Worst case analysis for a general class of on-line lot-sizing heuristics

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Abstract

In this paper we analyze the worst case performance of heuristics for the classical economic lot-sizing problem with time-invariant cost parameters. We consider a general class of on-line heuristics that is often applied in a rolling horizon environment. We develop a procedure to systematically construct worst case instances for a fixed time horizon and use it to derive worst case problem instances for an infinite time horizon. Our analysis shows that any on-line heuristic has a worst case ratio of at least 2. Furthermore, we show how the results can be used to construct heuristics with optimal worst case performance for small model horizons.

1 Introduction

The economic lot-sizing (ELS) problem is a well-known problem in inventory management and is described as follows. Given the (deterministic) demand for a discrete and finite planning horizon, find a production plan that satisfies demand and minimizes total costs. Costs include

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setup cost for each time period production takes place and holding cost for each item carried over from a period to the next period.

Although the ELS problem can be solved in polynomial time, heuristics are often used to solve the problem. One reason is that exact algorithms (such as the algorithm by Wagner and Whitin (1958)) are difficult to understand and hence are often not used by practitioners. Furthermore, heuristics are often applied when the ELS problem needs to be solved in a rolling horizon environment. In such a situation heuristics may perform better than the Wagner-Whitin algorithm (see for example Stadtler (2000) and Van den Heuvel and Wagelmans (2005)). In a rolling horizon environment, lot-sizing heuristics can be considered as on-line algorithms, because decisions have to be taken while not all future demand information is known (a more formal definition of on-line heuristic will follow in Section 2).

Two methods are commonly used to measure the performance of heuristics. First, we have the empirical methods in which a simulation study is performed (see, e.g., Baker (1989), Fisher et al. (2001) and Simpson (2001)). The difficulty of a simulation study is to construct a representative testbed. Second, we have analytical methods which can be split into probabilistic and worst case analysis. Probabilistic methods analyze the expected performance of heuristics given the distribution of some problem parameters (see Axsäter (1988)). In worst case analysis one searches for a bound on the relative performance that holds for any problem instance (see Axsäter (1982), Bitran et al. (1984), Axsäter (1985) and Vachani (1992)).

In this paper we are interested in the worst case performance of heuristics for the ELS problem. As mentioned above several papers on this subject have appeared in the literature. Axsäter (1982) and Bitran et al. (1984) analyze the worst case performance of some specific lot-sizing rules. Vachani (1992) analyzes the worst case performance of seven heuristics, where also data dependence, such as the length of the time horizon and demand properties (constant and bounded demand), is taken into account. The paper that is closest to our research is Axsäter (1985). He shows that all on-line heuristics which use a specific type of decision rule have worst case ratio at least 2. A nice aspect of this result is that it applies to almost all popular heuristics.

Our research was motivated by the following natural questions. First, do there exist on-line heuristics with worst case performance smaller than 2? Second, can we construct problem instances with large performance ratio for a broader class of on-line heuristics than Axsäter (1985)? In this paper we will provide a negative answer to the first question and a positive answer to the second question. We will not only show that there exists no on-line heuristic with worst case performance smaller than 2, but we also show that the result can be applied to a broader range of heuristics. Although this means that we generalize the result of Axsäter (1985), we would like to emphasize that our approach is (necessarily) completely different than his. In fact, we believe that the actual contribution of this paper lies not only in the fact that we provide a worst case problem instance, but also in the description of the systematic way in which we have searched for this instance. This systematic way also led to the construction of optimal on-line heuristics for small model horizons.

The remainder of this paper is organized as follows. In Section 2 we formally introduce the economic lot-sizing problem and we define our class of on-line heuristics by a single property. In Section 3 we show how to systematically construct problem instances with a high performance ratio for a fixed time horizon. In Section 4 we present our main result and show that any on-line lot-sizing heuristic has worst case ratio at least 2. In Section 5 we use the analysis of Section 3 to construct heuristics with optimal worst case performance for small time horizons. In Section 6 we discuss several implications of the results. The paper is completed in Section 7 with the conclusion.

2 Definitions, problem formulation and observations

We start this section by describing the ELS problem mathematically. If we use the following notation

- T : model horizon
- d_t : demand in period t ($t = 1, \ldots, T$)
- K : setup cost
- h : unit holding cost
- x_t : production quantity in period t ($t = 1, \ldots, T$)
- I_t : ending inventory in period t ($t = 1, \ldots, T$),

then the ELS problem can be modeled as

$$\begin{array}{rlll} C^*(d,T) = & \min & \sum_{t=1}^{T} \left(K\delta(x_t) + hI_t\right) & \\ & \text{s.t.} & I_t = I_{t-1} - d_t + x_t & t = 1, \ldots, T \\ & & x_t, I_t \geq 0 & t = 1, \ldots, T \\ & & I_0 = 0, & \end{array}$$

where

$$\delta(x) = \begin{cases} 0 & \text{for } x = 0 \\ 1 & \text{for } x > 0. \end{cases}$$

First, note that we may assume w.l.o.g. that $K = 1$, as the objective function only depends on the ratio K/h. Furthermore, we may assume w.l.o.g. that $h = 1$. Namely, defining the variables $x'_t = hx_t$, $I'_t = hI_t$ and $d'_t = hd_t$ leads to the model

$$\begin{array}{rlll} C^*(d', T) = & \min & \sum_{t=1}^{T} \left(\delta(x'_t) + I'_t\right) & \\ & \text{s.t.} & I'_t = I'_{t-1} - d'_t + x'_t & t = 1, \ldots, T \\ & & x'_t, I'_t \geq 0 & t = 1, \ldots, T \\ & & I'_0 = 0. & \end{array}$$

This shows that, when considering the worst case performance of heuristics, it suffices to consider only problem instances with $K = h = 1$. This means that a problem instance is completely defined by a demand sequence $d = d_1, \ldots, d_T$. Finally, we may also assume w.l.o.g. that $d_1 > 0$ since otherwise this period can be ignored.

Let d be a problem instance and let $C^H(d)$ be the cost of a solution generated by some heuristic H for instance d. We define the performance ratio $r(d)$ of H for instance d as $r(d) = C^H(d)/C^*(d)$, where $C^*(d)$ is the optimal cost for this instance. Furthermore, the worst case ratio of H is defined as

$$\sup_{d \in I} r(d),$$

where I is the set of all problem instances. From the definitions it follows that the performance ratio is a measure for a particular problem instance d and the worst case ratio is a measure for a set of instances. Note that the performance ratio for any instance is a lower bound on the worst case ratio.

Axsäter (1985) considers a class of on-line heuristics where a setup is made in period $n + 1$ (with the previous setup in period 1) if

$$\sum_{t=1}^{k} a_{tk} d_t \leq 1 \text{ for } k = 2, \ldots, n \text{ and } \sum_{t=1}^{n+1} a_{t,n+1} d_t > 1, \tag{1}$$

where a_{tk} $(1 \leq t \leq k \leq T)$ are constants that depend on the specific heuristic. After the setup assignment to period $n + 1$, this period becomes period 1 and the procedure starts again. Axsäter (1985) proves that this class of heuristics has a worst case ratio of at least 2 (and this bound is tight for some heuristics) by considering nine different cases dependent on the properties of the constants a_{tk}.

As in Axsäter (1985) we consider a complete class of heuristics. Our general class of on-line heuristics is defined as follows:

Definition 1 *On-line lot-sizing heuristics make setup decisions period by period (so previously made decisions are fixed and cannot be changed) and setup decisions do not depend on future demand.*

The on-line property states that the decisions are made starting in period 1 and in every next period we decide whether to make a setup or not irrespective of future demand. This generalizes the class of Axsäter (1985) in two ways. First, instead of the term $\sum_{t=1}^{k} a_{tk} d_t$ in condition (1) we can have arbitrary functions $f_k(d_1, \ldots, d_k)$ changing the condition into $f_k(d_1, \ldots, d_k) \leq 0$ for $k = 2, \ldots, n$ and $f_{n+1}(d_1, \ldots, d_{n+1}) > 0$. Second, decisions for the on-line heuristics may depend on previous setups, whereas in the class of Axsäter (1985) decision only depend on the current (last) setup period.

The observation that w.l.o.g. $K = h = 1$ leads to some interesting insights. First, it is clear that every problem instance has minimal cost at most T: the cost of the trivial lot-for-lot (L4L) heuristic which has a setup in each period. Because the optimal solution has cost at least 1, the worst case ratio of L4L is at most T (the instance with $d_1 = 1$ and $d_t = 0$ $(t = 2, \ldots, T)$ has worst case ratio T). Furthermore, if $d_t \geq p > 0$ for all $t = 1, \ldots, T$ with $p < 1$, then the optimal solution has cost at least p in each period and the worst case ratio of L4L is at most $\frac{T}{Tp} = \frac{1}{p}$. Now look at Table 1 where we reproduced the summary of the worst case analysis on the seven heuristics by Vachani (1992, p. 805, Table 2). If we look at instances with a finite time horizon

Heuristic	T	$d_t = d$	$d_t \leq p, p > 0$	$d_t \geq p, p \leq 1$
Economic order quantity (EOQ)	∞	1.059	∞	∞
Periodic order quantity (POQ)	T	1.059	∞	∞
Silver-Meal (SM)	$\sqrt{T}/2\sqrt{2} \leq w \leq T$	1	∞	$\frac{1}{p}$
Least unit cost (LUC)	∞	1	∞	∞
Part period balancing (PPB)	$3T/(T+2)$	$\frac{3}{2}$	3	$3-2p$
Bitran-Magnanti-Yanasse (BMY)	$2T/(T+1)$	1	2	$2-p$
Freeland-Colley (FC)	$\sqrt{T}/2\sqrt{2} \leq w \leq T$	∞	∞	$\frac{1}{p}$

Table 1: Data dependent worst case ratios (w) of some heuristics

(column 2) and demand bounded from below (column 5), then it follows that in the worst case only PPB and BMY perform strictly better than the L4L heuristic. The other five heuristics perform as bad or even worse than the L4L heuristic on one of the two problem characteristics. So from a worst case analysis point of view these heuristics perform badly. Note that BMY is a heuristic with worst case ratio 2.

3 Constructing worst case examples for a finite horizon

Consider some on-line lot-sizing heuristic. In each period the heuristic 'decides' to start a new lot-size or to add the demand to the current lot-size. By this property worst case performance can be interpreted as a game between a heuristic and an adversary. In each period t the heuristic 'receives' some demand d_t from the adversary and the heuristic has to 'decide' whether to add demand d_t to the current production run (incurring holding cost) or to start a new one (incurring setup cost). Whereas the heuristic's objective is to minimize the performance ratio, the adversary tries to maximize the performance ratio.

3.1 A relaxed mathematical formulation

It is well known that given a demand sequence $d = d_1, \ldots, d_T$, a solution for the ELS problem is completely determined by its setup periods (the zero inventory ordering property). So a production plan can be represented by a vector $P \in \{0,1\}^T$ with $P_t = 1$ if t is a setup period and $P_t = 0$ otherwise. As we may assume w.l.o.g. that demand in period 1 is positive, $P_1 = 1$. Let $P(T)$ be the set of all production plans of T periods. Let $d = d_1, \ldots, d_T$ be a demand sequence and $P \in P(T)$ a production plan. Let $C(d, P, t)$ be the cost of the first t periods for demand sequence d in production plan P, i.e.,

$$C(d, P, t) = \sum_{i=1}^{t} \left(P_i + (i - p(i))d_t\right), \tag{2}$$

where $p(i)$ is the setup period preceding period i (or period i itself if i is a setup period). Then the performance ratio for instance d and plan P is defined as

$$\max_{t=1,\ldots,T} \frac{C(d, P, t)}{C^*(d, t)}.$$

Note that we take the maximum over all periods as every sequence $d_1, \ldots, d_t$ represents a problem instance for the ELS problem (the adversary can stop at any moment or the demand beyond period t can be set equal to zero).

Now consider a binary tree of depth T representing the set $P(T)$ (see Figure 1). In each node

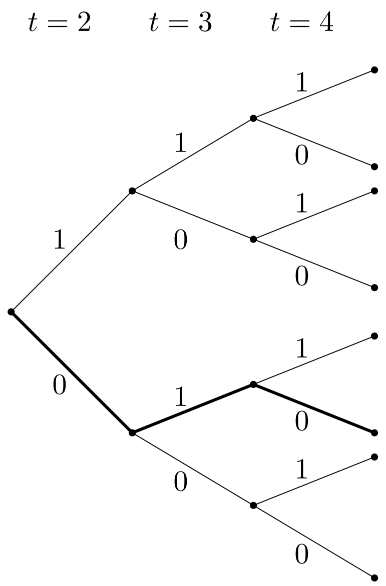

Figure 1: Representing production plans by a binary tree

of depth t one branch represents a new setup in period $t+1$ and the other branch represents a non-setup period. For example, the path in Figure 1 represents the plan $P = \{1, 0, 1, 0\}$. Note that, given a demand sequence, every heuristic has to choose a path (corresponding to a production plan) through the binary tree. So the tree reflects that decisions are made on a period by period basis. Hence the performance ratio $R(d, T)$ of any heuristic on demand sequence d of length T equals at least

$$R^*(d, T) = \min_{P \in P(T)} \max_{t=1,\ldots,T} \frac{C(d, P, t)}{C^*(d, t)}$$

and the worst case ratio of any heuristic equals at least

$$W^*(T) = \max_{d \in [0,1]^T} R^*(d, T) = \max_{d \in [0,1]^T} \min_{P \in P(T)} \max_{t=1,\ldots,T} \frac{C(d, P, t)}{C^*(d, t)} \tag{3}$$

as the worst case ratio is the maximum performance ratio over all problem instances. We only consider problem instances with demand $d_t \leq 1$. If we have an instance d with some demands strictly larger than 1 and if d' is the instance with demand 1 for these periods, then it is not difficult to see that $R^*(d, T) = R^*(d', T)$ and hence d can be ignored when evaluating (3).

We note that the above formulation is not a complete description of our original problem. This can be seen as follows. Assume that we have two demand sequences $d^1 = [d_1, d_2, d_3^1]$ and $d^2 = [d_1, d_2, d_3^2]$. Furthermore, assume that the performance ratios for each (partial) production plan are as shown in Figure 2, where (as in Figure 1) an upper branch represents a setup period.

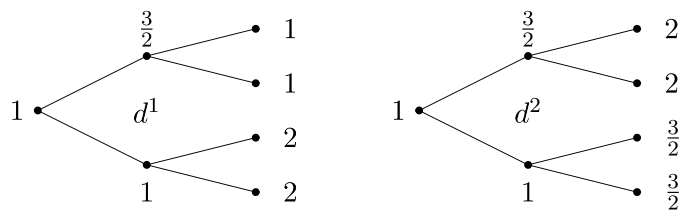

Figure 2: Performance ratio for d^1 and d^2

It follows from the figure that $R^*(d^1, 3) = R^*(d^2, 3) = \frac{3}{2}$. If d^1 and d^2 are the only possible problem instances, then from (3) it follows $W^*(3) = \frac{3}{2}$. However, the worst case performance of any on-line heuristic H is 2. Namely, if H generates no setup in period 2, then we give d_3^1 in period 3 leading to performance ratio 2. On the other hand, if H generates a setup in period 2, then we give d_3^2 in period 3 again leading to performance ratio 2. Hence the worst case ratio equals 2 in this particular case.

The problem of the mathematical formulation is that it is not possible that two branches arising from the same node have different remaining demand sequences. However, it is possible in the formulation to have zero demands as the remaining demand sequence, because we evaluate the performance ratio for all t-period production plans. This means that the problem formulation leads to lower bounds on the worst case performance of any on-line heuristic.

We have plotted the graph of the function $R^*(d, 5)$ with $d_2 = 0$ and $d_5 = \frac{1}{4}$ in Figure 3. It is clear that finding the demand sequence d that optimizes $R^*(d, 5)$ is not a nice concave maximization problem.

3.2 A special class of production plans

Because equation (3) is hard to analyze, we will consider a further relaxation of the problem. First we will derive a lower bound on the value $R^*(d, T)$. Define the set of production plans P^i $(i = 1, \dots, T)$ as follows

$$P_t^1 = \begin{cases} 1 & \text{for } t = 1, \\ 0 & \text{for } t = 2, \dots, T, \end{cases}$$

$$P_t^i = \begin{cases} 1 & \text{for } t = 1 \text{ and } t = i, \\ 0 & \text{for } t = 2, \dots, i-1, \end{cases} \quad \text{for } i = 2, \dots, T.$$

Note that P^i $(i = 2, \dots, T)$ is a production plan for i periods and P^1 is a plan for T periods. In Figure 4 production plans P^i $(i = 1, \dots, 4)$ are the paths from the root to the leafs in the

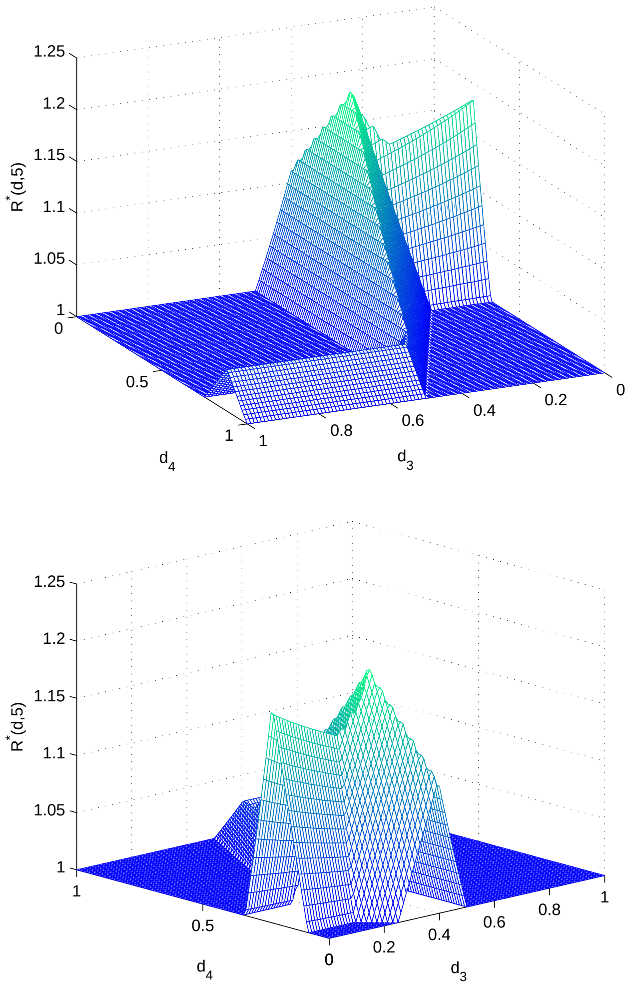

Figure 3: The graph of the function $R^*(d, 5)$ with $d_2 = 0$ and $d_5 = \frac{1}{4}$

tree. Define

$$r(d,1) = \frac{C(d,P^1,T)}{C^*(d,T)} \text{ and } r(d,i) = \frac{C(d,P^i,i)}{C^*(d,i)} \text{ for } i = 2,\ldots,T$$

and let

$$R(d,T) = \min_{i=1,\ldots,T} r(d,i).$$

So the values $r(d,i)$ represent the performance ratios of the leaf nodes in Figure 4 and $R(d,T)$ is the minimum performance ratio of these nodes.

Lemma 1 *For any instance $d_1, \ldots, d_T$ it holds*

$$R(d,T) \leq R^*(d,T).$$

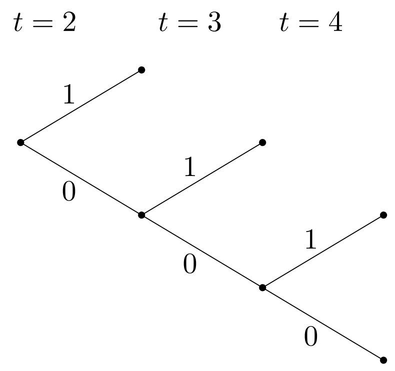

Figure 4: Production plans P^i $(i = 1, \dots, 4)$ represented by paths in a tree

Proof Let $P \in P(T)$. Then there exists a j for which P^j is a subpath of P. But then

$$r(d, j) \leq \max_{i=1,\dots,T} \frac{C(d, P, i)}{C^*(d, i)}$$

as the term at the left hand side is contained in the maximum at the right hand side. $\square$

Lemma 1 shows that, when using the special set of production plans, we find a lower bound on the performance ratio for d. The motivation for taking P^i $(i = 2, \dots, T)$ is that one expects that these plans lead to high costs because in general it is not profitable to have a setup in the last period. Plan P^1 is needed, because with this plan included, any production plan P has a plan P^i as subplan (and so without P^1 Lemma 1 does not hold). It is clear that for a fixed d the value $R(d, T)$ is a lower bound on $W^*(T)$. Furthermore, we define the lower bound $W(T)$ on $W^*(T)$ as

$$W(T) = \max_{d \in [0,1]^T} R(d, T) = \max_{d \in [0,1]^T} \min_{t=1,\dots,T} r(d, t). \tag{4}$$

Note that problem (4) is more tractable than problem (3) because the 'min max'-part is replaced by a 'min'-part. We will now derive some properties for a demand sequence that maximizes (4).

Lemma 2 *Let d be an instance that maximizes (4). Then the value of $d_T = \frac{1}{T-1}$.*

Proof First note that d_T only occurs in the calculation of $r(d, 1)$ and $r(d, T)$ because they contain the terms $C(d, P^1, T)$ and $C^*(d, T)$. The holding costs for d_T equal $(T - 1)d_T$ in plan P^1. If $p \geq 2$ is the setup period preceding period T in the optimal plan P^*, then the holding cost in this plan equals $(T - p)d_T$. (If $p = 1$, then $r(d, 1) = 1$ and hence $W(T) = 1$, which cannot be optimal.) Increasing d_T will increase the performance ratio $r(d, 1) = \frac{C(d, P^1, T)}{C^*(d, T)}$. However, the performance ratio $r(d, T) = \frac{C(d, P^T, T)}{C^*(d, T)}$ is decreasing in d_T. Therefore, $\min\{r(d, 1), r(d, T)\}$ is maximized when $C(d, P^1, T) = C(d, P^T, T)$, i.e., when $(T - 1)d_T = 1$ or $d_T = \frac{1}{T-1}$. $\square$

In the remainder of this paper we will assume that $d_T = \frac{1}{T-1}$ so that $r(d,1) = r(d,T)$ and hence

$$W(T) = \max_{d \in [0,1]^T} \min_{i=2,\ldots,T} r(d,i). \tag{5}$$

Another useful property of an optimal demand sequence can be found in the following lemma.

Lemma 3 *Let d be an instance with $d_{j-1} > 0$, $d_j = 0$ and $3 \leq j \leq T$. Then there exists an instance d' with $R(d',T) \geq R(d,T)$ and $d'_j > 0$.*

Proof Define an instance d' with $d'_2 = 0$ and $d'_{t+1} = \frac{t-1}{t} d_t$ for all $t < j$ and $d'_t = d_t$ for all $t > j$. So demand before period j is shifted one period and scaled. Clearly, $d'_j = \frac{j-2}{j-1} d_{j-1} > 0$. Let $i \leq j$. Then the holding cost for demand d'_{t+1} $(t < i)$ in P^{i+1} equals $t d'_{t+1} = t \frac{t-1}{t} d_t = (t-1) d_t$, which is the holding cost for demand d_t in P^i. Therefore, $C(d', P^{i+1}, i+1) = C(d, P^i, i)$ for $i \leq j$. Furthermore, because demand beyond period j is unchanged we also have $C(d', P^i, i) = C(d, P^i, i)$ for $i > j$.

Let P be the optimal plan for some i-period problem for instance d. Now shift all setup periods before period j (except for period 1) one period further. We will use this plan for the $(i+1)$-period problem with demand d' if $i < j$ and for the i-period problem with demand d' if $i > j$. Clearly, the setup costs for both plans are equal. Furthermore, let $t < j$ and let p be the setup period before period t in plan P. Then the holding cost for demand d_t equals $(t-p)d_t$ and the holding cost for demand d'_{t+1} equals $((t+1)-(p+1))d'_t = (t-p)\frac{t-1}{t} d_t \leq (t-p) d_t$. Using similar arguments one can show that holding cost for d'_t equals at most the holding cost for d_t for $t > j$. Therefore, $C^*(d,i) \geq C^*(d', i+1)$ for $i < j$ and $C^*(d,i) \geq C^*(d',i)$ for $i > j$.

Using the above (in)equalities it follows that

$$\begin{aligned} r(d,i) = \tfrac{C(d,P^i,i)}{C^*(d,i)} &\leq \tfrac{C(d',P^{i+1},i+1)}{C^*(d',i+1)} = r(d', i+1) \qquad \text{for } i < j \\ r(d,i) = \tfrac{C(d,P^i,i)}{C^*(d,i)} &\leq \tfrac{C(d',P^i,i)}{C^*(d',i)} = r(d',i) \qquad \text{for } i > j. \end{aligned}$$

Furthermore, $r(d,j) \geq r(d,j-1)$ because $C(d,P^j,j) \geq C(d,P^{j-1},j-1)$ and $C^*(d,j) = C^*(d,j-1)$ since $d_j = 0$. Now the lemma follows because

$$R(d,T) = \min_{i=2,\ldots,T} r(d,i) \leq \min_{i=2,\ldots,T} r(d',i) = R(d',T).$$

$\square$

The previous lemma shows that if we have an instance with a positive demand period followed by a zero demand period, then we can find an instance with larger performance ratio by shifting and scaling all the demand before this zero demand period by one period. Therefore, there exists a solution that maximizes (5) and has no positive demands followed by zero demands (except for the demand in period 1). Let d be a problem instance with $d_t = 0$ for $t = 2, \dots, n-1$. For this instance we have $r(d, i) = 2$ for $i = 2, \dots, n-1$ and hence

$$R(d, T) = \min_{i=n,\dots,T} r(d, i).$$

Lemma 4 *Let d^* be a local optimal solution of (5) with $d_t = 0$ for $t = 2, \dots, n-1$ and $d_t > 0$ for $t = n, \dots, T$. Then $r(d^*, i) = r(d^*, i+1)$ for $i = n, \dots, T-1$.*

Proof Assume that the lemma does not hold and let

$$r(d^*, u) = \min_{i=n,\dots,T}\{r(d^*, i)\} < \max_{i=n,\dots,T}\{r(d^*, i)\} = r(d, v).$$

Assume that $u < v$ (the case with $u > v$ can be proven analogously). We will construct an alternative solution with demand sequence $d' = d^* + \varepsilon$ such that $r(d^*, u) < r(d', u) \leq r(d', v)$ and $r(d', i) = r(d^*, i)$ for $i \in \{n, \dots, T\} \backslash \{u, v\}$. This means that the solution is not a local optimum which is a contradiction.

To achieve this, let $r_i = r(d^*, i)$ for $i = n, \dots, T$, keep the production plans fixed and consider the equations $r(d, i) = r_i$ in the variables $d_n, \dots, d_{T-1}$. So the function $r(d, i)$ is the ratio of two linear functions (see equation (2)), $r(d, i)$ is defined on the variables $d_n, \dots, d_i$ for $i = n, \dots, T-1$ and $r(d, T)$ is defined on the variables $d_n, \dots, d_{T-1}$. First, note that the optimal plan P^{*i} will not have a setup in period i, because moving the setup from period i to period $i-1$ will not increase the cost (recall $d_t \leq 1$). Therefore, $r(d, i)$ is strictly decreasing in d_i, because d_i appears in the denominator and it does not appear in the nominator. Similarly, $r(d, T)$ is strictly increasing in d_{T-1}, because d_{T-1} appears in the nominator with coefficient $T-2$.

Now let $d'_i = d^*_i$ for $i = n, \dots, u-1$ and let $d'_u = d^*_u + \varepsilon_u$ with $\varepsilon_u \in \mathbb{R}$ such that $r(d', u) > r(d^*, u)$. Note that $r(d', i) = r_i$ for $i = n, \dots, u-1$ and $r(d', i)$ may be changed for $i = u, \dots, T$. Let $d'_i = d^*_i + \varepsilon_i$ with ε_i such that $r(d', i) = r_i$ for $i = u+1, \dots, v-1$. Because the function $r(d, i)$ is strictly decreasing in d_i, the value $d'_i = d^*_i + \varepsilon_i$ is uniquely defined by the equation $r(d', i) = r_i$ for given $d'_n, \dots, d'_{i-1}$ and hence the values ε_i ($i = u+1, \dots, v-1$) exist. In a similar way we choose $d'_i = d^*_i + \varepsilon_i$ for $i = v, \dots, T-1$ such that $r(d', i) = r_i$ for $i = v+1, \dots, T$. Again such values ε_i ($i = v, \dots, T-1$) exist. Namely, start with an arbitrary value of ε_v. Then the values ε_i ($i = v+1, \dots, T-1$) are uniquely determined by the equations $r(d', i) = r_i$ for $i = v+1, \dots, T-1$. In general we will have that $r(d', T) \neq r_T$. This means that the choice of ε_v was not right and the right value of ε_v can be found by binary search since $r(d, i)$ is strictly decreasing in d_i. So summarizing, given an ε_u, the values ε_i for $i = u+1, \dots, T-1$ are uniquely determined by the equations $r(d', i) = r_i$ for $i \in \{u+1, \dots, T\} \backslash \{v\}$.

Finally, consider the value $r(d', v)$. If $r(d', v) \geq r(d^*, v)$, then we have proved the lemma as we have found a strictly better solution within an ε-environment which is a contradiction. If $r(d', v) < r(d^*, v)$, then we choose ε_u sufficiently small such that $r(d', v) \geq r(d', u) > r(d^*, u)$ and again we have found a better solution.

We end the proof with some remarks. First, if period u is not unique, then we can repeat the above procedure. Second, if there is some $d'_i < 0$ (which is not feasible), then ε_u must be chosen sufficiently small such that $d'_i \geq 0$. Third, it is possible that by the change from d^* to d' the optimal production plans will also change. In this case the denominator of $r(d', i)$ will be smaller and hence $r(d', i)$ will be larger which means that the proof still holds. $\square$

Let $W(n, T)$ be the maximum performance ratio of the demand sequence d that satisfies

$$W(n, T) = r(d, i) \text{ for } i = n, \dots, T, \; d_t > 0 \text{ for } t = n, \dots, T \text{ and } d_t = 0 \text{ for } t = 2, \dots, n-1. \quad (6)$$

Corollary 5 *For any model horizon T it holds $W(T) = \max_{1<n<T} W(n, T)$.*

Proof Immediate from Lemma 4. $\square$

Corollary 5 shows that we can find $W(T)$ by calculating all values $W(n, T)$. This means that we changed the optimization problem (5) into solving the system of equations in (6). In the following sections we will focus on how to find the values $W(n, T)$, but first we give two other corollaries.

Corollary 6 *For $1 < n < T$ we have $W(n+1, T+1) \geq W(n, T)$.*

Proof A T-period instance can be considered as a $(T+1)$-period instance with $d_{T+1} = 0$. Apply Lemma 3 to this instance. $\square$

Corollary 7 *For any model horizon T we have $W(T+1) \geq W(T)$.*

Proof Immediate from Corollary 6. $\square$

3.3 Finding the optimal demand sequence given the production plans

It follows from the previous section that if we can find the values $W(n, T)$, then we can find the value $W(T)$. The difficulty of finding $W(n, T)$ is that the optimal production plans and the demand sequence have to be determined simultaneously. For example, a change in the demand sequence may cause a change in the optimal production plans. In this section we will derive an approach to calculate a demand sequence that satisfies (6), assuming that the optimal production plans are known.

Assume that we have a demand sequence with $d_t = 0$ for $t = 2, \ldots, n-1$. If the optimal plans are known, then by Lemma 4 a local optimal demand sequence can be found by solving the system in (6). Given the plans P^i and the optimal production plans P^{*i} for each horizon $i = n, \ldots, T$, it follows from (2) that both the nominator and the denominator of $r(d,i) = \frac{C(d,P^i,i)}{C(d,P^{*i},i)}$ are linear functions in the variables $d_n, \ldots, d_{T-1}$. Now by 'cross-multiplying', system (6) is a system of multivariate quadratic equations. This is in general a hard problem, because by the method of repeated substitution one has to find the roots of univariate polynomials of high degree. Example 1 illustrates this.

Example 1 Consider a problem with $T = 5$ and $n = 3$ so that $d_2 = 0$ and $d_5 = \frac{1}{4}$. In Table 2 the production plans, corresponding costs and performance ratios are shown. From

i	P^i	$C(d,P^i,i)$	P^{*i}	$C(d,P^{*i},i)$	$r(d,i)$
3	$\{1,0,1\}$	2	$\{1,0,0\}$	$1+2d_3$	$\frac{2}{1+2d_3}$
4	$\{1,0,0,1\}$	$2+2d_3$	$\{1,0,1,0\}$	$2+d_4$	$\frac{2+2d_3}{2+d_4}$
5	$\{1,0,0,0,1\}$	$2+2d_3+3d_4$	$\{1,0,1,0,0\}$	$2+d_4+\frac{1}{2}$	$\frac{2+2d_3+3d_4}{\frac{5}{2}+d_4}$

Table 2: Performance ratios for $T = 5$ and $n = 3$

$r(d,3) = r(d,4)$ it follows that $d_4 = 2d_3^2 + 3d_3 - 1$. Substituting this in $r(d,3) = r(d,5)$ we have $12d_3^3 + 24d_3^2 + 3d_3 - 4 = 0$. Solving this equation we have $d_3^* \approx 0.328$, $d_4^* \approx 0.201$ and $r^* \approx 1.207$. Note that this problem can be solved exactly because there exists a closed formula for finding the root of a polynomial of degree 3. However, no closed formulas exist for polynomials of degree at least 5 and hence another approach is required for larger values of T.

We will derive a numerical procedure to find values d_t that satisfy (6). Assume that the optimal ratio equals r^*. This means there exists an instance d^* with $r(d^*,i) = r^*$ for $i = n, \ldots, T$. Furthermore, assume for the moment that the optimal plans corresponding to the i-period problem of d^*, say P^{*i}, are known (note that $C(d^*, P^{*i}, i) = C^*(d^*, i)$). If the value r^* is not known, we can start with an initial guess r. Then for fixed r the system

$$\frac{C(d,P^i,i)}{C(d,P^{*i},i)} = r \text{ for } i = n, \ldots, T$$

is a system of $m+1$ linear equations in m variables $(d_n, \ldots, d_{T-1})$ with $m = T-n$, which means it is overdefined. Define the residual of equation i by

$$e_i(r,d) = rC(d, P^{*i}, i) - C(d, P^i, i)$$

and the sum of squared residuals by

$$S(r,d) = \sum_{i=n}^{T} e_i(r,d)^2. \tag{7}$$

Because

$$e_i(r^*, d^*) = r^* C(d^*, P^{*i}, i) - C(d^*, P^*, i) = 0,$$

it follows $S(r^*, d^*) = 0$. Therefore, given the optimal production plans, the demand sequence d^* that solves (6) with corresponding ratio r^* is a solution of the problem of minimizing (7).

Clearly, given a fixed r, minimizing (7) is nothing but least squares fitting which is a relatively easy problem. Because the ratio r^* is at least 1 by definition and because of the existence of heuristics with worst case ratio 2, the value r^* can be found by a search procedure on the interval $[1, 2]$ given that the optimal production plans are known. (Note that we have not proved the existence of a unique solution in this interval.) We will call the above method which, for given plans, finds the demand sequence that satisfies (6) the least squares procedure (LSP).

3.4 An initial guess for the optimal plans

The procedure of the previous section cannot directly be applied, because the set of optimal production plans is not known. Therefore, we will construct a set P'^i $(i = n, \ldots, T)$ that serves as an 'approximation' for the set of optimal production plans. For ease of notation let

$$r'(d,i) = \frac{C(d, P^i, i)}{C(d, P'^i, i)} \text{ for } i = n, \ldots, T \text{ and } R'(d,T) = \min_{i=n,\ldots,T} r'(d,i).$$

Lemma 8 *For a given instance d with $d_t = 0$ $(t = 2, \ldots, n-1)$, $d_t > 0$ $(t = n, \ldots, T-1)$ and arbitrary plans P'^i $(i = n, \ldots, T)$, it holds that $R'(d,T) \leq R(d,T)$.*

Proof From the optimality of $C^*(d,i)$ it follows that $C(d, P'^i, i) \geq C^*(d,i)$. Therefore,

$$r'(d,i) = \frac{C(d, P^i, i)}{C(d, P'^i, i)} \leq \frac{C(d, P^i, i)}{C^*(d,i)} = r(d,i).$$

□

Note that starting with plans P'^i that are worse than P^i leads to $r'(d,i) < 1$. Therefore, we have to start with a reasonable guess. Let k be a fixed integer with $n < k \leq T$ and consider the set of production plans P'^i $(i = n, \ldots, T)$ defined as follows (for ease of notation we do not show the dependence on k of this set)

$$P'^i_t = \begin{cases} 1 & \text{for } t = 1 \\ 0 & \text{for } t = 2, \ldots, i, \end{cases} \qquad \text{for } i = n, \ldots, k-1 \tag{8}$$

$$P'^i_t = \begin{cases} 1 & \text{for } t = 1 \text{ and } t = n \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for } i = k, \ldots, T. \tag{9}$$

As for the plans P^i, the plan P'^i represents a plan for an i-period problem instance. The value k indicates that plans consisting of at least k periods have an additional setup in period n. We will come back on the choice of k in the next section. The motivation to take plans P'^i is that for small horizons $(t \leq k-1)$ it seems reasonable to have only a setup in period 1 and for larger horizons $(t \geq k)$ it seems reasonable to have an additional setup to reduce the holding costs. A lower bound $W'(n,T)$ on $W(n,T)$ can be found by solving the system

$$W'(n,T) = r'(d,i) \text{ for } i = n, \ldots, T. \tag{10}$$

Example 2 To illustrate the use of the sets P^i and P'^i consider a problem instance for $T = 3$. In this case $P^2 = \{1,1\}$, $P^3 = \{1,0,1\}$ and with $k = 3$ we have $P'^2 = \{1,0\}$, $P'^3 = \{1,1,0\}$. From Lemma 2 it follows that $d_3 = \frac{1}{2}$ and by (10)

$$W'(2,3) = \frac{2}{1+d_2} = \frac{2+d_2}{5/2}.$$

Solving this quadratic equation (note that we do not need the LSP of Section 3.3) we have $d_2 = \frac{1}{2}(\sqrt{21}-3) \approx 0.79$ and $W'(2,3) = \frac{1}{5}(1+\sqrt{21}) \approx 1.117$. So the instance $d_1 = 1, d_2 = \frac{1}{2}(\sqrt{21}-3), d_3 = \frac{1}{2}$ is a problem instance with performance ratio $\frac{1}{5}(1+\sqrt{21})$ and hence a lower bound on the worst case ratio for $T = 3$.

3.5 An iterative procedure to find worst case examples

In this section we will describe an iterative procedure in which the plans P'^i are updated in each iteration. We start with some initial guess for the optimal plans and calculate the optimal demand sequence using the least squares procedure. Now given this demand sequence, we can determine the 'real' optimal plans corresponding to this demand sequence. If these plans are different from our initial guess, a new iteration is performed starting with these new plans. The iterative procedure is schematically illustrated in Table 3.

In Step 1 we start with the initial guess P'^i_{old} $(i = n, \ldots, T)$ for the optimal production plans. Given these plans, we calculate the optimal demand sequence d^* and the corresponding

Iterative procedure to calculate $W'(n,T)$	
Step 1:	Start with some initial guess P'^{i}_{old} $(i = n, \dots, T)$
Step 2:	Calculate r^* and d^* given P'^{i}_{old} $(i = n, \dots, T)$ using the LSP
Step 3:	Calculate P'^{i}_{new} $(i = n, \dots, T)$ given d^*
	If $P'^{i}_{new} = P'^{i}_{old}$ $(i = n, \dots, T)$ Then
	Output: $W'(n,T) = r^*$ and d^*
	Stop
	Else
	$P'^{i}_{old} = P'^{i}_{new}$ $(i = n, \dots, T)$
	Go to Step 2
	End if

Table 3: Iterative procedure to calculate $W'(n,T)$

performance ratio r^* using the LSP of Section 3.3. In Step 3 we check whether the guess was right by calculating the optimal plans P'^{i}_{new} corresponding to d^*. If yes, the procedure is terminated. If not, then we go back to Step 2 and start with these plans. Note that by Lemma 8 in every iteration the performance ratio will increase. Because the number of plans is finite, the iterative procedure will terminate.

The iterative procedure (IP) of Table 3 was implemented in Visual Basic. When starting the IP, we have multiple initial guesses for P'^{i}_{old} as k in (8) and (9) can range from $n+1$ to T. Given some value of T and n we started the IP with all possible values of k and it turned out that the IP always terminated with the same plans P'^{i}_{new}. The values $W'(n,T)$ for $T = 3, \dots, 20$ and $n = 2, \dots, T-1$ can be found in Table 4. Below each performance ratio the minimum number of iterations needed before termination over all initial plans is shown. Note that the performance ratio is 1 if $T = 2$ or if $n = T$. In the latter case we only have two strictly positive demands (including the demand in period 1) which is similar to the case $T = 2$.

$T \setminus n$	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**
3	**1.117**																	
	1																	
4	1.150	**1.165**																
	1	1																
5	1.175	**1.207**	1.191															
	2	1	1															
6	1.197	1.232	**1.235**	1.207														
	3	1	1	1														
7	1.215	1.252	**1.274**	1.259	1.219													
	4	3	1	1	1													
8	1.226	1.270	1.293	**1.301**	1.276	1.227												
	4	2	1	1	1	1												
9	1.235	1.282	1.304	**1.320**	1.318	1.289	1.234											
	3	4	2	1	1	1	1											
10	1.241	1.291	1.317	1.330	**1.336**	1.331	1.299	1.239										
	5	4	3	1	1	1	1	1										
11	1.247	1.298	1.329	1.343	**1.353**	1.350	1.340	1.307	1.243									
	4	4	3	2	1	1	1	1	1									
12	1.252	1.305	1.337	1.354	1.364	**1.369**	1.363	1.346	1.314	1.246								
	5	4	3	3	1	1	1	1	1	1								
13	1.255	1.309	1.342	1.362	1.372	1.379	**1.381**	1.372	1.352	1.320	1.249							
	5	4	3	3	2	1	1	1	1	1	1							
14	1.258	1.312	1.347	1.367	1.378	1.385	**1.390**	1.389	1.380	1.356	1.324	1.251						
	5	4	4	4	2	1	1	1	1	1	1	1						
15	1.260	1.315	1.350	1.372	1.386	1.393	**1.398**	**1.398**	1.396	1.386	1.359	1.329	1.253					
	4	5	4	3	3	2	1	1	1	1	1	1	1					
16	1.262	1.317	1.352	1.375	1.391	1.400	1.404	**1.408**	1.407	1.401	1.391	1.362	1.332	1.255				
	5	5	6	4	3	2	1	1	1	1	1	1	1	1				
17	1.263	1.319	1.355	1.379	1.394	1.404	1.410	1.414	**1.416**	1.414	1.405	1.395	1.364	1.335	1.257			
	5	5	5	4	4	3	2	1	1	1	1	1	1	1	1			
18	1.264	1.320	1.357	1.381	1.397	1.407	1.414	1.418	1.421	**1.422**	1.419	1.408	1.399	1.367	1.338	1.258		
	5	6	5	5	4	3	2	1	1	1	1	1	1	1	1	1		
19	1.265	1.321	1.358	1.383	1.399	1.411	1.418	1.423	1.425	**1.427**	**1.427**	1.424	1.413	1.402	1.371	1.341	1.259	
	5	6	5	5	4	4	3	2	1	1	1	1	1	1	1	1	1	
20	1.266	1.322	1.360	1.384	1.401	1.413	1.422	1.427	1.430	1.432	**1.433**	1.430	1.427	1.417	1.405	1.374	1.343	1.260
	5	6	5	4	4	4	3	2	2	1	1	1	1	1	1	1	1	1

Table 4: Performance ratios and number of iterations for different model horizons

Table 4 shows some interesting results. First, it follows that $W'(n,T) \leq W'(n+1,T+1)$. As this property holds for $W(n,T)$ (see Lemma 3) and because the IP converges to the same solutions when starting with different initial guesses, it suggests that $W(n,T)$ and $W'(n,T)$ might be equal. Second, we see that for a fixed T the value of n that maximizes $W'(n,T)$ (the performance ratios in bold), say $n(T)$, is increasing in T. Furthermore, we see that for $n < n(T)$, $W'(n,T)$ is increasing in n, and for $n > n(T)$, $W'(n,T)$ is decreasing in n. Third, the minimum number of iterations shows that the initial guesses are reasonable. For n close to T we see that one of the initial guesses is the optimal one. Finally, we note that for large values of T we can find performance ratios close to $\frac{3}{2}$. For example, $W'(80,100) = 1.494$ and $W'(480,500) = 1.499$.

Again look at the graph of $R^*(d,5)$ in Figure 3. The function $R^*(d,5)$ has two local optima: $d_3^1 \approx 0.328$, $d_4^1 \approx 0.201$ with $R^*(d^1,5) \approx 1.207$ and $d_3^2 = 0$, $d_4^2 \approx 0.226$ with $R^*(d^2,5) \approx 1.191$. The performance ratios of these two solutions are equal to the ratios found by the IP ($W'(3,5)$ and $W'(4,5)$ in Table 4), which shows that the IP leads to the optimal solutions for $T = 5$ with $d_2 = 0$.

4 Worst case instances

Using the results of the previous section, we will present three worst case problem instances. First, we will derive a problem instance with three positive demand periods (including period 1) for which any on-line heuristic has a performance ratio of at least $\frac{1}{4}(\sqrt{17}+1) \approx 1.281$. Second, we will give a problem instance for which any on-line heuristic has a performance ratio of at least $\frac{3}{2}$ at the time period of the second setup (where the setup in period 1 is the first). We end this section with the main result of our paper and present a problem instance for which any on-line heuristic has a performance ratio of at least 2.

4.1 An instance with three positive demand periods

Theorem 9 *There exist a problem instance with three positive demand periods for which any on-line heuristic has a worst case ratio of at least* $\frac{1}{4}(\sqrt{17}+1) \approx 1.281$.

Proof Note that it is sufficient to show that

$$\lim_{T\to\infty} W(T,T+1) = \frac{1}{4}(\sqrt{17}+1) \approx 1.281.$$

Assume that demand in period T equals $d_T = \frac{c}{T-1}$ (for ease of notation we use $T-1$ in the denominator). Furthermore, by Lemma 2 we have $d_{T+1} = \frac{1}{T}$. It is not difficult to see that in the T-period problem it is optimal to have a setup only in period 1 for appropriate c, whereas plan P^T has setups in periods 1 and T. So we have $r(c,T) = \frac{2}{1+c}$. For the $(T+1)$-period problem it is optimal to have setups in periods 1 and T, whereas P^{T+1} has setups in periods 1 and $T+1$ implying that $r(c,T+1) = \frac{2+c}{2+\frac{1}{T}}$. Now the maximum of

$$W(T,T+1) = \max_c \min\left\{\frac{2}{1+c}, \frac{2+c}{2+\frac{1}{T}}\right\},$$

is attained for $c = \frac{1}{2}(\sqrt{17+8/T}-3)$ with

$$W(T,T+1) = \frac{\frac{1}{2}(\sqrt{17+8/T}+1)}{2+1/T} \to \frac{1}{4}(\sqrt{17}+1) \approx 1.281 \text{ as } T \to \infty.$$

$\square$

Note that for $T+1=3$, we have that $W(2,3) = \frac{1}{5}(\sqrt{21}+1)$ which is consistent with the value found in Example 2. Furthermore, it can be shown that there exist on-line heuristics that have a worst case ratio of $\frac{1}{4}(\sqrt{17}+1)$ for this instance and hence the problem instance can be called a worst case problem instance. The theorem also shows that the values on the diagonal of Table 4 tend to $\frac{1}{4}(\sqrt{17}+1)$.

4.2 An instance with at most two heuristic setup periods

The numerical results of the IP led to the construction of a problem instance with performance ratio $\frac{3}{2}$. In Table 5 we present the output of the IP for a 100-period problem instance with $d_{80}, \ldots, d_{100} > 0$. This problem instance has a performance ratio of 1.494. Note from the

t	80	81	82	83	84	85	86	87	88	89
$(t-1)d_t$	0.3391	0.2271	0.1520	0.1018	0.0682	0.0456	0.0306	0.0205	0.0137	0.0092
$\frac{(t-1)d_t}{(t-2)d_{t-1}}$	-	0.670	0.670	0.670	0.670	0.670	0.670	0.670	0.670	0.670
t	90	91	92	93	94	95	96	97	98	99
$(t-1)d_t$	0.0061	0.0041	0.0028	0.0018	0.0012	0.0016	0.0064	0.0242	0.0874	0.3017
$\frac{(t-1)d_t}{(t-2)d_{t-1}}$	0.670	0.670	0.670	0.670	0.670	1.305	3.976	3.781	3.608	3.454

Table 5: Problem instance generated from the IP with performance ratio 1.494

table that the ratio between the holding cost of two consecutive periods is approximately $\frac{2}{3}$ (for $t = 80, \ldots, 94$). We will present a problem instance with this property and we will show that the performance ratio of this instance tends to $\frac{3}{2}$ for T large. Define the sequence $x_t = \frac{1}{3}(\frac{2}{3})^t$ (note that $\frac{x_t}{x_{t-1}} = \frac{2}{3}$). In the proof we will use the following property of the sequence x_t.

Lemma 10 *Let $x_t = \frac{1}{3}(\frac{2}{3})^t$ for $t = 0, 1, \ldots$. Then for all $i = 0, 1, 2, \ldots$*

$$\frac{2 + \sum_{t=0}^{i-1} x_t}{1 + \sum_{t=0}^{i} x_t} = \frac{3}{2}.$$

Proof By induction. First, note that the lemma holds for $i = 0$ as $\frac{2}{1+1/3} = \frac{3}{2}$. Assume that the lemma holds for some $i \geq 1$ so that $4 + 2\sum_{t=0}^{i-1} x_t = 3 + 3\sum_{t=0}^{i} x_t$. Since $2x_i = 3x_{i+1}$ we have

$$4 + 2\sum_{t=0}^{i-1} x_t + 2x_i = 3 + 3\sum_{t=0}^{i} x_t + 3x_{i+1} \Leftrightarrow 4 + 2\sum_{t=0}^{i} x_t = 3 + 3\sum_{t=0}^{i+1} x_t \Leftrightarrow \frac{2 + \sum_{t=0}^{i} x_t}{1 + \sum_{t=0}^{i+1} x_t} = \frac{3}{2}.$$

□

Theorem 11 *There exists a problem instance for which any on-line heuristic generates at most 2 setup periods (including the first period) and for which the heuristic has worst case ratio at least $\frac{3}{2}$.*

Proof We will prove the theorem by showing that there exists a problem instance d with

$$\lim_{T\to\infty} R'(d, T) = \frac{3}{2}.$$

Define the demand sequence d with time horizon $T^2 + T + 1$ as follows: $d_1 = 1$, $d_t = 0$ for $t = 2, \ldots, T^2 - 1)$, $d_t = \frac{x_{t-T^2}}{t-1}$ for $t = T^2, \ldots, T^2 + T$ with x_t as in Lemma 10 and $d_{T^2+T+1} = \frac{1}{T^2+T}$. (Using the notation of the previous sections we have set T to $T^2 + T + 1$ and n to T^2.) First, for $i = T^2, \ldots, T^2 + T + 1$ we have

$$C(d, P^i, i) = 2 + \sum_{t=T^2}^{i-1} (t-1)d_t = 2 + \sum_{t=0}^{i-T^2-1} x_t$$

and

$$C(d, P^1, T^2 + T + 1) = 1 + \sum_{t=T^2}^{T^2+T+1} (t-1)d_t = 1 + \sum_{t=0}^{T} x_t + 1.$$

Now let P'^i be a production plan for the i-period problem ($i = T^2, \ldots, T^2 + T$) with only a setup in period 1 and let P'^{T^2+T+1} be a production plan with setups in periods 1 and T^2 (so using the notation of Section 3.4 we have set k to $T^2 + T + 1$). Then we have

$$C(d, P'^i, i) = 1 + \sum_{t=T^2}^{i} (t-1)d_t = 1 + \sum_{t=0}^{i-T^2} x_0$$

and

$$
\begin{aligned}
C(d, P'^{T^2+T+1}, T^2+T+1) &= 2 + \sum_{t=T^2+1}^{T^2+T+1} (t-T^2) d_t = 2 + \sum_{t=T^2+1}^{T^2+T} \frac{t-T^2}{t-1} x_{t-T^2} + \frac{T+1}{T^2+T} \\
&\leq 2 + \sum_{t=T^2+1}^{T^2+T} \frac{T}{T^2} x_{t-T^2} + \frac{1}{T} = 2 + \frac{1}{T} \sum_{t=1}^{T} \frac{1}{3} \left(\frac{2}{3}\right)^t + \frac{1}{T} \leq 2 + \frac{2}{T}.
\end{aligned}
$$

By Lemma 10 we have that

$$
r'(d, i) = \frac{2 + \sum_{t=0}^{i-T^2-1} x_t}{1 + \sum_{t=0}^{i-T^2} x_t} = \frac{3}{2} \text{ for } i = T^2, \ldots, T^2+T.
$$

Furthermore, because $\lim_{T\to\infty} \sum_{t=0}^{T} x_t = 1$ we have that $C(d, P^{T^2+T+1}, T^2+T+1) \to 3$, $C(d, P^1, T^2+T+1) \to 3$ and $C(d, P'^{T^2+T+1}) \to 2$ for $T \to \infty$ and hence $r'(d, T^2+T+1) \to \frac{3}{2}$ and $r'(d, 1) \to \frac{3}{2}$ for $T \to \infty$. In conclusion, demand sequence d is an instance with performance ratio $R'(d, T^2+T+1) \to \frac{3}{2}$ for $T \to \infty$. □

4.3 An instance for the general problem

In the previous section we found a problem instance with performance ratio $\frac{3}{2}$. The problem instance started with a sequence of zero demands followed by a sequence of positive demands. In this section we will build on this idea. After the last setup generated by the heuristic, we again extend the demand sequence by a sequence of zero demands followed by a sequence of positive demands. By repeating this procedure, it will follow that any on-line heuristic has a worst case ratio of at least 2. We will use the following lemma to prove our main result.

Lemma 12 *Let $x_i = \frac{1}{2}(\frac{1}{2})^i$ and $y_i = \sum_{j=0}^{i} x_i$ for $i = 0, 1, 2, \ldots$. Then for $i \geq 0$*

$$
\frac{1 + y_{i-1}}{y_i} = 2 \textit{ and } y_i \to 1 \textit{ for } i \to \infty.
$$

Proof By induction. □

Theorem 13 *Any on-line lot-sizing heuristic has worst case ratio at least 2.*

Proof Consider a partial demand sequence $d_0, \ldots, d_T$ with a solution generated by some on-line heuristic. We will extend this demand sequence such that the ratio between the cost increase of the heuristic solution and the cost increase of the optimal solution is arbitrarily close to 2 or strictly larger than 2.

Let m, n be integers and let $d_t = 0$ for $t = T+1, \ldots, T+m-1$, $d_t = \frac{x_{t-(T+m)}}{t-T}$ for $t = T+m, \ldots, T+m+n-1$, and $d_{T+m+n} = \frac{1}{m+n}$. This means we add $m-1$ zero demand periods and $n+1$ positive demands to the original demand sequence. We will specify the values for m and n later.

Let k be the first setup period of the heuristic after period T. We will consider the problem instance $d_0, \ldots, d_k$ as the new problem instance. Let $\Delta C_k^H = C^H(k) - C^H(T)$ ($\Delta C_k^* = C^*(k) - C^*(T)$) be the cost increase for the heuristic (optimal) solution, where $C^H(t)$ ($C^*(t)$) is the heuristic (optimal) cost for the $0, \ldots, t$-period problem. We will show that $\frac{\Delta C_k^H}{\Delta C_k^*}$ is arbitrarily close to 2 or larger than 2 for any k.

- $k \in \{T+1, \ldots, T+m-1\}$:
  As $\Delta C_k^H = 1$ and $\Delta C_k^* = 0$, clearly $\frac{\Delta C_k^H}{\Delta C_k^*} \geq 2$.

- $k \in \{T+m, \ldots, T+m+n-1\}$:
  Let $p = k - (T+m)$. First,

$$\Delta C_k^H \geq \sum_{t=T+m}^{k-1} (t-T)d_t + 1 = \sum_{t=T+m}^{k-1} (t-T)\frac{x_{t-(T+m)}}{t-T} + 1 = \sum_{t=0}^{p-1} x_t + 1 = y_{p-1} + 1.$$

  Second,

$$\Delta C_k^* \leq \sum_{t=T+m}^{k} t d_t = \sum_{t=T+m}^{k} t\frac{x_{t-(T+m)}}{t-T} = \sum_{t=0}^{p} \frac{t+T+m}{t+m} x_t \to y_p \text{ for } m \to \infty.$$

  Hence, by Lemma 12, $\frac{\Delta C_k^H}{\Delta C_k^*} \leq \frac{y_{p-1}+1}{y_p} = 2$ for $m \to \infty$.

- $k = T+m+n$ or k does not exist:
  First, $\Delta C_k^H \geq y_{n-1} + 1$. Now consider the optimal solution for the $(T+m+n)$-period problem. Construct a solution by taking the optimal solution of the T-period solution and adding a setup in period $T+m$. As this is an arbitrary solution,

$$\Delta C_k^* \leq 1 + \sum_{t=T+m}^{T+m+n} (t-(T+m))d_t = 1 + \sum_{t=0}^{n-1} t\frac{x_j}{t+m} + \frac{n}{m+n} \leq 1 + \frac{n}{m} y_{n-1} + \frac{n}{m+n}.$$

  Hence, $\frac{\Delta C_k^H}{\Delta C_k^*} \leq (y_{n-1}+1)/(1 + \frac{n}{m} y_{n-1} + \frac{n}{m+n}) \to \frac{y_{n-1}+1}{y_n} = 2$ for $m = n^2$ and $n \to \infty$.

It follows that we can extend any problem instance in such a way that $\frac{\Delta C_k^H}{\Delta C_k^*}$ is larger or arbitrarily close to 2 no matter where the next setup of the heuristic occurs. Given this newly constructed k-period problem instance, we can in turn extend this problem instance by setting $T = k$ and applying the procedure as described above. As the cost increase of the heuristic solution is twice as large as the cost increase of the optimal solution and the cost increase is at least 1, repeating the procedure will lead to a problem instance with worst case ratio arbitrarily close to 2. □

We remark that the proof is constructive in the sense that we can build a problem instance for an arbitrary on-line heuristic starting with a single period ($t = 0$). We can set the appropriate values for m and n, dependent on how close to 2 the performance ratio has to be. Finally, the instance is also a worst case instance as there exist on-line heuristics with worst case ratio 2.

5 Optimal on-line heuristics for $T = 3$ and $T = 4$

The ideas of Section 3 did not only result in the construction of a problem instance with worst case ratio 2, they are also the basis for the construction of an optimal on-line heuristic for a 3-period and 4-period horizon. An on-line heuristic is called optimal if there does not exist any other on-line heuristic with lower worst case performance. It is clear that we can construct an optimal heuristic for the case $T = 2$, because it is optimal to have a setup in period 2 if and only if $d_2 \geq 1$. This result can be generalized as follows.

Observation 14 *Assume that we have a T-period instance and a plan generated for the first $T-1$ periods with the last setup in period p. Then it is optimal to make a new setup in period T if and only if $d_T \geq \frac{1}{T-p}$.*

5.1 An optimal heuristic for $T = 3$

Example 2 shows that $\overline{d}_2 = \frac{1}{5}(\sqrt{21} - 3) \approx 0.791$ may be the threshold value whether or not to have a setup in period 2 for a 3-period problem instance. Based on this threshold value we construct a heuristic as follows.

Heuristic for $T = 3$ (H_3)
$P_1 = 1$
If $d_2 < \frac{1}{2}(\sqrt{21} - 3)$ Then $P_2 = 0$
If $d_3 < \frac{1}{2}$ Then $P_3 = 0$
Else $P_3 = 1$
Else $P_2 = 1$
If $d_3 < 1$ Then $P_3 = 0$
Else $P_3 = 1$

Table 6: Heuristic H_3 for $T = 3$

Proposition 15 *Heuristic H_3 has a worst case ratio of at most $\frac{1}{5}(1+\sqrt{21}) \approx 1.117$.*

Proof First, by Observation 14 instances have a performance ratio larger than 1 if there is a non-optimal decision in period 2.

- Assume we have an instance d with $d_2 < \overline{d}_2$ and the optimal solution has a setup in period 2. Then one can show that an instance with $d_3 = \frac{1}{2}$ will give the largest performance ratio. The performance ratio of this instance equals

$$\frac{2+d_2}{5/2} < \frac{2+\overline{d}_2}{5/2} = \frac{1}{5}(1+\sqrt{21}).$$

- Assume we have an instance d with $d_2 \geq \overline{d}_2$ and the optimal solution has no setup in period 2. Then one can show that an instance with $d_3 = 0$ will give the largest performance ratio. The performance ratio of this instance equals

$$\frac{2}{1+d_2} \leq \frac{2}{1+\overline{d}_2} = \frac{1}{5}(1+\sqrt{21}).$$

So the worst case ratio of H_3 is at most $\frac{1}{5}(1+\sqrt{21})$. □

Example 2 and Proposition 15 show that the worst case ratio of Heuristic H_3 equals $\frac{1}{5}(1+\sqrt{21})$ and this bound is tight.

In the literature there has also been some research on the worst case performance for lot-sizing heuristics with a finite time horizon. Vachani (1992) analyzed the performance bounds of several heuristics (not necessarily in the class of on-line heuristics). In Table 7 we summarize the results for the case $T = 3$. It follows from Table 7 that our simple heuristic outperforms all other heuristics. For the notations we refer to Vachani (1992). All performance bounds

Heuristic	EOQ	POQ	SM	LUC	PPB	BMY	FC	H_3
Performance bound	∞	3	$\frac{4}{3}$	∞	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{6}{5}$	$\frac{1}{5}(1+\sqrt{21})$

Table 7: Performance ratios of some heuristics for $T = 3$

can be derived from (the references to) the examples in Vachani (1992) except for SM. The performance bound for SM is derived from the following example.

Example 3 Consider an instance with $d_1 = 1$, $d_2 = 0$ and $d_3 = \frac{1}{4} + \varepsilon$ with $\varepsilon > 0$. Let $AC(t)$ be the average cost for the first t periods with only a setup in period 1. Because $AC(1) = 1$, $AC(2) = \frac{1}{2}$ and $AC(3) = \frac{3/2+2\varepsilon}{3} > \frac{1}{2} = AC(2)$, SM has a setup in periods 1 and 3 with total cost $C^H = 2$. However, it is optimal to have only a setup in period 1 with cost $C^* = \frac{3}{2} + 2\varepsilon$. Therefore, the performance ratio of this instance equals

$$\frac{C^H}{C^*} = \frac{2}{\frac{3}{2} + 2\varepsilon} \to \frac{4}{3} \text{ for } \varepsilon \to 0.$$

Finally, we note that the bound for PPB is smaller than the bound in Vachani (1992) which is $\frac{3T}{T+2} = \frac{9}{5} > \frac{3}{2}$. The claim in Vachani (1992) that the example in Bitran et al. (1984) yields a tight bound is not correct. Namely the example yields a bound of $\frac{3T}{T+3}$ when T is a multiple of 3. The example from Bitran et al. (1984) and the instance $d_1 = 1$, $d_2 = 1 - \varepsilon$, $d_3 = 2\varepsilon$ have a performance ratio of $\frac{3}{2}$ for $\varepsilon \to 0$.

5.2 An optimal heuristic for $T = 4$

In a similar way as for the case $T = 3$ we can construct a heuristic for the case $T = 4$. The heuristic is illustrated by the decision tree in Figure 5. Within each node one can find a node number and above each (relevant) node one can find the threshold value for whether or not to make a new setup. We will show that the heuristic has a worst case ratio of $\frac{1}{14}(3+\sqrt{177}) \approx 1.165$.

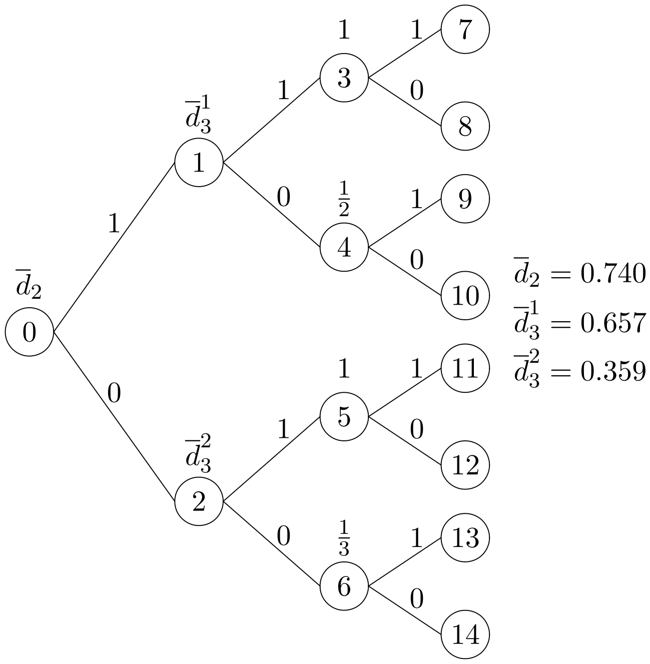

Figure 5: Heuristic H_4 represented by a decision tree

The construction of this heuristic is not as straightforward as for the case $T = 3$. The value $\overline{d}_2 \approx 0.740$[1] maximizes $W(2,4) \approx 1.150$ and the value $\overline{d}_3^2 = \frac{1}{22}(\sqrt{177} - 9) \approx 0.359$ maximizes $W(3,4) = \frac{1}{14}(3 + \sqrt{177}) \approx 1.165$. Furthermore, it can be verified that the production plans on the path through node 1 that maximize the performance ratio of the following nodes are heuristic plan $P^H = \{1,1,1\}$ (node 3) with the optimal plan $P^* = \{1,1,0\}$ and heuristic plan $P^H = \{1,1,0,1\}$ (node 9) with the optimal plan $P^* = \{1,0,1,0\}$. The value $\overline{d}_3^1 \approx 0.657$ maximizes $\min\{3/(2+d_3), (3+d_3)/(5/2+\overline{d}_2)\}$ at a value of approximately 1.129.

Proposition 16 *Heuristic H_4 has a worst case ratio of at most $\frac{1}{14}(3+\sqrt{177}) \approx 1.165$.*

Proof The proof consists of calculating the performance ratio at all nodes of the decision tree in Figure 5. That is, for each node we will consider all (relevant) optimal production plans and we will show that for each node the performance ratio will be at most $\frac{1}{14}(3+\sqrt{177}) \approx 1.165$.

- Node 1: $P^H = \{1,1\}$, $P^* = \{1,0\}$, $C^H/C^* = \frac{2}{1+d_2} \leq \frac{2}{1+\overline{d}_2} \approx 1.150$
- Node 2: $P^H = \{1,0\}$, $P^* = \{1,0\}$, $C^H/C^* = 1$
- Node 3: $P^H = \{1,1,1\}$

  - $P^* = \{1,0,1\}$, $C^H/C^* = \frac{3}{2+d_2} \leq \frac{3}{2+\overline{d}_2} \approx 1.095$
  - $P^* = \{1,1,0\}$, $C^H/C^* = \frac{3}{2+d_3} \leq \frac{3}{2+\overline{d}_3^1} \approx 1.129$

- Node 4: $P^H = \{1,1,0\}$

  - $P^* = \{1,0,0\}$, $C^H/C^* = \frac{2+d_3}{1+d_2+2d_3} \leq \frac{2}{1+\overline{d}_2} \approx 1.150$
  - $P^* = \{1,0,1\}$, $C^H/C^* = \frac{2+d_3}{2+d_2} \leq \frac{2+\overline{d}_3^1}{2+\overline{d}_2} \approx 0.970$ (So $P^* = \{1,0,1\}$ cannot be an optimal plan.)

- Node 5: $P^H = \{1,0,1\}$

  - $P^* = \{1,0,0\}$, $C^H/C^* = \frac{2+d_2}{1+d_2+2d_3} \leq \frac{2}{1+2\overline{d}_3^2} \approx 1.165$
  - $P^* = \{1,1,0\}$, $C^H/C^* = \frac{2+d_2}{2+d_3} \leq \frac{2+\overline{d}_2}{2+\overline{d}_3^2} \approx 1.162$

- Node 6: $P^H = \{1,0,0\}$

[1] Using the same approach as in Example 1, it can be shown that $\overline{d}_2$ is the positive root of the cubic equation $3x^3 + 12x^2 + 3x - 10 = 0$.

- $P^* = \{1, 0, 1\}$, $C^H/C^* = \frac{1+d_2+2d_3}{2+d_2} \leq \frac{1+2\overline{d}_3^2}{2} \approx 0.859$ (So $P^* = \{1, 0, 1\}$ cannot be an optimal plan.)

- $P^* = \{1, 1, 0\}$, $C^H/C^* = \frac{1+d_2+2d_3}{2+d_3} \leq \frac{1+\overline{d}_2+2\overline{d}_3^2}{2+\overline{d}_3^2} \approx 1.042$

For nodes 7–14 we note that the case with $P_4^* = 1$ is not of interest. Because the cost in period 4 of the optimal solution is always at least equal to the cost in period 4 of the heuristic solution, the performance ratios will be at most equal to the performance ratios of the 3-period problems (that is, the performance ratios corresponding to nodes 3–6).

- Nodes 7 and 11: Because both the heuristic and the optimal solution have a setup in period 4 (as $d_4 > 1$), the performance ratios will be smaller than the performance ratios of nodes 3 and 5, respectively.

- Node 8: $P^H = \{1, 1, 1, 0\}$, $P^* = \{1, 0, 1, 0\}$, $C^H/C^* = \frac{3+d_4}{2+d_2+d_4} \leq \frac{3}{2+\overline{d}_2} \approx 1.095$

- Node 9: $P^H = \{1, 1, 0, 1\}$, $P^* = \{1, 0, 1, 0\}$, $C^H/C^* = \frac{3+d_3}{2+d_2+d_4} \leq \frac{3+\overline{d}_3^1}{2+\overline{d}_2+\frac{1}{2}} \approx 1.129$

- Node 10: $P^H = \{1, 1, 0, 0\}$

  - $P^* = \{1, 0, 0, 0\}$, $C^H/C^* = \frac{2+d_3+2d_4}{1+d_2+2d_3+3d_4} \leq \frac{2}{1+\overline{d}_2} \approx 1.150$

  - $P^* = \{1, 0, 1, 0\}$, $C^H/C^* = \frac{2+d_3+2d_4}{2+d_2+d_4} \leq \frac{2+\overline{d}_3^1+2\cdot\frac{1}{2}}{2+\overline{d}_2+\frac{1}{2}} \approx 1.129$

- Node 12: $P^H = \{1, 0, 1, 0\}$

  - $P^* = \{1, 0, 0, 0\}$, $C^H/C^* = \frac{2+d_2+d_4}{1+d_2+2d_3+3d_4} \leq \frac{2}{1+2\overline{d}_3^2} \approx 1.165$

  - $P^* = \{1, 1, 0, 0\}$, $C^H/C^* = \frac{2+d_2+d_4}{2+d_3+2d_4} \leq \frac{2+\overline{d}_2}{2+\overline{d}_3^2} \approx 1.162$

- Node 13: $P^H = \{1, 0, 0, 1\}$

  - $P^* = \{1, 1, 0, 0\}$, $C^H/C^* = \frac{2+d_2+2d_3}{2+d_3+2d_4} \leq \frac{2+\overline{d}_2+2\overline{d}_3^2}{2+\overline{d}_3^2+2\cdot\frac{1}{3}} \approx 1.143$

  - $P^* = \{1, 0, 1, 0\}$, $C^H/C^* = \frac{2+d_2+2d_3}{2+d_2+d_4} \leq \frac{2+2\overline{d}_3^2}{2+\frac{1}{3}} \approx 1.165$

- Node 14: $P^H = \{1, 0, 0, 0\}$

  - $P^* = \{1, 1, 0, 0\}$, $C^H/C^* = \frac{1+d_2+2d_3+3d_4}{2+d_3+2d_4} \leq \frac{1+\overline{d}_2+2\overline{d}_3^2+3\cdot\frac{1}{3}}{2+\overline{d}_3^2+2\cdot\frac{1}{3}} \approx 1.143$

  - $P^* = \{1, 0, 1, 0\}$, $C^H/C^* = \frac{1+d_2+2d_3+3d_4}{2+d_2+d_4} \leq \frac{1+2\overline{d}_3^2+3\cdot\frac{1}{3}}{2+\frac{1}{3}} \approx 1.165$

$\square$

Note that if $d_2 > \overline{d}_2$, then we already know in the second period that the performance ratio will be smaller or equal than 1.150. This is due to the relatively high cost in period 2 (at least $\overline{d}_2$), which causes a relatively small performance ratio.

6 Discussion and implications of the results

In this section we will show that the main result of Section 4.3 does not only be apply to on-line heuristics, but also to a broader class of heuristics with a so-called look ahead-look back feature. This generalizes the result of Axsäter (1985) in another direction, besides the property that decisions may depend on all previous setup periods. Furthermore, we will show that the result also applies for on-line heuristics in a rolling horizon environment. Finally, we briefly discuss the worst case performance of on-line heuristics for the capacitated lot-sizing problem.

6.1 Look-ahead look-back heuristics

On-line heuristics are myopic in the sense that they do not take into account future demand. However, there is a broader class of heuristics which has a so-called look ahead-look back feature. When the decision is to make a setup in period t or not, there is an option to look back and look ahead a number of periods and to move the setup to one of those periods if an improvement can be made. Wemmerlöv (1983) proposes a variant of PPB where it is allowed to look ahead and look back one period in order to improve the current solution. Heuristics possessing the look ahead-look back feature can be considered as a compromise between the class of myopic on-line heuristics and the heuristics using the complete model horizon.

Consider an on-line heuristic with the additional property to look ahead and look back l periods. A slightly modified version of the worst case example of Section 4.3 shows that heuristics with the look ahead-look back feature also have worst case ratio at least 2. First, consider an arbitrary demand sequence $d = d_0, \ldots, d_T$. Define a demand sequence d' consisting of $(l+1)T+1$ periods with $d'_{(l+1)t} = d_t/(l+1)$ for $t = 0, \ldots, T$ and the remaining demands equal to zero. So we add l zero-demand periods between every two demand periods of the original sequence. Consider a solution with n setups in periods t_i $(i = 1, \ldots, n)$ for instance d. Then the solution for instance d' with setups in periods $(l+1)t_i$ $(i = 1, \ldots, n)$ has the same cost. Furthermore, it will never be optimal to have a setup in a zero demand period. Therefore, the optimal cost for sequence d and d' is equal.

Consider some on-line heuristic with the look ahead-look back feature that generates a solution for d'. Clearly, any heuristic of interest will only generate setups in positive demand periods. Assume that the heuristic generates the first setup in some period $t = p(l+1)$ with $p \in \mathbb{N}$. When looking back or looking ahead l periods, there are only zero-demand periods and hence cost will not decrease when moving the setup to one of these periods. Therefore, we can still apply the proof of Theorem 13 by adding l zero demand periods between any two demands of the original instance and scaling the demands with a factor $\frac{1}{l+1}$. Thus we have the following proposition.

Proposition 17 *Let H be an on-line heuristic with the additional property to look ahead and look back l periods for some fixed $l > 0$. Then H has worst case ratio at least 2.*

6.2 Rolling horizon environment

Often the demand for the complete horizon T is not known, but the demand for the first n periods is known (with $n \ll T$). In this case the lot-sizing problem for n periods is solved, the first lot-size decision is implemented and the horizon is rolled forward to the period where the next lot-size starts. Again it is assumed that the next n periods are known and the procedure is repeated. This is known as lot-sizing in a rolling horizon environment, where n is called the planning horizon. As the on-line heuristics use no future demand information, they are particularly suitable to be applied in a rolling horizon environment.

Consider a rolling horizon environment with a planning horizon of n periods. We can easily construct a problem instance with worst case performance arbitrarily large. Take the instance with $d_{tn} = \varepsilon$ for $t = 0, 1, 2, \ldots, N$ ($N \in \mathbb{N}$) and zero demands elsewhere. In period 0 any algorithm faces zero demands in all periods except for period 0 and hence a lot-size of ε is made in period 0. Now the horizon is rolled forward to period n and we are in the same situation as in period 0. So any heuristic will generate a solution with setups in periods tn for $t = 0, 1, 2, \ldots, N$. As it is optimal to have only a setup in period 0 for ε sufficiently small, the performance ratio for this instance becomes arbitrarily large for N sufficiently large.

Due to the rolling horizon environment no algorithm is able to construct a solution with a lot-size covering more than n periods, whereas it may be optimal to have lot-sizes that cover more than n periods. This means that the optimal solution can never be constructed by any algorithm. This is in contrast with the situation where the planning horizon is not bounded. In this case it is the fact that the heuristics make the setups in the wrong periods that cause the non-optimal behavior, while it is possible to construct a solution with the same setups as in the optimal solution.

In a rolling horizon environment it seems not fair to measure worst case performance by comparing the rolling horizon solution with the optimal solution over T periods. Therefore, Simpson (2001) proposes to measure the heuristic performance by comparing the heuristic solution with the optimal solution for which no lot-size covers more than n periods. Call this the n-optimal solution. Clearly, the worst case performance now depends on the length of the planning horizon n.

Consider the extreme case that $n = 1$. In this case both any heuristic and the n-optimal solution have a setup in each period. Using the alternative performance measure, each heuristic has worst case performance 1. Furthermore, consider the case $n = 2$ and the simple heuristic that makes a setup in each period t with $d_t > \frac{1}{2}$. It is not difficult to verify that the ratio of the cost of any 2-period lot-size in the 2-optimal solution is at most $\frac{3}{2}$ smaller than the cost of the same two periods in the heuristic solution. Therefore, the worst case performance of this simple heuristic is at most $\frac{3}{2}$. So for planning horizons $n = 1$ and $n = 2$ there are heuristics with worst case performance smaller than two when using the alternative performance measure.

On the other hand, consider a rolling horizon environment where n is relatively large. It will be clear that for n sufficiently large and using the instance of the proof of Theorem 13, the worst case performance of any on-line heuristic will be arbitrarily close to two (or larger). In fact, for n sufficiently large the rolling horizon environment changes to the 'on-line setting' as in the previous sections.

At first sight it seems counterintuitive that the larger the planning horizon (i.e., the more information available), the larger the worst case ratio. However, when using the alternative performance measure for small planning horizons, it is rather that the cost of the n-optimal solution is relatively high (compared to the 'real' optimal cost) than that the heuristics generate good solutions.

6.3 On-line capacitated lot-sizing heuristics

A natural question is whether the results in this paper can also be applied to on-line heuristics for the capacitated lot-sizing problem. In the capacitated lot-sizing problem there is only a limited amount of production capacity available in each period (Florian et al., 1980). It turns out that our results cannot be applied to this problem because of feasibility issues, and that the worst case performance can be arbitrarily large.

Assume that we have an on-line heuristic for the capacitated lot-sizing problem. As on-line heuristics do not use future demand information and future demand may be higher than future capacity, there is a feasibility issue. The only way to make sure that an on-line heuristic generates a feasible solution (assuming there exists a feasible solution) is to produce at capacity in each period, because cumulative demand may be equal to cumulative capacity. However, this leads to problem instances with arbitrarily high performance ratios, as producing at capacity is undesirable for instances where demand is relatively small compared to capacity.

7 Conclusion

In this paper we studied the worst case performance for a general class of on-line lot-sizing heuristics. On-line heuristics have the property that setup decisions are made on a period-by-period basis without taking into account future demand information. We developed a procedure to construct problem instances with a high performance ratio for a fixed horizon. The insights obtained from the analysis resulted in the construction of a problem instance with performance ratio 2. This means that any on-line heuristic has at least worst case performance 2, which generalizes a result from Axsäter (1985), who proves this result for a more restrictive class of heuristics. Furthermore, the analysis led to the construction of optimal on-line heuristics for 3- and 4-period horizons. A direction for future research is to find out whether we can construct optimal on-line heuristics for general model horizons.

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