A Preference Foundation for Rank-Additive Utility

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ABSTRACT. Many traditional conjoint representations are based on additive decomposability. An important generalization arises under rank-dependence, when such representations are restricted to cones with a fixed ranking of components from best to worst, leading to configural weighting, rank-dependent utility, and rank- and sign-dependent prospect theory. A new paradigm for representations was developed by Duncan Luce and others, allowing for basic rationality violations regarding the coalescing of events and other framing assumptions. In recent papers, Luce's approach called for a new, rank-additive, version of rank-dependent representations, where additive representations on different cones should be combined into one overall representation. This paper provides a preference foundation of rank-dependent additive utility. Thus, a complete preference foundation can be obtained of the recent models by Luce and others.

Key Words: rank-dependence, rank-additive utility, coalescing, joint independence

1. Introduction

Duncan Luce developed, jointly with several co-authors, an innovative paradigm for decision under uncertainty. Luce (2000) presented a complete description of this paradigm, with a short accessible account in Luce (1990). His paradigm deviates from the commonly used Savagean (1954) paradigm in several respects, and provides sophisticated models that can account for basic violations of rationality. The importance of modeling such violations has become increasingly understood during the last 20 years. Several recent papers by Luce et al. on preference foundations take a so-called rank-additive utility (RAU) model as point of departure (Luce & Marley 2005; Marley & Luce 2005; Marley, Luce, & Kocsis 2007). We refer to Luce et al. to designate this line of research. Their models also underly the RAM and TAX models developed by Birnbaum and his colleagues (Birnbaum 2007 and the references therein).

The preference foundation of the RAU model has as yet remained an open problem (Luce & Marley 2005, Section 2.1). Thus, the models of Luce et al. have not yet received complete preference foundations. This paper provides a preference foundation of RAU and, thereby, completes the preference foundation of the models by Luce et al.

2. Assumptions

$\mathcal{C}$ denotes a set of *outcomes*, and $\succcurlyeq$ a binary relation on $\mathcal{C}$. Π denotes a set of *partitions*. In works by Luce et al., a partition π is an ordered set (an array) of mutually exclusive events, where events can be logical statements or subsets of a universal event. The events need not be exhaustive, with the union of π's elements a nonuniversal event upon which decisions are conditioned. A partition is also called an experiment. In this paper, a partition can be any ordered index set containing finitely many, 2 or more, elements. For each π, $n(\pi) \geq 2$ denotes its number of elements. X contains $\mathcal{C}$ and all *rank-ordered* tuples of the form $x = (\pi, x_1, \ldots, x_{n(\pi)})$ with $x_1 \succcurlyeq \cdots \succcurlyeq x_{n(\pi)}$. We call x *$\pi$-related*. We will later give results for the case where this rank-ordering requirement is dropped, and X contains $\mathcal{C}$ and all tuples of the form $x = (\pi, x_1, \ldots, x_{n(\pi)})$.

In Luce et al.'s approach, $(\pi, x_1, \ldots, x_{n(\pi)})$ designates a gamble yielding outcome x_j if the j^{th} event of π is true. Formally, $(\pi, x_1, \ldots, x_{n(\pi)})$ is not a function from the partition to X as in the classical Savagean approach (1954), but it is treated as a general $n(\pi)+1$ tuple. Thus, Luce et al. allow for basic violations of rationality. For instance, for disjoint events A,B,C, $((A \cup B, C), \$100, \$0)$ can be treated differently than $((A,B,C), \$100, \$100, \$0)$. In this manner, violations of "coalescing" can be considered. In the Savagean approach, with events subsets of a set called state space, both objects are defined to be functions from $A \cup B \cup C$, assigning value \$100 to all elements of A and B, and \$0 to all elements of C. Then the two objects are identical by definition and there is no possibility to distinguish between them. For empirical studies of violations of coalescing, the generality of Luce et al.'s model is needed. To allow for the generality of Luce et al.'s approach, this paper treats partitions as general objects without restrictions imposed on them. Many interpretations are possible. Partitions could designate ordered sets of persons (or time points), with $(\pi, x_1, \ldots, x_{n(\pi)})$ designating allocations of money over these persons (or consumption on these time points) and with the implicit assumption that all persons not listed in π receive nothing (or no consumption on other time points). Partitions could be ordered sets of properties of persons such as kindness, honesty, age, and the x-s could designate scores regarding these properties.

We assume that $\succcurlyeq$ on $\mathcal{C}$ is extended to the whole set X and, for simplicity, denote the extension also by $\succcurlyeq$. The notation $\succ$, $\preccurlyeq$, $\prec$, and $\sim$ is as usual. We assume that *strong monotonicity* holds, implying that $(\pi, x_1, \ldots, x_{n(\pi)}) \succ (\pi, y_1, \ldots, y_{n(\pi)})$ whenever $x_j \succcurlyeq y_j$ for all j and $x_j \succ y_j$ for at least one j. In Luce et al.'s approach, strong monotonicity implies that null events are suppressed.

We assume that for each $x \in X$ there exists a *certainty equivalent* $\alpha \in \mathcal{C}$, defined by $\alpha \sim x$. *Idempotence*, requiring that $(\pi, \alpha, \ldots, \alpha) \sim \alpha$, is a natural assumption in some applications but Luce et al. also considered generalizations and, therefore, we do not assume it here. For example, if partitions contain mutually exclusive but not necessarily exhaustive events as in Luce's approach, and $(\pi, x_1, \ldots, x_{n(\pi)})$ designates a gamble conditional on the information that the event occurring is an element of the partition π, then idempotence is a natural condition. If, however, $(\pi,x_1,\ldots,x_{n(\pi)})$ designates a gamble with the implicit assumption that the outcome received is 0 for all events not contained in π, then idempotence is not a natural assumption.

To avoid cases of degeneracy, large cardinality of the equivalence classes in $\mathcal{C}$, and cases where different π's have no overlapping indifference classes so that they are unrelated, we assume that there exists an outcome α^0 such that for each π there is a π-related x, nonmaximal in the π-related n-tuples, with $x \sim \alpha^0$.[1] In the papers by Luce et al., α^0 can be what is called the neutral outcome.

We further assume that there exists a function $U: X \to \mathbb{R}$ that *represents* $\succcurlyeq$, i.e. $\succcurlyeq$ maximizes U. Hence, $\succcurlyeq$ is a *weak order*, meaning that it is *complete* ($x \succcurlyeq y$ or $y \succcurlyeq x$ for all $x,y \in X$) and transitive. We assume that for each fixed π there exist functions $V_1(\pi,.), \ldots, V_{n(\pi)}(\pi,.)$ from $\mathcal{C}$ to $\mathbb{R}$, and a function V_π on the π-related x such that

$$x = (\pi,x_1,\ldots,x_{n(\pi)}) \mapsto V_1(\pi,x_1) + \cdots + V_{n(\pi)}(\pi,x_{n(\pi)}) = V_\pi(x) \qquad (2.1)$$

represents $\succcurlyeq$ when restricted to the π-related x. Thus, there exist strictly increasing functions L_π such that

$$U(x) = L_\pi\big(V_\pi(x)\big) \qquad (2.2)$$

on the relevant domain. For each outcome $\alpha \in \mathcal{C}$ and partition π for which there exists a π-related x such that $\alpha \sim x$, we write $V_\pi(\alpha) = V_\pi(x)$. Thus, we have extended the domain of V_π to a part of $\mathcal{C}$. Eq. 2.2 continues to hold for this extension of V_π.

By strong monotonicity, every function V_j is strictly increasing in $\succcurlyeq$ in the usual sense. We further assume that the range of each function defined so far in this paper is a nondegenerate interval. It implies that all functions L_π have no "jumps" so that they must be continuous. We then also have connected preference topologies, and for each π preference is continuous w.r.t. the product topology on the π-related tuples.

In the preference foundation of rank-additive utility provided below, we will assume all conditions of this section. To obtain a complete preference foundation, a preference foundation of the assumptions of this section should be provided. The

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[1] This condition is satisfied under idempotence plus nontriviality (at least two nonindifferent objects).

existence of U with an interval as image is characterized by separability and connectedness of the order topology (Debreu 1964), a condition that in view of the existence of certainty equivalents needs to be imposed only on $\mathcal{C}$. For Eq. 2.1 and the assumptions about the functions therein, a preference foundation is in Wakker (1993), with generalizations in Chateauneuf & Wakker (1993). For brevity, we will not repeat them here, but refer the reader to those works.

It is easy to see that in Eq. 2.2 we can choose any real constants $\tau_1, \ldots, \tau_{n(\pi)}$, and any positive $\sigma > 0$, and then replace every $V_j(\pi,.)$ by $\tau_j + \sigma V_j(\pi,.)$. It can also be proved that this is the only freedom we have for this representation, so that the functions $V_j(\pi,.)$ are unique up to level and joint scale (Wakker 1993). The functions $V_1(\pi,.), \ldots, V_{n(\pi)}(\pi,.)$ are *joint interval scales*. The function $V_\pi(x)$ is an interval scale, being unique up to the level $\tau = \tau_1 + \cdots + \tau_{n(\pi)}$ and the scale σ.

3. Rank-Additive Utility

We are interested in the special case of Eq. 2.2 where the ordinal transformations L_π can be dropped.

DEFINITION 3.1. *Rank-additive utility* (*RAU*) holds if all functions L_π are the identity, so that we have

$$U(x) = V_1(\pi,x_1) + \cdots + V_{n(\pi)}(\pi,x_{n(\pi)}) = V_\pi(x). \qquad (3.1)$$

□

We provide a preference foundation of RAU. It will imply that all functions V_π coincide on common subdomains of $\mathcal{C}$.

Our preference foundation will be based on a variation of the tradeoff technique of Köbberling & Wakker (2003, 2004). A natural way to obtain a preference foundation for a decision model arises from considering ways to elicit the subjective quantities used in the model[2] from preferences in a parameter-free deterministic model, and then excluding inconsistencies in such measurements. We will next explain how U in Eq. 3.1 can be elicited from preferences.

For $x = (\pi, x_1, \ldots, x_{n(\pi)})$, $i \le n(\pi)$, and $\mu \in \mathcal{C}$, $\mu_i x$ denotes $(\pi, x_1, \ldots, x_{n(\pi)})$ with x_i replaced by μ. It is implicit in this notation that the replacement respects rank-ordering, so that $x_{i-1} \succcurlyeq \mu \succcurlyeq x_{i+1}$. U on $\mathcal{C}$ can be elicited from observations of the following kind:

$$\alpha \sim \mu_i x, \qquad \gamma \sim \mu_i y,$$
$$\beta \sim \nu_i x, \qquad \delta \sim \nu_i y. \tag{3.2}$$

We write $\alpha\beta \sim^* \gamma\delta$ if there exists a partition π, π-related x and y, outcomes μ and ν, and an index i such that Eq. 3.2 holds. In this notation we deliberately “forget” the partition π. The main point of the following discussion in fact amounts to establishing that this notation is useful. The following lemma shows how $\sim^*$ observations serve to measure V_π and, hence, U on $\mathcal{C}$ if RAU holds.

LEMMA 3.2. Assume $\alpha\beta \sim^* \gamma\delta$ as in Eq. 3.2, with π as specified there. Then $V_\pi(\alpha) - V_\pi(\beta) = V_\pi(\gamma) - V_\pi(\delta)$.

PROOF. By Eq. 2.1 both differences equal $V_i(\pi,\mu) - V_i(\pi,\nu)$. □

For the measurement of continuous monotonic interval scales on interval domains all that we need to observe is equalities of differences. For instance, if RAU holds, then we can scale $U(\alpha^0) = 0$, $U(\alpha^1) = 1$ for some arbitrary $\alpha^1 \succ \alpha^0$, and then a number of elicitations $\alpha^{z+1}\alpha^z \sim^* \alpha^z\alpha^{z-1}$ reveals $U(\alpha^z) = z$ for all integers z. Such measurements result, for instance, from pairs of indifferences

$$\alpha^{j+1} \sim \mu_i x, \qquad \alpha^j \sim \nu_i y$$

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[2] Such subjective quantities are, for instance, subjective probabilities and (subjective) utilities in subjective expected utility. For the RAU model they concern the various functions in Eq. 3.1.

where m such pairs of indifferences give m–1 elicitations $\alpha^{z+1}\alpha^{z} \sim^* \alpha^{z}\alpha^{z-1}$ and equalities of U differences. In this measurement, the μ-ν difference on the i^{th} coordinate of the partition has served as a gauge to peg out the "standard sequence" of the α^{j}'s that is equally spaced in U units. More refined measurements result from a number of elicitations $\beta^{z+1}\beta^{z} \sim^* \beta^{z}\beta^{z-1}$ with $\beta^{0} = \alpha^{0}$ and $\beta^{m} = \alpha^{1}$, which implies that $U(\beta^{z}) = z/m$ for all integers z.

RAU is obviously violated if the aforementioned measurements run into inconsistencies. If, for example, one partition π were to imply $\alpha\beta \sim^* \gamma\delta$, and another partition π´ were to imply $\alpha'\beta \sim^* \gamma\delta$ for an $\alpha' \succ \alpha$, then the implied $U(\alpha') - U(\beta) = U(\gamma) - U(\delta) = U(\alpha) - U(\beta)$ contradicts $U(\alpha') > U(\alpha)$, and RAU is violated in a deterministic model. A necessary condition for RAU is, consequently, that such violations be excluded. Similarly, we should not be able to improve one of β, γ, or δ above without breaking the relationship. As we will see, it suffices to exclude such inconsistencies for the special case of Eq. 3.2 with $\beta = \gamma$ (endogenous midpoint observations). Indeed, the sequence $\alpha^{z+1}\alpha^{z} \sim^* \alpha^{z}\alpha^{z-1}$ above concerned this special case.

DEFINITION 3.3. Under the assumptions of Section 2, *RAU-tradeoff consistency* holds if strictly improving an outcome in any $\alpha\beta \sim^* \beta\gamma$ breaks that relationship. □

Tradeoff consistency implies that standard sequences such as the α^{j} and β^{j} above will be consistent across different partitions π. It is similar to the standard sequence invariance condition of Krantz et al. (1971, §6.11.2).

THEOREM 3.4. RAU holds if and only if RAU-tradeoff consistency holds. □

The same result holds if we have additive representations as in Eq. 2.1 not only on rank-ordered sets but on full product sets. This follows as a corollary of Theorem 3.4, because full product sets are unions of rank-ordered sets.

[3] The notation ~*, with its "forgetting" of π, have then served to falsify RAU, but cannot be used to measure utility differences as in Lemma 3.2.

COROLLARY 3.5. If the domain of preference consists of $\mathcal{C}$ and all tuples of the form $x = (\pi,x_1,\ldots,x_{n(\pi)})$ without the restriction that $x_1 \succcurlyeq \cdots \succcurlyeq x_{n(\pi)}$, then still RAU holds if and only if RAU-tradeoff consistency holds. □

Further generalizations can be obtained. The set providing certainty equivalents need not be the same as the set of outcomes for gambles, and the outcome sets of gambles can depend on the partitions and events. Also more general domains can be considered. The main requirement is that these domains provide sufficient local richness to construct preference-neighborhoods of outcomes as in the proof below, where we can use the midpoint measurement of Eq. 3.2. Corollary 4.1 gives details.

4. Proof of Theorem 3.4

We demonstrated in the main text that RAU-tradeoff consistency is a necessary condition for the RAU model. We, henceforth, assume the condition and demonstrate that the RAU model is implied. Because the V_π's are interval scales, it will suffice to reduce the L_π functions to strictly increasing affine functions.

For every partition π, define $\mathcal{C}_\pi$ as the set of outcomes $\{\alpha\in\mathcal{C}$: there exists a π-related x with $x \sim \alpha\}$. In other words, $\mathcal{C}_\pi$ is the domain of V_π (in its extended sense) intersected with $\mathcal{C}$. Because the ranges of all functions are intervals, $\mathcal{C}_\pi$ is a preference interval in the sense that if it contains two outcomes, then it contains all outcomes in between. α^0 is contained in each $\mathcal{C}_\pi$. For every partition, we can choose the levels of the representations such that $V_\pi(\alpha^0) = 0$ because the representations in Eq. 2.1 are joint interval scales, and so we do.

Take any fixed partition π_f. For each $\alpha \in \mathcal{C}_{\pi_f}$, define $U(\alpha) = V_{\pi_f}(\alpha)$. Consider an arbitrary other partition π. Because $\mathcal{C}_\pi$ and $\mathcal{C}_{\pi_f}$ both contain α^0, both contain a strictly preferred outcome, and both are preference intervals, there is an outcome $\alpha_\pi \succ \alpha^0$ contained in both sets. Because V_π is an interval scale, we can choose its scale such that $V_\pi(\alpha_\pi) = V_{\pi_f}(\alpha_\pi)$ and so we do for each partition π.

We now compare two partitions π and π'. For each outcome λ in $\mathcal{C}_\pi \cap \mathcal{C}_{\pi'}$ that is neither minimal nor maximal in this set, we can find $\sigma \succ \lambda \succ \tau$ so close to λ, and an i, such that, for all outcomes α, β between σ and τ, we can have

$$\alpha \sim \mu_i x, \qquad \beta \sim \mu_i y,$$
$$\beta \sim \nu_i x,$$

for properly chosen π-related prospects, where also $\nu_i y$ is π-related. (4.1)

For the certainty equivalent γ of the latter prospect $\nu_i y$ (irrespective of whether γ is between σ and τ or not; we will only use the case where it is between), we have $\alpha\beta \sim^* \beta\gamma$ and, by Lemma 3.2, β is the V_π midpoint between α and γ. In this manner, for all α, β, γ between σ and τ such that β is the V_π midpoint of α and γ, we can construct the configuration of Eq. 3.2 with β for γ and γ for δ.

Imagine that we similarly have Eq. 4.1 satisfied for π' with respect to the same outcome λ. That is, we have $\sigma' \succ \lambda \succ \tau'$ so close to λ, and a j, such that for all outcomes α, β between σ' and τ', we can have

$$\alpha \sim \mu_j x', \qquad \beta \sim \mu_j y',$$
$$\beta \sim \nu_j x',$$

for properly chosen π'-related prospects, where also $\nu_i y'$ is π'-related. For the certainty equivalent γ of the latter prospect $\nu_i y'$ we have $\alpha\beta \sim^* \beta\gamma$ and β is the $V_{\pi'}$ midpoint between α and γ. In this manner, surely for all α, β, γ between σ' and τ' such that β is the $V_{\pi'}$ midpoint of α and γ, we can construct the configuration of Eq. 3.2 with $\gamma = \beta$ and $\delta = \gamma$.

Instead of σ and σ' we can take their minimum, and instead of τ and τ' we can take their maximum. That is, we can take $\sigma = \sigma'$ and $\tau = \tau'$. Then, by RAU tradeoff consistency, for all α, β, and γ between σ and τ, if β is a V_π-midpoint of α and γ, it must also be a $V_{\pi'}$ midpoint. (Sets of midpoints are, obviously, ~ equivalence classes.)

V_π and $V_{\pi'}$ are interval scales such that for each nonmaximal and nonminimal element in their common domain there is an open preference-neighborhood within which they have the same midpoints. It implies that the strictly increasing transformation that relates V_π and $V_{\pi'}$ on their common domain must have second derivative 0, so that it must be affine, which by continuity extends to the maximal and minimal outcomes in their common domain. Because V_π and $V_{\pi'}$ coincide with V_f at α^0 and at points strictly preferred to but close to α^0, they agree with each other at two or more points, so that they must be identical on their common domain. In this manner, all functions V_π coincide on their common domains, and they can be written as one function U. This function obviously represents preference on $\mathcal{C}$ and, hence, on X. The following corollary summarizes what was needed in the proof. In the definition of a rank-ordered set we used that the set of certainty equivalents and the sets of event-contingent outcomes are all the same. Ways to relax this point are also given in the corollary.

COROLLARY 4.1. The set of certainty equivalents and the domains of the functions $V_j(\pi,.)$ can all be different, and more general domains than rank-ordered sets can be considered. All that is needed is that all ranges of functions are intervals so that all functions are interval scales, and that we have the local richness of Eq. 4.1. □

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