lambda > 0$. As described in Section 6.2, a first proposal solution $\hat{\underline{S}}_{i+1}$ is generated by augmenting the value of $g_k(\underline{x})$ in $\underline{S}_i$ by a given amount, which will be called d here. If $d \leq \lambda$, the proposal solution will be accepted; otherwise, the proposal will not be accepted[1]. Then a new proposal solution is calculated by halving the value of d. If $(d/2) \leq \lambda$ the proposal solution is accepted. If not, the value of d is divided by 2^2 and so forth. Clearly, the proposal solution is accepted as soon as $(d/2^n) < \lambda$ which for $\lambda > 0$ occurs at a finite value n. Thus we

1) Whether d exceeds λ has to be judged by the decision maker. For this evaluation, he uses, among other things, the information presented in the potency matrices.

have shown that each new solution $\underline{S}_{i+1}$ is reached in a finite number of steps.

Next, we show that only a finite number of solutions has to be calculated before a final solution is obtained, in which the values of the respective goal variables differ less than some predetermined $\varepsilon > 0$ from the respective goal values in a most preferred solution. At each iteration of IMGP at least the value of one goal variable is raised. Because there is a finite number (m) of goal variables, it is sufficient to show that an arbitrary goal variable $g_k(\underline{x})$ reaches a most preferred value g_k^o (ignoring a small distance of at most ε_k) within a finite number of iterations. Assuming, that the decision maker has not defined any aspiration level for $g_k(\underline{x})$, we only know that $g_k^{min} < g_k^o < g_k^*$, where g_k^* again is the maximum value of $g_k(\underline{x})$ for $\underline{x} \in R$. As described in Section 6.2, a proposal solution is calculated as $\hat{g}_k = (g_k^*-g_k^{min})/2$. From the (correct) answer of the decision maker we can deduce whether $g_k^o < \hat{g}_k$ or $g_k^o < \hat{g}_k$. We then know that either $g_k^{min} < g_k^o < \hat{g}_k$ or $\hat{g}_k < g_k^o < g_k^*$. At the next iteration in which g_k is chosen to be raised, a new proposal solution is chosen exactly in the middle of the chosen region. Thus, the range in which g_k^o must be found is exactly halved each time the decision maker is consulted. This means that the ε-neighbourhood of g_k^o is reached when

$$(6.19) \qquad \frac{(g_k^*-g_k^{min})}{2^n} < \varepsilon_k,$$

where n is the number of times the decision maker gives his opinion on $\hat{g}_k$. In general, this ε_k-neighbourhood will be attained in less steps. This is because the aspiration levels which have been defined *a priori* may be of great help during the search procedure. Furthermore g_k^* is influenced by the values required for the other goal variables.

6.5. Concluding Remarks

In Chapters 3 and 5 we proposed several characteristics for describing interactive procedures, relating, among other things, to the class of problems which can be handled, the nature of the communication process between decision maker and decision model, and the technical properties.

First, we outline the type of problems which can be handled by interactive multiple goal programming. The convex set of feasible actions (R) is given and fixed over time. However, if this set changes over time, the interactive procedure should not be started all over again. This is because the solution obtained for the unchanged (old) problem can be used to make an advanced start. The set R needs to be convex. However, with the loss of some attractive properties of IMGP, mutually exclusive actions can also be handled (see Section 7.4). In IMGP, the goal variables are assumed to be measurable and known functions of the instrumental variables. The examples in Section 7.4 show how to include a goal variable which has to be measured on an ordinal scale. IMGP can be used (depending on the desires of the decision maker), to generate a unique final solution (within ε), an efficient solution or a satisficing solution. The method is not suitable for generating the complete set of efficient solutions. Instead, it aims at finding the efficient solution, which is considered (by the decision maker) to be the 'best' element within the set of all efficient solutions.

In IMGP, the communication process between decision maker and decision model has been structured in such a way that it has some attractive properties. The decision maker has only to provide a limited amount of information, although he has the option to give more information (and thus to command the interactive process) whenever he wishes to do so. Notably, if the decision maker has defined aspiration levels and pre-emptive priorities, these can be incorporated in the interactive process quite easily. Moreover, IMGP gives the decision maker the opportunity to reconsider this information during the interactive process. At each iteration, the method provides a large amount of information concerning the state of the problem. Depending on the decision maker at hand, this information may be translated in various ways. An important advantage of IMGP (and most other interactive procedures) is that the decision maker can give his preferences on the basis of well-specified solutions and is not obliged to answer any hypothetical question (cf. Chapter 5).

We can be very short about the technical properties of IMGP. In the computational phases of IMGP any solution procedure which meets the fairly modest requirements mentioned in sections 6.1 and 6.2 can be employed. As already mentioned in Section 6.4, IMGP converges within a finite number of iterations to a final solution which is known to exist and to be feasible. The computer time per iteration and the number of iterations needed to reach a final solution depends, among other things, on the problem to be solved and the solution procedure chosen.

Appendix 6.a. Suitable Starting Solutions

The minimal goal levels g_{j1}, $j = 1, \ldots, m$; included in the starting solution $\underline{S}_1$ (see (6.9)) determine the subset R_1 of the feasible region R in which the final solution $\underline{S}^*$ is to be found. Clearly, the size of R_1 co-determines the difficulty of the decision problem. Therefore the time spent on finding values for g_{j1}, that make R_1 as small as possible, may be worth-while. Of course the g_{j1} must be chosen so as not to exclude any of the most preferred solutions. Given R and given some elementary knowledge of the decision maker's preferences (see below), suitable values for g_{j1} can often be derived in a straightforward manner. These mathematically derived values for g_{j1} (which will be discussed in more detail below) may offer valuable insight into the decision problem. For instance by showing that certain solutions (those excluded by the g_{j1}) need not be evaluated because they are dominated by solutions included in R_1. On the other hand, the mathematically derived values for g_{j1} have often no meaning at all for the decision maker, because he may have defined higher values for the goal variable, which he considers as 'necessary conditions' to be met by the final solution(s).

In this appendix we consider the case in which the decision maker's preference ordering has the dominance property described in Section 6.1. This means that we have to find maximal values for the minimal goal levels g_{j1} in such a way as not to exclude any efficient solution. In other words we have to find for each goal variable g_j, $j = 1, \ldots, m$; its minimal value within the efficient set. The minimal value of a goal variable within the efficient set is sometimes but not always, equal to its minimal value at the points for which the other goal variables reach their maximum, as defined in (6.5). Dessouki et al. [1979] show that the minimal value within the efficient set can be found in the closed interval defined by the minimal value within the feasible region R and the minimal value defined by (6.5). The same authors propose a simple procedure to find the minimal goal values within the efficient set:

(i) Select an arbitrary vector of instruments $\underline{x}^1$ which yields an efficient solution and calculate the value of the goal variable concerned (say g_k).

(ii) Select $\varepsilon^1 > 0$ and try to find a vector $\underline{x}^2$ for which $g_k(\underline{x}^2) \leq g_k(\underline{x}^1) - \varepsilon^1$ and $\underline{g}(\underline{x}^2)$ is efficient.

(iii) If a vector $\underline{x}^2$ exists, which yields an efficient solution, select $\varepsilon^2 > \varepsilon^1$.
If no such $\underline{x}^2$ exists, select $\varepsilon^2 > \varepsilon^1$.
In both cases, repeat step (ii).

The minimal goal values defined by (6.5) as well as the associated instrument vectors are obtained in the first step of IMGP. Clearly, these vectors of instruments can be used to define $\underline{x}^1$ in the above procedure. Dessouki et al. (Ibid) implement the second step of their procedure by solving a quadratic programming problem. The same task can be performed by maximizing $g_k(\underline{x})$ subject to

$$g_k(\underline{x}) \leq g_k(\underline{x}^1) - \varepsilon^1, \text{ and} \qquad (6a.1)$$
$$\underline{x} \in R$$

and checking whether the resulting optimal solution $\underline{x}^2$ is efficient. Tests to check whether a given vector of instruments is efficient, are provided among others by Zionts and Wallenius [1980] and Wendell and Lee [1977].

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