Discount functions for fitting individual data

Han Bleichrodt, Kirsten I.M. Rohde, and Peter P. Wakker

Department of Economics, Erasmus University,

Rotterdam, The Netherlands

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Address for correspondence:

Peter P. Wakker

Econometric Institute

Erasmus University

P.O. Box 1738

3000 DR Rotterdam

The Netherlands

31-(0)10 - 408.12.65 (O)

31-(0)10 - 408.91.62 (F)

Wakker@few.eur.nl

Homepage: http:/people.few.eur.nl/wakker/index.html

Abstract

The commonly used hyperbolic and quasi-hyperbolic discount functions imply decreasing impatience, which is the prevailing empirical phenomenon in intertemporal choice, in particular for aggregate behavior. At the individual level there is much variation, however, and there will always be some individuals who exhibit increasing impatience. Hence, to fit data at the individual level, new discount functions are needed. This paper introduces such functions, with constant absolute (CADI) or constant relative (CRDI) decreasing impatience. These functions can accommodate any degree of decreasing or increasing impatience, which makes them sufficiently flexible for analyses at the individual level. The CADI and CRDI discount functions are the analogs of the well known CARA and CRRA utility functions for decision under risk.

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1 Introduction

Under stationarity, indifference between a small outcome received soon and a large outcome received later is preserved if both dates of receipt are delayed by a common amount. Stationarity reflects constant impatience, and is equivalent to time consistency under common assumptions. Empirical studies have found that stationarity is usually violated (Frederick, Loewenstein, and O'Donoghue 2002), with impatience mostly decreasing and not constant.

That is, delaying the aforementioned outcomes makes the decision maker less impatient and more willing to wait for the (large and) late outcome. Thus, the indifference turns into a preference for the late outcome, and stationarity is violated.

Generalized hyperbolic (Loewenstein and Prelec 1992) and quasi-hyperbolic (Phelps and Pollak 1968; Laibson 1997) discount functions were introduced so as to accommodate decreasing impatience. A drawback is that these functions can accommodate neither increasing impatience nor strongly decreasing impatience, as we will demonstrate later. Even though these restrictions may not be very problematic when fitting aggregate data concerning many individuals, they make it impossible to fit data at the individual level. To clarify this point, we first note that there is much variation at the individual level (Harrison, Lau, and Williams, 2002; Barsky, Juster, Kimball, and Shapiro, 1997; Tversky and Kahneman, 1992). Even if a minority, still a significant number of individuals will exhibit increasing impatience, or stronger decreasing impatience than the existing models can accommodate. For example, in Abdellaoui, Attema, & Bleichrodt (2007), 75% of the subjects exhibited decreasing impatience, 1% exhibited constant impatience, and 24% exhibited increasing impatience. If we were to use the currently existing discount models, then we would have to remove the latter subjects from the data set, which would generate an unacceptable bias. Hence, by the current state of the art, no quantitative analyses of intertemporal choice are possible at the individual level. The impossibility to fit data at the individual level is particularly undesirable in view of recent advancements in neuroeconomics, where typically only a few individuals can be analyzed.

To further clarify the above points, we compare intertemporal choice to choice under risk. Even while risk aversion is the prevailing empirical phenomenon, and is the common finding for aggregate behavior, at the individual level always some risk seekers are found.

Hence, to fit data at the individual level we need utility functions that can also accommodate risk seeking. The CARA (constant absolute risk averse) and CRRA (constant relative risk averse) functions can do so, and these functions have accordingly been widely used. Without utilities such as CARA and CRRA available, no quantitative analysis of risk attitude at the individual level would be possible. Such a problematic situation exists at present for intertemporal choice.

Recent empirical studies have suggested that, even at the aggregate level, hyperbolic and quasi-hyperbolic discounting have less explanatory power than was originally assumed. Indeed, Attema, Bleichrodt, Rohde, and Wakker (2006), Gigliotti and Sopher (2004), Read, Airoldi, and Loewenstein (2005), and Sayman and Öncüler (2006) found increasing impatience rather than decreasing impatience at the aggregate level. Onay and Öncüler (2007) and Chesson and Viscusi (2003) even found concave discounting, implying strongly increasing impatience. Hence, even at the aggregate level there is an interest in discount functions that can accommodate increasing impatience. Read's (2001) work casts further doubt on traditional views on discounting. All these findings violate hyperbolic and quasi-hyperbolic discounting and support the interest of finding new and more flexible functional forms for discounting.

This paper introduces two new classes of discount functions that can account for increasing impatience, and also for any degree of decreasing impatience. These classes are the intertemporal counterparts to the CARA and CRRA utility functions from decision under risk (or, more precisely, to state-dependent generalizations thereof as we will see later). Thus, any individual degree of decreasing or increasing impatience can be accounted for, in the same way as CARA and CRRA can account for any individual degree of risk aversion or risk seeking. A part of the CRRA family was introduced before by Ebert and

Prelec (2007). The part that they considered can only accommodate moderate degrees of decreasing impatience, which leaves problems for the purpose of fitting individual data.

The paper is organized as follows. Basic definitions are in Section 2. Section 3 analyzes the properties of generalized hyperbolic and quasi-hyperbolic discounting. Section 4 presents our new classes of discount functions, with their main empirical implications used to provide preference foundations in Sections 5 and 6. Section 7 derives further properties of our discount functions, demonstrating in particular that they can accommodate any degree of increasing or decreasing impatience. Section 8 shows that our analysis can also be used for decision under uncertainty. By taking logarithms of our discount functions when restricted to positive utilities, it can be seen that in a mathematical sense our results amount to generalizing CARA and CRRA utility from state-independent expected utility to state-dependent expected utility. We further generalize by allowing for zero and negative utility. Section 9 concludes, and proofs are in the appendix.

2 Discounting

Throughout this paper we study preferences $\succcurlyeq$ over *timed outcomes* $(t : x) \in T \times X$, where T is a nondegenerate subinterval of $[0, \infty]$ and X is the outcome set. A timed outcome $(t : x)$ is interpreted as the receipt of *outcome* x at *time* t. The preference notation $\succ$, $\sim$, $\preccurlyeq$, and $\prec$ is as usual.

We assume that preferences $\succcurlyeq$ can be represented by *discounted utility* (DU):

$$DU(t : x) = \varphi(t)U(x).$$

That is, $(t : x) \succcurlyeq (s : y)$ if and only if $DU(t : x) \geq DU(s : y)$. DU is the *discounted utility*, U is the *utility*, and φ is the *discount function*. Preference conditions for DU were given by

Roskies (1965), Krantz et al. (1971, Chapter 7), and Fishburn & Rubinstein (1982) (either for only gains or with φ for losses different than for gains). For brevity, we do not state these conditions but assume DU throughout. Time points are often denoted by s and t, where we usually have $s < t$ with s abbreviating soon.

Throughout this paper our main interest will concern the discount function. We assume $\varphi > 0$, with φ continuous and strictly decreasing. The discount function is often normalized through $\varphi(0) = 1$ in the literature, but in this paper it is more convenient not to commit to such a normalization, the more so as we allow T not to contain time point 0.

We assume that the utility image $U(X)$ contains a nondegenerate interval $(0, \epsilon)$ for an $\epsilon > 0$, implying that X must be infinite. Other than that, X and $U(X)$ can be general. For example, with U continuous, outcomes can be monetary with X containing a subinterval of the reals, or outcomes can be commodity bundles with X a convex subset of $\mathbb{R}^m$. With a slight abuse of notation, for outcomes x, y we write $x \succcurlyeq y$ if there exists a time point t such that $(t : x) \succcurlyeq (t : y)$. Because of positivity of φ, U represents $\succcurlyeq$ over the outcomes and the aforementioned preference at one time point t automatically implies it at all time points t. *Gains* are outcomes with positive utility and *losses* are outcomes with negative utility.

Strict decreasingness of φ implies *impatience*: for all $s < t$ and gains x, $(s : x) \succ (t : x)$. The condition implies a corresponding condition, with the implied preference reversed, for losses: for all $s < t$ and losses x, $(s : x) \prec (t : x)$. Impatience is assumed throughout this paper. In what follows, many preference conditions will be imposed only on gains. The corresponding preference conditions for losses, with implied preferences reversed, then automatically follow from DU.

Stationary, also called *constant impatience*, holds if

$$(s : x) \succcurlyeq (t : y) \iff (s + \tau : x) \succcurlyeq (t + \tau : y) \text{ for every } \tau > 0.$$

Thus, the indifference depends only on t and s through their difference $t - s$. The condition is defined only if τ and the other variables are such that all time points considered are contained in T. As a convention, for all preference conditions throughout this paper it will always be assumed implicitly that the time points considered are contained in T.

Decreasing impatience holds if for all $s < t$, $\sigma > 0$, and gains $x \prec y$,

$$(s : x) \sim (t : y) \Rightarrow (s + \sigma : x) \preccurlyeq (t + \sigma : y).$$

Increasing impatience holds if the implied preference always is the reverse. Under decreasing impatience, if we consider two equivalent timed outcomes, then delaying both outcomes equally long results in less distinction between the times of receipt so that the best outcome then decides. In this sense, decreasing impatience reflects diminishing sensitivity: a time difference becomes less important as it lies farther ahead in the future. This interpretation applies to both gains and losses. Prelec and Loewenstein (1991) discussed this interpretation and Ebert and Prelec (2007) provided empirical evidence supporting it.

We summarize the assumptions made.

Assumption 2.1 *[Structural Assumption]. $DU(t : x) = \varphi(t)U(x)$ represents $\succcurlyeq$ on $T \times X$ with T a subinterval of $[0, \infty)$, $\varphi : T \to (0, \infty)$, $U : X \to I\!R$, $U(X) \supset (0, \epsilon)$ for an $\epsilon > 0$, and φ continuous and strictly decreasing.* □

3 Hyperbolic and Quasi-Hyperbolic Discounting

Loewenstein and Prelec (1992) proposed generalized hyperbolic discounting to accommodate the various findings of decreasing impatience in the literature. There has been some confusion in the literature regarding the term hyperbolic discounting. To avoid confusion, Prelec (2004) used the term "actually-hyperbolic" instead of Loewenstein & Prelec's (1992) generalized hyperbolic. We will use the term hyperbolic discounting to designate generalized hyperbolic discounting. The *hyperbolic discount function* is given by

$$\varphi(t) = (1 + \alpha t)^{-\beta/\alpha},$$

with $\alpha > 0$ and $\beta > 0$.

Herrnstein (1961, p. 270) used the discount function $\varphi(t) = 1/t$. Harvey (1986) studied hyperbolic discounting with $\alpha = 1$, and Mazur (1987) and Harvey (1995) discussed hyperbolic discounting with $\alpha = \beta$. Harvey called the latter *proportional discounting.* Phelps and Pollak (1968) introduced *quasi-hyperbolic discounting*, where the discount function is

$$\varphi(t) = \beta\delta^t \text{ for some } \beta \leq 1 \text{ and } \delta > 0 \text{ for all } t > 0, \text{ and where } \varphi(0) = 1.$$

Laibson (1997) demonstrated the relevance of quasi-hyperbolic discounting for economic applications. Quasi-hyperbolic discounters have decreasing impatience only at present, and constant impatience throughout the future. The latter assumption adds to the tractability of quasi-hyperbolic discounting.

We will now demonstrate that hyperbolic and quasi-hyperbolic discounting impose serious restrictions on the degrees of increasing and decreasing impatience. It is well known that both models imply decreasing impatience, as can be verified from substitution. There is, unfortunately, no easy way to generalize these models to accommodate increasing impatience. Because of its importance, we display this point as an observation, and explain its claim thereafter in the main text.

Observation 3.1 *There is no natural way to extend the quasi-hyperbolic and hyperbolic discount functions to accommodate increasing impatience.* □

To obtain increasing impatience for hyperbolic discounting, we should set $\alpha < 0$. Then, however, discounting decreases "too fast," with $\varphi(t) = 0$ at $t = -1/\alpha$ and $\varphi(t)$ negative for larger t, which cannot be.

To obtain increasing impatience for quasi-hyperbolic discounting we should set $\beta > 1$, but then φ is increasing and not decreasing near $t = 0$ so that $\varphi(t) > \varphi(0)$ for small t, with the unwarranted implication that impatience is violated there. Thus, the claim of the above observation has been established.

Another drawback of hyperbolic and quasi-hyperbolic discounting is that these models can accommodate decreasing impatience only to a limited degree. We analyze this claim in some detail. Consider gains $x \prec y$, time points $0 < s < t$, and a "delay" $\sigma > 0$, with

$$(s : x) \sim (t : y) \quad \& \quad (s + \sigma : x) \sim (t + \sigma + \tau : y). \tag{1}$$

In the second indifference, the receipt of both outcomes has first been delayed by σ. Then τ is the extra time the decision maker is willing to wait for the better y. The larger τ, the more impatience has decreased by the σ increase in time. The extra delay τ for the last timed outcome can be taken as a measure of decreasing impatience, and τ can be interpreted as a time premium similar to risk premiums for decision under risk. Lower bounds for τ naturally result from impatience.

Lemma 3.2 *Assume Structural Assumption 2.1 and Eq. 1.*

(i) Impatience implies $\tau > -(t-s)$ *and* $\tau > -\sigma$*.*

(ii) Decreasing impatience implies $\tau \geq 0$*.*

(iii) Constant impatience (stationarity) implies $\tau = 0$*.*

(iv) Increasing impatience implies $\tau \leq 0$*.*

□

We now demonstrate that hyperbolic and quasi-hyperbolic discounting can only accommodate moderate degrees of decreasing impatience.

Observation 3.3 *Assume Structural Assumption 2.1 and Eq. 1. Hyperbolic and quasi-hyperbolic discounting can only accommodate moderate degrees of decreasing impatience. More precisely:*

(i) Hyperbolic discounting implies $0 < \tau < \frac{\sigma(t-s)}{s}$*.*

(ii) Quasi-hyperbolic discounting implies $\tau = 0$*.*

□

The general model of Section 2 does not impose an upper bound on τ in Eq. 1 as we will see in Section 7, but hyperbolic discounting and quasi-hyperbolic discounting do. The following example illustrates some limitations resulting from these upper bounds.

Example 3.4 Assume (next week : \$100) $\sim$ (in 2 weeks : \$105). Under hyperbolic discounting, (in 4 weeks : \$100) $\sim$ (in 9 weeks : \$105) cannot hold, because it would imply, with $\sigma = 3$ weeks, $t - s = 1$ week, $\tau = 4 > 3 = \frac{\sigma(t-s)}{s}$ in violation of Observation 3.3.[1] Although the above indifferences reflect a strong degree of decreasing impatience, they can be exhibited by some subjects, and hyperbolic discounting cannot describe such subjects. □

There is another way to see that hyperbolic discounting imposes strong limitations on the deviations from stationarity, which we turn to now. Prelec (2004) defined comparative decreasing impatience by comparing the values of τ in Eq. 1 for different persons. He showed that these comparisons reflect deviations from stationarity and vulnerability to arbitrage. Mathematically, they concern convexity of the logarithmic transform of the discount function. That is, more decreasing impatience corresponds to dominance at each time point t in terms of the following index of convexity (Prelec 2004, Proposition 2b):[2]

$$\gamma(t) = -\frac{[\ln \varphi(t)]''}{[\ln \varphi(t)]'} \ . \tag{2}$$

By straightforward algebraic manipulations, not given here, we obtain the following result.

Observation 3.5 *For quasi-hyperbolic discounting, $\gamma(t) = 0$ for all $t > 0$, and $\gamma(0)$ is undefined. For hyperbolic discounting (with $\alpha > 0$):*

$$\gamma(t) = \frac{\alpha}{1 + \alpha t} \ \in \ [0, \frac{1}{t}].$$

□

[1] By Observation 3.3, hyperbolic discounting always implies that (in 4 weeks : \$100) $\succ$ (in 9 weeks : \$105). Quasi-hyperbolic discounting implies the same, and even (in 4 weeks : \$100) $\sim$ (in 5 weeks : \$105).

[2] The index measures convexity of $ln(\varphi)$, a decreasing function. For increasing functions the index would measure concavity.

The result shows once again that there can be no increasing impatience (because $\gamma(t) \geq 0$) and that there is an upper bound to the degree of decreasing impatience with decreasing impatience vanishing if t tends to infinity.

Note that all limitations on discounting established in this section were irrespective of utility, i.e. they hold whatever the utility function is. They were, in fact, also independent of the actual discount rate, i.e. the power of the discount function. Prelec (2004, p. 515) argued for the plausibility of such independencies when studying deviations from stationarity. The limitations for hyperbolic and quasi-hyperbolic discounting demonstrated in this section will be overcome by the discount functions introduced in the following sections.

4 Constant Absolute and Constant Relative Decreasing Impatience for Discount Functions

This section introduces the two new discount functions of interest to us. Their logarithms will be the well-known CARA and CRRA families (see Section 8). For the first family, the index of Eq. 2 is constant as we will see, and it will be the parameter c.

Definition 4.1 *The discount function φ is a CADI function if there exist constants $r > 0$ and c, and a normalization constant $k > 0$, such that:*

for $c > 0$, $\varphi(t) = ke^{re^{-ct}}$;

for $c = 0$, $\varphi(t) = ke^{-rt}$;

for $c < 0$, $\varphi(t) = ke^{-re^{-ct}}$.

□

Lemma 4.2 *For the CADI discount functions, Prelec's (2004) measure is constant and is given by* $\gamma(t) = c$. □

The proof follows from substitution and is omitted. The parameter c is the Constant that indicates the Convexity of $ln(\varphi)$. Our family allows for any degree of decreasing impatience, simply because c can be any real number.

The discount function for $c = 0$ is the constant discounting introduced by Samuelson (1937). The parameter k is a scaling factor without empirical meaning in the sense that it does not affect preferences. The parameter r, which is the power of the discount function (given that we can set $k = 1$), determines the degree of discounting and is empirically relevant when streams of two or more outcomes are received. In this paper we restrict attention to receipts of single outcomes. Then the joint power of discounting and utility can be chosen freely without affecting preference. That is, if preferences over timed outcomes can be represented by $\varphi(t)U(x)$, then they can also be represented by $\varphi(t)^{\lambda}U(x)^{\lambda}$ whenever $\lambda > 0$. Thus, r has no empirical meaning as long as utility has not been specified.

The most convenient normalization of the parameters k and r, yielding $\varphi(0) = 1$, is as follows.

for $c > 0$, $\varphi(t) = e^{(e^{-ct}-1)}$;

for $c = 0$, $\varphi(t) = e^{-t}$;

for $c < 0$, $\varphi(t) = e^{1-e^{-ct}}$.

We next turn to our second family. Outside of $t = 0$, dominance at each time point t in terms of the index in Eq. 2 corresponds, obviously, with dominance at each time point t in terms of the following index:

$$\delta(t) = -t\frac{[\ln \varphi(t)]''}{[\ln \varphi(t)]'} \ . \tag{3}$$

Hence, this index can also serve as a measure of decreasing impatience. We will see in Section 8 that it corresponds to relative, instead of absolute, risk aversion of the logarithmic transform of the discount function. This index will be constant for our second family, and it will be the parameter d.

The first two cases below are defined only for $t > 0$ because φ tends to infinity for t tending to 0. A similar complication arises for CRRA functions, which also are not defined at 0 for negative powers and for power zero (which designates the logarithmic function). To incorporate the present time point into such models, we should not set $t = 0$ for the present but $t = \epsilon$ for some positive ϵ. This term ϵ can be compared to the initial wealth w that is often incorporated in CRRA utility functions $x \to \frac{(x+w)^r}{r}$, so that the functions are defined at $x = 0$ also for negative and zero power.

Definition 4.3 *The discount function φ is a CRDI function if there exist constants $r > 0$ and d, and a normalization constant $k > 0$, such that:*

for $d > 1$, $\varphi(t) = ke^{rt^{1-d}}$ (only if $0 \notin T$);

for $d = 1$, $\varphi(t) = kt^{-r}$ (only if $0 \notin T$);

for $d < 1$, $\varphi(t) = ke^{-rt^{1-d}}$.

□

Lemma 4.4 *For the CRDI discount functions, Prelec's (2004) measure $\gamma(t)$ is d/t. The measure $\delta(t)$ is constant and is d.* □

Samuelson's (1937) constant impatience results for $d = 0$. The functions exhibit decreasing impatience for $d > 0$ and increasing impatience for $d < 0$.

The CADI family results from the CRDI family applied to e^t instead of to t (and with c for $d - 1$). Herrnstein (1961) considered the CRDI discount function for $d = 1$ with $r = 1$. Hyperbolic discounting and proportional discounting evolved as generalizations of Herrnstein's discounting. The CRDI discount functions are also natural generalizations of Herrnstein's function. Ebert and Prelec (2007) introduced the subpart of the CRDI family for $d < 1$, i.e. the part with increasing impatience or moderately decreasing impatience.

A convenient normalization of the parameters k and r, yielding $\varphi(0) = 1$ when $t = 0$ is contained in the domain (in the case of $d < 1$), is as follows.

for $d > 1$, $\varphi(t) = e^{t^{1-d}}$ (only if $t = 0 \notin T$);

for $d = 1$, $\varphi(t) = t^{-1}$ (only if $t = 0 \notin T$);

for $d < 1$, $\varphi(t) = e^{-t^{1-d}}$.

5 Constant Absolute Decreasing Impatience: A Preference Condition

This section defines constant absolute decreasing impatience in terms of preferences. It will lead to a preference foundation for the CADI discount functions introduced in the preceding section.

Definition 5.1 *Constant absolute decreasing impatience (CADI)* holds for preferences if for all $s < t < l$, all τ, and all outcomes x, y, z[3]

$$\begin{array}{lllll} (s:x) & \sim & (t:y) & \text{and} & (s+\tau:y) \sim (t+\tau:z) \quad \text{and} \\ (t:x) & \sim & (l:y) & & \\ & & & \text{imply} & (t+\tau:y) \sim (l+\tau:z) \end{array}$$

□

Constant absolute decreasing impatience has the following interpretation. The first indifference suggests that a delay from s to t is compensated by the improvement from x to y. The third indifference, the one below the first, suggests that the delay from t to l (l refers to “late”) requires exactly the same compensation. Under CADI, this equality of delays in terms of required compensation should not be affected if there is a common extra delay of τ for everything involved.

The next lemma translates the definition of CADI into a condition for discount function ratios: equality of such ratios is not affected if a common delay is added to all time points. The proof of the lemma clarifies the CADI condition and, hence, is given in the main text.

Lemma 5.2 *The following two statements are equivalent.*

(i) CADI holds for preferences.

(ii) For all $s \leq t \leq l$ and τ

$$\frac{\varphi(s)}{\varphi(t)} = \frac{\varphi(t)}{\varphi(l)} \iff \frac{\varphi(s+\tau)}{\varphi(t+\tau)} = \frac{\varphi(t+\tau)}{\varphi(l+\tau)} \ . \tag{4}$$

[3]Impatience implies $x \prec y \prec z$ for gains and $x \succ y \succ z$ for losses. It is sufficient to impose the condition only for gains.

□

Proof of Lemma 5.2. We will consider logical relations between the following equalities:

$$\frac{U(y)}{U(x)} \stackrel{1}{=} \frac{\varphi(s)}{\varphi(t)} \stackrel{2}{=} \frac{\varphi(t)}{\varphi(l)} \stackrel{3}{=} \frac{U(y)}{U(x)}; \quad \frac{U(z)}{U(y)} \stackrel{4}{=} \frac{\varphi(s+\tau)}{\varphi(t+\tau)} \stackrel{5}{=} \frac{\varphi(t+\tau)}{\varphi(l+\tau)} \stackrel{6}{=} \frac{U(z)}{U(y)} \; .$$

Equalities $\stackrel{1}{=}$, $\stackrel{3}{=}$, $\stackrel{4}{=}$, and $\stackrel{6}{=}$ are equivalent to the upper left, lower left, upper right, and lower right equivalences in the definition of CADI, respectively.

First assume that CADI holds. We assume the left equality in Eq. 4 and derive the right one (the reversed implication is symmetric). Because the image of U contains an interval $(0, \epsilon)$ for $\epsilon > 0$, we can find outcomes $x \prec y \prec z$ such that equalities $\stackrel{1}{=}$ and $\stackrel{4}{=}$ hold. Because $\stackrel{2}{=}$ is assumed, $\stackrel{3}{=}$ follows. We have $\stackrel{1}{=}$, $\stackrel{3}{=}$, and $\stackrel{4}{=}$, i.e. the three antecedent indifferences in the definition of CADI. The fourth indifference there follows from CADI, implying $\stackrel{6}{=}$ and, hence, $\stackrel{5}{=}$. This establishes Eq. 4.

Next assume Eq. 4. To derive CADI, assume the three antecedent indifferences there, implying $\stackrel{1}{=}$, $\stackrel{3}{=}$, and $\stackrel{4}{=}$. Then $\stackrel{2}{=}$ follows and then, by Eq. 4, $\stackrel{5}{=}$. This and $\stackrel{4}{=}$ imply $\stackrel{6}{=}$ and, thus, the fourth indifference in CADI. CADI has been established. □

We now obtain the main result of this section.

Theorem 5.3 *Let the Structural Assumption 2.1 hold. The discount function is a CADI function if and only if CADI holds for preferences.* □

6 Constant Relative Decreasing Impatience: A Preference Condition

This section presents a preference condition for constant relative decreasing impatience.

Definition 6.1 *Constant relative decreasing impatience (CRDI)* holds for preferences if for all $s < t < l$, all $\lambda > 0$, and all outcomes x, y, z[4]

$$\begin{aligned} (s:x) &\sim (t:y) \quad \text{and} \quad (\lambda s:y) \sim (\lambda t:z) \quad \text{and} \\ (t:x) &\sim (l:y) \\ &\qquad\qquad \text{imply} \quad (\lambda t:y) \sim (\lambda l:z) \end{aligned}$$

□

The interpretation is similar to that of CADI, but with scalar multiplication of the time points by a factor $\lambda > 0$ rather than the addition of a constant τ. In other words, if the unit of time is changed from a day into a week without changing the numbers, then equality of time delays in terms of required outcome-compensation should be maintained. A slightly more restrictive preference condition was used by Ebert and Prelec (2007) to characterize the case of $d < 1$.

Lemma 6.2 *The following two statements are equivalent.*

(i) CRDI holds for preferences.

(ii) For all $0 \le s \le t \le l$ *and* $\lambda > 0$

$$\frac{\varphi(s)}{\varphi(t)} = \frac{\varphi(t)}{\varphi(l)} \iff \frac{\varphi(\lambda s)}{\varphi(\lambda t)} = \frac{\varphi(\lambda t)}{\varphi(\lambda l)} \ . \tag{5}$$

□

The proof of the lemma is very similar to the one of Lemma 5.2 and will not be given. It can be derived from the aforementioned proof through the transformation $t \to ln(t)$.[5]

[4] Impatience implies $x \prec y \prec z$ for gains and $x \succ y \succ z$ for losses. It is sufficient to impose the condition only for gains.

[5] Now the values $ln(t)$ play the role of t for CADI. The former can be negative (for $t < 1$), unlike the latter. This possible negativity does not require any modification of the analysis.

Theorem 6.3 *Let the Structural Assumption 2.1 hold. The discount function is a CRDI function if and only if CRDI holds for preferences.* □

7 Properties of CADI and CRDI discounting

We already saw that Prelec's (2004) measure $\gamma(t)$ is constant and can take any value for CADI discount functions. For CRDI, Prelec's measure can also be arbitrarily large or small, but always tends to plus or minus infinity for t tending to 0, while tending to 0 for t tending to infinity. The index $\delta(t) = t\gamma(t)$ is constant for CRDI functions and can take any value.

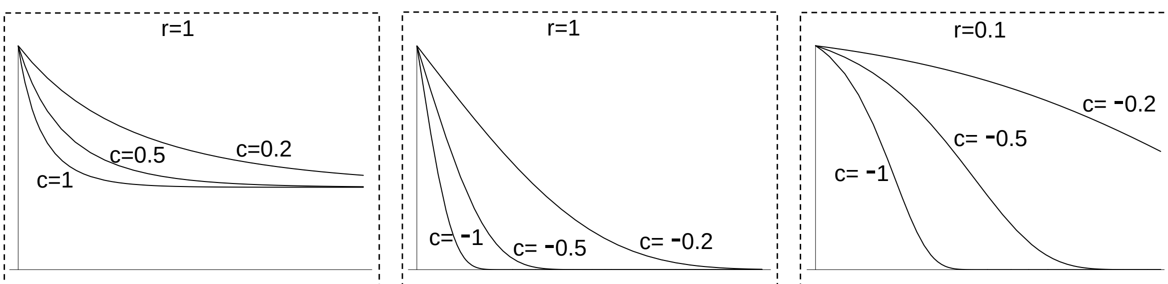

Figure 1: CADI discount functions

Figures 1 and 2 display some shapes of φ. Most shapes in the figures are convex, but some have concave parts initially, implying strongly increasing impatience there.

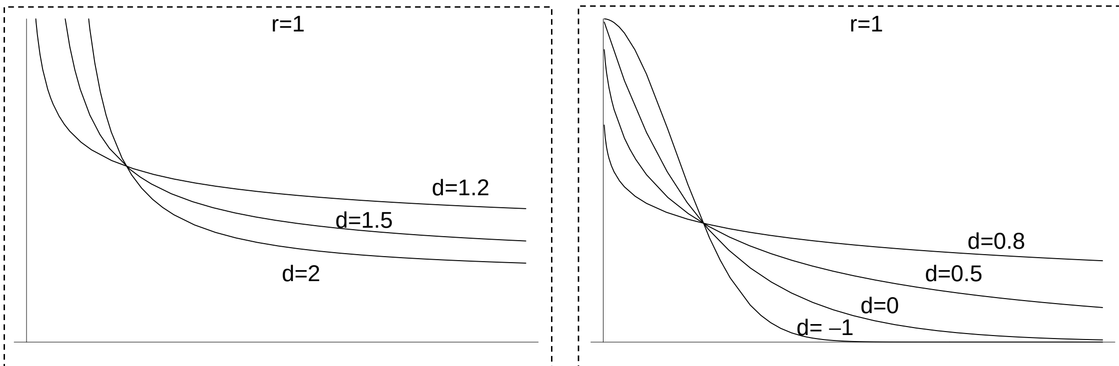

Figure 2: CRDI discount functions

The following observation shows that the CADI and CRDI discount functions can account for all values of τ in Eq. 1. Thus, contrary to the conventional discounting functions, they can handle any degree of increasing or decreasing impatience.(cf. Observation 3.3 and Example 3.4). The proof will also illustrate that larger parameters c for CADI discounting and larger parameters d for the CRRI model lead to more strongly decreasing impatience.

Observation 7.1 *Let the Structural Assumption 2.1 hold. For all $s < t, \sigma, \tau$ and $(s : x) \sim (t : y)$, there exists a CADI discount function, and also a CRDI discount function, such that Eq. 1 is satisfied.* □

A CADI discount function with $c < 0$ and appropriate r, and a CRDI discount function with $d < 1$ and appropriate r, can explain the results of Onay and Öncüler (2007) with a concave discount function on the relevant domain. When assuming discounted expected utility their results can only be explained by a concave discount function, which implies not only increasing impatience but even strongly so. Chesson and Viscusi (2003) similarly found that many of their subjects violate convex discounting. No traditional discount model can account for concave discounting.

8 An Application to State-Dependent Expected Utility

In this section we show that the results from the preceding sections can serve to resolve an open problem in decision under uncertainty: how to characterize state-dependent utility with power or exponential utility. Assume a *state space* containing two *states of nature*, $\{s_1, s_2\}$. Let T be a nondegenerate intervals in the reals, and let $X \subset I\!R$, with restrictions on X imposed through the images of functions U_2 and V_2 defined later. We allow in this section that not only X, but also T, can contain negative real numbers. The elements of T and X are *outcomes*. A pair $(t : x) \in T \times X$ (we maintain the notation with a colon instead of a comma), interpreted before as a timed outcome, is now interpreted as a *prospect* (also called act), i.e. a function on S that yields t if s_1 is true and x if s_2 is true. *Expected utility* holds if there exist probabilities p_1 and p_2, and a continuous strictly increasing utility function U, such that $p_1U(t)+p_2U(x)$ represents preferences. For decision under risk, the probabilities are objective and given, and for decision under uncertainty they may be subjective.

The following utility functions are most commonly used. To easily apply the families defined below to other functions than U (such as functions V_1 and V_2 later on), we define the families for a general function f. The CARA and CRRA functions result from the CADI and CRDI functions, respectively, by applying a logarithmic transformation, multiplying by -1, and substituting $-ln(k)$ for k.

Definition 8.1 *Let Y be an interval. $f : Y \to I\!R$ is a CARA function if there exist constants $r > 0$ and c, and a constant k, such that:*

for $c > 0$, $f(t) = k - re^{-ct}$;

for $c = 0$, $f(t) = k + rt$;

for $c < 0$, $f(t) = k + re^{-ct}$.

□

It is well known that the Pratt-Arrow absolute index of concavity (or "risk aversion") for these increasing functions, $-f''/f'$, is constant and is equal to c for the CARA family.

Definition 8.2 *Let Y be an interval in $I\!R^+ \cup \{0\}$. $f : Y \to I\!R$ is a CRRA function if there exist constants $r > 0$ and d, and a constant k, such that:*

for $d > 1$, $f(t) = k - rt^{1-d}$ (only if $0 \notin Y$);

for $d = 1$, $f(t) = k + r\ln(t)$ (only if $0 \notin Y$);

for $d < 1$, $f(t) = k + rt^{1-d}$.

□

It is also well known that the relative index of concavity (or "risk aversion") for the above increasing functions, $-tf''/f'$, is constant and is equal to d for the CRRA family.

Preference conditions to characterize the above families for state-independent expected utility, being constant absolute and constant relative risk aversion, are well known. We generalize these results to state dependence. Karni (1985) gave many examples where state dependence is warranted.

Definition 8.3 *State-dependent expected utility holds if there exist positive probabilities p_1 and p_2 adding up to 1, and continuous strictly increasing functions $U_1 : T \to I\!R$ and $U_2 : X \to I\!R$, such that $(t : x) \mapsto p_1U_1(t) + p_2U_2(x)$ represents preference.* □

Here U_1 and U_2 can be different on their common domain $T \cap X$. It is well known that, without further information, probability and utility cannot be separated under state dependence, and that from revealed preference we can only derive their products $V_1 = p_1U_1$ and $V_2 = p_2U_2$ (Nau 1995; Grant and Karni 2005). Thus, *state-dependent expected utility* is equivalent to an additive representation (Kreps 1988, Formula 7.13).

Definition 8.4 *An additive representation holds if there exist continuous strictly increasing functions $V_1 : T \to I\!R$ and $V_2 : X \to I\!R$ such that $(t : x) \mapsto V_1(t) + V_2(x)$ represents preference.* □

Karni (1983, 1987) showed that conditions to characterize risk attitudes under expected utility cannot be reasonably applied under state dependence. Miyamoto and Wakker (1996) demonstrated that an unqualified application of the usual risk-attitude conditions to characterize CARA and CRRA utility under state dependence implies state independence after all, so that these conditions cannot even be imposed in the presence of state dependence.

Karni (1983, 1987) presented comparative risk results for different utility functions that have the same reference set (a prospect is in the reference set if it has the same marginal utilities across different states). We show how the results of this paper can be used to characterize CARA and CRRA under general state dependence.

As a first step to link state-dependent utility to our analysis of DU, we transform state-dependent utility through the exponential function. Because this transformation is strictly increasing, the result is again a representation of preference. Then, with $\varphi(t) = e^{V_1(t)}$ and $U(x) = e^{V_2(x)}$, a multiplicative representation as in DU has resulted. The only difference is that now φ is strictly increasing rather than strictly decreasing as it was in DU. This difference does not affect the proofs of Lemmas 5.2 and 6.2, or of Theorems 5.3 and 6.3. Neither does the fact that now we only have gains x available (in the sense of $U(x) = e^{V_2(x)} > 0$), as long as the image of $e^{V_2(x)}$ contains an interval $(0, \epsilon)$ for some $\epsilon > 0$. This does require that V_2 and U_2 are unbounded below. A generalization that relaxes this restriction is conceivable, but we will not pursue it here.

The CADI preference condition was defined only for domains $T \subset I\!R^+ \cup \{0\}$, but can without any modification be extended to domains $T \subset I\!R$ that contain negative numbers. The CRDI preference condition can be defined only for domains $T \subset I\!R^+ \cup \{0\}$, and this will be assumed hereafter. We summarize the structural assumption needed and then obtain the first result of this section.

Assumption 8.5 *[Structural Assumption for State-Dependent Utility]. $\succcurlyeq$ is a binary relatrion on $T \times X$ with T a subinterval of $I\!R$.* □

Observation 8.6 *[CARA and CRRA for additive representations]. Let the Structural Assumption 8.5 hold. Assume an additive representation with* $V_2(X) \supset (-\infty, \epsilon')$ *for a real number* ϵ'*. Then CADI holds for preferences if and only if* V_1 *is CARA. For* $T \subset I\!R^+ \cup \{0\}$*, CRDI holds for preferences if and only if* V_1 *is CRRA.* □

The following result readily follows because the factor p_1 can always be incorporated into the constants k and r.

Theorem 8.7 *[CARA and CRRA for state-dependent expected utility]. Let the Structural Assumption 8.5 hold. Assume state-dependent expected utility with* $U_2(X) \supset (-\infty, \epsilon')$ *for a real number* ϵ'*. Then CADI holds for preferences if and only if* U_1 *is CARA. For* $T \subset I\!R^+ \cup \{0\}$*, CRDI holds for preferences if and only if* U_1 *is CRRA.* □

We can, obviously, obtain the same forms of V_2 and U_2 by imposing the corresponding structural condition and the CADI or CRDI preference condition on the second coordinate x. We can also obtain results for general state spaces with more than two states by fixing all outcomes except over two disjoint nonnull events, and then applying the techniques of this section.

9 Conclusion

This paper has introduced two new families of discount functions (CADI and CRDI). They are natural extensions of Samuelson's (1937) constant discounting and Herrnstein's (1961) hyperbolic discounting. They serve better to flexibly fit various patterns of intertemporal choice than hyperbolic and quasi-hyperbolic discounting do, by allowing any degree of increasing or decreasing impatience. Thus, the CADI and CRDI families are the first that can be used to fit data at an individual level. They can also explain findings at the aggregate level that could not be explained up to now, such as prevailing increasing impatience at the aggregate level (Attema, Bleichrodt, Rohde, and Wakker 2006; Gigliotti and Sopher 2004; Read, Airoldi, and Loewenstein 2005; Sayman and Öncüler 2006), and even of concave discounting (Onay and Öncüler 2007; Chesson and Viscusi 2003). We hope that CADI and CRDI discount functions will provide the same flexibility for intertemporal choice as the CARA and CRRA utility functions do for decision under risk.

10 Appendix; Proofs

Proof of Lemma 3.2. The implication of impatience follows because we must have $t+\sigma+\tau > s+\sigma$ and $t+\sigma+\tau > t$. The other results in the observation are straightforward.

□

Proof of Observation 3.3. For hyperbolic discounting it follows from algebraic manipulations that $\tau = \frac{\alpha\sigma(t-s)}{1+\alpha s}$. Because $\alpha > 0$, the bounds in the observation follow. The bound for quasi-hyperbolic discounting follows because all time points considered are positive and there quasi-hyperbolic discounting implies constant impatience. □

Proof of Theorem Observation 5.3. Eq. 4 entails that midpoints of $-ln(\varphi(t))$ are invariant with respect to the addition of a common term to all arguments. By Miyamoto (1983, Lemma 1), this is equivalent to $-ln(\varphi(t))$ being from the CARA family. A useful feature of Myamoto's lemma is that it allows for general intervals as domain. □

Proof of Theorem 6.3. Eq. 5 implies that $-ln(\varphi(e^t)$-midpoints are invariant with respect to the addition of a common term to all arguments. By Miyamoto (1983, Lemma 1), this is equivalent to $-ln(\varphi(e^t)$ being from the CARA family. It is equivalent to $-ln(\varphi)$ being from the CRRA family. □

Proof of Observation 7.1. Assume Eq. 1. It implies that $\frac{U(y)}{U(x)}$ equals

$$\frac{\varphi(s)}{\varphi(t)} = \frac{\varphi(s+\sigma)}{\varphi(t+\sigma+\tau)}. \tag{6}$$

As an aside, for all time points and discount functions we can take outcomes/utility such that the first indifference in Eq. 1 holds and, hence, the equality concerning the utility ratio above can be verified. To simplify upcoming formulas, we rewrite the equality as

$$ln(\varphi(s)) - ln(\varphi(t)) = ln(\varphi(s+\sigma)) - ln(\varphi(t+\sigma+\tau)). \tag{7}$$

We assume all variables in Eq. 1 fixed except τ. We show that for each τ proper discount functions can be given that generate that τ. Unfortunately, there exist no analytical solutions to the proper choices of parameters c or d. We, therefore, give a proof based on continuity. We may assume that $t < s + \sigma$, for otherwise we can interchange these two symbols in Eq. 7

We first consider CRDI utility. The parameters k, r do not affect preference and we assume that they are 1. For any smooth function f, a difference $f(y) - f(x)$ can be written as $y - x$ times the average derivative of f over the interval $[x, y]$. Thus, the ratio $\frac{(t+\sigma+\tau)-(s+\sigma)}{t-s}$ equals the [average derivative of $ln(\varphi)$ over the interval $[s, t]$] divided by [the average derivative of $ln(\varphi)$ over the interval $[(s + \sigma), (t + \sigma + \tau)]$].

We first show that for arbitrarily large d, τ becomes arbitrarily large. For large d,

$ln(\varphi)$ becomes very convex. By convexity, the above ratio of average derivatives exceeds the ratio of derivatives at t and $s+\sigma$. The latter is $(t/(s+\sigma))^{-d}$ which tends to infinity if d does. For $\frac{(t+\sigma+\tau)-(s+\sigma)}{t-s}$ to tend to infinity, τ must tend to infinity.

We next show that for very small (very negative) d, τ becomes arbitrarily close to $-(t-s)$, which, given $t < s+\sigma$ (Lemma 3.2), is the lowest value τ can take. For very negative d, $ln(\varphi)$ becomes very concave. By concavity, the ratio of average derivatives is less than the ratio of derivatives at t and $s+\sigma$. The latter is $(t/(s+\sigma))^{-d}$ which tends to 0 if d tends to $-\infty$. For $\frac{(t+\sigma+\tau)-(s+\sigma)}{t-s}$ to tend to 0, τ must tend to $-(t-s)$.

We have seen that large d generate arbitrarily large τ and very negative d imply τ to get arbitrarily close to its lower bound $-(t-s)$. The ratio of average derivatives over the two intervals mentioned depends on d in a continuous manner (as does any normalized-φ difference), also at $d = 1$. Hence, for each τ between its supremum and its infimum there exists a d that generates that τ.[6]

The proof for CADI discount functions is completely analogous, the only change being that all time points are replaced by their exponents. The exponent of $t+\sigma+\tau$ tending to infinity or to its infimum corresponds with τ doing so. □

[6]It can be seen that the existing d is unique, because the ratio of average derivatives over the two intervals mentioned depends monotonically on d. This can be seen by dividing all derivatives by the derivative at t. In the interval $[s,t]$ the derivatives divided by that at t are increasing in d, and in the interval $[(s+\sigma),(t+\sigma+\tau)]$ they are decreasing. This monotonicity also shows that greater values of d lead to stronger decreasing impatience.

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