Prospect Theory with Reference Points in the Opportunity Set

Han Bleichrodt

Department of Economics, Erasmus University, H13-27, P.O. Box 1738, 3000 DR Rotterdam, the Netherlands, bleichrodt@few.eur.nl.

November 2006

Abstract

Many empirical studies have shown that preferences are reference-dependent, implying loss aversion. Because reference-dependence is a relatively new concept, there exists little theory on it. The theories that are available generally take the reference point as given and then impose the traditional assumptions such as completeness of preferences. Reference-dependence raises, however, new problems that do not occur in reference-independent theories. This paper argues that in the empirically realistic case where the reference point is always an element of the decision maker's opportunity set, reference-dependent preferences are, according to the principles of revealed preference, necessarily incomplete. A new model of decision under uncertainty is presented that extends prospect theory to cover this case. The paper also gives preference foundations for two special cases: one in which utility is decomposed into a normative and a psychological component and one in which loss aversion is constant. The latter case has frequently been used in empirical research and in applied decision analysis.

Subject classifications: Decision analysis: theory. Utility/preference: theory.
Area of review: Decision analysis.

1. Introduction

Traditional economic and operational models assume that agents behave according to expected utility. Evidence abounds, however, that people systematically violate expected utility. An important cause of deviations from expected utility is that preferences are reference-dependent. People do not evaluate outcomes as absolute amounts, as expected utility assumes, but as gains and losses relative to a reference point and are more sensitive to losses than to gains. Reference-dependence is empirically well-established (e.g. Kahneman et al. 1990, Tversky and Kahneman 1991, Barberis et al. 2001) and it is an important factor in explaining people's attitudes towards risk (Rabin 2000). There is also growing evidence that reference-dependence can explain a variety of field data (Hardie et al. 1993, Benartzi and Thaler 1995, Camerer et al. 1997, Camerer 2000, Pennings and Smidts 2003). A formal theory of reference-dependence is prospect theory (Kahneman and Tversky 1979, Tversky and Kahneman 1992), currently the main descriptive theory of decision under uncertainty.

Because reference-dependence is a relatively new concept, there exists little theory on it. The theories that exist generally take the reference point as fixed and certain and then apply the traditional axioms of decision theory, such as completeness of preferences (Luce and Fishburn 1991, Tversky and Kahneman 1992, Wakker and Tversky 1993, Zank 2001, Köbberling and Wakker 2003). Important advancements were made by Sugden (2003), who allowed that the reference point could be uncertain, and by Schmidt (2003), who analyzed the impact of shifts in the reference point. As was formally shown by Wakker (2005, Observation 4.4 and Theorem 4.5), reference-dependence can only explain the commonly observed deviations from expected utility when the reference point shifts across decisions. See, for example, Popescu and Wu (forthcoming) for an application of shifting reference points in dynamic pricing. Schmidt (2003) and Sugden (2003) also imposed all the traditional assumptions of decision theory, in particular completeness of preferences.

Reference-dependence raises new problems that are not present in reference-independent theories. This paper focuses on one of these problems, that reference-dependence makes the assumption of complete preferences less plausible and often requires incompleteness of preferences. As will be explained below, this is a consequence of the fact that in many decision situations, the reference point is always one of the options that are available to the decision maker. This often happens because the status quo is taken as the reference point and retention of the status quo is always possible. Consider the example of an investor who contemplates buying stocks. Then the status quo, i.e., buying no stocks, is always one of the available options. If the status quo is not readily available, which happens for instance in decisions between uncertain medical treatments involving unknown health states, people often take one of the alternatives as their reference point.

For example, Robinson, Loomes, and Jones-Lee (2001) asked people to compare the certainty of impaired health with a risky treatment offering a chance p of an improvement in health and a chance 1−p of a deterioration in health. They observed that people took the certain health state as their reference point and evaluated the outcomes of the treatment as gains and losses relative to this reference point. A similar finding for monetary outcomes was reported by Hershey and Schoemaker (1985) who observed that in a choice between a sure monetary amount and a risky prospect people took the sure amount as their reference point and evaluated the outcomes of the prospect as deviations from this reference point. Further empirical evidence of people taking one of the available options as their reference point is in Johnson and Schkade (1989), Stalmeier and Bezembinder (1999), Morrison (2000), Robinson et al. (2001), Bleichrodt et al. (2001), and van Osch et al. (2004). Let me emphasize here that my point is not that in every decision situation the reference point will always be in the decision maker's opportunity set. I only claim that there are decision situations of considerable interest where this is the case and, hence, this issue is worth exploring.

If the reference point always belongs to the decision maker's opportunity set, then reference-dependent preferences must be incomplete. This follows from the principles of revealed preference, the basic primitive of economics and decision theory, which require that preferences are derived from observable choices. To see this point assume that three acts f, g, and r all belong to the decision maker's opportunity set and that the reference point r is better than both f and g. For instance, consider a patient who has to make a choice between his current treatment, his reference point, or adopting one of two new treatments f and g. Suppose that there are two possible states of nature the probabilities of which are unknown, either the patient has disease A or disease B. Suppose further that the reference treatment r gives 10 additional years of life if the patient turns out to have disease A and 15 additinal years of life if he turns out to have disease B. For treatment f these numbers are 8 if A and 12 if B and for treatment g they are 10 if A and 9 if B. Then the decision maker will always choose r over f and g, so that we can never observe a choice between f and g from reference point r, and revealed preference is incomplete regarding f,g *given reference point r*. Assuming completeness of reference-dependent preferences in such a case would require a resort to introspective preferences not based on observable choice and would deviate from the revealed preference paradigm. Some authors distinguish between revealed and psychological preferences (e.g. Danan 2003, Mandler 2005). I do not follow this approach. This paper takes choice as the basic primitive and derives preferences directly from binary choice, as is common in economics and decision theory.

Because all existing characterizations of prospect theory assume complete preferences, they cannot handle this empirically important situation and extensions of prospect theory that allow for incomplete preferences are required. The purpose of this paper is to present such extensions. The paper makes use of some representation results for incomplete reference-dependent preferences that were derived in Bleichrodt (2005). It starts by giving a preference foundation for a general model of incomplete reference-dependent preferences. As in Sugden (2003), in this model the reference point can be any act, certain or uncertain. As in Schmidt (2003), the reference point can shift across decision contexts.

Then two special cases will be considered. In the first model, utility is decomposed into two terms: a basic utility, which can be interpreted as the normative component of utility or the standard economic concept of utility and a function reflecting the impact of additional psychological factors on utility, in particular loss aversion. While loss aversion is general in the first special case, in the second model it is captured by a single parameter. Tversky and Kahneman (1991) referred to this case as *constant loss aversion*. Constant loss aversion is commonly assumed in empirical research on reference-dependence and in applications of prospect theory.

There are two ways in which the results of this paper can be interpreted. One interpretation is that the paper derives representations for incomplete preferences on a full Cartesian product. An alternative interpretation is that the paper derives representations of binary preferences on a subset of a Cartesian product, being a subset where the preference relation is complete. A common misunderstanding in the literature is that representation theory on subsets of Cartesian products does not differ from representation theory on full Cartesian products. This is not so, and several authors have pointed out that the restriction of representation theorems to subsets of Cartesian products poses many extra complications (Falmagne 1981, Fishburn 1970, p.74, Krantz et al. 1971, p.276, Shapiro 1979, and Wakker 1989, Remark III.7.8). Wakker (1993a) cited common misunderstandings about the nontrivial nature of extensions to subsets from economics, psychology, operations research, and functional equations and, in a complex analysis, demonstrated that even for well-behaved subsets, connected with full-dimensional interior, the extra complexities are already large.

The focus of this paper is on reference-dependence and it does not consider nonadditive decision weighting, the other deviation from expected utility modeled by prospect theory. Consequently, the paper should not be interpreted as an attempt to explain all observed regularities in decision under uncertainty. Incorporating nonadditive decision weighting into the models studied in this paper would substantially complicate the already long and complex proofs. Schmidt et al. (2005) proposed an extension of Sugden's (2003) model to incorporate nonadditive decision weighting, but gave no preference foundations.

Several studies besides Schmidt (2003), Sugden (2003), and Schmidt et al. (2005) have recently proposed new theories of reference-dependence in decision under uncertainty. Masatlioglu and Ok (2005) studied cases where status quo bias is rational and were mainly concerned with riskless choice. They also considered incompleteness of preferences, but in their paper status quo bias follows from indecisiveness of the decision maker. The present paper gives another reason why studying incompleteness of preference relations is important: in many decision contexts incompleteness follows from reference-dependence. Reference-dependence requires a different form of incompleteness than studied by Masatlioglu and Ok and, consequently, their paper is rather different from mine. Sagi (2006) explored the impact of imposing a "no-regret condition" on reference-dependent preferences and suggested a model that satisfies this condition. Apesteguia and Ballester (2006) studied decisions where the reference point, if it exists, is an element of the decision maker's choice set, as opposed to an element of the opportunity set which is considered here, and also allowed for specific shifts in the reference point. They did not consider incompleteness of preferences though and their analysis is restricted to riskless choice. Closer to the present paper is Köszegi and Rabin (forthcoming), who proposed a model that bears some resemblance to the model characterized in Section 4, but did not give preference foundations.

Köszegi and Rabin (forthcoming) also presented a theory of how reference points are formed. Like previous studies that gave preference foundations for prospect theory, I will assume that the reference point is exogenously given. That said, the tools of this paper may also prove useful in giving preference foundations for models with endogenous reference points because such models will most likely entail assuming a type of incompleteness of preferences that resembles the type of incompleteness modeled in this paper. To illustrate, if people behave as suggested by Hershey and Schoemaker (1985), they will take €100 as their reference point in a comparison between €100 for sure and a prospect giving €200 with probability p and nothing otherwise. Consequently, a preference between €100 and €200 with probability p judged from any other reference point than €100 cannot be observed.

The paper proceeds as follows. Section 2 gives notation and states the main assumptions. Section 3 gives a preference foundation for the general reference-dependent utility model with incomplete preferences. Section 4 characterizes the special case in which utility is decomposed into a normative and a psychological component. Section 5 characterizes constant loss aversion. Section 6 concludes. All proofs are in the appendix.

2. Notation and Assumptions

2.1. Notation

Consider an individual who has to make a decision under uncertainty: there is a finite number, n, of *states of the nature*, exactly one of which will occur. Probabilities for the states of nature may but need not be known.The restriction to finite states is made for simplicity; the extension to infinite states is discussed in Appendix B. There the case of decision under risk is also considered, i.e. the case where probabilities are objectively given. The set of states is denoted by $\mathcal{S}$, elements of which are denoted by i,j,k. $\mathcal{C}$ denotes a set of consequences or *outcomes*, elements of which are denoted by α,β,γ. The individual's decision problem is to choose between acts. Each act is an n-tuple of outcomes, one for each state of the world. Formally, an *act* is a function from $\mathcal{S}$ to $\mathcal{C}$. The set of acts is denoted as $\mathcal{F} = \mathcal{C}^n$. $(f_1,\ldots,f_n)$ denotes the act which results in outcome f_j if state of nature j obtains. By $\alpha_j f$ denote the act f with f_j replaced by $\alpha \in \mathcal{C}$, i.e., $\alpha_j f = (f_1,\ldots,f_{j-1},\alpha,f_{j+1},\ldots,f_n)$.

Let $r \in \mathcal{F}$ denote a *reference act*. Each act can serve as a reference act. Hence, the reference act can yield different reference outcomes for different states of nature, which is contrary to most previous characterizations of prospect theory in which the reference act yields the same outcome for each state of nature. A preference relation $\succcurlyeq_r$ is defined over $\mathcal{F}$, where f $\succcurlyeq_r$ g means "f is preferred to g judged from reference act r." The common notation $\succ_r$ is used for strict preference and $\sim_r$ for indifference; $\prec_r$ and $\preccurlyeq_r$ denote reversed binary relations.

A *constant act* yields the same outcome for each state of nature: act f is constant if there is some $\alpha \in \mathcal{C}$ such that $f(s) = \alpha$ for all $s \in \mathcal{S}$. This will be written $f = \alpha$. Given that outcomes are identified with constant acts, the preference relation also applies to outcomes and we can write $\alpha \succcurlyeq_r \beta$ in case $f \succcurlyeq_r g$ with $f = \alpha$ and $g = \beta$. I will assume that the preference relation over constant acts is independent of the reference act and, hence, I will simply write $\alpha \succcurlyeq \beta$ to denote the preference relation over outcomes. The assumption that the preference relation over outcomes is reference-independent seems plausible for outcomes that are one-dimensional such as money amounts and life duration. For multi-dimensional outcomes the assumption is less obvious.

2.2. Preference conditions

As has been mentioned in the introduction, if the reference act always belongs to the decision maker's opportunity set, then it follows from the revealed preference paradigm that the preference relation $\succcurlyeq_r$ cannot be complete. If r is the reference act and r is preferred to both f and g then the preference between f and g judged from reference act r cannot be observed. On the other hand, if at least one of the acts f and g is at least as good as the reference act r then a preference between f and g judged from r can be observed. Hence, this paper considers a special type of incompleteness: loosely speaking, "above" the reference act preferences are complete, but "below" the reference act they do not exist. The next definition formalizes the above argument.

Definition 1. For a given reference act r, *r-upper completeness* holds if (i) $r \sim_r r$ and (ii) for all $f,g \in \mathcal{F}$, if $f \succcurlyeq_r r$ or $g \succcurlyeq_r r$, then either $f \succcurlyeq_r g$ or $g \succcurlyeq_r f$; if $r \succ_r f$ and $r \succ_r g$ then neither $f \succcurlyeq_r g$ nor $g \succcurlyeq_r f$.

Completeness of preferences above the reference act may be too strong. Indecisiveness of the decision maker may, for example, lead to some incompleteness of preferences above the reference act. This paper does not consider such incompleteness. To handle it, the tools of this paper have to be combined with those in Maccheroni (2004), for example, who provided an axiomatic analysis of incompleteness due to indecisiveness for non-expected utility.

For $r \in \mathcal{F}$, let B_r be the set of pairs of acts for which, judged from r, a preference can be observed. That is, $B_r = \{(f,g) \in \mathcal{F} \times \mathcal{F}: f \succcurlyeq_r r \text{ or } g \succcurlyeq_r r\}$. State j is *nonnull with respect to r* if there exist $(\alpha_j f, f) \in B_r$ such that $\alpha_j f \succ_r f$. Intuitively, nonnullness means that a state matters to the decision maker. If the decision maker would always be indifferent between $\alpha_j f$ and f, judged from r, regardless of α then apparently he does not care what happens under state j. The assumption that there exist at least three states that are nonnull with respect to r is crucial for what follows. For convenience, assume throughout that for all r, all states are nonnull with respect to r.

Definition 2. *Weak monotonicity* holds if for all $(f,g) \in B_r$, $f_j \succcurlyeq g_j$ for all j implies that $f \succcurlyeq_r g$.

Weak monotonicity says that for acts f and g for which a preference can be observed from reference point r, if for each state the outcome under f is at least as attractive as the outcome under g then f is at least as attractive as g judged from r. This seems intuitively plausible.

Definition 3. *Solvability* holds if for all acts f,g,r, with $g \succcurlyeq_r r$ and for all states j there exists an outcome α such that $\alpha_j f \sim_r g$.

Solvability is a strong condition that will imply unbounded utility. It may be possible to weaken solvability to a restricted form as in Definition 6.12 in Krantz et al. (1971). Most likely this weakening would require adding additional assumptions; it is not pursued further in this paper.

2.3. Tradeoff consistency

Consider the following two indifferences $\alpha_j f \sim_r \beta_j g$ and $\gamma_j f \sim_r \delta_j g$ and let $r_j = \mu$. The interpretation of these two indifferences is that judged from μ receiving α instead of β is an equally good tradeoff as receiving γ instead of δ, because both exactly offset the receipt of g_i instead of f_i for all states i distinct from j. We may say that these indifferences imply that the strength of preference of α over β is equal to the strength of preference of γ over δ. Let us now formalize the above interpretation. For outcomes $\alpha,\beta,\gamma,\delta$ write

$$\alpha\beta \sim^{*}_{\mu} \gamma\delta$$

if there exist acts f and g and a reference act r with $r_j = \mu$ such that

$$\alpha_j f \sim_r \beta_j g \text{ and } \gamma_j f \sim_r \delta_j g.$$

Note that even though the $\sim^{*}_{\mu}$ relations may be interpreted as measuring strength of preferences, they are entirely defined in terms of observable choices and do not rely on unobservable subjective judgments. More detailed discussions of relations that are similar to the $\sim^{*}_{\mu}$ relations can be found in Köbberling and Wakker 2003; 2004).

To be interpretable as strength of preference relations, the $\sim^{*}_{\mu}$ relations must be "well-behaved": when the strength of preference of α over β is equal to the strength of preference of γ over δ and also to the strength of preference of γ over δ' then δ and δ' should be equally attractive. The next condition formalizes this intuition.

Definition 4. *Tradeoff consistency* holds if improving an outcome in any $\sim^{*}_{\mu}$ relationship breaks that relationship. That is, if $\alpha\beta \sim^{*}_{\mu} \gamma\delta$ and $\alpha\beta \sim^{*}_{\mu} \gamma\delta'$ both hold then we must have $\delta \sim \delta'$.

Tradeoff consistency is a central condition in the derivation of the results of this paper. It ensures two things, first that the preference relations $\succcurlyeq_r$ have additive representations and, second, that the utility functions for different states can be taken proportional.

An important advantage of using tradeoff consistency as a condition in axiomatizations, besides its intuitive appeal, is that it is easily tested empirically. Measurements of utility by the tradeoff method (Wakker and Deneffe 1996) provide direct tests of tradeoff consistency. Many empirical studies have used the tradeoff method to measure utility and to test the validity of decision models in decision under uncertainty, showing that such measurements are feasible and easily performed (e.g. Abdellaoui 2000, Abdellaoui et al. 2005, Bleichrodt and Pinto 2000, Bleichrodt and Pinto 2005, Etchart-Vincent 2004, Fennema and van Assen 1999).

I finally introduce some conditions of a technical nature. Consider the order topology on $\mathcal{C}$, which is generated by the sets $\{\alpha\in\mathcal{C}: \alpha \succ \beta\}$ and $\{\alpha\in\mathcal{C}: \alpha \prec \beta\}$, where $\beta\in\mathcal{C}$. Let $\mathcal{F}$ be endowed with the product topology.

Definition 5. *Preference continuity* holds if for all acts f and r, the sets $\{g\in\mathcal{F}: g \succcurlyeq_r f\}$ and $\{g\in\mathcal{F}: g \preccurlyeq_r f\}$ are closed in $\mathcal{F}$.

A function U is a *ratio scale* if it can be replaced by a function V if and only if there exists a positive σ such that $V = \sigma U$. A function U is an *interval scale* if it can be replaced by a function V if and only if there exists a positive σ and a real τ such that $V = \sigma U + \tau$.

3. Reference-Dependent Utility

The section will give preference foundations for a general model of reference-dependent utility with incomplete preferences. First, consider the case where the reference act is given, i.e., the case where the same reference act is used in different decision contexts. This case corresponds to decision situations in which the status quo is taken as the reference act. For a given reference act r, *reference-dependent utility (RU) holds for $\succcurlyeq_r$* if there exist positive subjective probabilities p_j that sum to one and a *utility function* U: $\mathcal{C}\times\mathcal{C} \to \mathbb{R}$, such that

a. $(f,g)\in B_r$ iff $\sum_{j=1}^{n} p_jU(f_j,r_j) \geq 0$ or $\sum_{j=1}^{n} p_jU(g_j,r_j) \geq 0$

b. for all $(f,g)\in B_r$, $f \succcurlyeq_r g$ iff $\sum_{j=1}^{n} p_jU(f_j,r_j) \geq \sum_{j=1}^{n} p_jU(g_j,r_j)$

c. $U(r_j,r_j) = 0$ for all j.

d. for all $j \in \{1,\ldots,n\}$, $U(\cdot, r_j)$ represents $\succcurlyeq$ for every r_j: for all $\alpha,\beta \in C$, $U(\alpha,r_j) \geq U(\beta,r_j)$ iff $\alpha \succcurlyeq \beta$.

e. U is continuous in its first argument and the range of U is $\mathbb{R}$

Theorem 1: Consider a given reference alternative $r \in \mathcal{F}$. Let there be at least three states of nature, which are all nonnull with respect to r. The following two statements are equivalent for $\succcurlyeq_r$:

1. The order topology on C is connected, $\succcurlyeq_r$ is transitive and satisfies r-upper completeness, weak monotonicity, solvability, preference continuity, and tradeoff consistency.
2. reference-dependent utitlity holds for $\succcurlyeq_r$. Furthermore, the p_j are uniquely determined and U is a ratio scale. □

The next step is to extend Theorem 1 to $\{\succcurlyeq_r: r \in \mathcal{F}\}$, i.e. the case where the reference act can vary across decision contexts. As mentioned in the introduction, shifts in the reference point are necessary to explain the common deviations from expected utility. The subjective probabilities p_j and the utility function $U(\cdot,r_j)$ in Theorem 1 depend on the given reference act r. Without further restrictions the model is too general to yield predictions. Hence, new conditions will be imposed that ensure that $U(\cdot,r_j)$ and the p_j's are independent of r. First, r-upper completeness is extended to all r.

Definition 6. *Upper completeness* holds if r-upper completeness holds for all r.

It is plausible to assume that an outcome is viewed as less attractive the better is the reference point from which it is judged. This assumption corresponds to the utility function U being *decreasing in* *its second argument*, i.e., for all $\alpha,\beta,\delta \in \mathcal{C}$, $U(\delta,\alpha) \le U(\delta,\beta)$ iff $\beta \preccurlyeq \alpha$. The condition that is introduced next implies this.

Definition 7. *Reference monotonicity* holds when for all acts f and r and for all outcomes α,β,δ, $\beta \preccurlyeq \alpha$ iff $\delta_j f \sim_{\alpha_j r} \alpha_j r$ implies $\delta_j f \succcurlyeq_{\beta_j r} \beta_j r$.

The next condition is central in the derivation of Theorem 2 and implies that the subjective probabilities p_j are independent of the reference act and that we can choose the utility functions $U(\cdot,r_j)$ such that $U(\cdot,r_j) = U(\cdot,r'_j)$ for all reference alternatives r and r′ for which $r_j = r'_j$.

Definition 8. *Neutral independence* holds if for all $f,g \in \mathcal{F}$ and for all $\alpha,\beta \in \mathcal{C}$, $\alpha_j f \sim_{\alpha_j r} \alpha_j g$ implies $\beta_j f \sim_{\beta_j r} \beta_j g$.

Neutral independence says that indifferences are not affected when an outcome under a given state that is the same both for the acts under comparison and for the reference act is changed into another common outcome. Alternatively stated, if for a given state there are no deviations from the reference point then these “no-deviations” are ignored in indifference judgments, regardless of what the common outcomes are.

We are now in a position to state the extension of Theorem 1 to $\{\succcurlyeq_r : r \in \mathcal{F}\}$. Reference-dependent utility (RU) holds if RU holds for all $\succcurlyeq_r$, if U is decreasing in its second argument, and if the subjective probabilities p_j and the utility function U are independent of the reference act r.

Theorem 2: Let there be at least three states of nature, which are all nonnull with respect to every $r \in \mathcal{F}$. The following two statements are equivalent for $\{\succcurlyeq_r : r \in \mathcal{F}\}$:

1. The order topology on $\mathcal{C}$ is connected, $\succcurlyeq_r$ is transitive and satisfies upper completeness, weak monotonicity, reference monotonicity, solvability, preference continuity, tradeoff consistency, and neutral independence.
2. Reference-dependent utility holds. Furthermore, the p_j are uniquely determined and U is a ratio scale.

□

4. Separating "Normative Utility" and "Psychological Utility"

Next a special case of RU will be derived in which the utility function U can be decomposed as

$$U(f_j,r_j) = F(u(f_j) - u(r_j)), \tag{1}$$

with $u:\mathcal{C}\to\mathbb{R}$ continuous and F continuous and strictly increasing and $F(0) = 0$. Moreover, u is an interval scale and F is a ratio scale.

In this decomposition, u is a *basic utility function* (Köbberling and Wakker 2005), which expresses the individual's attitude towards outcomes. The basic utility function is reference-independent and may be interpreted as the normative component of utility. The function F models the impact of additional psychological factors, such as numerical sensitivity and loss aversion. Hence, (1) separates normative and psychological utility. Köszegi and Rabin (forthcoming) called F a "universal gain-loss function." Sugden (2003) referred to (1) as a satisfaction-change decomposition and showed that it can account for several important empirical deviations from expected utility. Fishburn (1992, Theorem 2) proposed a slightly more general model for additive skew-symmetric nontransitive preferences, in which the functions F and u are state-dependent.

To derive (1) two new conditions must be imposed. Suppose that $\alpha \succ \beta \succ \gamma$ and that $U(\alpha,\beta) = U(\kappa,\lambda)$ and $U(\beta,\gamma) = U(\lambda,\mu)$. That is, the gain of getting α instead of β is as attractive as the gain of getting

κ instead of λ and the gain of getting β instead of γ is as attractive as the gain of getting λ instead of μ. Then it seems plausible that the gain of getting α instead of γ should also be equally attractive as the gain of getting κ instead of μ. That is, $\alpha\gamma$, the "concatenation" of the preference intervals $\alpha\beta$ and $\beta\gamma$, should match $\kappa\mu$, the concatenation of the preference intervals $\kappa\lambda$ and $\lambda\mu$, in the sense that the gain of getting α instead of γ is as attractive as the gain of getting κ instead of μ. The next condition ensures this.

Definition 9. The *concatenation condition* holds if for all j, if $\alpha_j f \sim_{\beta_j r} \beta_j r$, $\kappa_j f \sim_{\lambda_j r} \lambda_j r$, $\beta_j g \sim_{\gamma_j r} \gamma_j r$, $\kappa_j g \sim_{\mu_j r} \mu_j r$, and $\alpha_j h \sim_{\gamma_j r} \gamma_j r$ then also $\kappa_j h \sim_{\mu_j r} \mu_j r$.

The second new condition is of a technical nature and extends preference continuity.

Definition 10. *Reference act continuity* holds if for all acts $f \in \mathcal{F}$ the sets $\{r \in \mathcal{F}: f \succcurlyeq_r r\}$ and $\{r \in \mathcal{F}: f \preccurlyeq_r r\}$ are both closed in $\mathcal{F}$.

Theorem 3: Let the conditions of Theorem 2 hold. Then the following two statements are equivalent.

1. the concatenation condition, and reference act continuity hold.
2. Reference-dependent utility holds with $U(f_j,r_j) = F(u(f_j) - u(r_j))$. □

Sugden (2003, Theorem 2) also gave a preference foundation for the decomposition (1). His result is less general than Theorem 3. First, Sugden (2003) assumed richness of the state space, which is not assumed here. Second, the outcome set is $\mathbb{R}^+$ in Sugden (2003), whereas the set of outcomes in this paper is more general, being any set whose order topology is connected. This greater generality allows for the application of Theorem 3 to qualitative outcomes such as health states, for example. Finally, also for $\mathcal{C} = \mathbb{R}^+$, Sugden's conditions imply our conditions, but the reverse is not true, as we prove in the appendix. The concatenation condition is also easier to test empirically than Sugden's conditions S3 (gain/loss symmetry) and S4 (gain/loss additivity) because it is entirely defined in terms of indifferences. As mentioned before, the main difference with Sugden (2003) is, however, conceptual: if the reference act always belongs to the individual's opportunity set then Sugden's result cannot be applied because it then uses unobservable inputs.

5. Constant Loss Aversion

Loss aversion is general in (1). For empirical purposes and for applications, it is, however, often desirable to restrict loss aversion. Indeed, in most empirical analyses loss aversion is characterized by a single parameter (e.g. Bleichrodt et al. 2001, Schmidt and Traub 2002, Pennings and Smidts 2003). Preference foundations for this case will be given in this Section.

Constant loss aversion (Tversky and Kahneman 1991) holds if in Theorem 2 $U(f_j,r_j) = u(f_j) - u(r_j)$ when $f_j \succcurlyeq r_j$ and $U(f_j,r_j) = \lambda(u(f_j) - u(r_j))$ when $f_j \preccurlyeq r_j$. The parameter λ captures the attitude towards gains and losses and can be interpreted as a loss aversion coefficient. Constant loss aversion is the special case of Theorem 3 where F is both linear for gains and linear for losses, but may be different for gains than for losses. Constant loss aversion models have been proposed by Tversky and Kahneman (1991), Tversky and Kahneman (1992), Shalev (2000), and Köbberling and Wakker (2005), but none of these papers provided preference foundations for constant loss aversion.

Consider two acts f and g and suppose that for some state of nature j the outcomes of both acts are gains when judged from reference act r. An implication of constant loss aversion is that if r_j is changed into ρ_j but the change is such that f_j and g_j remain gains while for all other states of nature the reference act does not change, i.e. $r_i = \rho_i$ for all $i \neq j$, then the preference between f and g is not affected by the change in the reference point. Hence, we may say that under constant loss aversion, reference-independence holds for gains. Similarly, reference-independence then holds for losses.

Let us now formalize the above implication.

Definition 11. *Reference-independence for gains* holds if for all r, for all ρ_j, for all f,g $\in B_r \cap B_{\rho_j r}$, and for all j, if $f_j \succcurlyeq r_j$, $f_j \succcurlyeq \rho_j$, $g_j \succcurlyeq r_j$, and $g_j \succcurlyeq \rho_j$, then $f \succcurlyeq_r g$ iff $f \succcurlyeq_{\rho_j r} g$. *Reference-independence for losses* holds if for all r, for all ρ_j, for all f,g $\in B_r \cap B_{\rho_j r}$ and for all j, if, $r_j \succcurlyeq f_j$, $\rho_j \succcurlyeq f_j$, $r_j \succcurlyeq g_j$, and $\rho_j \succcurlyeq g_j$, then $f \succcurlyeq_r g$ iff $f \succcurlyeq_{\rho_j r}$ g.

The next theorem shows that reference-independence for gains and reference-independence for losses are not only necessary for constant loss aversion but also sufficient if we impose these two conditions on top of the assumptions of Theorem 3.

Theorem 4. Let the conditions of Theorem 3 hold. Then the following two statements are equivalent.

1. reference-independence for gains and reference-independence for losses both hold.
2. constant loss aversion holds. □

6. Conclusion.

Prospect theory has emerged as the dominant descriptive theory of decision under uncertainty. A crucial difference between prospect theory and expected utility is the existence of a reference point. In many decision situations the reference point is one of the acts that are available to the decision maker. This paper has argued that, according to the revealed preference paradigm, this requires that reference-dependent preferences must be incomplete. Existing axiomatic models of prospect theory assume complete preferences and, hence, they cannot handle this case. I have proposed a generalization of prospect theory in which the reference point is one of the available acts. Hence, the paper extends prospect theory to new and empirically realistic decision situations. Two special cases were considered: one model that separates the normative and psychological components of utility. This separation seems particularly useful for prescriptive applications. The other model is more useful for empirical research and practical applications of prospect theory and takes loss aversion as constant and characterized by a single parameter. These two models are able to explain important behavioral deviations from expected utility and I hope that the characterizations given in this paper will contribute to the study and applications of reference-dependence in economics, operations research, and decision theory.

Appendix: Proofs.

Proof of Theorem 1.

The proof that statement 2 implies statement 1 is similar to the proof of Theorem 3.1. in Bleichrodt (2005). The set of ~ equivalence classes of $\mathcal{C}$ is homeomorphic to the range of U and, hence, it is connected. Suppose that $f \succcurlyeq_r g$ and $g \succcurlyeq_r h$. By parts (a) and (b) of the definition of RU for $\succcurlyeq_r$, $(f,g) \in B_r$ and $(g,h) \in B_r$ and, hence, $(f,h) \in B_r$. Transitivity follows from part (b). r-upper completeness follows from the definition of B_r p and from parts (a) and (c). For $(f,g) \in B_r$, if $f_j \succcurlyeq g_j$ for all j, then because U represents preferences over outcomes, $U(f_j) \geq U(g_j)$ for all j and thus $\sum_{j=1}^{n} p_j U(f_j,r_j) \geq \sum_{j=1}^{n} p_j U(g_j,r_j)$. Because $(f,g) \in B_r$, part (b) of the definition of RU for $\succcurlyeq_r$ implies that $f \succcurlyeq_r g$ and thus weak monotonicity follows. Solvability holds because $U(\mathcal{C}) = \mathbb{R}$.

Continuity of U in its first argument implies continuity of $\sum_{j=1}^{n} p_j U(f_j,r_j)$ in the f_j's. Suppose that $r \succ_r f$. Then $\{g \in \mathcal{F}: g \preccurlyeq_r f\} = \emptyset$, which is closed. Because $V = \sum_{j=1}^{n} p_j U(f_j,r_j)$ is continuous, the inverse of V of the closed subset $[0,\rightarrow)$ in $\mathbb{R}$ is closed. This inverse is $\{g \in \mathcal{F}: g \succcurlyeq_r r\}$. If $f \succ_r r$ and $V(x) = \sum_{j=1}^{n} p_j U(f_j,r_j) = c > 0$, then by continuity of V, the inverses of V of the closed subsets $[0,c]$ and $[c,\rightarrow)$ in $\mathbb{R}$ are both closed. These inverses are $\{g \in \mathcal{F}: g \preccurlyeq_r f\}$ and $\{g \in \mathcal{F}: g \succcurlyeq_r f\}$. Preference continuity follows.

The derivation of tradeoff consistency is slightly different from the proof of Theorem 3.1. in Bleichrodt (2005). If $\alpha\beta \sim^{*}_{\mu} \gamma\delta$ then there exist acts f and g and a state j with $r_j = \mu$ such that $\alpha_j f \sim_r \beta_j g$ and $\gamma_j f \sim_r \delta_j g$. Hence, $p_j U(\alpha,\mu) + \sum_{i \neq j} p_i U(f_i,r_i) = p_j U(\beta,\mu) + \sum_{i \neq j} p_i U(g_i,r_i)$ and $p_j U(\gamma,\mu) + \sum_{i \neq j} p_i U(f_i,r_i) = p_j U(\delta,\mu) + \sum_{i \neq j} p_i U(g_i,r_i)$, which together with $p_j > 0$ gives $U(\alpha,\mu) - U(\beta,\mu) = U(\gamma,\mu) - U(\delta,\mu)$. Suppose we also have $\alpha\beta \sim^*_\mu \gamma\delta'$. Then there exist acts f′ and g′ and a state k with $r_k = \mu$ such that $\alpha_k f' \sim_r \beta_k g'$ and $\gamma_k f' \sim_r \delta'_j g'$. By a similar line of argument as above we obtain $U(\alpha,\mu) - U(\beta,\mu) = U(\gamma,\mu) - U(\delta',\mu)$. It follows that $U(\delta',\mu) = U(\delta,\mu)$ and, because U represents $\succcurlyeq$, $\delta' \sim \delta$, which establishes tradeoff consistency.

We now show that statement (1) implies statement (2). Let $r\in\mathcal{F}$. By setting $X_j = \mathcal{C}$ for all j, it is easily verified that all conditions in statement 1 of Theorem 3.1 in Bleichrodt (2005) are satisfied. Our tradeoff consistency condition is in fact a strengthening of the tradeoff consistency condition that is used there. It follows that there exist functions V_j: $\mathcal{C}\times\mathcal{C}\to\mathbb{R}$ such that

a. $(f,g)\in B_r$ iff $\sum_{j=1}^n V_j(f_j,r_j) \geq 0$ or $\sum_{j=1}^n V_j(g_j,r_j) \geq 0$

b. for all $(f,g)\in B_r$, $f \succcurlyeq_r g$ iff $\sum_{j=1}^n V_j(f_j,r_j) \geq \sum_{j=1}^n V_j(g_j,r_j)$

c. $V_j(r_j,r_j) = 0$ for all j.

d. for all $j\in\{1,\ldots,n\}$, V_j represents $\succcurlyeq$: for all $f_j,g_j \in\mathcal{C}$, $V_j(f_j,r_j) \geq V_j(g_j,r_j)$ iff $f_j \succcurlyeq g_j$.

e. The V_j are continuous and their range is $\mathbb{R}$

Furthermore, the V_j are joint ratio scales.

It remains to show that the V_j can be taken proportional. To prove this we need the following lemma, which was proved in Bleichrodt (2005). The preference relation satisfies *strong monotonicity* if for all $(f,g)\in B_r$, for all states of nature j, $f_j \succcurlyeq g_j$ and for at least one state of nature i, $f_i \succ g_i$ then $f \succ_r g$.

LEMMA 1: If $\succcurlyeq_r$ satisfies transitivity, upper completeness, and weak monotonicity, then tradeoff consistency implies strong monotonicity.

LEMMA 2: Assume that the V_j: $\mathcal{C}\times\mathcal{C} \to \mathbb{R}$, j = 1,…,n, are as defined above. Then every V_i is proportional to every other V_j.

Proof. Assume that the reference act is such that $r_i = r_j = \mu$. Let α, β, and γ be arbitrary outcomes with $\alpha \succ \gamma$. By solvability and because there are at least three states which are all nonnull, we can find an act f such that $\gamma_i f \sim_r r$. By strong monotonicity (Lemma 1) $\alpha_i f \succ_r r$. By solvability and because there are at least three states of nature, we can find an outcome δ and an act g such that $\alpha_i f \sim_r \beta_i g$ and $\gamma_i f \sim_r \delta_i g$. Hence, $\alpha\beta \sim^*_\mu \gamma\delta$. By strong monotonicity and transitivity it follows that $\beta \succ \delta$.

The indifference $\alpha_i f \sim_r \beta_i g$ implies that

$$V_i(\alpha,\mu) + \sum_{j\neq i} V_j(f_j,r_j) = V_i(\beta,\mu) + \sum_{j\neq i} V_j(g_j,r_j)$$

or

$$V_i(\alpha,\mu) - V_i(\beta,\mu) = \sum_{j\neq i} V_j(g_j,r_j) - \sum_{j\neq i} V_j(f_j,r_j)$$

Similarly, the indifference $\gamma_i f \sim_r \delta_i g$ implies that

$$V_i(\gamma,\mu) - V_i(\delta,\mu) = \sum_{j\neq i} V_j(g_j,r_j) - \sum_{j\neq i} V_j(f_j,r_j)$$

Hence, $V_i(\alpha,\mu) - V_i(\beta,\mu) = V_i(\gamma,\mu) - V_i(\delta,\mu)$.

By solvability and because there are other states than i, we can find acts f′ and g′ such that $\gamma_j f' \sim_r \delta_j g' \sim_r r$. By strong monotonicity, $\alpha_j f' \succ_r r$ and $\beta_j g' \succ_r r$. By tradeoff consistency and strong monotonicity, it follows that $\alpha_j f' \sim_r \beta_j g'$. For suppose that $\alpha_j f' \prec_r \beta_j g'$. By solvability there exists a β' such that $\alpha_j f' \sim_r \beta'_j g'$. By transitivity, $\beta'_j g' \prec_r \beta_j g'$, and, hence, by weak monotonicity $\beta' \prec \beta$. But then we have $\alpha\beta \sim^*_\mu$

$\gamma\delta$ and $\alpha\beta' \sim^*_\mu \gamma\delta$ but $\beta' \nsim \beta$, which violates tradeoff consistency. $\alpha_j f' \succ_r \beta_j g'$ is excluded by a similar argument.

The indifferences $\alpha_j f' \sim_r \beta_j g'$ and $\gamma_j f' \sim_r \delta_j g'$ imply that

$$V_j(\alpha,\mu) - V_j(\beta,\mu) = V_j(\gamma,\mu) - V_j(\delta,\mu).$$

Hence, $V_i(\alpha,\mu) - V_i(\beta,\mu) = V_i(\gamma,\mu) - V_i(\delta,\mu)$ implies $V_j(\alpha,\mu) - V_j(\beta,\mu) = V_j(\gamma,\mu) - V_j(\delta,\mu)$. Hence, V_i and V_j order utility equalities the same. Because V_i and V_j both represent $\succcurlyeq$ and have range $\mathbb{R}$, they must also order utility differences the same. This and the fact that the functions V_i and V_j are continuous in their first argument on a connected domain implies that V_i and V_j are related by a positive linear transformation. They are 0 at μ, hence they are related by a scale factor. The scale factor is positive by strong monotonicity. Q.E.D.

By Lemma 2, we have $V_j = \sigma_j V_i$ with $\sigma_j > 0$ because i and j are nonnull. Because i and j were arbitrary it follows that all V_j are proportional. Define $U = V_i$ for some state i and define $p_j = \sigma_j/(\sum_{j=1}^{n}\sigma_j)$. Because $\sigma_i = 1$, the denominator exceeds zero. Uniqueness of the p_j follows from the uniqueness properties of the V_j. Theorem 1 has been established.

□

Proof of Theorem 2.

Assume that statement (2) holds. For transitivity, r-upper completeness, weak monotonicity, solvability, preference continuity, and tradeoff consistency, the proof is the same as in Theorem 1. If $\beta \preccurlyeq \alpha$ then $U(\delta,\beta) \geq U(\delta,\alpha)$ because U is decreasing in its second argument. Hence, $\delta_j f \sim_{\alpha_j r} \alpha_j r$ implies $\delta_j f \succcurlyeq_{\beta_j r} \beta_j r$.

Suppose next that $\delta_j f \sim_{\alpha_j r} \alpha_j r$ implies $\delta_j f \succcurlyeq_{\beta_j r} \beta_j r$. Then $U(\delta,\beta) \geq U(\delta,\alpha)$ and it follows that $\beta \preccurlyeq \alpha$ because U is decreasing in its second argument. Hence, reference monotonicity holds.

If $\alpha_j f \succcurlyeq_{\alpha_j r} \alpha_j g$ then $p_j U(\alpha,\alpha) + \sum_{i\neq j} p_i U(f_i,r_i) \geq p_j U(\alpha,\alpha) + \sum_{i\neq j} p_j U(g_i,r_i)$. Because $U(\alpha,\alpha) = U(\beta,\beta) = 0$, also $p_j U(\beta,\beta) + \sum_{i\neq j} p_i U(f_i,r_i) \geq p_j U(\beta,\beta) + \sum_{i\neq j} p_j U(g_i,r_i)$ and, thus, $\beta_j f \succcurlyeq_{\beta_j r} \beta_j g$, which establishes neutral independence.

Assume next statement (1). By the proof of Theorem 3.2 in Bleichrodt (2005), there exist functions V_j: $\mathcal{C} \times \mathcal{C} \rightarrow \mathbb{R}$ which are independent of the reference act r such that

a. $(x,y) \in B_r$ iff $\sum_{j=1}^{n} V_j(x_j,r_j) \geq 0$ or $\sum_{j=1}^{n} V_j(y_j,r_j) \geq 0$

b. for all $(x,y) \in B_r$, $x \succcurlyeq_r y$ iff $\sum_{j=1}^{n} V_j(x_j,r_j) \geq \sum_{j=1}^{n} V_j(y_j,r_j)$

c. $V_j(r_j,r_j) = 0$ for all j.

d. for all $j \in \{1,\ldots,n\}$, V_j represents $\succcurlyeq$: for all $f_j,g_j \in \mathcal{C}$, $V_j(f_j,r_j) \geq V_j(g_j,r_j)$ iff $f_j \succcurlyeq g_j$.

e. for all $j \in \{1,\ldots,n\}$, V_j is decreasing in its second argument: for all $\alpha,\beta,\delta \in \mathcal{C}$, $V_j(\delta,\beta) \geq V_j(\delta,\alpha)$ iff $\beta \preccurlyeq \alpha$.

f. The V_j are continuous in their first argument and their range is $\mathbb{R}$

Furthermore, the V_j are joint ratio scales. Proportionality of the V_j follows from Lemma 2. As in Theorem 1, define $U = V_i$ for some state i and define $p_k = \sigma_k/(\sum_{j=1}^{n}\sigma_k)$ for all k.

□

Proof of Theorem 3:

Throughout the proof we will use the assumption that each act can serve as a reference act without further mentioning. That statement (2) implies statement (1) is easily verified. Assume that statement (1) holds. Let U be as in Theorem 2. Define $\succcurlyeq'$ on $\mathcal{C} \times \mathcal{C}$ by $\alpha\beta \succcurlyeq' \gamma\delta$ if $U(\alpha,\beta) \geq U(\gamma,\delta)$. $\succcurlyeq'$ is clearly transitive. Let $\alpha,\beta \in \mathcal{C}$. Let r be an arbitrary reference act with $r_j = \beta$. By solvability there exists an act f such that $\alpha_j f \sim_r r$. Hence, $U(\alpha,\beta)$ is defined for each pair $\alpha\beta$ and it follows that $\succcurlyeq'$ is complete. From the definition of U it is further immediately verified that for all $\alpha,\beta \in \mathcal{C}$, $\alpha\alpha \sim' \beta\beta$. We can define a preference relation over $\mathcal{C}$ by $\alpha \succcurlyeq \beta$ if $\alpha\beta \succcurlyeq' \beta\beta$. Since this implies that $U(\alpha,\beta) \geq U(\beta,\beta) = 0$ and, by Theorem 2, U is representing over outcomes, it follows that this derived preference relation over outcomes is identical to the preference relation over outcomes defined in the main text.

LEMMA 11: For all $\alpha,\beta,\gamma,\delta \in \mathcal{C}$, if $\alpha\gamma \succcurlyeq' \beta\gamma$ then $\alpha\delta \succcurlyeq' \beta\delta$, and if $\gamma\alpha \succcurlyeq' \gamma\beta$ then $\delta\alpha \succcurlyeq' \delta\beta$.

Proof. This follows immediately from the facts that $U(\cdot,r_j)$ represents $\succcurlyeq$ over outcomes for every r_j and that U is decreasing in its second argument. Q.E.D.

LEMMA 12: For all $\alpha,\beta,\gamma,\kappa,\lambda,\mu \in \mathcal{C}$, if $\alpha\beta \sim' \kappa\lambda$ and $\beta\gamma \sim' \lambda\mu$ then $\alpha\gamma \sim' \kappa\mu$.

Proof. Suppose that $\alpha\beta \sim' \kappa\lambda$ and $\beta\gamma \sim' \lambda\mu$. Hence, $U(\alpha,\beta) = U(\kappa,\lambda)$ and $U(\beta,\gamma) = U(\lambda,\mu)$. Let $r \in \mathcal{F}$. By solvability and the fact that there are at least three states of nature, we can find acts f, g, and h such that $\alpha_j f \sim_{\beta_j r} \beta_j r$, $\beta_j g \sim_{\gamma_j r} \gamma_j r$, and $\alpha_j h \sim_{\gamma_j r} \gamma_j r$. Because $U(\alpha,\beta) = U(\kappa,\lambda)$ and $U(\beta,\gamma) = U(\lambda,\mu)$, it follows that $\kappa_j f \sim_{\lambda_j r} \lambda_j r$ and that $\lambda_j g \sim_{\mu_j r} \mu_j r$. By the concatenation condition, it follows that $\kappa_j h \sim_{\mu_j r} \mu_j r$. Hence, $U(\alpha,\gamma) = U(\kappa,\mu)$ and thus $\alpha\gamma \sim' \kappa\mu$. Q.E.D.

The following two lemmas would follow easily from continuity of $U(\alpha,\beta)$. Unfortunately, such continuity is not easily established. Continuity of U in each of its variables follows from preference continuity and reference act continuity, but that is too weak to obtain these lemmas. Hence, we give independent derivations.

Because $U(\alpha,\beta)$ is continuous in each of its variables and because $U(\alpha,\beta)$ is representing preferences over outcomes and is decreasing in its second variable, $U(\alpha,\beta)$ is continuous and, consequently, $\succcurlyeq'$ is continuous. The following lemma follows straightforwardly from the continuity of U and the fact that C is connected. For a proof see Krantz et al. (1971, pp. 309-310) or Wakker (1989, Lemma III.3.3).

LEMMA 13: For all $\alpha,\beta,\gamma,\kappa,\mu \in C$, $\kappa\gamma \succcurlyeq' \alpha\beta \succcurlyeq' \mu\gamma$ implies the existence of a λ with $\lambda\gamma \sim' \alpha\beta$ and $\gamma\kappa \succcurlyeq' \alpha\beta \succcurlyeq' \gamma\mu$ implies the existence of a λ with $\gamma\lambda \sim' \alpha\beta$.

Let α^1 and α^0 be two outcomes such that $\alpha^1 \nsim \alpha^0$. Let $r\in\mathcal{F}$. By solvability and the fact that there are states $i\neq j$, we can find an act $f\in\mathcal{F}$ such that $(\alpha^1)_j f \sim_{(\alpha^0)_j r} (\alpha^0)_j r$. We then proceed inductively to define α^{k+1} as $(\alpha^{k+1})_j f \sim_{(\alpha^k)_j r} (\alpha^k)_j r$, k = 1, 2,….. By solvability, α^{k+1} exists. By Theorem 2 we have $U(\alpha^{k+1},\alpha^k) = U(\alpha^1,\alpha^0)$ and thus $\alpha^{k+1}\alpha^k \sim' \alpha^1\alpha^0$, for all k = 1,…… The sequence $\alpha^0, \alpha^1,\ldots,\alpha^k,\ldots$ is a *standard sequence*.

If $\alpha^1 \succ \alpha^0$ then $U(\alpha^1,\alpha^0) > 0$. Because $U(\alpha^1,\alpha^0) = U(\alpha^{k+1},\alpha^k)$, $U(\alpha^{k+1},\alpha^k) > 0$ and $\alpha^{k+1} \succ \alpha^k$. Hence, if $\alpha^1 \succ \alpha^0$ then the standard sequence is *increasing*: $\alpha^{k+1} \succ \alpha^k$, for all nonnegative integers k. Similarly, if $\alpha^1 \prec \alpha^0$ then the standard sequence is *decreasing*: $\alpha^{k+1} \prec \alpha^k$, for all nonnegative integers k. A standard sequence is *bounded* if there exist $\beta,\gamma \in C$ with $\beta \succcurlyeq \alpha^k \succcurlyeq \gamma$ for all α^k in the standard sequence.

LEMMA 14: Every bounded standard sequence is finite.

Proof. We assume that the standard sequence is increasing. Let $\{\alpha^k\}$, $k\in\mathbb{N}$ be an infinite standard sequence. For $j\in\mathbb{N}$, let $L_j = \{\beta\in C: \beta \prec \alpha^j\}$. Let $L = \cup L_j$. Each L_j is open by assumption, and hence L is open. L is clearly nonempty, for example α^1 is in L. Let $\beta \in -L$. Then $\beta \succcurlyeq \alpha^k$ for all $k\in\mathbb{N}$. Because $\alpha^{k+1}\alpha^k \sim' \alpha^1\alpha^0 \sim' \alpha^k\alpha^{k-1}$. We have by transitivity $\alpha^{k+1}\alpha^k \sim' \alpha^k\alpha^{k-1}$. Similarly, we obtain $\alpha^k\alpha^{k-1} \sim' \alpha^{k-1}\alpha^{k-2}$. By

Lemma 12, $\alpha^{k+1}\alpha^{k-1} \sim' \alpha^k\alpha^{k-2}$. Repeatedly using Lemma 12, we obtain $\alpha^{k+1}\alpha^1 \sim' \alpha^k\alpha^0$. The preference $\alpha^{k+1} \succ \alpha^k$ implies by Lemma 11 that $\alpha^{k+1}\alpha^0 \succ' \alpha^k\alpha^0$. By transitivity, $\alpha^{k+1}\alpha^0 \succ' \alpha^{k+1}\alpha^1$. By Lemma 11, $\beta\alpha^0 \succ' \beta\alpha^1$.

Hence, we have $\beta\alpha^0 \succ' \beta\alpha^1 \succcurlyeq' \alpha^{k+1}\alpha^1 \sim' \alpha^k\alpha^0$. By Lemma 13, there exists a $\gamma \in C$ such that $\gamma\alpha^0 \sim' \beta\alpha^1$. γ is also in $-$L because $\gamma\alpha^0 \succcurlyeq' \alpha^k\alpha^0$ and hence $\gamma \succcurlyeq \alpha^k$, for all $k \in \mathbb{N}$. Because by transitivity $\beta\alpha^0 \succ' \gamma\alpha^0$, $\beta \succ \gamma$. Thus, $\{\alpha \in C: \alpha \succ \gamma\}$ is an open subset of C that contains β and is included in $-$B. Hence, $-$B is open because it is the union of open sets. Because C is connected it cannot be the union of two disjoint open sets. Because L is open and nonempty, $-$L must be empty. Hence, if a standard sequence is infinite then it cannot be bounded above. A similar proof shows that an infinite standard sequence cannot be bounded below. In other words, any bounded standard sequence must be finite. Q.E.D.

The above results establish that $\succcurlyeq'$ satisfies all the conditions in Theorem 1 in Köbberling (2006). Her definition of solvability follows from Lemma 13, Archimedeanity follows from Lemma 14, her definition of weak separability follows from Lemma 11, and weak ordering of $\succcurlyeq'$ and neutrality were established in the text above Lemma 11. Her definition of the concatenation condition follows from Lemma 12. Theorem 1 in Köbberling (2006) ensures that there exists an u: $C \to \mathbb{R}$ such that for all $\alpha,\beta,\gamma,\delta \in C$, $\alpha\beta \succcurlyeq' \gamma\delta$ iff $u(\alpha) - u(\beta) \geq u(\gamma) - u(\delta)$, with u unique up to positive linear transformations. Hence, by the definition of $\succcurlyeq'$, $U(\alpha,\beta) \geq U(\gamma,\delta)$ iff $u(\alpha) - u(\beta) \geq u(\gamma) - u(\delta)$. Because C is connected and $\succcurlyeq'$ is continuous, we obtain that u is continuous by the proof of Theorem 5.3 in Wakker (1988).

Because there are nonnull states of nature, it follows that $\Delta[u] = \{u(\alpha) - u(\beta): \alpha,\beta \in C\}$ is nondegenerate. For all $\alpha,\beta \in C$, define F on $\Delta[u]$ by $F(u(\alpha) - u(\beta)) = U(\alpha,\beta)$. F is clearly strictly increasing. Because u is continuous, $u(\alpha) - u(\beta)$ is continuous both in α and in β and represents preferences over outcomes, $u(\alpha) - u(\beta)$ is continuous. Because $u(\alpha) - u(\beta)$ and U are continuous real-valued functions on connected topological spaces and F is strictly increasing, we obtain by Theorem 2.1 in

Wakker (1991) that F is continuous. Because $U(\alpha,\alpha) = 0$, it follows that $F(0) = 0$. The uniqueness properties of F follow from the uniqueness properties of u and U. This completes the proof of Theorem 3.

Proof that Sugden's (2003) conditions imply the conditions in Theorem 3

We finally show that Sugden (2003) conditions imply the conditions of Theorem 3. For ease of comparison with Sugden (2003) we express the derivations in terms of U. Reference-act continuity follows from his condition S2 (consequence-space continuity). Suppose that $\beta \preccurlyeq \alpha$. Because U represents preferences over outcomes, for all $\delta \in \mathcal{C}$, $U(\alpha,\delta) \geq U(\beta,\delta)$. By Sugden's S3 (gain/loss symmetry), $U(\delta,\beta) \geq U(\delta,\alpha)$. If $U(\delta,\beta) \geq U(\delta,\alpha)$ then by S3, $U(\alpha,\delta) \geq U(\beta,\delta)$ and, hence, $\beta \preccurlyeq \alpha$ because U represents preferences over outcomes.

Sugden's condition S4 (gain/loss additivity) is equivalent to the strong crossover condition in difference measurement (Suppes and Winet 1955, Scott and Suppes 1958, Debreu 1958, Pfanzagl 1968). Köbberling (2006) showed that the strong crossover condition implies the concatenation condition. She also showed that the reverse implication does not hold (p. 390).

□

Proof of Theorem 4.

That statement (2) implies statement (1) is easily verified. Assume that statement (1) holds. An element $\alpha \in \mathcal{C}$ is *maximal* if for all $\beta \in \mathcal{C}$, $\alpha \succcurlyeq \beta$. An element $\alpha \in \mathcal{C}$ is *minimal* if for all $\beta \in \mathcal{C}$, $\alpha \preccurlyeq \beta$. It follows immediately from U being unbounded that

LEMMA 15: $\mathcal{C}$ contains no maximal or minimal elements.

Let $r \in \mathcal{F}$. Let $f_i \succ r_i$, $f_j \succ r_j$, $f_k = r_k$, $k \neq i,j$. Such an act f can be constructed by Lemma 15. By strong monotonicity, $f \succ_r r$. Select γ such that $f_i \succ \gamma \succ r_i$. Such a γ exists by connectedness of the order topology. By solvability, there exists an outcome δ such that $\gamma_i\delta_j r \sim_r f$. By strong monotonicity $\delta \succ f_j$. Let $g = \gamma_i\delta_j r$.

Let $r'_j = r_j$ for all $j \neq i$, and let $r'_i \prec r_i$. By Lemma 15, r' can be constructed. Because $f_i \succ g_i = \gamma \succ r_i \succ r'_i$, by reference-independence for gains we obtain $f \sim_{r'} g$. The indifference $f \sim_r g$ gives by Theorem 3 and cancellation of common terms:

$$p_iF(u(f_i) - u(r_i)) + p_jF(u(f_j) - u(r_j)) = p_iF(u(\gamma) - u(r_i)) + p_jF(u(\delta) - u(r_j))$$

The indifference $f \sim_{r'} g$ gives by Theorem 3 and cancellation of common terms:

$$p_iF(u(f_i) - u(r'_i)) + p_jF(u(f_j) - u(r_j)) = p_iF(u(\gamma) - u(r'_i)) + p_jF(u(\delta) - u(r_j)).$$

Let $x = u(f_i) - u(r_i)$, $y = u(\gamma) - u(r_i)$, and $\varepsilon = u(r_i) - u(r'_i)$. Then the above two equalities yield after some rearranging: $F(x) - F(y) = F(x+\varepsilon) - F(y+\varepsilon)$. Because $u(\mathcal{C}) = \mathbb{R}$ and because the above preferences can be constructed for all x,y, and ε thanks to solvability, $F(x) - F(y) = F(x+\varepsilon) - F(y+\varepsilon)$ holds for all x, y, and ε and, hence, F must be linear: $F(x) = ax + b$. Since F is increasing, $a > 0$ and since $F(0) = 0$, $b = 0$.

A similar line of argument shows that $F = bx$ for losses, i.e. $x < 0$. We can rescale utility such that $a = 1$. It then follows that $\lambda = \frac{b}{a}$ and constant loss aversion results.

□

Appendix B: Extension to Infinite State Spaces and to Decision under Risk

Let $\mathcal{S}$ be a general state space endowed with an algebra whose elements are called *events*. An *act* is a finite-valued function from $\mathcal{S}$ to $\mathcal{C}$, where the inverse of each outcome is an event. Acts are denoted by $(E_1:f_1,\ldots,E_n:f_n)$ meaning outcome f_j is received if event E_j obtains with the E_js partitioning $\mathcal{S}$.

First consider a fixed partition of $\mathcal{S}$. This fixed partition is isomorphic to the sets studied before and, hence, all theorems derived before can be applied. The representations can be made to coincide on all finite partitions by choosing a common normalization of utility, a common refinement for each pair of partitions, and applying the uniqueness results of Theorems 1 and 2. Because every pair of acts is measurable with respect to a sufficiently fine partition, one representation is obtained for the preferences between all acts.

The $\sim_\mu^*$ relations depend on the partitions chosen and should formally be related to these. However, choosing common refinements of partitions reveals that these dependencies can be omitted and $\sim_\mu^*$ relations can be defined as the union of all corresponding partition-dependent $\sim_\mu^*$ relations. Tradeoff consistency can then be defined with respect to the partition-independent $\sim_\mu^*$ relation thus defined. Extensions to acts with infinitely many outcomes can be obtained through the techniques of Wakker (1993b).

Decision under Risk

In the case of decision under risk preferences are defined over prospects $(p_1:f_1,\ldots,p_n:f_n)$ giving outcome f_j with probability p_j. The probabilities p_j are objectively given and are not derived from preferences. To relate decision under risk to decision under uncertainty, take the interval [0,1] as state space with events being subintervals. Probabilities correspond to lengths of intervals. Each act then generates a prospect over the outcomes and preferences over acts determine preferences over prospects. Because a prospect can be generated by different acts, acts that generate the same probability distribution over outcomes must be indifferent. In this manner, the results above can also be applied to give preference foundations for decision under risk.

Acknowledgements:

I thank Mohammed Abdellaoui, Stefan Trautmann, and, in particular, Peter P. Wakker for their comments on previous drafts. Discussions with Ulrich Schmidt led to the research question of this paper. Financial support was provided by the Netherlands Organization for Scientific Research (NWO).

References

Abdellaoui, M. 2000. Parameter-free elicitation of utilities and probability weighting functions. *Management Science* **46** 1497-1512.

Abdellaoui, M., F. Vossmann, M. Weber. 2005. Choice-based elicitation and decomposition of decision weights for gains and losses under uncertainty. *Management Science* **51** 1384-1399.

Apesteguia, J., M. A. Ballester. 2006. A theory of reference-dependent behavior. *Working Paper, Universitat Pompeu Fabra*.

Barberis, N., M. Huang, T. Santos. 2001. Prospect theory and asset prices. *Quarterly Journal of Economics* **66** 1-53.

Benartzi, S., R. H. Thaler. 1995. Myopic loss aversion and the equity premium puzzle. *Quarterly Journal of Economics* **110** 73-92.

Bleichrodt, H. 2005. Reference-dependent utility with shifting reference points and incomplete preferences. *Working Paper, Erasmus University Rotterdam*.

Bleichrodt, H., J. L. Pinto. 2000. A parameter-free elicitation of the probability weighting function in medical decision analysis. *Management Science* **46** 1485-1496.

Bleichrodt, H., J. L. Pinto. 2005. The validity of QALYs under non-expected utility. *Economic Journal* **115** 533-550.

Bleichrodt, H., J. L. Pinto, P. P. Wakker. 2001. Using descriptive findings of prospect theory to improve the prescriptive use of expected utility. *Management Science* **47** 1498-1514.

Camerer, C. F. 2000. Prospect theory in the wild: Evidence from the field. D. Kahneman and A. Tversky, eds. *Choices, values and frames*. Cambridge University Press, New York, 288-300.

Camerer, C. F., L. Babcock, G. F. Loewenstein, R. H. Thaler. 1997. Labor supply of New York city cabdrivers: One day at a time. *Quarterly Journal of Economics* **112** 407-442.

Danan, E. 2003. A behavioral model of individual welfare. *Working Paper, Université de Paris*.

Debreu, G. 1958. Stochastic choice and cardinal utility. *Econometrica* **26** 440-444.

Etchart-Vincent, N. 2004. Is probability weighting sensitive to the magnitude of consequences? An experimental investigation on losses. *Journal of Risk and Uncertainty* **28** 217-235.

Falmagne, J.-C. 1981. On a recurrent misuse of a classical functional equation result. *Journal of Mathematical Psychology* **23** 190-193.

Fennema, H., M. van Assen. 1999. Measuring the utility of losses by means of the trade-off method. *Journal of Risk and Uncertainty* **17** 277-295.

Fishburn, P. C. 1970. *Utility theory for decision making*. Wiley, New York.

Fishburn, P. C. 1992. Additive differences and simple preference comparisons. *Journal of Mathematical Psychology* **36** 21-31.

Hardie, B. G. S., E. J. Johnson, P. S. Fader. 1993. Modeling loss aversion and reference dependence effects on brand choice. *Marketing Science* **12** 378-394.

Hershey, J. C., P. J. H. Schoemaker. 1985. Probability versus certainty equivalence methods in utility measurement: Are they equivalent? *Management Science* **31** 1213-1231.

Johnson, E. J., D. A. Schkade. 1989. Bias in utility assessments: Further evidence and explanations. *Management Science* **35** 406-424.

Kahneman, D., J. L. Knetsch, R. H. Thaler. 1990. Experimental test of the endowment effect and the coase theorem. *Journal of Political Economy* **98** 1325-1348.

Kahneman, D., A. Tversky. 1979. Prospect theory: An analysis of decision under risk. *Econometrica* **47** 263-291.

Köbberling, V. 2006. Strength of preference and cardinal utility. *Economic Theory* **27** 375-391.

Köbberling, V., P. P. Wakker. 2003. Preference foundations for nonexpected utility: A generalized and simplified technique. *Mathematics of Operations Research* **28** 395-423.

Köbberling, V., P. P. Wakker. 2004. A simple tool for qualitatively testing, quantitatively measuring, and normatively justifying Savage's subjective expected utility. *Journal of Risk and Uncertainty* **28** 135-145.

Köbberling, V., P. P. Wakker. 2005. An index of loss aversion. *Journal of Economic Theory* **122** 119-131.

Köszegi, B., M. Rabin. forthcoming. A model of reference-dependent preferences. *Quarterly Journal of Economics.*

Krantz, D. H., R. D. Luce, P. Suppes, A. Tversky. 1971. *Foundations of measurement, vol. 1*. Academic Press, New York.

Luce, R. D., P. C. Fishburn. 1991. Rank- and sign-dependent linear utility models for finite first-order gambles. *Journal of Risk and Uncertainty* **4** 29-59.

Maccheroni, F. 2004. Yaari's dual theory without the completeness axiom. *Economic Theory* **23** 701-714.

Mandler, M. 2005. Incomplete preferences and rational intransitivity of choice. *Games and Economic Behavior* **50** 255-277.

Masatlioglu, Y., E. A. Ok. 2005. Rational choice with status quo bias. *Journal of Economic Theory* **121** 1-29.

Morrison, G. C. 2000. The endowment effect and expected utility. *Scottish Journal of Political Economy* **47** 183-197.

Pennings, J. M. E., A. Smidts. 2003. The shape of utility functions and organizational behavior. *Management Science* **49** 1251-1263.

Pfanzagl, J. 1968. *Theory of measurement*. Wiley, New York.

Popescu, I., Y. Wu. forthcoming. Dynamic pricing strategies with reference effects. *Operations Research*.

Rabin, M. 2000. Risk aversion and expected-utility theory: A calibration theorem. *Econometrica* **68** 1281-1292.

Robinson, A., G. Loomes, M. Jones-Lee. 2001. Visual analog scales, standard gambles, and relative risk aversion. *Medical Decision Making* **21** 17-27.

Sagi, J. S. 2006. Anchored preference relations. *Journal of Economic Theory* **130** 283-295.

Schmidt, U. 2003. Reference dependence in cumulative prospect theory. *Journal of Mathematical Psychology* **47** 122-131.

Schmidt, U., C. Starmer, R. Sugden. 2005. Explaining preference reversal with third-generation prospect theory. *CeDex Discussion Paper 2005-19*.

Schmidt, U., S. Traub. 2002. An experimental test of loss aversion. *Journal of Risk and Uncertainty* **25** 233-249.

Scott, D., P. Suppes. 1958. Foundational aspects of theories of measurement. *Journal of Symbolic Logic* **23** 113-128.

Shalev, J. 2000. Loss aversion equilibrium. *International Journal of Game Theory* **29** 269-287.

Shapiro, L. 1979. Necessary and sufficient conditions for expected utility maximizations: The finite case with a partial order. *Annals of Statistics* **7** 1288-1302.

Stalmeier, P. F. M., T. G. G. Bezembinder. 1999. The discrepancy between risky and riskless utilities: A matter of framing? *Medical Decision Making* **19** 435-447.

Sugden, R. 2003. Reference-dependent subjective expected utility. *Journal of Economic Theory* **111** 172-191.

Suppes, P., M. Winet. 1955. An axiomatization of utility based on the notion of utility differences. *Management Science* **1** 259-270.

Tversky, A., D. Kahneman. 1991. Loss aversion in riskless choice: A reference-dependent model. *Quarterly Journal of Economics* **56** 1039-1061.

Tversky, A., D. Kahneman. 1992. Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and Uncertainty* **5** 297-323.

van Osch, S. M. C., P. P. Wakker, W. B. van den Hout, A. M. Stiggelbout. 2004. Correcting biases in standard gamble and time tradeoff utilities. *Medical Decision Making* **24** 511-517.

Wakker, P. P. 1988. The algebraic versus the topological approach to additive representations. *Journal of Mathematical Psychology* **32** 421-435.

Wakker, P. P. 1989. *Additive representations of preferences: A new foundation of decision analysis*. Kluwer Academic Publishers, Dordrecht, The Netherlands.

Wakker, P. P. 1991. Continuity of transformations. *Journal of Mathematical Analysis and Applications* **162** 1-6.

Wakker, P. P. 1993a. Additive representations on rank-ordered sets II. The topological approach. *Journal of Mathematical Economics* **22** 1-26.

Wakker, P. P. 1993b. Unbounded utility for Savage's "Foundation of statistics," And other models. *Mathematics of Operations Research* **18** 446-485.

Wakker, P. P. 2005. Reference dependence, initial wealth, and utility of income: A reply to cox & sadiraj's criticism of Rabin's calibration theorem. *Working Paper, Erasmus University Rotterdam*.

Wakker, P. P., D. Deneffe. 1996. Eliciting von Neumann-Morgenstern utilities when probabilities are distorted or unknown. *Management Science* **42** 1131-1150.

Wakker, P. P., A. Tversky. 1993. An axiomatization of cumulative prospect theory. *Journal of Risk and Uncertainty* **7** 147-176.

Zank, H. 2001. Cumulative prospect theory for parametric and multiattribute utilities. *Mathematics of Operations Research* **26** 67-81.