Koopmans' Constant Discounting: A Simplification and an Extension to Incorporate General Economic Growth

Han Bleichrodt, Kirsten I.M. Rohde, & Peter P. Wakker*

Department of Economics, Erasmus University, Rotterdam, The Netherlands

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Abstract

Koopmans provided a well-known preference foundation for discounted utility, the most widely used model for intertemporal optimization. There were, however, some problems in his analysis. For example, there was an unforeseen implication of bounded utility. For some domains solutions have been advanced in the literature, primarily when particular production processes impose time-dependent restrictions on consumption. This paper completely resolves the problems mentioned, irrespective of what the restrictions on consumption are. It obtains complete flexibility concerning the utility functions that can be used and concerning the conceivable economic growth. This paper, thus, provides a complete preference foundation of discounted utility, and clarifies the appeal of Koopmans' intuitive axioms.

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1 Introduction

Preference foundations give conditions on preferences that are necessary and sufficient for the applicability of some theoretical choice model, such as the discounted utility model for intertemporal consumption programs, the topic of this paper. Preference foundations, thus, express the empirical content of a model directly in terms of observables. One of the most appealing and well-known preference foundations is the one of Koopmans (1960, 1972) for discounted utility. This model, originally introduced by Samuelson (1937), is often considered to be normative. It is the almost exclusively used model in prescriptive applications of intertemporal optimization and in policy recommendations. For such applications, preference foundations serve to justify (or criticize) the appropriateness of the model used.

In descriptive applications, preference foundations serve to demonstrate how a model can be falsified or corroborated through observed choices. Indeed, many recent empirical studies have examined, and mostly falsified, the preference conditions of discounted utility. Discounted utility is commonly taken as an initial benchmark, a proper variation of which is to be developed (Frederick, Loewenstein, & O'Donoghue 2002; Laibson 1997; Rubinstein 2003).

As we will explain later, the intuitive part of Koopmans' preference foundation is exceptionally appealing and efficient. While his paper has been widely cited, the appeal and efficiency of his intuitive axioms have not always been fully understood. Koopmans' analysis is, unfortunately, obscured by technical digressions and some inaccuracies. It cannot always be decided with certainty what technical conditions Koopmans assumed (see appendix). Koopmans claimed his result only for bounded consumption programs (Koopmans 1960, Postulate P5; Koopmans 1972, Postulate P5$'$), but speculated on extensions to other programs if his axiom P5/P5$'$ is dropped (Koopmans 1960, Section 10*; Koopmans 1972, Section 6*). However, even if this axiom is dropped, then his remaining conditions still imply that utility must be bounded (see Example 10 in the appendix). Most utility functions considered in the literature, such as CRRA and CARA utilities, are unbounded from at least one side if no restrictions are imposed on their domain (e.g., $I\!R$ for CARA). Koopmans' utility bounds, therefore, impose undesirable restrictions on the outcomes for such cases. In particular for unbounded time horizons it is undesirable to exclude unbounded growth and continuing technological progress. It is pessimistic to a priori impose upper bounds on the utility levels attainable for all the future (see Koopmans, 1972, following Postulate P5$'$; de Hek & Roy 2001).

Several extensions of Koopmans' preference foundation have been advanced in the literature. Those extensions usually were not the main topic, but were obtained as a by-product of studies into other more complex questions. Also Koopmans's (1960) himself presented his preference foundation as a by-product. He derived impatience from continuity for general recursive functionals, and presented this as his main result. His axiomatization of discounted utility has, however, become the most famous result of his paper, and it is the topic that we focus on.

Our work mainly builds on extensions of Koopmans' axiomatization by Dolmas (1995), Harvey (1986), and Streufert (1990), who resolved Koopmans' problems for particular domains of application. Primarily by relaxing Koopmans' topological assumptions, we will completely resolve the problems of unbounded utility. That is, first, we allow every utility function, and, second, our domain has maximal flexibility concerning the unbounded programs considered. For every utility function we are free to incorporate (or exclude) every program that generates unbounded utility in the future as long as its discounted utility is well-defined and finite. Examples will illustrate our results, and their logical relations with preceding results in the literature.

Koopmans' characterization of the discounted-utility presentation is one of the most appealing and important preference foundations known today. By focusing on this presentation and by resolving the technical problems in Koopmans' analysis, we achieve maximal generality and simplicity. The main purpose of our paper is to help clarify and popularize Koopmans' intuitive axioms.

2 Koopmans' Intuitive Preference Conditions

X, the set of all conceivable consumptions, is any convex subset of $I\!R^m$. For example, X may be $I\!R^m_+$.[1] A *(consumption) program* $x = (x_1, x_2, \ldots)$ yields consumption $x_t \in X$ in period $t \in I\!N$. Preferences over consumption programs are denoted by $\succcurlyeq$, with $\succ, \sim, \preccurlyeq$, and $\prec$ as usual. We will specify the requirements for the preference domain F later.

[1]All results of this paper remain valid if X is a topologically separable arc-connected topological space; see the appendix. It can, for instance, be a set of nonquantified health states.

The main reason for the complications in Koopmans' analysis is that he did not allow for a restricted domain of programs on which the preference relation is defined and on which completeness and other preference conditions are imposed. With unbounded utility, there can be programs with infinite or undefined discounted utility, and imposing completeness in combination with other preference conditions on such programs is problematic.

A classical illustration of the above problems concerns Savage (1954) who, like Koopmans, imposed completeness and other axioms on all programs ("acts" in his model). An unforeseen implication, discovered by Fishburn, was that Savage's axioms imply boundedness of utility (Savage, 1972, 2nd edition, footnote on p. 80). Koopmans' axioms similarly have this unforeseen implication (see Example 10). Koopmans, Diamond, & Williamson (1964, Section 8*) speculated on problems as just specified and, to avoid them, explicitly restricted their analysis to bounded utility. Koopmans (1972), however, defined a preference relation and its conditions on the full domain again and without boundedness restriction, adding the latter only for particular results, and speculating on dropping it.

Our way to handle unbounded utility follows approaches by Dolmas (1995), Harvey (1986), Streufert (1990), and Wakker (1993a). Like these authors, we avoid the aforementioned problems by considering preferences and their conditions only on restricted domains of programs, using only conditions expressed entirely in terms of observables. The new condition in our solution through which we obtain complete generality for unbounded programs, tail robustness (defined later), can be interpreted in a two-fold manner. For each given domain we completely characterize the preferences that generate well-defined and finite discounted utility on this domain. Conversely, for each given preference relation we completely characterize all domains to which the discounted utility representation can be extended.

Discounted utility holds on a domain of programs if there exist a *utility function* $u : X \to I\!R$ and a *discount factor* $0 < \rho < 1$ such that every consumption program x in the domain is evaluated through the well-defined and finite value

$$\sum_{t=1}^{\infty} \rho^{t-1} u(x_t). \tag{1}$$

This summation is called the *discounted utility* (DU) of x. A program is preferred if and only if it is has the highest discounted utility. Our term discounted utility entails what sometimes has been called constant or exponential discounting. The evaluation implies the common assumption of *weak ordering* of $\succcurlyeq$, meaning that $\succcurlyeq$ is *complete* ($x \succcurlyeq y$ or $y \succcurlyeq x$ for all x, y, possibly both) and transitive. Many papers that do not assume additive separability over time do not use a function as u above, and then often use the term utility for what we call discounted utility. In our set-up it is most convenient to let utility designate the "instant" utility u.

For programs for which the summation in Eq. 1 is not defined or is infinite, we continue to use the term DU informally and we say that "DU is not well-defined" or "DU is infinite," as the case may be. Before defining the preference conditions for our characterization of DU, we introduce some notation. For a program $x = (x_1, x_2, \ldots)$ and a consumption α, αx denotes the program $(\alpha, x_1, x_2, \ldots)$ where the first consumption is α and then the consumptions of x follow, all delayed by one period. The procedure can be repeated, as in $\alpha\beta x = (\alpha, \beta, x_1, x_2, \ldots)$, etc.

Preferences over consumptions agree with preferences over constant programs in the sense that $\alpha \succcurlyeq \beta$ if and only if $(\alpha, \alpha, \ldots) \succcurlyeq (\beta, \beta, \ldots)$. Koopmans (1972, p. 84) discussed (a small variation of) the following condition. *Monotonicity* holds if $x \succcurlyeq y$ whenever $x_t \succcurlyeq y_t$ for all t, with strict preference $x \succ y$ whenever $x_t \succ y_t$ for some t. In our result this condition need not be imposed because it is implied by the other conditions, mainly stationarity. To avoid triviality and preferences entirely determined by the tail behavior of programs, we will, like Koopmans (1960 Postulate 2), assume that period 1 is *sensitive*, i.e. $\alpha x \succ \beta x$ for some α, β, x.

The most interesting axioms in preference foundations are the intuitive axioms characteristic of the model considered. Koopmans assumed, essentially, the following two intuitive axioms. *Initial-tradeoff independence* holds if

$$\alpha\beta x \succcurlyeq \gamma\delta x \text{ if and only if } \alpha\beta y \succcurlyeq \gamma\delta y \tag{2}$$

for all programs x, y and all consumptions $\alpha, \beta, \gamma, \delta$. That is, tradeoffs between today and tomorrow are not affected by future consumption. The condition amounts to separability of the first two periods. A set of periods is *separable* if preferences over consumptions in these periods, while keeping consumptions in other periods fixed, are independent of the level at which those other consumptions are kept fixed.

Koopmans' second intuitive axiom, *stationarity*, holds if

$$\alpha x \succcurlyeq \alpha y \text{ if and only if } x \succcurlyeq y \tag{3}$$

for all programs x, y and all consumptions α. That is, a preference is not affected if a common first consumption is dropped, and the timing of all other consumptions is advanced by one period. By repeated application, it implies that for a preference between two programs all initial periods with common consumption can be dropped, and the first period of different consumption can be taken as the initial period.

Koopmans formulated his intuitive axioms equivalently but slightly differently, with for instance stationarity imposed only for one initial consumption α and then separability of $\{2, 3, \ldots\}$ added, which is equivalent to our stationarity imposed for all initial consumptions α.

3 Preference Conditions for Unbounded Time Horizons

The difficulties in Koopmans' analysis are due to the technical conditions. In his model as well as in ours, programs are infinite-dimensional objects, and then topological considerations can be complex (Becker & Boyd 1997, Section 3.1). We avoid such complexities by imposing topological conditions (continuity) only on finite-dimensional subspaces. For two further implications of infinite-dimensional continuity that are used in proofs of other papers, namely constant-equivalence and tail-robustness (defined later), it will be both more appealing and more general to state these as explicit axioms, rather than to derive them from stronger infinite-dimensional topological assumptions.

An *ultimately constant* program x is such that $\alpha = x_t = x_{t+1} = \cdots$ for all $t > T$, for some consumption α and some period T. By $x_T\alpha$, for some general program x, period T, and consumption α, we denote the ultimately constant program $(x_1, \ldots, x_T, \alpha, \alpha, \ldots)$. For each period T, X_T is the set of ultimately constant programs of the form $x_T\alpha$, i.e., all ultimately constant programs that are constant over all $t > T$. X_T can be considered a $T+1$ dimensional product space, specified by $T+1$ tuples $(x_1, \ldots, x_T, \alpha)$. *Ultimate-continuity* holds if $\succcurlyeq$ is continuous on each set X_T.

To extend the discounted utility evaluation to unbounded programs, we, contrary to Koopmans, do not impose a preference relation on all programs, but only on those that will have (finite) discounted utility. We want to achieve maximal generality of the domain of unbounded programs considered, and allow for any subset of the set of all such programs in our theorem. To achieve our purpose, we have to solve a mathematical problem that has hampered many papers dealing with infinite-dimensional evaluations, and that is explained next.

A typical example of the aforementioned mathematical problem is the derivation of subjective expected utility in DeGroot (1970, Chapter 7). Having derived the evaluation on bounded programs ("acts"), he explicitly used utility and the expected utility functional to define the domain of all programs with finite expected utility (denoted $\mathcal{P}_E$ in his Section 7.10), and then went on to establish the preference axiomatization of expected utility in terms of preference conditions on this extended domain. This procedure is undesirable because utility and expected utility are theoretical constructs and are related to observables only in complex ways. Hence, they should not be used explicitly in preference foundations. In fact, if expected utility can be used explicitly in the definition of the domain and in preference conditions, then its preference axiomatization becomes a tautology because we can then simply state expected utility maximization directly as a preference axiom. A similar problem as in DeGroot's analysis arose in Hübner & Suck's (1993) extension of Koopmans' discounted utility to unbounded programs. They used a condition about the interior of circles of convergence that explicitly uses both the discount factor and the utility function.

The problem to be solved is that we should find conditions that imply well-defined and finite discounted utility for the preferences and programs considered, but that are stated entirely in terms of observables (preferences) without any explicit use of utility or discounted utility. A first result in this direction was obtained by Dolmas (1995). For each program $\omega \in I\!R^m_{++}$, define A_ω as the set of all programs x for which there exists $\lambda > 0$ such that $|x_t| \leq |\lambda \omega_t|$ for all t. Dolmas considered domains of the form A_ω. He imposed preference conditions that allow for particular unbounded utility functions and, thus, relaxed the boundedness restriction. Sets A_ω naturally arise if, due to limitations of production, consumption cannot grow faster asymptotically than through ω times an initial input. Thus, Dolmas covered an important class of economic problems. He imposed continuity with respect to a modified supnorm that made his space of programs homeomorphic (equivalent in a topological sense) to the set of all bounded programs. He then derived discounted utility on this extended domain from a strengthening of Koopmans' axioms. In Example 5 we will demonstrate how we can generalize Dolmas' domain.

Streufert (1990, 1993) did not take preference as a primitive, but the utility $U(x)$ of programs x and an aggregator function W such that utility is the aggregation of present consumption x_1 and future utility: $U(x_1, x_2, \ldots) = W(x_1, U(x_2, x_3, \ldots))$. Thus, his results are not directly comparable to ours. His results present conditions under which the utility function U is the only solution to the above aggregation equation given the aggregator function W. Streufert considered the programs that can be generated by a particular production function, thus relaxing the boundedness restriction. In Streufert (1990), he introduced a biconvergence continuity condition in terms of the production function and the utility function, and derived discounted utility on the extended domain from a strengthening of Koopmans' axioms. In Streufert (1993), he used a tail insensitivity condition that is more general than biconvergence. Most of his conditions can be restated for preference relations (Becker & Boyd 1997, Section 3.3.3), and this is how we will use his conditions for preference foundations, the topic of our paper.

Conditions that can identify all programs with finite discounted utility, whatever the (unbounded) utility function, were used by Harvey (1986) for the present context of summation over discrete periods, and by Wakker (1993a) for integrals over general spaces for the context of decision under uncertainty. The approach of this paper will combine these two approaches with those of Dolmas (1995) and Streufert (1990, 1993), using two preference conditions introduced next.

A program x satisfies *constant-equivalence* on a domain if there exists an equivalent constant program in that domain. This condition was derived from topological conditions in Diamond (1965, Lemma on p. 172), Dolmas (1995 p. 324), Harvey (1986, p. 1136 second para) and Koopmans (1960, Eq. 17 and Section 10). It was immediate, given that W and U are primitives, in Streufert (1990, 1993). The main condition in our analysis is the following.

DEFINITION 1 A program x is *tail robust* if, for all outcomes β: if $x \succ \beta$ $(x \prec \beta)$ then there exists a t such that $x_T\beta \succ \beta$ $(x_T\beta \prec \beta)$ for all $T \geq t$.

In words, a sufficiently remote future does not affect preference much. An example where the remote future does affect preference in a way violating tail robustness, and demonstrating the necessity of a condition such as tail robustness, can be found in Becker & Boyd (1997, Section 3.3.3, Example 7). Tail robustness extends Streufert's (1990) biconvergence condition (when redefined for preferences as in Becker & Boyd 1997, Section 3.3.3) to the case where no upper bound (time-dependent and generated by a production function), and no lower bound 0, need to be available for consumption. It similarly extends Streufert's (1993) tail insensitivity to the case of unbounded utility (by not considering every replacement of tails but considering only β-tail replacements). In the special case of an upper and lower bound for consumption and under usual assumptions such as monotonicity, our tail robustness can be seen to be equivalent to Streufert's conditions. Dolmas' limited monotonicity axiom A6 and his impatience axiom A7 (used also in Becker & Boyd, Section 3.3.3) are related, and similarly concern cases where a time-dependent upper bound $\lambda\omega$ and a lower bound 0 are available for consumption.

The two conditions just defined deliver the proper restrictions on preferences and programs considered, entirely in terms of observables. A program will turn out to have well-defined finite discounted utility if and only if it satisfies constant-equivalence and tail robustness. These definitions combine and generalize the C^* definition of Harvey (1986), truncation-robustness of Wakker (1993a), and biconvergence and tail insensitivity of Streufert (1990, 1993).

4 A Preference Foundation for Discounted Utility

In the characterizing Statement (ii) in the following theorem, (a) states usual preference conditions, with continuity only in a simple finite-dimensional version, (b) gives the conditions to ensure well-defined finite discounted utility, and (c) gives Koopmans' intuitive conditions. The uniqueness up to unit and level in the following theorem means that any constant can be added to utility, and utility can be multiplied by any positive number.

THEOREM 2 *[Preference Foundation for Discounted Utility]. Let $\succcurlyeq$ be defined on a domain F of programs that contains all ultimately constant programs. Then the following two statements are equivalent.*

(i) Discounted utility holds on F, where the utility function is continuous and not constant.

(ii) $\succcurlyeq$ satisfies

(a) weak ordering, sensitivity of the first period, and ultimate-continuity;

(b) constant-equivalence and tail robustness;

(c) initial-tradeoff independence and stationarity.

Furthermore, the discount factor in Statement (i) is unique, and utility is unique up to unit and level. □

Some of the above conditions can be relaxed on particular domains. A program x is *bounded* if there exist consumptions μ, ν such that $\mu \succcurlyeq x_t \succcurlyeq \nu$ for all t.

OBSERVATION 3 *[Relaxing Conditions on Particular Domains] In Theorem 2, monotonicity is implied by the other conditions. If F contains only ultimately constant programs, then constant-equivalence can be dropped and tail robustness can be replaced by monotonicity. Tail robustness can also be replaced by monotonicity if F contains only bounded programs.* □

The results in the rest of this section illustrate the generality of our approach. Corollary 4 shows that the only restriction for the domain F is that all ultimately constant programs be present. Other than that, we have complete flexibility regarding the domain, and we can incorporate any set of programs with well-defined and finite DU.

COROLLARY 4 *[Maximal Generality of Domain] Assume discounted utility with given u and ρ. Then the set F in Theorem 2 can be any subset of the set of all programs with well-defined and finite DU that contains all ultimately constant programs.* □

We next show that our theorem can handle programs that are excluded by some domains considered in the literature. If utility u is bounded, then Theorem 2 can handle the domain of all programs, which is obviously more general than the domain A_ω defined in Section 3 and considered in Dolmas (1995) or Becker & Boyd (1997), or other domains bounded by a production process (Streufert 1990). The following example concerns unbounded utility.

EXAMPLE 5 [Domains More General than Dolmas' (1995) Set A_ω]. We show that our approach can deal with examples more general than sets A_ω. Assume that all conditions and conclusions in Theorem 2 hold, with u the identity and $X = I\!R_+$ ($m = 1$), so that u is unbounded above. Take some fixed program ω with well-defined (finite) discounted utility and all $\omega_t > 0$, and take as preference domain A_ω. Define the program x as follows, with $t_0 = 0$. For each j, take $t_j > t_{j-1}$ such that $\rho^{t_j - 1} u(\omega_{t_j}) < \frac{2^{-j}}{j}$. Such a t_j always exists because the discounted utility of ω is well-defined. Define $x_{t_j} = j\omega_{t_j}$ for all j, and $x_t = 0$ for all remaining t. Then the discounted utility of x is less than $\sum_{j=1}^{\infty} 2^{-j} = 1$ and x can be incorporated into the domain F of Theorem 2. There is, however, no $\lambda > 0$ such that $|x_t| \leq |\lambda \omega_t|$ for all t. Hence the program is not contained in A_ω. □

The following example shows that tail robustness is a generalization of Streufert's (1990) biconvergence and Streufert's (1993) tail insensitivity.

EXAMPLE 6 [Generalizing Streufert's Biconvergence and Tail Insensitivity]. Let discounted utility hold and let u be unbounded. Let F consist of all programs with well-defined and finite DU. There is no upper bound to the conceivable utility of consumption for any time point. Hence, Streufert's (1990) biconvergence and Streufert's (1993) tail insensitivity cannot be satisfied: These conditions require that for each given strict preference a time t exists such that however we change the tails of the programs after t, the preference is not affected. In our case, however, for every strict preference and every time point, no matter how far remote, the preference can be reversed by changing the consumption on that remote time point because the conceivable u values for that remote time point are unbounded. □

Restricted domains of conceivable consumption as in the following example, taken from Dolmas (1995, Theorem 1) and Becker & Boyd (1997, Section 3.2, Example 1), naturally arise from limitations on future production or from restrictions on future borrowing (Dolmas 1995, p. 322). Tail robustness then identifies the preference relations that have well-defined and finite DU on the domain considered. *Homotheticity* means that preference is invariant under multiplication by a positive constant.

EXAMPLE 7 [Restricting Preferences through Tail Robustness]. Assume that $\alpha > 1$, and $X = I\!R_+$ ($m = 1$). F contains all programs x with $sup_t\{x_t/\alpha^t\} < \infty$. That is, it is A_ω with $\omega_t = \alpha^t$ for all t. The parameter α represents the growth rate of capital in the optimal accumulation model or the growth rate of the endowment in an exchange economy. Assume that all conditions of Theorem 2 are satisfied, so that all programs in F satisfy tail robustness. Assume further that homotheticity holds. Then there exists $s > 0$ such that $u(\beta) = \beta^s$. This follows from Becker & Boyd (1997, p. 81), with s necessarily positive because 0 is contained in u's domain. Further, $\rho\alpha^s < 1$, as follows from finiteness of discounted utility of ω. □

In the above example, tail robustness implies the inequality $\rho\alpha^s < 1$ of Becker & Boyd (1997, Section 3.2, Example 1) and Boyd (1990, Section 4.1). That is, growth (in utility units) is dominated by impatience (discounting); see also Streufert's (1993, p. 83) interpretation of biconvergence. Conversely, the inequality implies tail robustness, so that tail robustness provides a preference foundation for the inequality to hold.

Tail robustness can serve as a tractable tool for constructing a domain of preference on which DU is well-defined and finite. The following observation illustrates this application. It is similar in spirit to domain constructions in Dubra & Ok (2002).

EXAMPLE 8 [Constructing the Preference Domain through Tail Robustness]. Assume that all conditions of Theorem 2 are satisfied. We consider a program x not contained in F, and wonder whether we can incorporate x in the domain of discounted-utility preference by adding an indifference $x \sim (\alpha, \alpha, \ldots)$ for some outcome α, combined with extension through weak ordering. Denote the extended domain as $F' = F \cup \{x\}$. Then the DU representation still holds, with $DU(x)$ well-defined, finite, and equal to $DU(\alpha)$ if and only if x satisfies tail robustness (proved in the appendix). In particular, tail robustness and weak ordering imply all other conditions in Statement (ii) of Theorem 2.

A convenient aspect of the above extension of domain is that we need to verify tail robustness of x only with respect to the ultimately constant programs. That is, tail robustness is satisfied on $F' = F \cup \{x\}$ if and only if it is on $F^{uc} \cup \{x\}$ where F^{uc} denotes the set of ultimately constant programs. This follows immediately from the definition of tail robustness which, apart from the preference of x, involves only ultimately constant programs. □

The example has demonstrated that tail robustness identifies the new programs that can be incorporated. The other preference conditions then automatically follow. To verify whether x can be incorporated into the domain, we only need to relate x to the ultimately constant programs, and need not consider any other unbounded program, which enhances the tractability of the domain construction.

5 Related Literature

The efficiency of Koopmans' intuitive axioms, in particular stationarity, is exceptional. To illustrate it, we sketch the proof of Theorem 2. First, by repeated application, stationarity implies independence of preference from every common first part of programs, not just if related to the first period. Then, because of stationarity, separability of the first two periods implies separability of every pair of consecutive periods. Stationarity also implies separability of the "tail of periods" $\{2, 3, \ldots\}$, and then of every tail of periods $\{t, t+1, \ldots\}$. These separabilities together imply complete separability of all periods (Gorman 1968, Streufert 1995) and, hence, an additively separable evaluation

$$x \mapsto V_1(x_1) + V_2(x_2) + \cdots \ , \tag{4}$$

given appropriate continuity. Other papers in the literature used stronger separability than of merely $\{1, 2\}$, as we do; see Table 1.

Stationarity further implies that preferences over consumptions are the same in each period, so that the component-functions are ordinally the same:

$$V_t = f_t(V_1) \text{ for a strictly increasing } f_t. \tag{5}$$

Several papers in the literature used conditions other than stationarity to obtain Eq. 5; see Table 1. Stationarity further implies that all functions evaluating consumption in different periods order differences in the same way, so that they can be taken proportional, leading to the general period-dependent discounting

$$V_t = \rho_t V_{t-1} \text{ for } \rho_t > 0. \tag{6}$$

Table 1 indicates that several papers used conditions other than stationarity to obtain this implication. For decision under uncertainty, this implication is crucial for the step from state-dependent expected utility to state-independent expected utility. It, thus, is crucial for defining subjective probabilities (Grant & Karni 2005). Stationarity implies, furthermore, constant discounting with $\rho_t = \rho$ independently of t. Several papers first derived general, period-dependent, discounting as in Eq. 6 from separate preference conditions, and then could use weaker versions of stationarity to obtain constant discounting. See Harvey (1986, 1995, (partial) absolute timing preference), Bleichrodt & Gafni (1996, p. 53), Meyer (1976, p. 480), and Wakker (1989, the $\$\alpha$ condition in Statement (ii), p. 88). The partial results in the last column of Table 1, regarding the possibility of unbounded utility, were discussed in Sections 3 and 4.

The consumption sets X in the literature were mostly less general than in this paper. They were: $A_\omega \subset I\!R^m_+$ in Dolmas (1995) and Becker & Boyd (1997 Section 3.3.3); an interval in Harvey (1986, 1995), Meyer (1976), and Streufert (1990); a connected subset of $I\!R^m$ in Koopmans (1960, 1972). For finitely many periods, some papers considered more general consumption sets, but then could not use Gorman's (1968) theorem and had to assume full-force separability. Thus, the consumption set was connected in Fishburn & Edwards (1997), Bleichrodt & Gafni (1996), and Wakker (1989). The consumption set only had to satisfy a solvability condition, which is more general than any of the aforementioned restrictions, in Krantz et al. (1971) for finitely many periods and in Hübner & Suck (1993) for infinitely many periods. Some further restrictions are linear utility in Harvey (1995, Theorem 3.2) and Meyer (1976), and convex preferences over consumption in Becker & Boyd (1997, P3). Fishburn & Edwards (1997) did allow infinitely many periods, but only compared programs that differ in at most finitely many periods (implied by the overtaking criterion) so that, essentially, only finite-dimensional considerations arise. Fishburn & Rubinstein (1982) also considered an infinite set of conceivable periods, but confined its attention to single consumption. Then no issues of intertemporal separability or diverging sums of utility arise, and this study is not easily compared to the other models considered here. Lancaster (1963), who incorporated decisions at different time points considered only single consumptions, as did Fishburn & Rubinstein (1982).

The above papers, scattered in journals in economics, management science, and psychology, and written when internet was not yet available, obtained their extensions of Koopmans' theorem independently. With the exception of Fisburn & Edwards (1997) referring to Harvey (1995) and Streufert (1993), and Becker & Boyd (1997) referring to Streufert (1990) and Dolmas (1995), none of the above papers referred to any of the other contributions for infinitely many periods other than Koopmans'. It is remarkable

Infinitely many periods

	Eq. 4, additivity, through separability of:	Eq. 5, monotonicity, through:	Eq. 6, proportionality, through:	bounds on utility?
Becker & Boyd '97, p. 83	{1}	K4		partly
Dolmas '95, Theorem 2	{1}	A5		partly
Harvey '86, Theorem 4.a	all {t,t+1}	increasingness	equal tradeoffs	
Harvey '86, Theorem 4.b	all {t,t+1}	increasingness		
Harvey '95, Theorem 3.2	all {t,t+1}	increasingness	condition (e)	
Hübner & Suck '93, Theorem 3	all finite and cofinite sets	monotonicity		partly
Koopmans '60, Eq. 47	{1}, {1,3,4,…}			yes
Koopmans '72, Prop. 3	{1}, {2}			yes
Streufert '90, Theorem G	assumed	assumed	assumed	partly
Our paper, Theorem 2				
	Finitely many periods			
Bleichrodt & Gafni '96, Theorem 2.1	all (derived from CCI)	CCI-condition	CCI-condition	
Fishburn '70, Theorem 7.5	all			
Fishburn & Edwards '97, Theorem 3	all			
Krantz et al. '71, Theorem 6.15.ii	all			
Meyer '76, Theorem 9.1	all {t,t+1}	dominance	linearity	
Wakker '89, Theorem IV.4.4	all (derived from CCI)	CCI-condition + Eq. IV.3.1	CCI-condition	

TABLE 1. Extra nontechnical conditions besides those in Theorem 2 (stationarity and separability of {1,2}) assumed in the literature for constant discounted utility.

that mere separability of $\{1, 2\}$ plus stationarity implies all the conditions of the works listed in Table 1. In this comparison we should emphasize that the other papers were not designed to optimally characterize discounted utility, as is the purpose of our paper. The other papers usually sought to generalize Koopmans' discounted utility, or examined aggregation problems, and discounted utility was then obtained only as a by-product.

In Theorem 2, no restriction is imposed on the utility function u. Further, the theorem allows for great flexibility regarding the domain F. The only restriction is that F should contain all ultimately constant programs. F can, for instance, contain all programs with finite discounted utility, as in Harvey (1986), but it can also contain any arbitrary subset of the latter given the restriction mentioned (Corollary 4). For each program x that is not ultimately constant, we can check if it can be added to the preference domain by verifying tail robustness (Example 8). This checking involves, besides x, only ultimately constant programs.

6 Conclusion

Many generalizations and applications of Koopmans' discounted utility have been developed. Preference foundations were then sometimes obtained as a corollary and by-product, which is also how Koopmans original result was obtained. This paper has focused on the preference foundation of discounted utility, the most popular result of Koopmans (1960, 1972), resolving a number of technical problems in his original analysis, and leading to an efficient and general version of Koopmans' result.

We are not aware of any preference condition in the literature that is so appealing and simple to comprehend, and at the same time so powerful in its implications, as Koopmans' stationarity. We hope that this paper can help to clarify and popularize Koopmans' intuitive axioms.

Appendix A. Proofs

The following example prepares for the proof. Similar examples were presented in Streufert (1990) and Becker & Boyd (1997, Section 3.3.2, Example 6). The example shows that tail robustness cannot be omitted in Theorem 2. It may clarify Koopmans' (1972) discussion of his monotonicity postulates P5 and P5$'$.

EXAMPLE 9 The preference domain F consists of all ultimately constant programs. Evaluations are through

$$(x_1, \ldots, x_T, \alpha, \alpha, \ldots) \mapsto \sum_{i=1}^{T} \rho^{i-1} u(x_i) + \rho^T u(\alpha)/(1-\rho) \tag{7}$$

with u continuous and not constant, and $\rho > 1$. For $\rho < 1$ the above evaluation agrees with DU. For $\rho > 1$ as considered here, DU is not defined, but the above evaluation is. To ensure that the above formula is well-defined, we should verify that different ways of splitting up an ultimately constant program into the first part and its constant-tail part lead to the same evaluation. They do because, if $x_{T+1} = \alpha$, then

$$\sum_{i=1}^{T} \rho^{i-1} u(x_i) + \rho^T u(\alpha)/(1-\rho) = \sum_{i=1}^{T+1} \rho^{i-1} u(x_i) + \rho^{T+1} u(\alpha)/(1-\rho),$$

etc. All conditions of Statement (ii) in Theorem 2, except tail robustness, are satisfied. Besides tail robustness, also monotonicity and impatience (preference for early receipt of outcomes with higher u value) are violated. We obtain an evaluation of α equal to $u(\alpha)+\rho u(\alpha)+\rho^2 u(\alpha)/(1-\rho) = u(\alpha)/(1-\rho)$. It is not increasing, but decreasing, in $u(\alpha)$ for $\rho > 1$. Preferences satisfy a weaker monotonicity condition, i.e. *finite monotonicity*, which means that replacing any finite number of consumptions x_t by other consumptions with higher u value always improves the program. By sensitivity of period 1, u is not constant; and there are $u(\gamma) > u(\beta)$. Then $\gamma \prec \beta$. This preference and $\gamma x \succ \beta x$ violate monotonicity. It is like preferring cake to bread for any finite number of days, but preferring bread to cake for an infinite lifetime. Tail robustness would require that $\gamma_T\beta \prec \beta$ should hold for all T sufficiently large. However, by finite monotonicity, $\gamma_T\beta \succ \beta$ for all T.

Even if utility of consumption is bounded, the evaluation of programs is not (cf. $\gamma_T\beta$ for T increasing). There still does not exist a best program, so that Koopmans' (1972) Postulate P5$'$ is still violated.

This example is an alternative to the example of Burness (1976, p. 505), who showed that the very existence of a utility indicator need not imply impatience. Our example revealed a more fundamental problem, i.e. a violation of monotonicity. By imposing the supremum norm, all conditions of Burness (1976) are satisfied. □

PROOF OF THEOREM 2. The following proof and, consequently, Theorem 2, are valid for every topologically separable[2] and arc-connected consumption space X. For each x, $DU(x) = \sum_{t=1}^{\infty} \rho^{t-1} u(x_t)$ whenever defined. First assume that Statement (i) holds. Tail robustness follows because of convergence of the summations in $DU(x)$. There exists a constant-equivalent for each x, with utility strictly between $limsup(u(x_t))$ and $liminf(u(x_t))$ if these two are different, mainly because of connectedness of $u(X)$. All other conditions in Statement (ii) easily follow.

For the reversed implication, we assume henceforth that Statement (ii) holds. We fix an arbitrary outcome θ throughout, at which many functions below will be normalized to be 0. Initial-tradeoff independence amounts to separability of the set of periods $\{1, 2\}$, with a preference $\alpha\beta x \succcurlyeq \gamma\delta x$ independent of x. By stationarity, it implies that a preference $\mu\alpha\beta x \succcurlyeq \mu\gamma\delta x$ is independent of μ and x, i.e. $\{2, 3\}$ is separable. Similarly, by repeated application of stationarity, separability of all sets $\{i, i+1\}$ follows. Stationarity implies that a preference $\alpha x \succcurlyeq \alpha y$ is the same as between x and y and, hence, is independent of α, so that separability of $\{2, 3, \ldots\}$ follows. Similarly as above, stationarity then implies separability of $\{3, 4, \ldots\}$, and then of all "tail" sets $\{i, i+1, \ldots\}$. Similarly, sensitivity of period 1 and stationarity imply sensitivity of all periods t.

Consider a period $T > 1$. By the separabilities just established, and the other conditions such as sensitivity of at least three periods (in fact all) and continuity, Gorman (1968) implies that we have complete separability (i.e., separability of all subsets of components) over every $T + 1$ dimensional space X_T, and an additive evaluation

$$(x_1, \ldots, x_T, \alpha, \alpha, \ldots) \mapsto V_{1,T}(x_1) + \cdots + V_{T,T}(x_T) + R_T(\alpha) \tag{8}$$

on each such set, where all functions used are continuous. We may set $V_{j,T}(\theta) = 0 = R_T(\theta)$ for all j. The function

$$(x_1, \ldots, x_T, \alpha, \alpha, \ldots) \mapsto V_{1,T+1}(x_1) + \cdots + V_{T,T+1}(x_T) + V_{T+1,T+1}(\alpha) + R_{T+1}(\alpha),$$

obtained from X_{T+1}, is an alternative additive evaluation over X_T. By the usual uniqueness results of additive evaluations, we may, inductively with respect to T, set

$$\text{for all } T, \text{ and all } j \leq T, V_{j,T+1} = V_{j,T} \text{ and } V_{T+1,T+1}(\alpha) + R_{T+1}(\alpha) = R_T(\alpha). \tag{9}$$

[2] We add the adjective "topological" throughout to distinguish this condition, not defined here, from the separability preference condition.

From the first equality it follows that $V_{j,T}$ is independent of T for $j \leq T$, and we can drop the subscript T.

By stationarity and consideration of programs γx for a fixed γ, $V_2(x_1) + \ldots + V_{T+1}(x_T) + R_{T+1}(\alpha)$ evaluates the same preferences over X_T as does $V_1(x_1) + \ldots + V_T(x_T) + R_T(\alpha)$. By the usual uniqueness results, there exists a $\rho_T > 0$ such that $V_{j+1} = \rho_T V_j$ for all $j \leq T$ and

$$R_{T+1} = \rho_T R_T. \tag{10}$$

$\rho_T = V_2/V_1$ is independent of T because V_1 and V_2 are, and we drop the subscript T. Writing $u = V_1$ and $R = R_2/\rho^2$, we have obtained an evaluation

$$(x_1, \ldots, x_T, \alpha, \alpha, \ldots) \mapsto \sum_{i=1}^{T} \rho^{i-1} u(x_i) + \rho^T R(\alpha) \tag{11}$$

with u continuous. Note that consumptions are ordered the same for every period, which comprises most of monotonicity. The last equality in Eq. 9 implies that $\rho^T R(\alpha) = \rho^T u(\alpha) + \rho^{T+1} R(\alpha)$, or

$$R(\alpha) = u(\alpha) + \rho R(\alpha). \tag{12}$$

$\rho = 1$ cannot be: By Eq. 12, then $u(\alpha) = 0$ for all α. Constantness of u violates sensitivity of period 1, and $\rho = 1$ cannot be indeed. Hence,

$$R = u/(1 - \rho). \tag{13}$$

$\rho > 1$ cannot be either because it violates tail robustness (and monotonicity); see Example 9. We conclude that, besides $0 < \rho$, also $\rho < 1$. Eq. 13 implies $R(\alpha) = \sum_{i=1}^{\infty} \rho^{i-1} u(\alpha)$. Substituting this in Eq. 11 yields discounted utility for the ultimately constant programs.

To extend the evaluation to general, possibly unbounded, programs, consider a general program x and its constant-equivalent α. If there exists a consumption β with $x \sim \alpha \succ \beta$ then, by tail robustness, $x_T\beta \succ \beta$ for all T sufficiently large. By the evaluation for ultimately constant programs, $DU(x_T\beta) > DU(\beta)$ for all T sufficiently large. Hence, $liminf_T(\sum_{t=1}^{T} \rho^{t-1} u(x_t)) \geq DU(\beta)$. Because this holds for all $\beta \prec \alpha$, the $liminf$ also is not less than $DU(\alpha)$. If there exists no β as above, then α is the worst consumption, and by the DU representation and its implication of monotonicity on the ultimately constant programs, the above $liminf$ again is not less than $DU(\alpha)$. Similarly,

$limsup_T \sum_{t=1}^{T}(\rho^{t-1}u(x_t))$ is not more than $DU(\alpha)$. It follows that $DU(x) = DU(\alpha)$ and, hence, $DU(x)$ evaluates x.

Note that we nowhere made any assumption about which of the not ultimately constant programs are contained in F, and complete flexibility of domain has been maintained. By standard uniqueness results for additive evaluations (Gorman 1968), the additive evaluations such as in Eq. 8 have the functions $V_{j,T} = \rho^{j-1}u$ unique up to level and common unit. This implies the uniquess result regarding u and $\rho = (V_2(.) - V_2(\theta))/(V_1(.) - V_1(\theta))$. Our result is a special case of an additively decomposable representation on an infinite product space. General results on this topic are in Streufert (1995) and Wakker & Zank (1999).

□

PROOF OF OBSERVATION 3. Monotonicity follows from the discounted utility representation. For ultimately constant programs, constant-equivalence was not used in the proof of Theorem 2, and tail-robustness was used only to show that $\rho > 1$ cannot be. The latter is also excluded by monotonicity, as indicated in the proof of Theorem 2; see also Example 9. To show that tail robustness can be replaced by monotonicity for bounded programs, assume that $\mu \succcurlyeq x_t \succcurlyeq \nu$ for all t, and that α is the constant-equivalent of x. Then, by monotonicity, $x_T\mu \succcurlyeq x \sim \alpha \succcurlyeq x_T\nu$ for all T. By discounted utility for ultimately constant programs, $DU(x_T\mu) \geq DU(\alpha) \geq DU(x_T\nu)$ for all T. Because $DU(x_T\mu)$ and $DU(x_T\nu)$ converge to each other, they converge to $DU(\alpha)$. They also converge to $DU(x)$, which, hence, is equal to $DU(\alpha)$ and evaluates x. □

PROOF OF EXAMPLE 8. That extendability of the representation implies tail robustness of x follows immediately from the implication (i) $\Rightarrow$ (ii) in Theorem 2. Therefore, we assume x and α as in the example with x tail robust, and we show that the discounted utility representation can be extended to x, in which proof we use, besides x, only ultimately constant programs.

If we had, for contradiction, $DU(\alpha) > liminf_T \sum_{t=0}^{T} \rho^{t-1}u(x_t)$, then we could take an outcome β with DU value strictly between $DU(\alpha)$ and the $liminf$. Then, by $x \sim \alpha \succ \beta$ and tail robustness, $\sum_{t=0}^{T} \rho^{t-1}u(x_t) + \sum_{t=T+1}^{\infty} \rho^{t-1}u(\beta) > DU(\beta)$ for all T sufficiently large. It implies $\sum_{t=0}^{T} \rho^{t-1}u(x_t) > \sum_{t=0}^{T} \rho^{t-1}u(\beta)$ for all T sufficiently large and, thus,

$liminf_T \sum_{t=0}^{T} \rho^{t-1} u(x_t) \geq DU(\beta)$. This contradicts that $DU(\beta)$ is strictly between $DU(\alpha)$ and the $liminf$. $\square$

Appendix B. Example

For general domains with all programs incorporated, it is not very clear how preference relations can be constructed that satisfy all the axioms of Koopmans (1960). The following example illustrates for a natural case that the axioms of Koopmans, also without his axioms P5 or P5$'$, imply that utility must be bounded.

EXAMPLE 10 Assume that $X = I\!R_+$, the set of nonnegative monetary outcomes. Assume, as in Koopmans (1960), that $\succcurlyeq$ is defined on the whole set of programs $\Pi_{i=1}^{\infty} X$, and that there exists a function $U : \Pi_{i=1}^{\infty} X \to I\!R$ that evalutes $\succcurlyeq$, i.e. a consumption program is preferred if and only if it has the higher U value, where further U satisfies *uniform continuity on equivalence classes* with respect to the supnorm. That is, for each program x and each $\epsilon > 0$ there exists $\delta > 0$ such that $|U(y) - U(x)| < \epsilon$ as soon as the supnorm distance of y to the equivalence-class of x is less than δ (his Postulate 1). Koopmans also assumed that the range of U is an interval I_U (3* in Section 3), using this assumption heavily throughout his analysis. We further assume that discounted utility applies to the set F of all programs for which discounted utility is finite, with u continuous and strictly increasing. F^c, the complement of F, contains all programs with infinite discounted utility. Because for every program $z \in F^c$ and $x \in F$ we can make a program $y \in F$ with z dominating y in every period and $y \sim x$, by monotonicity and transitivity every program in F^c must be strictly preferred to every one in F. Hence, $U(F^c)$ exceeds $U(F)$. We conclude that $U(F)$ and $U(F^c)$ partition the interval I_U where $U(F)$ comprises the lower part and $U(F^c)$ the upper part.

Take $x \in F$. Then we can strictly improve x_1, leading to a strictly better program. Apparently $U(F)$ does not contain its upper bound. Next take $x \in F^c$. There must be a t with $x_t > 0$. Then we can strictly lower x_t some, leading to a strictly worse program. Apparently $U(F^c)$ does not contain its lower bound. By connectedness, either $U(F)$ or $U(F^c)$ is empty, so that either F or F^c is empty. F contains all constant programs and, therefore, F^c is empty. This implies that u must be bounded. If u were unbounded, we could construct x with $u(x_t) > 1/\rho^{t-1}$ in F^c so that x has infinite discounted utility, contradicting emptiness of F^c. □

Section 8* of Koopmans, Diamond, & Williamson (1964) suggested that arc-connectedness together with a "finite interior diameter" of X implies boundedness of utility. The above example suggests that connectedness alone already is irreconcilable with unbounded utility.

The status of Koopmans' boundedness condition (Koopmans 1960, Postulate P5; Koopmans 1972, Postulate P5$'$) is not clear. Possibly he had in mind that all programs preferred more than a prespecified best program, and programs preferred less than a prespecified worst program, be dropped from the domain, and that all other programs be retained. Such an approach does not work well because then additive representation theorems of Debreu (1960) or Gorman (1968), used in the proofs, can no longer be applied. About the latter point there have been many misunderstandings in the literature (Wakker 1993b). The only way to have a best and worst program, and at the same time have the full product structures required for the theorems of Debreu (1960) and Gorman (1968), is to have a best and worst consumption, and let the best and worst programs correspond with these consumptions in all periods. Then all pograms can be included, and the domain contains enough subspaces isomorphic to full product spaces to apply known proof techniques. Utility then has to be bounded automatically, which explains claims about Koopmans' analysis with P5/P5$'$ assumed in the main text.

Koopmans (1972, Section 6*) briefly discussed the extension of his results to unbounded progams with finite discounted utility. There are however, apparently, typos in his text, to the extent that we are unable to guess what he may have had in mind.

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