Integrated Market Selection and Production Planning: Complexity and Solution Approaches

Wilco van den Heuvel[*†] O. Erhun Kundakcioglu[‡] Joseph Geunes[‡§]
H. Edwin Romeijn[‡§] Thomas C. Sharkey[‡¶]
Albert P.M. Wagelmans[*]

Econometric Institute Report EI 2007-45

October 2007

Abstract

Emphasis on effective demand management is becoming increasingly recognized as an important factor in operations performance. Operations models that account for supply costs and constraints as well as a supplier's ability to influence demand characteristics can lead to an improved match between supply and demand. This paper presents a new class of optimization models that allow a supplier to select, from a set of potential markets, those markets that provide maximum profit when production/procurement economies of scale exist in the supply process. The resulting optimization problem we study possesses an interesting structure and we show that although the general problem is $\mathcal{NP}$-complete, a number of relevant and practical special cases can be solved in polynomial time. We also provide a computationally very efficient

[*]Econometric Institute, Erasmus University Rotterdam, P.O. Box 1738, 3000 DR Rotterdam, The Netherlands; e-mail: wvandenheuvel@few.eur.nl, wagelmans@few.eur.nl.

[†]The work of this author was done while he was a visiting scholar at the Department of Industrial and Systems Engineering at the University of Florida. He would like to thank the faculty, students, and staff for their kind hospitality.

[‡]Department of Industrial and Systems Engineering, University of Florida, P.O. Box 116595, Gainesville, Florida 32611-6595; e-mail: erhun@ufl.edu, geunes@ise.ufl.edu, romeijn@ise.ufl.edu, sharkeyt@ufl.edu.

[§]The work of this author was supported by the National Science Foundation under Grant No. DMI-0355533.

[¶]The work of this author was supported by a Graduate Research Fellowship of the National Science Foundation.

and intuitively attractive heuristic solution procedure that performs extremely well on a large number of test instances.

1 Introduction

Production/procurement planning problems are among the most well studied problems in the operations literature. The majority of research in this area focuses on managing the supply side of the problem when capacities and/or nonlinear cost structures lead to problems containing a combinatorial structure. Much of this past research provides efficient solution methods for minimizing the supply cost incurred in order to meet forecasted demand. Recent research recognizes the importance of accounting for the levers suppliers have for managing or *shaping* demand in order to ensure the most profitable match between supply and demand (see, e.g., Lapide (2006) and Ettl et al. (2006)). Spurred on by the development of research in the area of revenue management in the past 20 years (see Talluri and van Ryzin (2004) for a thorough discussion of this field), a number of recent papers have considered the pricing lever that suppliers can use to help shape demand within procurement planning contexts (see, e.g., Gilbert (1999, 2000), Biller et al. (2005), Chen and Simchi-Levi (2004a,b), Chan et al. (2006), Van den Heuvel and Wagelmans (2006), Geunes et al. (2006), Deng and Yano (2006)). We note that these more recent works built on the foundations provided by Thomas (1970, 1974), Florian and Klein (1971), and Kunreuther and Schrage (1973).

Pricing can be considered as a (partial) determinant of the set of demands a supplier implicitly *selects* for a given product. That is, if a supplier decides to sell an item in a given market or through a given sales channel, then the combination of consumer utility functions and reservation prices in the market determine those consumers who ultimately purchase the item. Thus, pricing serves as a critical lever that a supplier can use for demand shaping. At a higher level in the decision hierarchy, however, a supplier must often first make an explicit selection of the set of markets or sales channels in which it will offer the item. This paper focuses on this higher level demand shaping decision, and its role in determining the best match between supply and demand when the production/procurement process involves economies of scale in output.

The class of problems we consider can be stated simply and concisely: Given a set of potential markets[1] for an item and the sales (and revenue) levels implied by each market in every period over

[1]Note that the term "market" may be used interchangeably with "sales channels" or "customers."

a finite horizon, determine a subset of markets that maximizes net profit, given that all demand for each selected market must be satisfied. Clearly, when supply costs are linear, each market's profitability can be determined in isolation, and the resulting problem is easily solved. However, economies of scale in production/procurement for an item can lead to a high degree of combinatorial complexity in determining a subset of markets that maximizes a supplier's net profit from sales. In fact, we will show that when supply cost structures are consistent with those found in the often studied and well-solved Wagner-Whitin problem (Wagner and Whitin, 1958), the resulting integrated market selection and production planning problem is $\mathcal{NP}$-complete. The optimization problem that results under these supply cost structures serves as the focus of this paper.

This optimization problem has a particularly interesting structure, as a number of its practical special cases are polynomially solvable. For instance, if we are given a set of M markets that must be served, then the resulting problem (after a preprocessing step in which the aggregate demand of all markets is determined in $\mathcal{O}(MT)$ time) takes the form of a standard uncapacitated lot sizing problem, which can be solved in $\mathcal{O}(T \log T)$ in general (where T is the planning horizon length) and $\mathcal{O}(T)$ under stationary costs (see, e.g., Wagelmans et al. (1992)). On the other hand, if we are given a prescribed set of procurement periods, then we simply need to determine, for each market, whether the average revenue per demand unit of the market exceeds the average cost per demand unit of the procurement plan (which, after a preprocessing step in which these average costs per demand unit are determined in $\mathcal{O}(MT)$ time, can be done in $\mathcal{O}(M)$ time, where M is the number of potential markets). When the supplier has flexibility in simultaneously determining which markets to serve and an optimal corresponding procurement plan, however, the resulting problem cannot in general be solved in polynomial time (unless $\mathcal{P} = \mathcal{NP}$). Despite this negative result, we will explore a number of special cases of the problem that may occur in practice and can be solved in polynomial time. Moreover, we provide a new heuristic solution approach for the problem and demonstrate the effectiveness of this approach on a large number of test instances. Thus, the contributions of this paper include (i) the definition, formulation, and modeling of this new, interesting, and relevant problem class; (ii) characterization of the worst-case complexity of this problem class; (iii) identification of a number of polynomially solvable special cases; and (iv) providing an intuitively attractive heuristic solution method for the problem with excellent performance results.

The remainder of this paper is organized as follows. The following section formally defines and formulates the integrated market selection and procurement planning problem, which serves as the focus of this paper. Section 3 demonstrates the $\mathcal{NP}$-completeness of this problem, while Section 4 provides polynomial-time solution approaches for a number of special cases of the problem. Section 5 describes our heuristic solution approach, and Section 6 validates the performance of this heuristic on a broad test set of problem instances. In Section 7 we provide concluding remarks and directions for future work on this class of problems.

2 Formal problem description

The market selection problem (MSP) can be described as follows. We are given a number of markets and a revenue associated with each market. Each market has a known (deterministic) demand for every period of a discrete and finite time horizon. There is a single manufacturer that has the possibility to either satisfy the demand of a market or to reject a market. If the manufacturer decides to select a market, the demand of this market has to be satisfied in all periods. If the manufacturer rejects a market, then no demand from this market needs to be satisfied and the revenue is lost.

Given a selection of markets, the manufacturer faces a production planning problem. The cost of a production plan consists of a fixed setup cost for every period with positive production, production cost for each unit produced, and holding cost for each item in inventory at the end of a period. The objective of the manufacturer is to maximize total profit, i.e., to make a selection of markets that maximizes the revenues minus the production cost associated with the demands of the selected markets.

If we use the following notation:

Data:

M : number of markets,

R_m : revenue of market m $(m = 1, \ldots, M)$,

T : number of time periods,

d_t^m : demand of market m in period t $(m = 1, \ldots, M;\ t = 1, \ldots, T)$,

K_t : setup cost in period t $(t = 1, \ldots, T)$,

p_t : unit production cost in period t $(t = 1, \ldots, T)$,

h_t : holding cost in period t $(t = 1, \ldots, T)$,

$c_{i,t}^m$: variable production and holding cost to satisfy demand d_t^m by production in period i, i.e. $c_{i,t}^m = d_t^m(p_i + \sum_{j=i}^{t-1} h_j)$ $(i, t = 1, \ldots, T,\ i \leq t;\ m = 1, \ldots, M)$,

Decision variables:

$x_{i,t}^m$: proportion of demand d_t^m satisfied by production in period i $(i, t = 1, \ldots, T,\ i \leq t;\ m = 1, \ldots, M)$,

y_t : 1 if there is a setup in period t, 0 otherwise $(t = 1, \ldots, T)$,

z_m : 1 if market m is selected, 0 otherwise, $(m = 1, \ldots, M)$,

then the problem can be formulated as the following mixed integer program (MIP):

$$
\begin{aligned}
(P) \quad \max \quad & \sum_{m=1}^{M} R_m z_m - \sum_{t=1}^{T} \left(K_t y_t + \sum_{i=1}^{t} \sum_{m=1}^{M} c_{i,t}^m x_{i,t}^m \right) && && (1) \\
\text{s.t.} \quad & \sum_{i=1}^{t} x_{i,t}^m = z_m, && t = 1, \ldots, T,\ m = 1, \ldots, M, && (2) \\
& x_{i,t}^m \leq y_i, && i, t = 1, \ldots, T,\ i \leq t;\ m = 1, \ldots, M, && (3) \\
& x_{i,t}^m \geq 0, && i, t = 1, \ldots, T,\ i \leq t;\ m = 1, \ldots, M, && (4) \\
& y_t \in \{0, 1\}, && t = 1, \ldots, T, && (5) \\
& z_m \in \{0, 1\}, && m = 1, \ldots, M. && (6)
\end{aligned}
$$

The objective function maximizes total revenue minus total production cost. The first set of constraints ensures that demand is satisfied when a market is selected. Constraints (3) are the setup forcing constraints: production in period i can only occur if there is a setup in that period. Constraints (4) ensure that production is nonnegative, while constraints (5) and (6) represent the binary setup and market selection decisions.

For a given selection of markets or a *market selection* for short, i.e., a vector $z \in \{0, 1\}^M$, the problem boils down to a classical lot-sizing problem (Wagner and Whitin, 1958), where demand in a period equals the sum of the demands of the selected markets in that period. Federgruen and Tzur (1991), Wagelmans et al. (1992) and Aggarwal and Park (1993) show that the classical lot-sizing problem can be solved in $\mathcal{O}(T \log T)$ time in case of general cost parameters and in $\mathcal{O}(T)$ time in the case of non-speculative motives, i.e., $p_t + h_t \geq p_{t+1}$ for $t = 1, \ldots, T - 1$. The polynomial solvability of the MSP for a given market selection is also captured in model (P). Namely, for a given market selection the associated subproblem of formulation (P) is equivalent to the so-called plant location formulation. Krarup and Bilde (1977) show that the linear relaxation of this formulation has integer y-variables in an optimal solution. This then, in turn, implies that in the presence of the binary constraints (6), the binary constraints (5) can be relaxed.

On the other hand, given any production plan (i.e., given the y-variables), the problem is also easy to solve. As the zero-inventory property holds, one can calculate the variable production and holding cost for every market. Then, only markets for which the revenues exceed the variable production costs are selected. This implies in addition that, in the presence of the binary constraints (5), the binary constraints (6) can be relaxed.

In conclusion, the problem is easy to solve if either a selection of markets (the z-variables) or a production plan (the y-variables) is fixed. Moreover, one of the constraint sets (5) and (6) can be relaxed. Unfortunately, however, we cannot relax both constraint sets simultaneously, so the problem is not easily solvable in general.

3 $\mathcal{NP}$-completeness and approximation results

In this section we will show that the market selection problem is strongly $\mathcal{NP}$-complete. Proving this result requires the following steps:

1. We first construct a special case of the MSP in which costs and revenues have a special structure and where, in each period, at most one market has positive demand.

2. We will then show that, for this special case of the MSP, a solution with positive profit exists if and only if a selection of markets exists with the property that all subsequences of consecutive periods with positive demand are of even length. We will call this property the *even subsequence property*.

3. Finally, we will show that the problem of identifying a set of markets satisfying the even subsequence property is $\mathcal{NP}$-complete through a reduction from the 3SAT problem.

After showing that the MSP is strongly $\mathcal{NP}$-complete, we will end the section with a discussion of approximation results.

3.1 A special case of the MSP

The class of instances of the MSP that we employ is based on an instance of the uncapacitated lot-sizing problem with $d_t = 1$, $K_t = 2$, $h_t = 1$, and $p_t = 0$ for $t = 1, \ldots, T$.

When the horizon T is infinite, it is optimal to produce 2 units in every odd period, as this minimizes the average cost per time period. In other words, defining a subplan as an interval for which demands are satisfied by the same production period, then every subplan in an optimal solution consists of 2 periods, and the average cost per unit demand equals $\frac{3}{2}$. It is easy to see that the same solution structure yields an optimal solution when T is finite and even, in which case the total cost optimal cost is given by $C(T) = \frac{3}{2}T$ and the average cost per unit demand by $\bar{C}(T) = \frac{3}{2}$.

Now consider a problem instance with a finite horizon T which is given by $T = 2n + 1$ for some $n \in \mathbb{N}$, i.e., T is odd. An optimal solution for this instance consists of n two-period subplans and 1 subplan that supplies demand in a single period. The cost of this solution is

$$C(T) = \frac{3}{2}T + \frac{1}{2}. \tag{7}$$

(Note that a solution with $n - 1$ subplans consisting of two periods and 1 subplan consisting of 3 periods has the same cost.) The average cost per unit demand for this instance equals

$$\bar{C}(T) = \frac{3}{2} + \frac{1}{2T} > \frac{3}{2}.$$

Based on this lot-sizing problem, we define the following class of instances of the MSP with markets $\mathcal{M} = \{1, \ldots, M\}$:

- the planning horizon T is odd;
- $K_t = 2$, $h_t = 1$, $p_t = 0$ for $t = 1, \ldots, T$;
- for each period t there is exactly one market $m \in \mathcal{M}$ such that $d_t^m = 1$, and $d_t^{m'} = 0$ for all $m' \neq m$;
- $R_m = r \sum_{t=1}^{T} d_t^m$ with $r = \bar{C}(T)$.

For this class of problem instances, we will next show that an optimal solution exists containing positive profit if and only if a selection of markets exists that satisfies the even subsequence property, i.e., the property that all subsequences of consecutive periods with positive demand are of even length. Now let $\mathcal{M}' \subset \mathcal{M}$ denote a set of selected markets and $\Pi(\mathcal{M}')$ the corresponding profit. Let $d^{\mathcal{M}'}$ be the demand sequence induced by $\mathcal{M}'$, i.e., $d_t^{\mathcal{M}'} = \sum_{m\in\mathcal{M}'} d_t^m$. Then a market selection $\mathcal{M}'$ satisfies the even subsequence property if, for any $s < t$ with $d_s^{\mathcal{M}'} = d_{t+1}^{\mathcal{M}'} = 0$ and $d_i^{\mathcal{M}'} = 1$ for $i = s+1, \ldots, t$ we have that $t - s$ is even (where, for convenience, we let $d_0^{\mathcal{M}'} = d_{T+1}^{\mathcal{M}'} = 0$).

We have the following result:

Theorem 1 *Consider a problem instance of the market selection problem as described above. Then a selection of markets has strictly positive profit if and only if it satisfies the even subsequence property.*

Proof Note that all possible solutions for this market selection problem instance fall into one of the following four categories:

- *select no markets, i.e., $\mathcal{M}' = \emptyset$:*

  Clearly, $\Pi(\mathcal{M}') = 0$.

- *select all markets, i.e., $\mathcal{M}' = \mathcal{M}$:*

  In this case the profit equals

$$\Pi(\mathcal{M}') = \sum_{m=1}^{M} R_m - C(T) = \sum_{m=1}^{M} r \sum_{t=1}^{T} d_t^m - C(T) = rT - C(T) = 0.$$

- *$\mathcal{M}' \notin \{\emptyset, \mathcal{M}\}$ and $\mathcal{M}'$ does not satisfy the even subsequence property:*

  Let k be the number of periods in the selection with unit demand, i.e., $k = \sum_{t=1}^{T} \sum_{m\in\mathcal{M}'} d_t^m$. As there is at least one odd subsequence of unit demands, it follows from (7) that the total cost $C(\mathcal{M}')$ equals at least $\frac{3}{2}k + \frac{1}{2}$. Therefore,

$$\Pi(\mathcal{M}') = \sum_{m\in\mathcal{M}'} R_m - C(\mathcal{M}') \leq rk - \left(\frac{3}{2}k + \frac{1}{2}\right) < rT - \left(\frac{3}{2}T + \frac{1}{2}\right) = 0.$$

- *$\mathcal{M}' \notin \{\emptyset, \mathcal{M}\}$ and $\mathcal{M}'$ satisfies the even subsequence property:*

  Again, let k be the number of periods in the selection with unit demand. As we have only even subsequences of unit demands, the optimal lot-sizing solution is to have only subplans of 2 periods. This means that the average cost per unit demand equals $\frac{3}{2}$, and the optimal profit equals:

$$\Pi(\mathcal{M}') = \sum_{m\in\mathcal{M}'} R_m - C(\mathcal{M}') = rk - \frac{3}{2}k = k\left(r - \frac{3}{2}\right) > 0.$$

This proves the desired result. $\square$

3.2 Reduction from 3SAT

Consider the following decision version of the market selection problem:

Market Selection Decision (MSD) problem: Is there a *non-empty* set of markets with profit strictly larger than B?

Note that Theorem 1 implies that when $B = 0$, for problem instances of the MSP as described in Section 3.1, solving the MSD problem is equivalent to determining whether a selection of markets exists that satisfies the even subsequence property. Using a reduction from the 3SAT problem to this decision problem, we next show the following result:

Theorem 2 *MSD is strongly $\mathcal{NP}$-complete.*

Proof It is easy to see that MSD $\in \mathcal{NP}$ since a nondeterministic algorithm needs only guess a selection of markets and check in polynomial time whether that selection has a profit larger than B. In the following reduction we set $B = 0$ and construct a problem instance as described in Section 3.1. To show that MSD is $\mathcal{NP}$-complete it is sufficient to prove that MSD is $\mathcal{NP}$-complete for this class of instances. Furthermore, it follows from Theorem 1 that MSD is $\mathcal{NP}$-complete for this class of instances if the corresponding problem of finding a market selection that satisfies the even subsequence property is $\mathcal{NP}$-complete.

The classical 3SAT problem is described as follows: Given a collection $C = \{c_1, c_2, \ldots, c_m\}$ of clauses on a finite set U of variables such that $|c_i| = 3$ for $1 \leq i \leq m$, is there a truth assignment for U that satisfies all the clauses in C? This problem is known to be $\mathcal{NP}$-complete (Garey and Johnson, 1979).

Before we proceed with the reduction, we make the following observations for any instance of the 3SAT problem. First, note that if u is a variable in U, then u and $\bar{u}$ are *literals* over U, and we will denote a generic literal by l. Then each clause c_i can be represented as a disjunction of 2 *subclauses* α_i and β_i, where $|\alpha_i| = 2$ and β_i is a literal. Without loss of generality we let $\alpha_i = l_1^i \vee l_2^i$ and $\beta_i = l_3^i$ for each clause of the form $c_i = l_1^i \vee l_2^i \vee l_3^i$. Therefore, a truth assignment should satisfy

$$u_j \oplus \bar{u}_j = 1, \qquad j = 1, 2, \ldots, |U| \tag{8}$$

$$l_1^i \vee l_2^i = \alpha_i, \qquad i = 1, 2, \ldots, |C| \tag{9}$$

$$\alpha_i \vee l_3^i = 1, \qquad i = 1, 2, \ldots, |C| \tag{10}$$

(where the $\vee$ operator corresponds to the logical "OR" operator, while the $\oplus$ operator corresponds to the logical "Exclusive OR" operator).

Now suppose we are given an instance of the 3SAT problem. We will construct an instance of the MSD problem that uses the cost and revenue parameters provided in Section 3.1. The construction of this instance of the MSD will create a market for each literal and subclause in the 3SAT problem, where the selection of a market corresponds to a truth assignment for the corresponding literal or subclause. In addition, we will also create two "dummy markets" that are needed to ensure equivalence with 3SAT. More formally, we create the following markets:

(i) Create 2 markets for each variable in U and its negation, i.e., m_{u_j} and $m_{\bar{u}_j}$ ($j = 1, \ldots, |U|$).

(ii) Create 1 market that corresponds to a *subclause* α of each clause in C, i.e., m_{α_i} ($i = 1, \ldots, |C|$).

(iii) Create 2 markets m_s and m_n. (We will later show that m_s must be selected and m_n must not be selected in a market selection with the even subsequence property.)

Given these markets, we construct a specially structured demand matrix such that a truth assignment exists for the instance of 3SAT if and only if the MSD problem we construct contains a profitable selection of markets (i.e., satisfying the even subsequence property). Note that, since in each period there is exactly one market with positive (unit) demand, we can characterize all demands through a vector of market indices that indicate the market that has positive demand in the corresponding period.

(i) For the first 3 periods introduce unit demands for markets

$$[m_n \; m_s \; m_s], \tag{11}$$

i.e., $d_1^{m_n} = 1$, $d_2^{m_s} = 1$, $d_3^{m_s} = 1$ (and the remaining demands in periods 1, 2, and 3 are zero).

(ii) For each variable u_j, $j = 1, 2 \ldots, |U|$, add 4 periods with a unit demand for markets

$$[m_{u_j} \; m_s \; m_{\bar{u}_j} \; m_n], \tag{12}$$

(iii) For each clause c_i, $i = 1, 2 \ldots, |C|$, add 8 periods with a unit demand for markets

$$[m_{l_1^i} \; m_{\alpha_i} \; m_{l_2^i} \; m_{l_1^i} \; m_{\alpha_i} \; m_{l_2^i} \; m_n \; m_n], \tag{13}$$

and 8 additional periods with a unit demand for markets

$$[m_{\alpha_i}\ m_s\ m_{l_3^i}\ m_{\alpha_i}\ m_s\ m_{l_3^i}\ m_n\ m_n]. \tag{14}$$

We next need to show that, given the instance of MSD we have defined, a selection of markets satisfying the even subsequence property exists if and only if a truth assignment exists for the corresponding 3SAT problem. With this in mind, we call a market selection *feasible* if and only if it satisfies the even subsequence property. Initially, we assume that m_s is selected and m_n is not selected for any feasible solution, and later verify that this must indeed hold. Under this assumption, observe that the demand pattern in (12) satisfies a feasible market selection if and only if the exclusive disjunction requirement in (8) is satisfied. The pattern in (13) then ensures that m_{α_i} is selected if and only if $m_{l_1^i}$ or $m_{l_2^i}$ or both are selected, which corresponds to the requirement in (9). Similarly, the pattern in (14) guarantees a feasible market selection if and only if (10) is satisfied by either selecting m_{α_i} or $m_{l_3^i}$ or both. Therefore, we can conclude that, assuming m_s must be selected and m_n must not be selected for any feasible solution, the MSD problem instance has a feasible market selection if and only if the 3SAT instance has a "true" assignment.

It remains to show that m_s must be selected and m_n must not be selected for any feasible solution.

- Assume that m_n is selected. From (11), m_n appears in the first period followed by a demand from m_s, therefore m_s has to be selected as well. Selection of m_n and m_s together forces all introduced markets to be selected as a result of (12), (13) and (14). However, this solution is *infeasible* because the total number of periods is not even.

- Assume that neither m_s nor m_n is selected. Then (12), (13) and (14) require that none of the markets are selected and the result is an *empty* set, which is also *infeasible*.

In conclusion, if 3SAT has a "true" assignment, then only m_s must be selected among these two markets and the MSD problem results with a feasible market selection. The presented reduction is polynomial and hence the decision version of the market selection problem is $\mathcal{NP}$-complete. □

An example of the demand matrix resulting from the above reduction for a single clause is given in Table 1. Note that the cost parameters are time-invariant in the proof. This means that we are considering the least general case for the cost parameters in terms of time-dependency. Furthermore, the market revenue per demand unit r_m is also constant for all markets (i.e., $r_m = r = R_m / \sum_{t=1}^{T} d_t^m$).

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
u_1				1												1			1												
$\bar{u}_1$						1																									
u_2								1										1			1										
$\bar{u}_2$										1																					
u_3												1												1			1				
$\bar{u}_3$														1																	
α_1																	1			1						1			1		
d_s		1	1		1				1				1												1			1			
d_n	1						1				1				1							1	1							1	1

Table 1: Demand matrix resulting from the clause $u_1 \vee u_2 \vee u_3 = \alpha_1 \vee u_3$.

The $\mathcal{NP}$-completeness proof of Theorem 2 is based on instances of the MSP in which the demand matrix is very sparse. However, by making a slight modification to the proof, it can be shown that the MSP is also $\mathcal{NP}$-complete if we restrict ourselves to problem instances in which each market has positive demand in each period. In particular, for the class of problem instances of Section 3.1, replace the zero demands by a positive value $\varepsilon > 0$. It can be verified that a market selection has a strictly positive profit if and only if the following properties are satisfied: (i) the selection has a unit demand in period 1, (ii) the selection satisfies the even subsequence property, and (iii) ε is sufficiently small (for example, $\varepsilon = (r - \frac{3}{2})/(MT^2)$). In the reduction, we then add two other time periods at the start of the planning horizon with unit demands for market m_s. This is necessary because we do not want to have a setup in period 1 to satisfy a demand of ε, as this causes a negative profit for a market selection satisfying the even subsequence property. With this modification, the problem instance preserves the property that any feasible truth assignment for the 3SAT instance corresponds to a market selection that satisfies the even subsequence property (and vice versa). Furthermore, because only market selections with properties (i)–(iii) have a strictly positive profit and m_s has a unit demand in period 1, the class of problem instances with only strictly positive demands is $\mathcal{NP}$-complete.

3.3 Approximation results

As we will show next, the proof of Theorem 2 in fact implies that it is unlikely that the solution to the MSP can be approximated efficiently. An algorithm is called an $(1-\varepsilon)$-approximation algorithm if, for any problem instance, the algorithm finds a solution with profit Π_A that satisfies

$$\Pi_A \geq (1-\varepsilon)\Pi^*,$$

where $\Pi^* \geq 0$ is the optimal profit and $\varepsilon > 0$.

Theorem 3 *There exists no polynomial-time* $(1-\varepsilon)$*-approximation algorithm for the MSP for any* $0 < \varepsilon < 1$ *unless* $\mathcal{P} = \mathcal{NP}$*.*

Proof Consider a problem instance for the MSP reduced from a 3SAT instance as described in the proof of Theorem 2. Recall that any solution for the MSP instance with profit $\Pi > 0$ corresponds to a feasible truth assignment for the 3SAT instance and that a solution with $\Pi \leq 0$ corresponds to an infeasible truth assignment. Assume there exists a polynomial-time $(1-\varepsilon)$-approximation algorithm A for some $0 < \varepsilon < 1$. If $\Pi^* = 0$, then the 3SAT instance is unsatisfiable and the profit found by algorithm A satisfies $\Pi_A \geq (1-\varepsilon)\Pi^* = 0$ and hence $\Pi^A = 0$. If $\Pi^* > 0$, then the 3SAT instance is satisfiable and the profit of the solution found by algorithm A satisfies $\Pi_A \geq (1-\varepsilon)\Pi^* > 0$. But this means that algorithm A determines in polynomial time whether the 3SAT instance is satisfiable, which is a contradiction unless $\mathcal{P} = \mathcal{NP}$. □

Theorem 3 shows that the problem is inapproximable for any $0 < \varepsilon < 1$. Note that the algorithm that selects the empty set of markets has profit $\Pi = 0$ and hence is a 0-approximation. The inapproximability result is in contrast with the result of Levi et al. (2005). They consider a number of inventory and facility location models with market selection which includes the MSP. However, instead of maximizing profit, they minimize the following cost function:

$$\sum_{m=1}^{M} R_m(1-z_m) + \sum_{t=1}^{T}\left(K_t y_t + \sum_{i=1}^{t}\sum_{m=1}^{M} c_{i,t}^m x_{i,t}^m\right). \tag{15}$$

In this objective function the lost revenue of a non-selected market is considered an opportunity cost. Levi et al. (2005) develop a polynomial-time 1.582-approximation algorithm for the MSP.

At first sight, this approximation result seems to be inconsistent with the inapproximability result of Theorem 3. However, this is not the case. Let $\Gamma(\mathcal{M}')$ be the cost for a market selection

$\mathcal{M}' \subset \mathcal{M}$ using (15) and let $\Pi(\mathcal{M}')$ be the profit for this market selection using (1). Then the relationship

$$\Gamma(\mathcal{M}') + \Pi(\mathcal{M}') = \sum_{m=1}^{M} R_m$$

holds. Assume there exists a $(1+\varepsilon)$-approximation for the minimization problem for some $\varepsilon > 0$. For a given MSP instance let $\mathcal{M}^A$ be the market selection found by this algorithm and let $\mathcal{M}^*$ be the optimal market selection. Then

$$\frac{\Gamma(\mathcal{M}^A)}{\Gamma(\mathcal{M}^*)} \leq 1 + \varepsilon \Leftrightarrow \frac{\Pi(\mathcal{M}^A)}{\Pi(\mathcal{M}^*)} \geq (1+\varepsilon) - \varepsilon \frac{\sum_{m=1}^{M} R_m}{\Pi(\mathcal{M}^*)}.$$

As $\left(\sum_{m=1}^{M} R_m\right)/\Pi(\mathcal{M}^*)$ can be arbitrarily large, the $(1+\varepsilon)$-approximation algorithm for the cost minimization problem does not provide any performance guarantee for the profit maximization problem.

4 Polynomially solvable cases

Although the general MSP is $\mathcal{NP}$-hard, in this section we will present some practical special cases for which the MSP can be solved in polynomial time. We will use different techniques and results from the literature to solve these cases. For example, one case can be solved by finding a maximum-flow/minimum-cut in an appropriately defined network, while other cases will be solved through dynamic programming (DP).

4.1 Seasonal demand

Consider a set of markets that only differ in demand volume but that experience the same seasonal demand fluctuations, i.e., $d_t^m = \sigma_t d_m$ where σ_t is the seasonal coefficient for period t and d_m represents the demand volume of market m. We will call this the seasonal demand case and we will show that this case can be solved in polynomial time.

To this end, consider the following parametric lot-sizing problem. First, let the base demand vector be given by $\sigma = (\sigma_1, \ldots, \sigma_T)$. Then let $C_S(d)$ be the cost of a lot-sizing problem with demand vector σd. It is not difficult to see that the lot-sizing cost increases linearly in d for a given production plan. Because $C_S(d)$ is the lower envelope of the cost functions of all production plans, it is piecewise linear and concave.

This means that for the seasonal case we can rewrite the MSP as

$$\begin{array}{lll}(\mathrm{P}_S) & \max & \displaystyle\sum_{m=1}^{M} R_m z_m - C_S\left(\sum_{m=1}^{M} d_m z_m\right) \\ & \text{s.t.} & z_m \in \{0,1\}, \qquad\qquad\qquad m = 1,\ldots,M.\end{array}$$

Huang et al. (2005) show that for any concave function C_S, the optimal solution to (P_S) can be found in polynomial time. In particular, they show that if we re-order the markets in nonincreasing order of the ratio R_m/d_m, an optimal solution to (P_S) can be found among the market selections $\{1,\ldots,m\}$ for $m = 0,\ldots,M$. Since the ordering of the markets takes $\mathcal{O}(M \log M)$ time and because we have to solve M lot-sizing problems, the total running time to solve the seasonal demand case is $\mathcal{O}(M(\log M + T \log T))$.

4.2 Market-specific prices

We next consider instances of the MSP where the demand for market m in period t is given by

$$d_t^m = \alpha_t - \beta_t p_m$$

where α_t and β_t are time-dependent coefficients of a linear price-demand response curve and p_m is the price in market m. Note that we consider each of the market prices to be fixed, i.e., they are not themselves decision variables in our problem. Note also that, more generally, p_m could represent any market-dependent but stationary function of price, allowing for more general price-demand response curves than the linear one presented above.

The aggregate demand in period t corresponding to a given market selection z is given by

$$\sum_{m=1}^{M} (\alpha_t - \beta_t p_m)\, z_m = \alpha_t \sum_{m=1}^{M} z_m - \beta_t \sum_{m=1}^{M} p_m z_m.$$

Once again we consider a parametric lot-sizing problem, in this case one where the demand in period t is given by $\alpha_t k - \beta_t p$, and where the parameters are k and p. Let $C_P(k,p)$ be the corresponding lot-sizing cost function. The MSP can then be written as

$$\begin{array}{lll}(\mathrm{P}_P) & \max & \displaystyle\sum_{m=1}^{M} R_m z_m - C_P\left(\sum_{m=1}^{M} z_m, \sum_{m=1}^{M} p_m z_m\right) \\ & \text{s.t.} & z_m \in \{0,1\}, \qquad\qquad\qquad m = 1,\ldots,M.\end{array}$$

Now suppose that we know that, in the optimal solution to (P_P), exactly $\bar{k}$ markets are selected. The problem then reduces to

$$
\begin{aligned}
(\mathrm{P}_P(\bar{k})) \quad \max \quad & \sum_{m=1}^{M} R_m z_m - C_P\left(\bar{k}, \sum_{m=1}^{M} p_m z_m\right) \\
\text{s.t.} \quad & \sum_{m=1}^{M} z_m = \bar{k}, \\
& z_m \in \{0,1\}, \qquad\qquad m = 1,\ldots,M.
\end{aligned}
$$

As in Section 4.1, it is easy to see that $C_P(\bar{k}, \cdot)$ is a concave function. The algorithm of Sharkey et al. (2007) now solves this problem in $\mathcal{O}\left(M^2(\log M + T\log T)\right)$ time (where the term $T\log T$ represents the time required to solve a lot-sizing problem). This result immediately implies that, by solving ($\mathrm{P}_P(\bar{k})$) for $\bar{k} = 1,\ldots,M$, we can find the optimal solution to (P_P) in $\mathcal{O}\left(M^3(\log M + T\log T)\right)$ time.

However, we can reduce the running time of this algorithm by a factor of M by employing the similarities between the M problems of the form ($\mathrm{P}_P(\bar{k})$) that need to be solved. This yields a generalization of the approach of Sharkey et al. (2007) to solve (P_P). This result is summarized in the following theorem:

Theorem 4 *The MSP with market-specific prices can be solved in* $\mathcal{O}\left(M^2(\log M + T\log T)\right)$ *time.*

Proof See the Appendix. □

4.3 Infinite holding cost

Assume that the manufacturer does not want to or cannot hold inventory, for example because holding costs are high or there is no storage space available. We will call this case the infinite holding cost case. For a given selection of markets, the setups will occur in the periods where at least one of the selected markets has a strictly positive demand. Furthermore, for each market, the variable production cost can be subtracted from the revenue because any demand is satisfied from the period in which it occurs and hence the variable cost is known. This means that a problem instance only consists of demand, revenue and setup cost parameters. The profit $\Pi(\mathcal{M}')$ for some subset of markets $\mathcal{M}' \subset \mathcal{M}$ equals

$$\Pi(\mathcal{M}') = \sum_{m\in\mathcal{M}'} R_m - \sum_{\{t: d_t^m > 0, m\in\mathcal{M}'\}} K_t.$$

It turns out that the class of infinite holding cost problems is equivalent to a class of selection problems in the literature (see Hochbaum (2004) for a survey on this class of problems). This class of selection problems can be described as follows. We are given a set of items $\mathcal{I}$ and a collection of subsets $S_j \subset \mathcal{I}$ with $j \in \mathcal{J}$. There is a cost c_i associated with each item $i \in \mathcal{I}$ and a benefit b_j associated with each subset S_j ($j \in \mathcal{J}$). The items corresponding to some subcollection $\mathcal{J}' \subset \mathcal{J}$ are $\{i \in S_j : j \in \mathcal{J}'\}$. The profit of subcollection $\mathcal{J}'$ equals the benefits of the subsets minus the cost of the items in the subsets, i.e.,

$$\sum_{j \in \mathcal{J}'} b_j - \sum_{\{i \in S_j : j \in \mathcal{J}'\}} c_i.$$

The objective is to find a subcollection $\mathcal{J}'$ that maximizes the profit.

We will show that an infinite holding cost instance of the MSP corresponds to an instance from the class of selection problems. First, the time periods form the set $\mathcal{I}$ and the markets form the set $\mathcal{J}$. Second, the cost c_i is set to the setup cost of the period corresponding to i and the benefit b_j is set to the revenue of the market corresponding to j. Finally, the set S_j consists of the periods for which the market corresponding to j has strictly positive demands. So every infinite holding cost instance can be transformed to an instance from the class of selection problems. It is not difficult to see that the reverse also holds.

The class of selection problems can be solved in polynomial time by solving a max-flow/min-cut problem on an appropriately defined network. When applied to the infinite holding cost case, the network consists of $\mathcal{O}(M+T)$ nodes and $\mathcal{O}(MT)$ arcs. As the max-flow/min-cut problem can be solved in polynomial time, the infinite horizon MSP can be solved in polynomial time. For more details on the class of selection problems we refer to Hochbaum (2004) and its references.

4.4 Selection of orders spanning up to two consecutive periods

Geunes et al. (2006) consider a joint pricing and lot-sizing model where, in each period, the demand to be satisfied depends on the price level. The goal is to find a sequence of prices and a production plan that maximize total profit. They assume that the revenue function in each period is piecewise-linear and concave, and show that the problem can be interpreted as an order selection problem. In turn, this means that this model is a special case of the MSP where each market (or order) has exactly one period with positive demand. Geunes et al. (2006) solve this problem using DP by using the fact that an optimal solution consists of a series of subplans. For a given subplan, only the orders for which the revenue per demand unit exceeds the variable cost per demand unit are satisfied. The total running time of this approach is $\mathcal{O}(MT^2)$. This of course immediately implies that that the MSP with exactly one positive demand period for each market can be solved in $\mathcal{O}(MT^2)$ time.

An interesting extension of this case allows for orders to span two consecutive periods. That is, in this case each "order" corresponds to a market with positive demand in two consecutive periods. Let t_1^m be the first period with positive demand for market m and let $t_2^m = t_1^m + 1$. Let $1 \leq s < t \leq T$ for the remainder of this section. We will consider two types of subplans: (i) a "classical" subplan (s,t) consisting of periods $[s,t]$ with a setup in period s and no other setup, and (ii) a subplan (s,t) consisting of periods $[s,t]$ with a setup in period s and in period t. (We use $[s,t]$ as a shorthand notation for $\{s,\ldots,t\}$.)

Let

$$\mathcal{M}^1_{s,t} = \{m \in \mathcal{M} : s \leq t_1^m < t_2^m \leq t\} \text{ and } \mathcal{M}^2_t = \{m \in \mathcal{M} : t_2^m = t\}$$

be the markets that may be selected for a type 1 subplan (s,t) and the markets that have their second positive demand in period t, respectively. A market $m \in \mathcal{M}^1_{s,t}$ is selected for a type 1 subplan (s,t) if $\pi^{m,1}_{s,t} \equiv R_m - (d^m_{t_1^m} c_{s,t_1^m} + d^m_{t_2^m} c_{s,t_2^m}) > 0$ and a market $m \in \mathcal{M}^2_{s,t}$ is selected for a type 2 subplan if $\pi^{m,2}_{s,t} \equiv R_m - (d^m_{t_1^m} c_{s,t_1^m} + d^m_{t_2^m} c_{t,t}) > 0$. If we let $\pi^1_{s,t}$ $(\pi^2_{s,t})$ be the profit corresponding to a type 1 (type 2) subplan (s,t), then

$$\pi^1_{s,t} = \sum_{m \in \mathcal{M}^1_{s,t}} (\pi^{m,1}_{s,t})^+ - K_s$$

and

$$\pi^2_{s,t} = \sum_{m \in \mathcal{M}^1_{s,t-1}} (\pi^{m,1}_{s,t-1})^+ + \sum_{m \in \mathcal{M}^2_t} (\pi^{m,2}_{s,t})^+ - K_s.$$

To prevent "double counting" the setup cost in the recursion, the setup cost in period t for a type 2 subplan is not contained in $\pi^2_{s,t}$.

Note that a type 1 subplan (s,t) can be followed by a type 1 subplan $(t+1,u)$ or a type 2 subplan $(t+1,u)$ with $t+1 < u$, and that a type 2 subplan (s,t) can be followed by a type 1 subplan (t,u) or a type 2 subplan (t,u) with $t < u$. By defining the variables Π^1_t (Π^2_t) as the maximum profit for periods $[1,t]$ with the last subplan of type 1 (type 2), the recursion formulas become

$$\Pi_t^1 = \max\left\{\max_{1\le s<t}\{\Pi_{s-1}^1 + \pi_{s,t}^1\}, \max_{1\le s<t}\{\Pi_s^2 + \pi_{s,t}^1\}, 0\right\} \text{ for } t = 1,\dots,T$$

and

$$\Pi_t^2 = \max\left\{\max_{1\le s<t}\{\Pi_{s-1}^1 + \pi_{s,t}^2\}, \max_{1\le s<t}\{\Pi_s^2 + \pi_{s,t}^1\}\right\} \text{ for } t = 1,\dots,T,$$

where the recursion is initialized by $\Pi_0^1 = \Pi_0^2 = 0$. The optimal solution equals $\Pi^* = \max\{\Pi_T^1, \Pi_T^2 - K_T\}$. It is not difficult to verify that all values $\pi_{s,t}^1$ and $\pi_{s,t}^2$ can be calculated in $\mathcal{O}(MT^2)$ time. As the double recursion takes $\mathcal{O}(T^2)$ time, the total running time is $\mathcal{O}(MT^2)$.

A natural question is whether the above approach can be generalized to the case where every market has a fixed number of consecutive positive demand periods, say k. Unfortunately, when $k > 2$ the demands of a given market (or order) may be satisfied with more than 2 subplans, which is in contrast to the cases where $k = 1$ or $k = 2$ discussed above. We leave the question of whether the extension to general (but fixed) k can be solved in polynomial time as an issue for further research.

4.5 Staircase demand matrix

The final special case that we will consider deals with a situation in which, in each period, there is positive demand in only one market, and each market has positive demand in a consecutive sequence of periods. (Note that this is a special case of the class of instances for which the corresponding MSP was shown to be $\mathcal{NP}$-complete.) This situation may occur, for example, if we face demands in a single market only, but the planning horizon is partitioned into a collection of time intervals with the property that if we satisfy demand in a period we have to satisfy demand in all periods that are in the same interval. These intervals could correspond to seasons, so that this constraint would, for example, say that we should either satisfy demand for the entire summer season or not satisfy demand in any of the summer months. Now if we view each time interval as a distinct "market", we obtain the more general situation sketched above. This class of problems has the attractive property that, in contrast with the general MSP, there is a natural ordering for the markets based on the demands.

More formally, let market m have positive demand in periods $t_1^m, \dots, t_2^m$. Furthermore, assume that $t_2^m < t_1^{m+1}$ for $m = 1, \dots, M-1$. This ensures that for two different markets the positive demand periods do not have any "overlap" and the positive values in the demand matrix form a "staircase." As in the previous sections we will solve this problem by DP. Define $\Pi(m,t)$ ($m = 1,\ldots,M$, $t = 1,\ldots,t_2^m$) as the maximum profit when markets $1,\ldots,m$ are considered, and where the last setup occurs in period t. To develop the recursion, we will need some more notation. Let $c_q^V(s,t)$ be the variable production and holding cost to satisfy demand in periods $[s,t]$ from production in period q and let $c^{LS}(s,t)$ the optimal lot-sizing cost for the problem consisting of periods $[s,t]$.

Consider the following two cases to calculate $\Pi(m,t)$:

- $1 \leq t < t_1^m$:

  In this case we only have to determine whether the variable costs are less than the revenue for the market as the setup cost is already taken into account. Therefore,

$$\Pi(m,t) = \Pi(m-1,t) + \left(R_m - c_t^V(t_1^m, t_2^m)\right)^+.$$

- $t_1^m \leq t \leq t_2^m$:

  Let s be the first setup in $[t_1^m, t_2^m]$. Furthermore, let q be the last setup period before this interval ($q = 0$ if it does not exist). With this information we can calculate all relevant costs: the variable cost to satisfy demand in periods $[t_1^m, s-1]$, the cost of the lot-sizing problem in periods $[s, t-1]$, and the variable cost to satisfy demand in periods $[t, t_1^m]$. Therefore, the recursion becomes

$$\Pi(m,t) = \max_{0 \leq q < t_1^m, t_1^m \leq s \leq t} \{\Pi(m-1,q) + (R_m - (c_q^V(t_1^m, s-1) + c^{LS}(s,t-1) + c_t^V(t, t_2^m)))^+ - K_t\}.$$

The recursion is initialized by $\Pi(0,t) = -\infty$ ($t = 1,\ldots,T$) and $\Pi(m,0) = 0$ ($m = 0,\ldots,M$). Furthermore, for the recursion to be valid, we let $c_s^V(t,t-1) = c^{LS}(t,t-1) = 0$ ($t = 1,\ldots,T$) and $c_0^V(s,t) = \infty$ for $s \neq t_1^m$ ($m = 1,\ldots,M$). The optimal profit equals

$$\Pi^* = \max_{t=0,\ldots,T} \{\Pi(M,t)\}.$$

Note that we can calculate the values $c_q^V(s,t)$ ($1 \leq q \leq s \leq t$) and $c^{LS}(s,t)$ ($1 \leq s \leq t$) in $\mathcal{O}(T^3)$ and $\mathcal{O}(T^2 \log T)$, respectively. Given these values and given a fixed m and t, the calculation of $\Pi(m,t)$ for $1 \leq t < t_1^m$ and for $1 \leq t < t_1^m$ takes $\mathcal{O}(1)$ and $\mathcal{O}(T^2)$ time, respectively. Therefore, the total running time of the DP is $\mathcal{O}(MT^3)$.

5 Heuristics

Because the MSP is $\mathcal{NP}$-complete, it is very unlikely that there exist efficient algorithms to solve the problem to optimality. Moreover, the inapproximability result of Theorem 3 shows that we cannot hope for a polynomial time approximation algorithm. Therefore, to find good solutions for large instances in a reasonable amount of time we have to restrict ourselves to heuristics. In the following sections we propose two heuristics.

5.1 An iterative algorithm

Recall from Section 2 that given a selection of markets, the MSP is easy to solve, and given a production plan, the MSP is also easy to solve. This suggests the following iterative algorithm (IA). Start with an initial production plan y. Given this production plan, find the optimal set of markets, say $z = z(y)$. Given this selection of markets, determine the optimal production plan, say $y = y(z)$. Repeat this procedure until it converges. The idea of iterating between two types of variables is based on Kunreuther and Schrage (1973) and Van den Heuvel and Wagelmans (2006), who developed an algorithm that iterates between prices and production plans for a joint lot-sizing and pricing model.

Note that in each iteration the profit strictly improves until identical production plans are found in two consecutive iterations. As the total number of production plans and market selections are finite, the algorithm terminates after a finite number of iterations. In a single iteration we have to find the optimal selection of markets given a production plan and an optimal production plan given a selection of markets. Given a production plan, a market m is selected if the variable production and holding cost are less than the revenue. For a single market, this takes $\mathcal{O}(T)$ time and hence $\mathcal{O}(MT)$ in total. Given a selection of markets, we have to solve a lot-sizing problem, which can be done in $\mathcal{O}(T \log T)$ time. Therefore, each iteration takes $\mathcal{O}(T(M + \log T))$ time.

It is not clear in advance how many iterations the IA will take. Clearly, this depends on the starting production plan y (among other factors). In the implementation we used T initial production plans $y^{(n)}$ ($n = 1, \ldots, T$) where plan $y^{(n)}$ has n subplans (approximately) equally spaced over the periods. Formally, plan $y^{(n)}$ has setups in periods $t = 1 + \left\lfloor \frac{i-1}{n} T \right\rfloor$ for $i = 1, \ldots, n$.

Furthermore, for instances with zero demands, we include some additional starting production plans. If t is the first non-zero demand period for some market m, then we add starting production plans with the first setup in t, and $T-t-i+1$ (for $i=0,\dots,T-t$) setups evenly distributed over periods $t,\dots,T$ as described before. These production plans may be good starting points because an optimal solution will not have a setup period before the first non-zero demand period (in the case of time-independent cost parameters). So for an instance with only non-zero demands we have T starting points and for instances with zero demands we have $\mathcal{O}(MT)$ starting points.

5.2 A rounding procedure

As mentioned in Section 3.3, Levi et al. (2005) developed a polynomial-time approximation algorithm for the MSP with (15) as the objective function. The approximation algorithm works as follows. First, the LP relaxation of (P) is solved (with (15) as the objective function). We then sort the markets in nondecreasing order of their z-value in the optimal solution of the LP relaxation, say z_m^{LP}, $m=1,\dots,M$. In a similar spirit as the algorithm for solving the seasonal demand case (Section 4.1), this algorithm then chooses the best among the market selections $\{1,\dots,m\}$ for $m=0,\dots,M$. Levi et al. (2005) show that this is a 1.582-approximation algorithm in terms of the cost of the solution. Although, as we mentioned before, this procedure does not provide a worst-case performance guarantee for the maximization version of the MSP, we will nevertheless apply the rounding procedure and empirically test its performance. Since, for each of the $\mathcal{O}(M)$ candidate solutions, we effectively round the market selection variables to 0 or 1 according to some threshold, we will refer to this algorithm as the rounding procedure (RP). Note that in the RP we have to solve one LP, sort M numbers, and solve at most M lot-sizing problems, and hence the RP takes polynomial time.

6 Computational study

In this section we present the results of a computational study that tests the performance of the heuristics in terms of solution quality and computation time. Since Theorem 2 shows that the MSP is hard even when the cost parameters are time-invariant, we restrict ourselves to such instances; in particular, we set $p_t=p=0$ and $h_t=h=1$ throughout. We tested the heuristics on three sets of problem instances with randomly generated demands according to a

A. stationary demand distribution;

B. demand distribution that follows a seasonal pattern;

C. demand distribution that allows for a high proportion of periods with zero demand.

All computations were performed on a 3.4 GHz Pentium IV desktop computer with 2.0 Gb RAM and the Windows XP operating system. Furthermore, we used CPLEX 10.1 with default settings to solve (P) and its LP-relasation. As we mentioned before, we can relax one of the sets of binary constraints (5) and (6). Based on preliminary tests, we concluded that CPLEX could solve (P) most efficiently when constraints (5) are relaxed for instances in which $T > M$ and when constraints (6) are relaxed when $M \geq T$.

6.1 Problem set A: Stationary demands

In Set A, the demands are generated from an integer uniform distribution over the interval $[0, 2\bar{d}]$, which we denote by $\mathrm{U}[0, 2\bar{d}]$. To obtain comparable problem instances for different values of M and T, we let the parameters K_t and R_m depend on M and T. In particular, we set $K_t = K = \alpha M \bar{d}$, where α is a parameter that characterizes the class of problem instances. We next discuss the implications of this choice of α on the total cost in the case that all markets are selected. The average demand per period equals $M\bar{d}$. In an EOQ model environment the optimal solution has a setup in every $n = \sqrt{2\alpha}$ periods. In turn, a subplan of n periods in a discrete environment has inventory and setup costs equal to

$$\frac{1}{2}n(n-1)M\bar{d} + \alpha M\bar{d}.$$

As there are $\frac{T}{n}$ subplans and $\alpha = \frac{1}{2}n^2$, the total cost equals approximately $\frac{1}{2}MT\bar{d}(2n-1)$ if all markets are selected. Finally, revenues are generated from $\mathrm{U}[0, 2\bar{R}]$ with $\bar{R} = \frac{1}{2}T\bar{d}(2n-1)$. By using these parameters the profit will be approximately equal to zero when all markets are selected (and exactly zero when no market is selected).

We generated 25 instances with $\alpha \in \{2, 5, 8, 11\}$ and $(M, T) \in \{(40, 40), (80, 40), (40, 80)\}$. The most important statistics on the quality of the heuristic solutions can be found in Table 2. Column 'NS' presents the number of times the optimal solution consists of a "non-straightforward" market selection, i.e., neither the empty selection nor the selection that includes all markets. The columns under 'IG' provide the average (avg) and maximum (max) integrality gap (which is defined as the difference between the optimal values of the LP relaxation and the IP as a *fraction* of the optimal

M	T	α	NS	LP	IG avg	IG max	avg dev (%) IA	avg dev (%) RP	max dev (%) IA	max dev (%) RP	non-opt IA	non-opt RP	IA-RP
40	40	2	17	2	5.99	85.2	0.01	17.3	0.14	100.0	1	16	16 - 0
		5	20	2	0.44	2.7	0.00	5.4	0.00	82.9	0	11	11 - 0
		8	19	1	0.23	1.1	0.00	1.1	0.00	11.2	0	9	9 - 0
		11	24	0	0.52	3.5	0.00	0.6	0.00	5.3	0	10	10 - 0
80	40	2	19	3	0.74	3.6	0.00	26.8	0.00	100.0	0	19	19 - 0
		5	24	0	1.16	12.9	0.00	11.9	0.00	100.0	0	24	24 - 0
		8	19	2	0.69	4.3	0.00	0.5	0.00	4.7	0	12	12 - 0
		11	21	0	0.37	2.4	0.00	1.0	0.00	6.1	0	15	15 - 0
40	80	2	14	5	0.19	1.1	0.00	9.2	0.00	75.3	0	13	13 - 0
		5	19	2	10.06	212.2	0.00	7.7	0.00	100.0	0	16	16 - 0
		8	18	4	0.32	1.9	0.00	0.1	0.03	1.7	1	4	3 - 0
		11	18	4	0.40	2.4	0.04	0.6	0.97	10.3	1	8	7 - 0

Table 2: Set A: Performance of the IA and RP heuristic.

value of the IP). The columns 'avg dev' and 'max dev' gives the average and maximum relative deviation from the optimal profit (in %). Finally, the columns under 'non-opt' present the number of instances for which each of the heuristics does not find an optimal solution. Finally, a cell with value a - b in column 'IA-RP' means that there are a instances for which the IA finds a strictly better solution than the RP and b instances for which the RP finds a strictly better solution than the IA.

Table 2 shows that the instances vary in terms of optimal solution characteristics. Most of the instances have an optimal solution that is a "non-straightforward" market selection. However, the integrality gap varies widely: in some instances the LP relaxation is tight (and gives the optimal solution), while in other instances the integrality gap is very large. Overall, Table 2 shows that the IA performs extremely well on problem instances in Set A. Only in 3 out of 300 instances does it fail to find the optimal solution, and the the maximum deviation is less than 1%. It clearly outperforms the RP, especially on instances with lower values of α. For some instances, the RP finds a solution whose deviation from optimality is 100%, which means that the RP finds a solution with profit zero while there does exist a profitable solution. Finally, we see that the RP never finds a better solution than the IA.

Table 3 presents the running times of the heuristics and CPLEX. The last two columns present the average and maximum number of iterations that the IA takes for a single starting point (i.e., for a single initial production plan). We see that the average and maximum number of iterations for the IA increases slightly as the size of the instances increases. With respect to running time, the IA finds a solution within 0.1 s for all instances, while the RP takes more than 20 minutes in the worst case. Furthermore, the optimal solutions are found within 10 minutes by solving the MIP for $(M, T) = (40, 40)$, but for $(M, T) = (80, 40)$ and $(M, T) = (40, 80)$ there are instances for which solving the MIP takes more than half an hour and almost 12 hours, respectively. Overall, we can conclude that the IA outperforms the RP both in terms of solution quality and running time for this set of instances.

Finally, Table 4 shows the consequence of using (15) instead of (1) as the objective function. We see that both the integrality gap and the deviation from optimality are much smaller in Table 4

			avg time (s)			max time (s)			IA It	
M	T	α	IA	RP	MIP	IA	RP	MIP	avg	max
40	40	2	0.013	9.9	140	0.016	11.0	475	2.91	9
		5	0.013	14.5	159	0.032	17.7	376	3.10	7
		8	0.016	18.0	170	0.016	20.9	519	3.18	6
		11	0.013	20.3	202	0.016	24.3	478	3.02	6
80	40	2	0.017	47	571	0.032	59	1,673	2.90	10
		5	0.020	69	1,182	0.032	73	2,150	3.19	7
		8	0.020	92	994	0.032	180	2,114	3.20	8
		11	0.019	151	982	0.032	410	1,843	3.14	6
40	80	2	0.054	196	2,907	0.078	264	16,734	3.08	9
		5	0.062	274	6,711	0.094	327	36,473	3.34	8
		8	0.063	286	4,337	0.093	1024	16,400	3.40	8
		11	0.064	319	7,357	0.094	1303	43,154	3.36	8

Table 3: Set A: Running time of the IA and RP heuristic.

			IG		avg dev (%)		max dev (%)	
M	T	α	avg	max	IA	RP	IA	RP
40	40	2	0.027	0.051	0.000	0.478	0.009	1.968
		5	0.019	0.034	0.000	0.155	0.000	0.972
		8	0.018	0.036	0.000	0.100	0.000	0.759
		11	0.019	0.035	0.000	0.055	0.000	0.440
80	40	2	0.025	0.046	0.000	0.964	0.000	2.192
		5	0.028	0.038	0.000	0.318	0.000	0.808
		8	0.020	0.036	0.000	0.029	0.000	0.167
		11	0.020	0.028	0.000	0.049	0.000	0.389
40	80	2	0.017	0.042	0.000	0.636	0.000	2.466
		5	0.026	0.048	0.000	0.180	0.000	0.710
		8	0.018	0.039	0.000	0.009	0.002	0.178
		11	0.021	0.043	0.003	0.025	0.070	0.226

Table 4: Set A: Performance of the IA and RP heuristic for the minimization MSP.

compared to Table 2. This can be explained by the fact that $\Gamma(\mathcal{M}') = \sum_{m=1}^{M} R_m - \Pi(\mathcal{M}')$. This relation implies that, in the minimization problem, the deviation from optimality equals

$$\frac{\Gamma(\mathcal{M}^H) - \Gamma(\mathcal{M}^*)}{\Gamma(\mathcal{M}^*)} = \frac{\Pi(\mathcal{M}^*) - \Pi(\mathcal{M}^H)}{\sum_{m=1}^{M} R_m - \Pi(\mathcal{M}^*)} \leq \frac{\Pi(\mathcal{M}^*)}{\sum_{m=1}^{M} R_m - \Pi(\mathcal{M}^*)},$$

where $\mathcal{M}^H$ is the set of markets selected by some heuristic that considers the empty set as a candidate solution (hence $\Pi(\mathcal{M}^H) \geq 0$). As (in general) $\sum_{m=1}^{M} R_m$ is large relative to $\Pi(\mathcal{M}^*)$, the deviation from optimality will be small for the minimization problem. For example, the instances in the $\mathcal{NP}$-completeness proof satisfy $\Pi(\mathcal{M}^*) \leq \frac{1}{2}$ and $\sum_{m=1}^{M} R_m = \frac{3}{2}T + \frac{1}{2}$. Hence, the deviation from optimality is smaller than $\frac{1}{3T}$ and tends to zero as $T \to \infty$.

6.2 Problem set B: Seasonal demands

The good performance of the IA may be caused by the choice of the initial production plans. Note that in the prior test set, expected demand is constant over the time periods for any selection of markets. The optimal solution for a lot-sizing problem with time-invariant demand and cost parameters consists of subplans that differ at most one period in length, and the initial production plans we use exactly have this property. To see whether the IA still performs well for problems where the expected demand is not constant over time, we generate demands with a seasonal component in Set B. The parameters are the same as in Set A, except for demand which is generated by

$$d_t^m = \lfloor \varepsilon_t^m + a(1 + \cos(2\pi(t + k_m)/c)) \rceil \,, \tag{16}$$

where $\varepsilon_t^m \sim \mathrm{U}[0, 2\bar{d}]$, a is the amplitude of the seasonal component, k_m is the starting period of the cycle for market m, c is the cycle length, and $\lfloor x \rceil$ is the integer closest to x. In the experiments we set $\bar{d} = 5$, $a = 5$ and $c = 10$. Furthermore, we generated one set with $k_m = 0$ and one set with $k_m \sim \mathrm{U}[0, c-1]$ $(m = 1 \ldots, M)$. In the first case the seasonal patterns of the different markets are aligned, while this does not hold for the second case. Finally, we also generated a set of instances with stationary demands from U[5, 15], the distribution obtained after removing the deterministic component in (16).

Tables 5 and 6 show the performance of the heuristics for Set B with $(M, T) = (40, 40)$ and

				IG		avg dev (%)		max dev (%)		non-opt		
type	α	NS	LP	avg	max	IA	RP	IA	RP	IA	RP	IA - RP
Stat	2	14	2	0.62	6.7	0.000	15.3	0.000	100.0	0	14	14 - 0
	5	20	2	2.24	42.4	0.000	9.0	0.000	100.0	0	16	16 - 0
	8	19	2	0.28	1.7	0.000	1.2	0.000	24.9	0	8	8 - 0
	11	24	0	0.40	2.4	0.000	0.5	0.000	8.6	0	9	9 - 0
Seas	2	17	2	0.992	6.0	0.000	18.0	0.000	100.0	0	16	16 - 0
Fixed	5	20	2	1.664	30.0	0.000	5.9	0.000	100.0	0	12	12 - 0
Peak	8	22	3	0.725	8.8	0.000	0.3	0.000	2.7	0	6	6 - 0
	11	18	5	0.175	1.0	0.000	0.1	0.000	1.2	0	4	4 - 0
Seas	2	19	0	0.417	1.9	0.000	16.2	0.000	52.7	0	18	18 - 0
Rnd	5	21	2	0.540	7.1	0.000	0.8	0.000	10.0	0	10	10 - 0
Peak	8	22	1	0.297	3.9	0.000	0.9	0.000	4.9	0	9	9 - 0
	11	25	0	0.211	2.1	0.006	0.7	0.147	16.2	1	4	4 - 1

Table 5: Set B: Performance of the IA and RP heuristic.

		avg time			max time			IA It	
type	α	IA	RP	MIP	IA	RP	MIP	avg	max
Stat	2	0.004	10.6	220	0.016	12.5	925	2.54	8
	5	0.008	14.5	284	0.016	18.3	668	2.97	6
	8	0.006	18.1	236	0.016	20.0	694	3.02	6
	11	0.004	22.3	268	0.016	43.8	634	2.50	7
Seas	2	0.007	9.3	181	0.016	12.4	943	2.75	9
Fixed	5	0.011	12.0	286	0.016	15.6	1178	2.96	7
Peak	8	0.011	14.1	216	0.016	19.2	608	2.98	8
	11	0.011	15.2	178	0.016	19.3	549	2.82	5
Seas	2	0.004	10.9	104	0.016	13.4	193	2.77	6
Rnd	5	0.008	12.1	148	0.016	16.0	296	2.72	7
Peak	8	0.008	13.1	94	0.016	16.2	231	2.84	8
	11	0.007	14.5	102	0.016	20.0	177	2.39	7

Table 6: Set B: Running time of the IA and RP heuristic.

with each row the result of 25 instances. The column 'type' shows the demand type, where 'Stat' is the stationary case, 'Seas Fixed Peak' is the seasonal case with $k_m = 0$, and 'Seas Rnd Peak' is the seasonal case with k_m chosen randomly. Table 5 shows that there is not much difference in solution quality between the stationary and seasonal demand case for both the IA and the RP. The IA still performs very well: only in 1 out of 300 instances it does not find the optimal solution. Interestingly, the RP finds a better solution for this particular instance. As in Table 2, the deviation from optimality for the RP is high for low values of α. Table 6 shows that the running times of the RP are lower for the two seasonal cases than for the stationary case. This also holds for the running time to solve the MIP.

6.3 Problem set C: Sparse demand matrix

The last set of instances has the property that, for each market, there is a high fraction of periods with zero demand, motivated by the fact that the $\mathcal{NP}$-completeness proof of the MSP is based on instances with a sparse demand matrix. We obtain Set C from Set A by replacing a market's demand in a period by zero with some probability ρ, and with probability $1-\rho$ it is generated from $U[0,2\bar{d}]$. We generated 100 instances with the cost parameters as in Sets A and B, $(M,T)=(40,40)$, and $\rho \in \{0.5, 0.75, 0.9\}$. The results of the tests can be found in Tables 7 and 8.

First, we see that the IA still performs quite well in terms of average deviation from optimality. However, there is a single instance (with $\rho = 0.9$ and $\alpha = 8$) for which the IA cannot find a profitable solution while one exists. Furthermore, we see that the performance of the IA worsens as ρ increases, both in terms of deviation from optimality and in terms of number of non-optimal solutions. The RP shows a decrease in performance as ρ increases from 0.5 to 0.75, but there is no clear difference between the cases $\rho = 0.75$ and $\rho = 0.9$. Finally, we see that for most cases the IA finds better solutions than the RP, but for $\rho = 0.9$ there is also a significant number of instances for which the RP finds a better solution.

Table 8 shows that the running time of the RP and the time to solve the MIP drop dramatically as ρ increases. This can be explained by the fact that, if $d_t^m = 0$, then $x_{i,t}^m = 0$ for $i = 1, \ldots, t$, which means that the number of variables and constraints in the MIP decreases considerably as

				IG		avg dev (%)		max dev (%)		non-opt		
ρ	α	NS	LP	avg	max	IA	RP	IA	RP	IA	RP	IA - RP
0.50	2	76	14	0.47	15.6	0.00	7.41	0.00	100.0	0	56	56 - 0
	5	81	5	0.77	28.1	0.03	3.48	3.05	60.9	1	52	52 - 0
	8	86	9	0.28	2.4	0.04	1.27	3.68	17.2	2	50	50 - 0
	11	83	10	0.22	2.5	0.00	2.19	0.30	48.3	1	51	50 - 0
0.75	2	90	9	0.26	3.0	0.19	10.01	8.43	100.0	7	65	65 - 1
	5	89	9	0.27	6.7	0.09	5.00	3.27	100.0	6	67	67 - 1
	8	90	10	0.30	4.2	0.22	5.23	5.76	100.0	9	65	65 - 1
	11	91	15	0.21	4.9	0.06	4.14	3.90	73.7	3	58	57 - 0
0.90	2	100	15	0.08	1.3	1.35	6.05	11.63	90.9	43	67	55 - 18
	5	100	20	0.09	0.8	1.01	4.78	12.06	54.5	30	66	59 - 9
	8	99	21	0.28	20.6	1.71	7.53	100.00	100.0	23	66	61 - 6
	11	100	22	0.10	1.4	0.58	5.44	9.57	72.1	19	64	58 - 4

Table 7: Set C: Performance of the IA and RP heuristic.

		avg time (s)			max time (s)			IA It	
ρ	α	IA	RP	MIP	IA	RP	MIP	avg	max
0.50	2	0.04	2.66	19.42	0.09	3.55	124	3.34	10
	5	0.04	3.61	32.71	0.06	5.17	113	3.32	10
	8	0.04	3.85	28.25	0.06	5.52	90	3.35	8
	11	0.04	4.17	29.03	0.08	5.42	111	3.33	8
0.75	2	0.06	0.93	5.62	0.13	1.17	11.6	3.17	10
	5	0.06	1.09	7.66	0.08	1.55	33.8	3.21	10
	8	0.06	1.17	7.63	0.09	1.67	25.5	3.17	9
	11	0.06	1.17	6.45	0.11	1.67	24.0	3.07	8
0.90	2	0.08	0.48	1.22	0.14	0.59	3.08	2.71	8
	5	0.08	0.50	1.35	0.11	0.64	3.23	2.80	8
	8	0.08	0.51	1.35	0.14	0.63	3.39	2.77	7
	11	0.08	0.52	1.37	0.11	0.63	3.67	2.72	8

Table 8: Set C: Running time of the IA and RP heuristic.

ρ increases. Although neither heuristic performs very well for instances with high values of ρ, the good news is that these instances can be solved to optimality by CPLEX. For example, it takes only a couple of seconds to solve an instance with $\rho = 0.9$.

7 Summary and concluding remarks

In this paper, we studied a new class of production planning problems with demand flexibility in the form of market selection decisions. We showed that, in general, the class of problems is $\mathcal{NP}$-complete and, in addition, that it is highly unlikely that there exists a polynomial-time algorithm with a constant worst-case guarantee. We provide polynomial-time solution approaches for several special cases that may occur in practice. In addition, we introduce a heuristic for solving the problem and perform extensive computational tests to compare its performance with a rounding procedure from the literature and with a commercial MIP solver. These results show that our heuristic finds near-optimal solution extremely fast, except when the demand patterns are unrealistically small. Our future research will focus on identifying additional classes of problem instances that are polynomially solvable, as well as the development of models and algorithms for problems in which production planning is integrated with market selection and pricing.

References

A. Aggarwal and J. K. Park. Improved algorithms for economic lot-size problems. *Operations Research*, 14:549–571, 1993.

S. Biller, L. M. A. Chan, D. Simchi-Levi, and J. Swann. Dynamic pricing and the direct-to-customer model in the automotive industry. *Electronic Commerce Journal (dynamic pricing special issue)*, 5(2):309–334, 2005.

L. M. A. Chan, D. Simchi-Levi, and J. Swann. Pricing, production, and inventory strategies for manufacturing with stochastic demand and discretionary sales. *Manufacturing & Service Operations Management*, 8(2):149–168, 2006.

X. Chen and D. Simchi-Levi. Coordinating inventory control and pricing strategies with random demand and fixed ordering cost: The finite horizon case. *Operations Research*, 52(6):887–896, 2004a.

X. Chen and D. Simchi-Levi. Coordinating inventory control and pricing strategies with random demand and fixed ordering cost: The infinite horizon case. *Mathematics of Operations Research*, 29(3):698–723, 2004b.

S. Deng and C. A. Yano. Joint production and pricing decisions with setup costs and capacity constraints. *Management Science*, 52(5):741–756, 2006.

M. Ettl, P. Huang, K. Sourirajan, T. R. Ervolina, and G. Y. Lin. Supply and demand synchronization in assemble-to-order supply chains. Technical report, IBM Research Report #RC23923, March 28 2006.

A. Federgruen and M. Tzur. A simple forward algorithm to solve general dynamic lot sizing models with n periods in $\mathcal{O}(n \log n)$ or $\mathcal{O}(n)$ time. *Management Science*, 37:909–925, 1991.

M. Florian and M. Klein. Deterministic procurement planning with concave costs and capacity constraints. *Management Science*, 26:669–679, 1971.

M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W.H. Freeman and Company, New York, 1979.

J. Geunes, H. E. Romeijn, and K. Taaffe. Requirements planning with pricing and order selection flexibility. *Operations Research*, 54(2):394–401, 2006.

S. M. Gilbert. Coordination of pricing and multiple-period production for constant priced goods. *European Journal of Operational Research*, 114:330–337, 1999.

S. M. Gilbert. Coordination of pricing and multiple-period production across multiple constant priced goods. *Management Science*, 46:1602–1616, 2000.

D. S. Hochbaum. Selection, provisioning, shared fixed costs, maximum closure, and implications on algorithmic methods today. *Management Science*, 50(6):709–723, June 2004.

W. Huang, H. E. Romeijn, and J. Geunes. The continuous-time single-sourcing problem with capacity expansion opportunities. *Naval Research Logistics*, 52:193–211, 2005.

J. Krarup and O. Bilde. Plant location, set covering and economic lot size: An $O(mn)$-algorithm for structuredproblems. In L. Collatz and W. Wetterling, editors, *Numerische Methoden bei Optimierungsaufgaben, Band 3 (Optimierung bei graphentheoretischen and ganzzahligen Problemen)*, volume 36 of *International Series of Numerical Mathematics*, pages 155–180. Birkhänser Verlag, 1977.

H. Kunreuther and L. Schrage. Joint pricing and inventory decisions for constant priced items. *Management Science*, 19:732–738, 1973.

L. Lapide. Align your organization to manage demand. *Supply Chain Strategy: A Newsletter from the MIT Supply Chain Lab*, 2(7), July/August 2006.

R. Levi, J. Geunes, H. E. Romeijn, and D. B. Shmoys. Inventory and facility location models with market selection. In M. Jünger and V. Kaibel, editors, *IPCO 2005, LNCS 3509*, pages 111–124. Springer-Verlag Berlin Heidelberg, 2005.

T. C. Sharkey, H. E. Romeijn, and J. Geunes. A class of nonlinear nonseparable continuous knapsack and multiple-choice knapsack problems. Technical report, Department of Industrial and Systems Engineering, University of Florida, 2007.

K. T. Talluri and G. J. van Ryzin. *The theory and practice of revenue management*. Kluwer Academic Publishers, Norwell, MA, 2004.

J. Thomas. Price production decisions with deterministic demand. *Management Science*, 16:747–750, 1970.

J. Thomas. Price and production decisions with random demand. *Operations Research*, 22(3): 513–518, 1974.

W. van den Heuvel and A. P. M. Wagelmans. A polynomial time algorithm for a deterministic joint pricing and inventory model. *European Journal of Operational Research*, 170(2):463–480, 2006.

A. P. M. Wagelmans, C. P. M. van Hoesel, and A. Kolen. Economic lot sizing: An $\mathcal{O}(n \log n)$ algorithm that runs in linear time in the Wagner-Whitin case. *Operations Research*, 40:S145–S156, 1992.

H. M. Wagner and T. M. Whitin. Dynamic version of the economic lot size model. *Management Science*, 5:89–96, 1958.

Appendix

Theorem 4 *The MSP with market-specific prices can be solved in* $\mathcal{O}\left(M^2(\log M + T \log T)\right)$ *time.*

Proof We will employ the similarities between the M problems $(\mathrm{P}_P(k))$ $(k = 1, \ldots, M)$ that need to be solved to specialize the approach of Sharkey et al. (2007). First, note that all extreme points of the continuous relaxation of the feasible region of each of these problems is integral, and applying the approach of Sharkey et al. (2007) to the relaxation of $(\mathrm{P}_P(k))$ for some fixed k yields an integral solution. Therefore, we can, without loss of optimality, relax the integrality constraints; we will refer to the relaxed problems as $(\mathrm{R}_P(k))$. Sharkey et al. (2007) construct a collection of candidate solutions indexed by

$$\Delta = \{(i, j) : i < j \text{ and } p_i \neq p_j\}$$

and denoted by $x^{(i,j)}$ ($(i,j) \in \Delta$) that is guaranteed to contain an optimal solution to $(\mathrm{R}_P(k))$. In particular, for each $(i,j) \in \Delta$, they use the complementary slackness conditions of a linear program associated with $(\mathrm{R}_P(k))$ to first construct partial solutions as follows. Denote the solution to the system

$$\begin{aligned} \lambda + \gamma p_i &= r_i \\ \lambda + \gamma p_j &= r_j \end{aligned}$$

by

$$\gamma^{(i,j)} = \frac{r_i - r_j}{p_i - p_j} \qquad \text{and} \qquad \lambda^{(i,j)} = \frac{r_j p_i - r_i p_j}{p_i - p_j}.$$

Then a corresponding partial solution is given by

$$\begin{aligned} x_\ell^{(i,j)} &= 0 \qquad\qquad \text{if } r_\ell < \lambda^{(i,j)} + \gamma^{(i,j)} p_\ell \\ x_\ell^{(i,j)} &= 1 \qquad\qquad \text{if } r_\ell > \lambda^{(i,j)} + \gamma^{(i,j)} p_\ell. \end{aligned}$$

Finally, set

$$\begin{aligned} I_0^{(i,j)} &= \{\ell : x_\ell^{(i,j)} = 0\} \\ I^{(i,j)} &= \{\ell : r_\ell = \lambda^{(i,j)} + \gamma^{(i,j)} p_\ell\} \\ I_1^{(i,j)} &= \{\ell : x_\ell^{(i,j)} = 1\}. \end{aligned}$$

Interestingly, these partial candidate solutions are *independent* of the value of k, and (as is shown in Sharkey et al. (2007)), the time required to determine the sets $I_0^{(i,j)}$, $I^{(i,j)}$, and $I_1^{(i,j)}$ for all $(i,j) \in \Delta$ is $O(M^2 \log M)$.

It now remains to complete the candidate solutions for each value of k by determining the values of the variables in $I^{(i,j)}$; we will refer to the corresponding solutions as $x^{(i,j)}(k)$. These solutions can be found by solving the subproblems

$$\begin{aligned} (\mathrm{SP}_P(k)) \quad \max \quad & \sum_{\ell \in I_1^{(i,j)}} r_\ell + \sum_{\ell \in I^{(i,j)}} r_\ell x_\ell - C_P\left(k, \sum_{\ell \in I_1^{(i,j)}} p_\ell + \sum_{\ell \in I^{(i,j)}} p_\ell x_\ell\right) \\ \text{s.t.} \quad & \sum_{\ell \in I^{(i,j)}} x_\ell = k - |I_1^{(i,j)}| \\ & 0 \le x_\ell \le 1 \qquad\qquad \ell \in I^{(i,j)}. \end{aligned} \tag{17}$$

It is shown in Sharkey et al. (2007) that $(\mathrm{SP}^{(i,j)}(k))$ can be solved by first solving two linear knapsack problems, namely,

$$
\begin{array}{llll}
(\mathrm{KP}(k)) & \max\ (\text{or } \min) & \displaystyle\sum_{\ell \in I^{(i,j)}} p_\ell x_\ell & \\
& \text{s.t.} & \displaystyle\sum_{\ell \in I^{(i,j)}} x_\ell = k - |I_1^{(i,j)}| & \\
& & 0 \leq x_\ell \leq 1 & \ell \in I^{(i,j)}
\end{array}
$$

and selecting the solution of these two problems with the best objective function value to $(\mathrm{SP}_P(k))$.

It is important to note that $(\mathrm{SP}^{(i,j)}(k))$ only has a feasible solution, and thus only needs to be considered, for $|I_1^{(i,j)}| \leq k \leq |I_1^{(i,j)}| + |I^{(i,j)}|$. We will proceed by first adapting the algorithm from Sharkey et al. (2007) by, for each $(i,j) \in \Delta$, solving for all relevant solutions $x^{(i,j)}(k)$ consecutively, i.e., for $k = I_1^{(i,j)}|, \ldots, |I_1^{(i,j)}| + |I^{(i,j)}|$. Suppose now that we have found a solution for some value $|I_1^{(i,j)}| \leq k < |I_1^{(i,j)}| + |I^{(i,j)}|$ and consider the value $k+1$. Assuming that we have recorded the sorting of the variables in the linear knapsack problems $(\mathrm{KP}(k))$ that led to their optimal solutions, then we need to only add in the next highest (or next lowest) value in the sorting in order to determine the solutions to $(\mathrm{KP}(k+1))$. We therefore conclude that, for a fixed $(i,j) \in \Delta$, we can solve for all candidate solutions $x^{(i,j)}(k)$ in $\mathcal{O}\left(|I^{(i,j)}| \log |I^{(i,j)}| + |I^{(i,j)}| T \log T\right)$ time, which follows from the fact that (i) we need to sort the variables in $I^{(i,j)}$ to solve all problems of the form $(\mathrm{KP}(k))$, and (ii) we need to evaluate the objective function value to $(\mathrm{SP}_P(k))$ for $\mathcal{O}(|I^{(i,j)}|)$ solutions, each of which requires the solution of a lot-sizing problem.

Typically, we can expect to have $|I^{(i,j)}| = 2$ for all $(i,j) \in \Delta$. If this is the case, we obtain the desired result immediately from the fact that $|\Delta| = \mathcal{O}(M^2)$. If, however, we may have $|I^{(i,j)}| > 2$ the situation may be more complex. In general, let us define the sets $\Delta_i = \{j : (i,j) \in \Delta\}$ for $i = 1, \ldots, M$. Now if, for a given i, no variable occurs in $I^{(i,j)}$ for more than one $j \in \Delta_i$ we have that

$$
\mathcal{O}\left(\sum_{j \in \Delta_i} |I^{(i,j)}|\right) = \mathcal{O}(M)
$$

so that the desired result would follow again. Finally, Sharkey et al. (2007) show that, for a given i, the only variables that can occur in more than one of the sets $I^{(i,j)}$ are the ones for which the revenue/price pair is identical to that of market i. Denoting, for each i, the set of such markets by

D_i, we obtain

$$\sum_{j \in \Delta_i} |I^{(i,j)}| = \mathcal{O}\left(M + M|D_i|\right)$$

and, since it is easy to see that the sets D_i are disjoint, we obtain the desired result:

$$\begin{aligned}\mathcal{O}\left(\sum_{i=1}^{M} \sum_{j \in \Delta_i} \left(|I^{(i,j)}| \log |I^{(i,j)}| + |I^{(i,j)}| T \log T\right)\right) &= \mathcal{O}\left((\log M + T \log T) \sum_{i=1}^{M} \sum_{j \in \Delta_i} |I^{(i,j)}|\right) \\ &= \mathcal{O}\left(M^2 (\log M + T \log T)\right).\end{aligned}$$

□