1. Introduction

In the multi-factor method risk is viewed as a multi-dimensional concept, in which the unexpected change in a performance variable, e.g. cash flow, is assumed to be related to unexpected changes of risk factors, such as the interest rate and the oil price. The relation between the unexpected change in a performance variable and the unexpected risk factor change is called the sensitivity of the unexpected performance change for an unexpected change of the risk factor. Subsequently, the vector containing the sensitivities is called the risk profile. Although the multi-factor method has already been applied many times, see e.g. Berry, Burmeister and McElroy (1988) and Spronk and Van der Wijst (1987), the sensitivity concept itself has received much less attention.

This paper tries to fill this gap. First, in Section 2, the multi-factor concept is briefly discussed. Section 3, then, gives definitions of concepts used in the multi-factor method, such as sensitivity, unexpected performance, unexpected risk factor change and instruments. The latter are discussed in more detail in Section 4, after which Section 5 focusses on the consequences for the definition of unexpected performance. Section 6 concludes and summarizes the relations between the different concepts.

2 The multi-factor concept

In the multi-factor framework, each entity "a" is assumed to have an unexpected stochastic return[1] $\Delta\tilde{R}_U$, which is related to unexpected changes of m risk factors $(\Delta\tilde{f}_{U1},..,\Delta\tilde{f}_{Um}) \equiv \Delta\tilde{f}_U$. Hence[2] $\Delta\tilde{R}_U = g(\Delta\tilde{f}_U)$. As a result, the probability density function of $\Delta\tilde{R}_U$ depends on the multi-variate probability density function of $\Delta\tilde{f}_U$, which is denoted by $h(\Delta\tilde{f}_U)$.

In the multi-factor framework, the following assumption is often made with respect to the functions $g(\Delta\tilde{f}_U)$: $g(\Delta\tilde{f}_U) = b.\Delta\tilde{f}_U$, where the vector $b \equiv b_1,..,b_m$, is called the risk profile of the entity, and b_j is the sensitivity for an unexpected change of risk factor j. In this paper we restrict ourselves to only one risk factor, hence both $\Delta\tilde{f}_U$ and b are scalars.

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[1] *In this paper, stochasts are indicated by a tilde, and realized values by an asterisk.*

[2] *Normally, $g(\Delta\tilde{f}_U)$ also contains a specific error term. However, without loss of generality, this term is neglected in this paper.*

In reality, b often depends on $\Delta\tilde{f}_U$. For example, the upward sensitivity, the sensitivity for an unexpected increase in the value of the risk factor, might be different from the downward sensitivity, the sensitivity for an unexpected decrease in risk factor.

3. Working definitions

In the following, we first specify the time horizon used in the model. Next, the concepts performance measure, risk factor, sensitivity, instruments, and manipulated sensitivity are defined.

Time

In this paper we concentrate on a one period model from t_0 to t_1.

Performance measure

In this paper, performance is measured in money units. Accordingly, the dimension of the performance measure is a currency unit, such as dollars.

Unexpected performance

Unexpected performance is assumed to be a stochastic variable, whose stochastic character is related to the stochastic character of unexpected changes of risk factors. As in the preceding, unexpected performance is denoted by $\Delta\tilde{R}_U$.

Risk factor

Risk factors are stochastic variables, which can not be controlled by the individual, but which might influence performance. Examples of risk factors are the interest rate and the oil price. The value of the risk factor is expressed in its natural unit. The interest rate, for instance, will be measured in percentage points and the oil price in dollars per barrel. Accordingly, the dimension of the risk factor is the natural unit, as well.

Unexpected change of the risk factor

In this paper, we concentrate on unexpected changes of risk factors. As in Section 2, the unexpected change of a risk factor is denoted by $\Delta\tilde{f}_U$, which is defined as follows:[3]

[3] *Since we are only using a one-period model, we will write $\Delta\tilde{f}_U$ instead of $\Delta\tilde{f}_U(t_1)$ in the remainder of this paper.*

$$\Delta \tilde{f}_U = \tilde{f}(t_1) - E(\tilde{f}(t_1)) \qquad (1)$$

The corresponding ex post unexpected factor change is:

$$\Delta f^*_{Uep} = f^*(t_1) - E(\tilde{f}(t_1)) \qquad (2)$$

where $f^*(t_1)$ stands for the realized value of the risk factor at t_1 and $E(\tilde{f}(t_1))$ stands for the expected factor value of $\tilde{f}$ at t_1.

By definition, the expected ex ante unexpected factor change is zero. Hence,

$$\Delta f_{Uea} \equiv \int_{-\infty}^{\infty} h(\Delta \tilde{f}_U)\ \Delta \tilde{f}_U\ d\Delta \tilde{f}_U = 0 \qquad (3)$$

where $h(\Delta \tilde{f}_U)$ stands for the probability density function of $\Delta \tilde{f}_U$.

Sensitivity

The sensitivity describes the relation between unexpected performance and an unexpected factor change. It is defined as the quotient of unexpected performance and unexpected risk factor change, and is denoted by "s". As a consequence, the dimension of the sensitivity is ((money units)/(natural unit)). If the sentivity is known, unexpected performance can be computed by multiplying the unexpected factor change with the sensitivity.

To illustrate, if someone borrows \$ 1000.- then his sensitivity is \$ 1000.- per unit change in the interest rate. Hence, an unexpected increase of 2% of the interest rate, results in an unexpected performance of -\$ 1000.- * 2% = -\$ 20.-.

The sensitivity sometimes depends on the unexpected factor change, in which case we write $s(\Delta \tilde{f}_U)$ instead of "s".

The ex post sensitivity is defined as the quotient of realized unexpected performance and realized unexpected factor change. Assume the unexpected realized factor change to be Δf^*_U, and unexpected performance to be equal to ΔR^*_U. Then, the ex post sensitivity s_{ep} is calculated as follows.

$$s^*_{ep} = \Delta R^*_U / \Delta f^*_U \qquad (4)$$

The computation of the ex ante sensitivity is more complex. First, for every possible value of the unexpected factor change the sensitivity is calculated by taking the quotient of the unexpected performance (given this unexpected factor change) and the unexpected factor change itself. Next, these sensitivities are weighted with the probabilities of the occurrences of the corresponding unexpected factor changes. For this reason, we will speak about the expected ex ante sensitivity. In mathematics, this expected ex ante sensitivity s_{ea} is calculated as follows.

$$s_{ea} = \int_{-\infty}^{\infty} s(\Delta\tilde{f}_U)\ h(\Delta\tilde{f}_U)\ d\Delta\tilde{f}_U \tag{5}$$

It is readily seen that the ex post and ex ante sensitivity are equal in case the sensitivity does not depend on the unexpected factor change. Generally, however, the ex post and ex ante sensitivity differ.

Instruments

Instruments enable the decision maker to change sensitivities. In Section 4, they will be discussed in greater detail.

Manipulated sensitivity

The manipulated sensitivity is the resulting sensitivity, when the influence of instruments on the sensitivity is also taken account of. The manipulated sensitivity is denoted by "b", and might also depend on Δf_U in which case we write $b(\Delta\tilde{f}_U)$.

The definition of the ex ante and ex post manipulated sensitivity is similar to the corresponding definitions of the sensitivity. Hence, we define in case instruments are applied:

$$b^*_{ep} = \Delta R^*_U/\Delta f^*_U \tag{6}$$

$$b_{ea} = \int_{-\infty}^{\infty} b(\Delta\tilde{f}_U)\ h(\Delta\tilde{f}_U)\ d\Delta\tilde{f}_U \tag{7}$$

Thus, the relation between unexpected performance and unexpected factor change without application of instruments is called sensitivity, and is denoted by "s", whereas the relation between unexpected performance and unexpected factor change when instruments are applied is called manipulated sensitivity, and is denoted by "b".

4. Instruments

In this section, we discuss three different kinds of instruments, subsequently investigate the way in which they effect the influence of an unexpected factor change on unexpected performance, and finally classify them.

The first kind of instruments to be discussed are level changing instruments, denoted by λ. As their name indicates, they change the level of the sensitivity at t_0. In this paper, we have chosen to represent their influence on the sensitivity in an additive way. Hence, the resulting manipulated sensitivity when level changing instruments are applied is b = (s+λ).

Another kind of instruments are the transformation instruments, which transform the unexpected factor change. Their influence will be denoted by $T(\Delta \tilde{f}_U)/\Delta \tilde{f}_U$. Application of these instruments leads to the following manipulated sensitivity: $b = s.(T(\Delta \tilde{f}_U)/\Delta \tilde{f}_U)$.

Finally, we can also distinguish flexibility instruments, represented by $e(\Delta \tilde{f}_U)$. They cause the manipulated sensitivity to be $b = e(\Delta \tilde{f}_U).s$. Of course, the manipulated sensitivity when all three instruments are applied is $b = (s+\lambda).e(\Delta \tilde{f}_U).(T(\Delta \tilde{f}_U)/\Delta \tilde{f}_U)$. Examples illustrating the influence of the various instruments can be found in the following subsection.

4.1 The impact of the instruments

The following example illustrates how instruments affect unexpected performance and the manipulated sensitivity. In this example, we make use of a one-period model from t_0 to t_1. We assume that the interest rate is the only risk factor, that its expected value at t_1 is 6%, and that the probability density function (p.d.f.) of an unexpected change in the interest rate at t_1 resembles the one presented in Figure 1. Furthermore, we suppose that someone borrowed \$ 10.- at the variable interest rate described above. Accordingly, the (non-manipulated) sensitivity, "s", has the value of -\$ 10.-, and is independent of the unexpected change of the interest rate. This sensitivity is shown in the upper picture of Figure 2. The middle picture shows the relation between unexpected performance and the unexpected factor change, and the lower picture the corresponding p.d.f. of unexpected performance, which can be obtained from the middle picture and the p.d.f. of unexpected factor change.

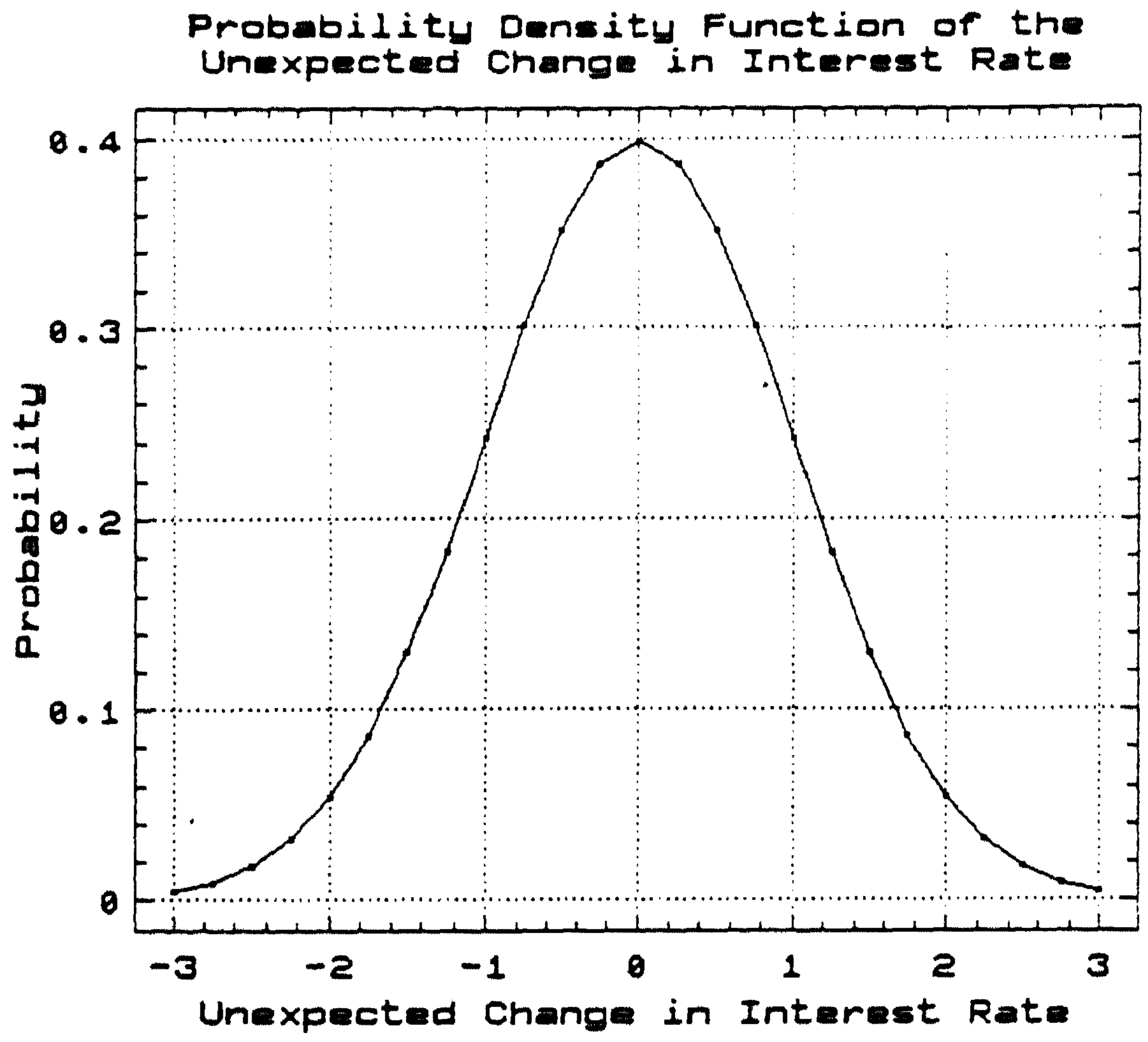

Figure 1: The probability density function of the unexpected change of the interest rate

Let us now investigate the influence of the three kinds of instruments on both the sensitivity and unexpected performance. Suppose that the borrower agrees to pay a fixed interest rate over \$ 4.–. In fact, this can be seen as the application of a level changing instrument with λ = \$ 4.–. As a result, the sensitivity is increased from -10 to -6, see the upper picture of Figure 3. It is easily seen from the middle picture that decreasing the absolute value of the level of the sensitivity makes the linear relation between unexpected performance and unexpected factor change less steep, and leads to a reduction of the variance of unexpected performance (Figure 3, lower picture).

To illustrate how flexibility instruments work, it is assumed that the sensitivity can be adapted to the realized value of the unexpected factor change at t_1 and, hence, is increased in case of a rise in the factor change and decreased in the opposite case (Figure 4, upper picture). (To bring the flexibility instrument into action, actually another decision is also required, as will be explained in Subsection 4.2.) As a result, the linear relation between unexpected performance and the unexpected change of the risk factor is split (Figure 4, middle picture), and the p.d.f. of unexpected performance is distorted to the right (Figure 4, lower picture). Thus increasing the possibility of higher unexpected performance and decreasing the probability of a loss.

Finally, in order to show how transformation instruments work, we assume that a cap at 7 % has been bought. The corresponding transformation function

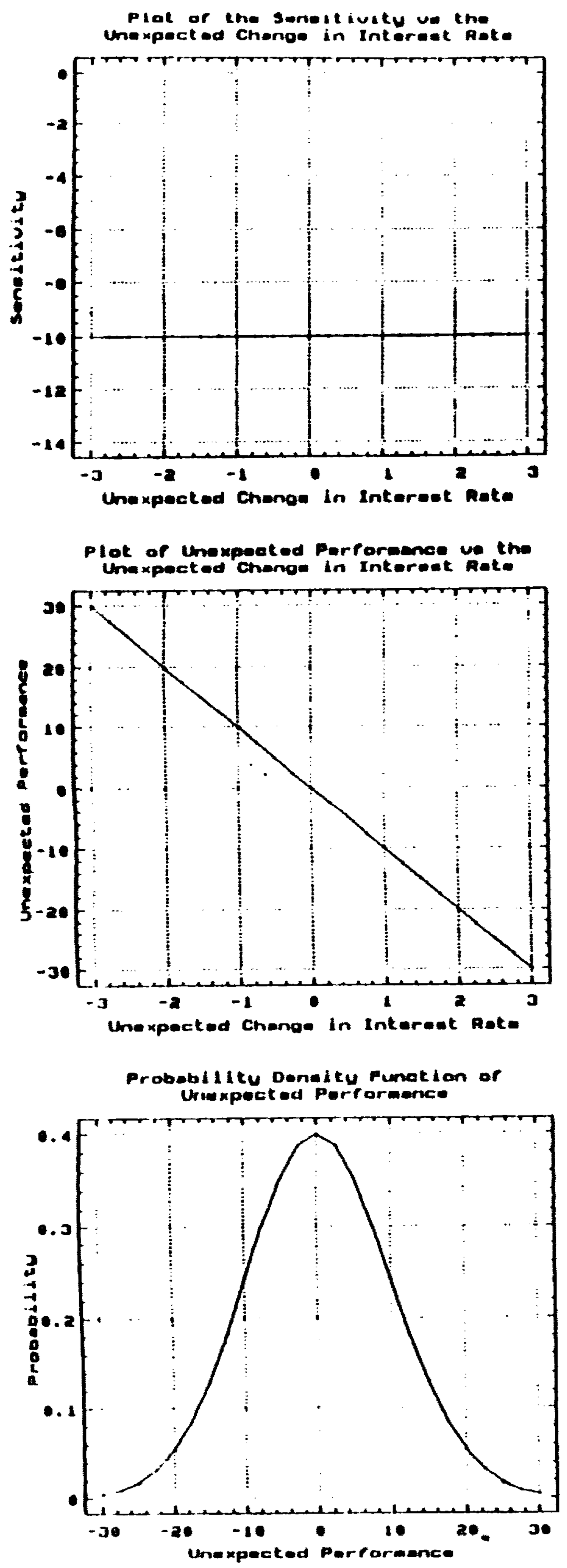

Figure 2

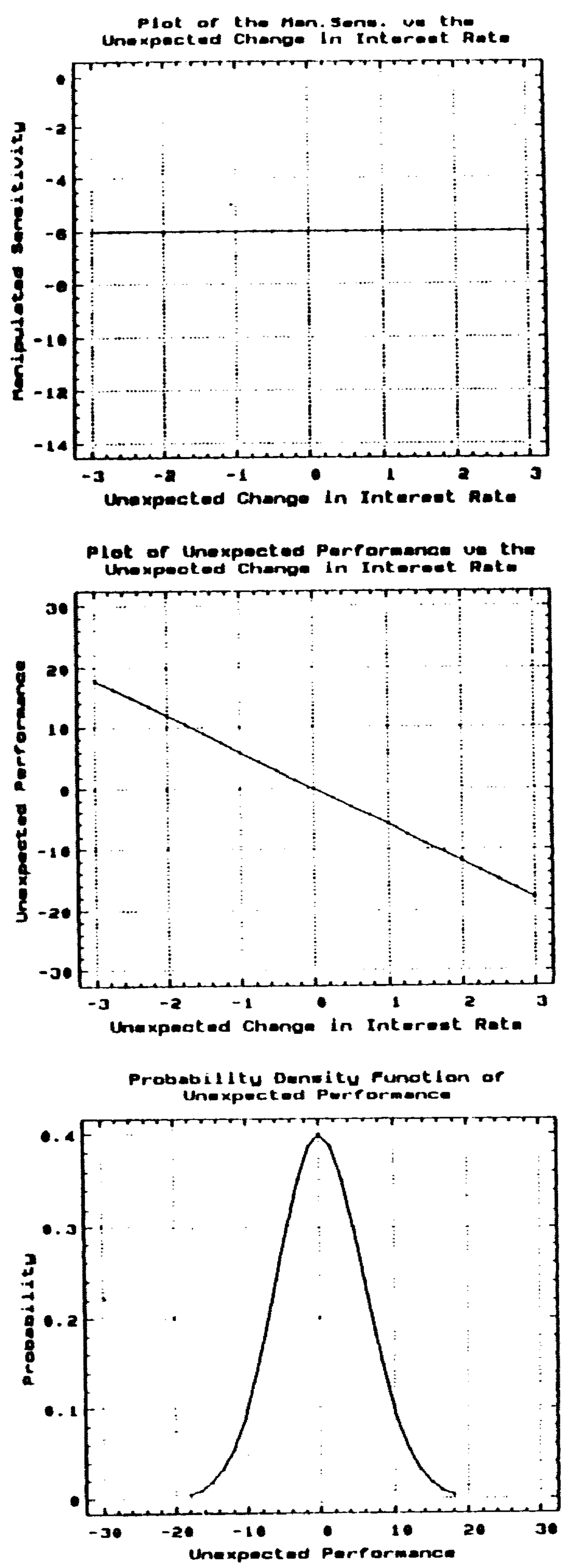

Figure 3

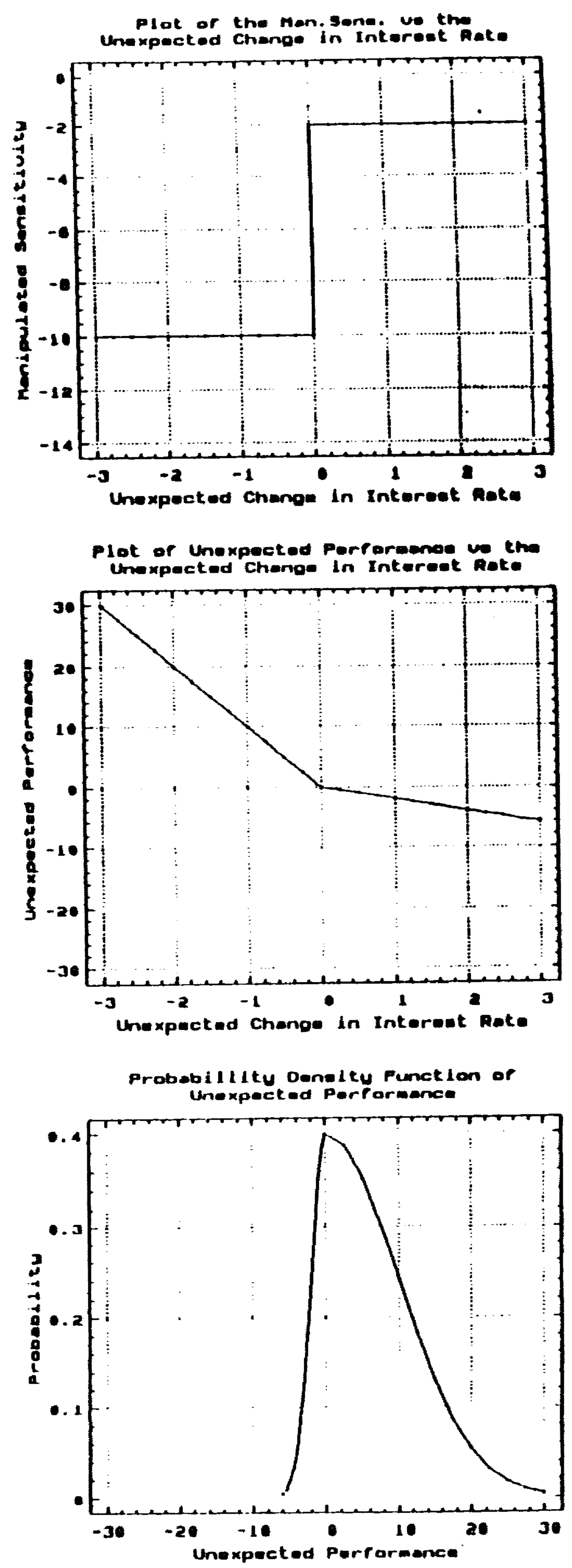

Figure 4

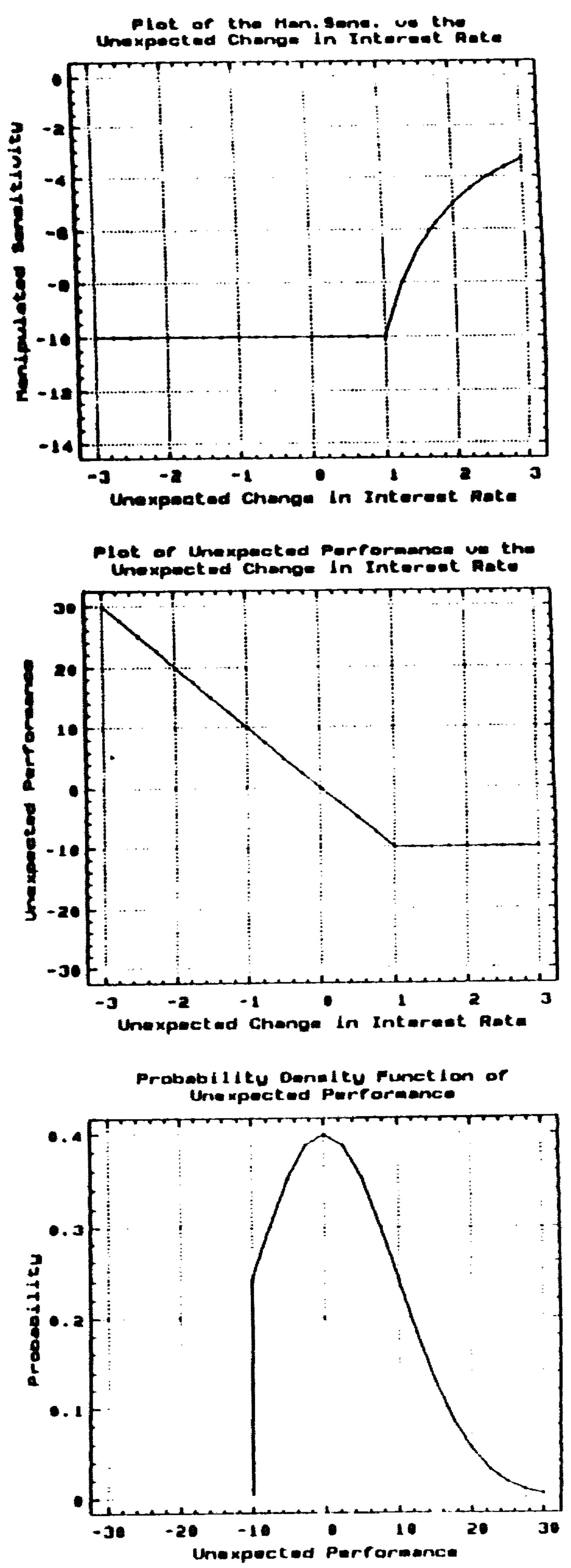

Figure 5

is as follows:

$$T(\Delta f_U)/\Delta f_U = 1 \text{ iff } \Delta f_U < (7\%-6\%) = 1\%$$

$$= (7\%-6\%)/\Delta f_U \text{ iff } \Delta f_U \geq (7\%-6\%) = 1\%$$

With respect to the manipulated sensitivity there holds:

$$b(\Delta f_U) = (T(\Delta f_U)/\Delta f_U).s$$

It is seen that in case of changes of more than 1% the absolute value of the sensitivity is gradually decreasing (see the upper picture of Figure 5), and that unexpected performance as a function of the unexpected factor change becomes constant (middle picture). The result is that the p.d.f. of unexpected performance is cut off (lower picture).

4.2 Differences between the various instruments

An important distinction between the various instruments concerns the decision moments involved in applying them. In the one period model from t_0 to t_1, the decision to buy the instrument has to be made at t_0. Accordingly, all instruments have an anticipatory character. However, flexibility instruments also have a reactive character, because they have to be brought into action at t_1. Of course, both decisions at t_0 and t_1 are made taking into account the costs of the various instruments, as well. Figure 6 summarizes the foregoing.

t_0 ————————————————	t_1
decision to buy	decision to bring into action
-level changing instrument	-flexibility instrument
-transformation instrument	
-flexibility instrument	

Figure 6: Decision chronology concerning the application of the various instruments

The instruments also differ in the way in which they affect the influence of an unexpected factor change on unexpected performance. The influence of level changing instruments is the same for all unexpected risk factor changes and, hence, independent of the unexpected change of the risk factors. However, the influence of the transformation and flexibility instruments is dependent on the unexpected change of the risk factor. For example, a cap only affects the influence of the unexpected factor change on unexpected performance, if the unexpected change is higher than the level determined by the cap. The same holds for flexibility instruments. The potential flexibility will only be used for certain unexpected risk factor changes. It is clear that application of these two instruments results in manipulated sensitivities that are dependent on the unexpected factor changes. This could also be seen in the Figures 3, 4 and 5: the manipulated sensitivity in case a level changing instrument is applied still is a horizontal line, in contrast to the manipulated sensitivity when transformation or flexibility instruments are applied.

Another distinction between the three instruments is illustrated in Figure 7. This figure shows that transformation instruments influence the manipulated sensitivity by transforming the unexpected change of the risk factor, whereas level changing and flexibility instruments influence the manipulated sensitivity by changing the level of the sensitivity.

Unexpected Change of the Risk factor	→	-Transformation Instrument	↘	
				Manipulated Sensitivity
Level of the Sensitivity	→	-Flexibility Instrument -Level changing Instrument	↗	

Figure 7: The difference between transformation and flexibility increasing instruments

5. Unexpected performance revised

Now that we have examined the influence of instruments on the sensitivity, we can go on by defining the concept "unexpected performance". We distinguish between situations in which instruments are applied and not, and between the ex post and the ex ante case.

In case no instruments are applied unexpected performance is calculated as follows in the ex post case. Note that we make use of the already defined concept sensitivity.

$$\Delta R^*_{Uep} = s(\Delta f^*_U).\Delta f^*_U \qquad (8)$$

In the ex ante case, the definition becomes:

$$\Delta R_{Uea} = \int_{-\infty}^{\infty} s(\Delta\tilde{f}_U).\Delta\tilde{f}_U.h(\Delta\tilde{f}_U)\ d\Delta\tilde{f}_U \qquad (9)$$

It can be easily seen that ΔR_{Uea} generally differs from zero, if the sensitivity depends on the unexpected factor change. In case the sensitivity does not depend on the unexpected factor change unexpected performance equals zero.

$$\Delta R_{Uea} = \int_{-\infty}^{\infty} s.\Delta\tilde{f}_U.h(\Delta\tilde{f}_U)\ d\Delta\tilde{f}_U = s \int_{-\infty}^{\infty} \Delta\tilde{f}_U.h(\Delta\tilde{f}_U)\ d\Delta\tilde{f}_U = s.0 = 0 \qquad (10)$$

In other words, if unexpected performance is a linear function of the unexpected change of the risk factor, then the expected ex ante unexpected performance is zero. The expected ex ante unexpected performance is not equal to zero by definition, because unexpected performance is not defined as realized minus expected performance, but as that part of the realized performance, which is due to unexpected factor changes.

In case instruments are applied unexpected performance is calculated as follows. Note that we make use of the already defined concept manipulated sensitivity.

$$\Delta R^*_{Uep} = b(\Delta f^*_U).\Delta f^*_U = (T(\Delta f^*_U)/\Delta f^*_U).e(\Delta f^*_U).(s+\lambda).\Delta f^*_U =$$

$$= T(\Delta f^*_U).e(\Delta f^*_U).(s+\lambda) \qquad (11)$$

And in the ex ante case:

$$\Delta R_{Uea} = \int_{-\infty}^{\infty} T(\Delta\tilde{f}_U).e(\Delta\tilde{f}_U).(s+\lambda).h(\Delta\tilde{f}_U)\ d\Delta\tilde{f}_U \qquad (12)$$

Again, there holds that if $(T(\Delta\tilde{f}_U)/\Delta\tilde{f}_U).e(\Delta\tilde{f}_U).(s+\lambda)$ is a constant term, then the expected ex ante unexpected performance equals zero.

6. Summary and conclusion

Although the multi-factor method has already been used many times, clear definitions of the concepts used in this method are scarce. For this reason, this paper extensively defines the relevant concepts, such as sensitivity, unexpected risk factor change, unexpected performance and instruments.

It can be concluded that when analysing risks, we are interested in the ipulated sensitivity of a performance measure for unexpected changes of a factor. The manipulated sensitivity relates an unexpected factor change unexpected performance, and depends both on the sensitivity and the :uments applied to it such as level changing, flexibility, and ısformation instruments.

$\rightarrow$ $(T(\Delta f_U)/\Delta f_U).e(\Delta f_U).(\lambda+s)$ $\rightarrow$ ΔR_U

expected ctor Change)	(Manipulated Sensitivity)	(Unexpected Performance)

$\uparrow$

Instruments applied
-Level changing instruments, λ
-Transformation instruments, $T(\Delta f_U)/\Delta f_U$
-Flexibility increasing instruments, $e(\Delta f_U)$

$\uparrow$

Sensitivity, s

ure 8: The relation between the used concepts

Figure 8 the relation between the different concepts is shown. In the ex ;t case we substitute the realized unexpected change of the factor value, ereas in the ex ante case we make use of the probability density function the unexpected change of the factor value.