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## SOLUTIONS AND MULTISOLUTIONS FOR BARGAINING GAMES

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### ABSTRACT

In this paper bargaining solutions are considered, which assign to each 2-person bargaining game one or more points of the weak Pareto boundary. All those solutions and multisolutions are described, which have the property of independence of irrelevant alternatives. Also all closed multisolutions with the restricted monotonicity property are characterized with the aid of monotonic multicurves.

Keywords: Bargaining game, independence of irrelevant alternatives, restricted monotonicity.

### 1. INTRODUCTION

A bargaining game is a pair  $(S, d)$ , where  $S$  is a compact and convex subset of  $\mathbb{R}^2$  (the payoff space) and  $d$  is an element of  $S$  (the disagreement point) with the property that there is an  $x \in S$  with  $x_1 > d_1$  and  $x_2 > d_2$ . The set of all bargaining games is denoted by  $\mathcal{B}$ . An element  $(S, d) \in \mathcal{B}$  corresponds to the following game situation: two players may cooperate and agree upon choosing a point  $s \in S$ , which has utility  $s_i$  for player  $i$  ( $i=1,2$ ), or they may not cooperate. In the latter case they are punished by getting point  $d$ , which has utility  $d_i$  for player  $i$ . The theory of bargaining games started with the paper of Nash (1950). His solution for the bargaining problem is a function  $\phi : \mathcal{B} \rightarrow \mathbb{R}^2$  with some nice properties, including the following one: for each  $(S, d) \in \mathcal{B}$ ,  $\phi(S, d)$  is an element of the Pareto set

$P(S) := \{x \in S; \forall_{y \in S} [y \geq x \Rightarrow y = x]\}$   
of  $S$ . The paper of Nash was followed by many other ones in which alternative solutions for the bargaining problem have been proposed. We only mention here the papers of Harsanyi and Selten (1972) and of Kalai and Smorodinsky (1975). For more background information we refer to the books of Rauhut, Schmitz and Zachow (1979) and Roth (1979). In this paper we are interested in multifunctions  $\phi : \mathcal{B} \rightarrow \mathbb{R}^2$ , which assign to

each bargaining game  $(S, d)$  a non-empty subset  $\phi(S, d)$  of the weak Pareto set

$$W(S) := \{x \in S; \forall y \in \mathbb{R}^2 [y > x \Rightarrow y \notin S]\} \text{ of } S.$$

DEFINITION 1.1. A multifunction  $\phi : \mathbb{B} \rightarrow \mathbb{R}^2$  is called a multisolution of the bargaining problem if the following properties hold:

- (P.1) For each  $(S, d) \in \mathbb{B}$  and each  $x \in \phi(S, d)$ , we have  $x \geq d$  (Individual rationality).
- (P.2) For each  $(S, d) \in \mathbb{B}$  and each  $x \in \phi(S, d)$ , we have  $x \in W(S)$  (Pareto optimality).
- (P.3) If  $(S, d) \in \mathbb{B}$  and  $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a map of the form  

$$(x_1, x_2) \mapsto (a_1 x_1 + b_1, a_2 x_2 + b_2)$$
where  $a_1, a_2, b_1, b_2 \in \mathbb{R}$  and  $a_1 > 0, a_2 > 0$ , then  
 $\phi(A(S), A(d)) = A(\phi(S, d))$  (Independence of equivalent utility representations).

A multisolution  $\phi : \mathbb{B} \rightarrow \mathbb{R}^2$  with the property that  $\phi(S, d)$  consists of exactly one point for each  $(S, d) \in \mathbb{B}$  is called a solution. If  $\phi(S, d) \subset W(S)$  for each  $(S, d) \in \mathbb{B}$ , then  $\phi$  is called a strong multisolution.

In section 2 of this paper we consider solutions and in section 3 multisolutions  $\phi$  with the following so called IIA-property [Cf. Nash (1950), Kaneko (1980)]:

- (P.4) For all  $(S, d)$  and  $(T, d)$  in  $\mathbb{B}$  with  $S \subset T$  and  $\phi(T, d) \cap S \neq \emptyset$  we have  $\phi(S, d) = \phi(T, d) \cap S$  (Independence of irrelevant alternatives).

The solution given by Nash (1950) is the unique solution, having the IIA-property and the symmetry property.

De Koster, Peters, Tijs and Wakker (1982) characterized all strong solutions having the IIA-property. These include the non-symmetric Nash solutions, introduced by Harsanyi and Selten (1972). One of the main purposes of this paper is to characterize all solutions and all multisolutions with the IIA-property. This is done in the sections 2 and 3 of this paper. For  $(S, d) \in \mathbb{B}$ , let  $S_d := \{x \in S; x \geq d\}$ , and let  $u(S_d) = (u_1(S_d), u_2(S_d))$  be the utopia-point of  $S_d$ , defined by  $u_i(S_d) := \max_{x_i} \{x_i; (x_1, x_2) \in S_d\}$  for  $i=1, 2$ . Interesting

are multisolutions  $\phi$  with the restricted monotonicity property or RM-property, which we state now.

(P.5) For all  $(S, d)$  and  $(T, e)$  in  $\underline{B}$  with  $d = e$ ,  $\phi(S_d) = \phi(T_e)$  and  $S \subset T$  we have  $\phi(S, d) \subset \phi(T, e) - \mathbb{R}_+^2$  and  $\phi(T, e) \subset \phi(S, d) + \mathbb{R}_+^2$  (Restricted monotonicity).

Note that  $\phi(S, d) \subset \phi(T, e) - \mathbb{R}_+^2$  means that for each  $x \in \phi(S, d)$  there is a point  $y \in \phi(T, e)$ , which is at least as good for both players i.e.  $y \geq x$ .

For a solution with the RM-property the inclusions in (P.5) are equivalent to  $\phi(S, d) \subset \phi(T, e)$  (if we identify the one-point set  $\phi(S, d)$  with the contained point). In Peters and Tijs (1982) it is proved that the RM-property is equivalent to individual monotonicity. The solution proposed by Kalai and Smorodinsky (1975) is the unique symmetric solution with the RM-property. In Peters and Tijs (1982) all strong RM-solutions were characterized. The second main purpose of this paper is to characterize all closed multisolutions with the RM-property, where we call a multisolution  $\phi : \underline{B} \rightarrow \mathbb{R}^2$  closed, if  $\phi(S, d)$  is a closed subset of  $\mathbb{R}^2$  for each  $(S, d) \in \underline{B}$ . It appears that there exist only strong RM-multisolutions (Proposition 4.1) and that the closed RM-multisolutions correspond to certain monotonic multicurves.

## 2. THE SOLUTIONS WITH THE IIA-PROPERTY

For  $t \in (0, 1)$ , let  $F^t : \underline{B} \rightarrow \mathbb{R}^2$  be the strong solution, which assigns to an  $(S, d) \in \underline{B}$  the unique point of  $P(S_d)$  where the function  $(x_1, x_2) \mapsto (x_1 - d_1)^t (x_2 - d_2)^{1-t}$  on  $S_d$  attains its maximum. Let  $F^0 : \underline{B} \rightarrow \mathbb{R}^2$  ( $F^1 : \underline{B} \rightarrow \mathbb{R}^2$ ) be the strong solution, which assigns to  $(S, d) \in \underline{B}$  the unique point in  $P(S_d)$  with maximal second coordinate (maximal first coordinate). One of the results in de Koster, Peters, Tijs and Wakker (1982) is the following

THEOREM 2.1. (i)  $\{F^t : t \in [0, 1]\}$  is the family of all strong bargaining solutions with the IIA-property.

(ii) A strong IIA-solution  $\phi : \underline{B} \rightarrow \mathbb{R}^2$  equals  $F^t$  iff  $\phi(T, 0) = (t, 1-t)$ , where  $T := \text{conv}\{(0,0), (1,0), (0,1)\}$ .

Now we want to introduce some new (weak) IIA-solutions.

Therefore we need the following notation. For  $(S, d) \in \underline{B}$ , let  
 $\bar{p}(S, d) :=$  the point in  $P(S_d)$  with maximal second coordinate,  
 $\underline{p}(S, d) :=$  the point in  $P(S_d)$  with maximal first coordinate,  
 $\bar{w}(S, d) :=$  the point in  $W(S_d)$  with minimal first coordinate,  
 $\underline{w}(S, d) :=$  the point in  $W(S_d)$  with minimal second coordinate.

Note that  $F^0(S, d) = \bar{p}(S, d)$  and  $F^1(S, d) = \underline{p}(S, d)$  for  $(S, d) \in \underline{B}$ .

Let  $\underline{U}^1 := \{(S, d) \in \underline{B}; \underline{w}_2(S, d) = d_2\}$ ,  $\underline{V}^1 := \underline{B} \setminus \underline{U}^1$ ,  
 $\underline{U}^2 := \{(S, d) \in \underline{B}; \bar{w}_1(S, d) = d_1\}$ ,  $\underline{V}^2 := \underline{B} \setminus \underline{U}^2$ .

Now let  $W^0, W^1, M^0, M^1$  be the solutions defined by

$$\begin{aligned} W^0(S, d) &:= \bar{w}(S, d) \text{ for each } (S, d) \in \underline{B}, \\ W^1(S, d) &:= \underline{w}(S, d) \text{ for each } (S, d) \in \underline{B}, \\ M^0(S, d) &:= \bar{w}(S, d) \text{ if } (S, d) \in \underline{U}^2 \text{ and} \\ M^0(S, d) &:= \bar{p}(S, d) \text{ if } (S, d) \in \underline{V}^2, \\ M^1(S, d) &:= \underline{w}(S, d) \text{ if } (S, d) \in \underline{U}^1 \text{ and} \\ M^1(S, d) &:= \underline{p}(S, d) \text{ if } (S, d) \in \underline{V}^1. \end{aligned}$$

It is straightforward to show that  $W^0, W^1, M^0, M^1$  are IIA-solutions. Hence, the family of IIA-solutions contains the set  $\{F^t; 0 \leq t \leq 1\} \cup \{W^0, W^1, M^0, M^1\}$ . The main purpose of this section is to prove the following theorem with the aid of a string of lemmas.

**THEOREM 2.2.**  $\{F^t; 0 \leq t \leq 1\} \cup \{W^0, W^1, M^0, M^1\}$  is the family of all bargaining solutions with the IIA-property.

**LEMMA 2.3.** Let  $\phi : \underline{B} \rightarrow \mathbb{R}^2$  be an IIA-solution. Let  $(S, d) \in \underline{B}$ . Then  $\phi(S, d) \in P(S_d) \cup \{\bar{w}(S, d), \underline{w}(S, d)\}$ .

**PROOF.** In view of (P.3) we suppose w.l.o.g. that  $d = 0$ . Let  $s \in W(S_d)$ ,  $s \notin P(S_d) \cup \{\bar{w}(S, d), \underline{w}(S, d)\}$ . We want to prove that  $s \neq \phi(S, d)$ . There are two cases: (i)  $s_2 = \bar{p}_2(S, d), 0 \leq \bar{w}_1(S, d) < s_1 < \bar{p}_1(S, d)$ , (ii)  $s_1 = p_1(S, d), 0 \leq \underline{w}_2(S, d) < s_2 < p_2(S, d)$ . We consider only the first case. Let  $\alpha := s_1^{-1} \bar{p}_1(S, d)$  if  $(S, 0) \in \underline{U}^2$ , and  $\alpha := \min\{(\bar{w}_1(S, d))^{-1} s_1, s_1^{-1} \bar{p}_1(S, d)\}$  if  $(S, 0) \in \underline{V}^2$ . Let  $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the map with

$A(x_1, x_2) = (\alpha x_1, x_2)$ . Note that  $\alpha > 1$ . Let  $T$  be the triangle with vertices  $d = 0$ ,  $\bar{w}(S, d)$  and  $s$ . Then  $T \subset S$  and

$A(T) = \text{conv}\{0, A(\bar{w}(S, d)), A(s)\} \subset S$  and  $s \in A(T)$ . Suppose for a moment that  $\phi(S, 0) = s$ . Then, by the IIA-property,

$\phi(A(T), 0) = \phi(S, 0) = s$  and  $\phi(T, 0) = \phi(S, 0) = s$ . But by (P.3),

we obtain  $\phi(A(T), 0) = A(\phi(T, 0)) = A(s) = (as_1, s_2) \neq (s_1, s_2)$ , a contradiction. Hence,  $\phi(S, 0) \neq s$ .  $\square$

For each  $q \in [0, 1]$  the triangle with vertices  $(0, 0)$ ,  $(1, q)$  and  $(1, 1)$  will be denoted by  $T^1(q)$  in the following and  $\text{conv}((0, 0), (q, 1), (1, 1))$  by  $T^2(q)$ . In view of lemma 2.3 we have for all  $q \in [0, 1]$ :

$$\phi(T^2(q), 0) = (q, 1) \text{ or } \phi(T^2(q), 0) = (1, 1). \quad (2.1)$$

LEMMA 2.4. Let  $\phi : \underline{B} \rightarrow \mathbb{R}^2$  be an IIA-solution. Then

- (i) if for some  $q \in (0, 1)$ ,  $\phi(T^1(q), 0) = (1, q)$ , then  $\phi(T^1(r), 0) = (1, r)$  for all  $r \in (0, 1)$ ,
- (ii) if for some  $q \in (0, 1)$ ,  $\phi(T^2(q), 0) = (q, 1)$ , then  $\phi(T^2(r), 0) = (r, 1)$  for all  $r \in (0, 1)$ .

PROOF. We only prove (ii). Let  $q \in (0, 1)$  and  $\phi(T^2(q), 0) = (q, 1)$ .

First take  $r \in (0, q)$ . Then  $T^2(q) \subset T^2(r)$ . So IIA and

$\phi(T^2(q), 0) \neq (1, 1)$  imply:  $\phi(T^2(r), 0) \neq (1, 1)$ . But then, by

(2.1),  $\phi(T^2(r), 0) = (r, 1)$ . Now take  $r \in (q, 1)$  and define

$A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by  $A(x_1, x_2) = (r^{-1}q x_1, x_2)$ . Then

$(q, 1) = A(r, 1) \in A(T^2(r)) \subset T^2(q)$ . Hence, by the IIA-property:

$\phi(A(T^2(r)), A(0)) = \phi(T^2(q), 0) = (q, 1)$  and by (P.3):

$\phi(A(T^2(r)), A(0)) = A(\phi(T^2(r), 0))$ .

Then  $A(\phi(T^2(r), 0)) = (q, 1)$ , so  $\phi(T^2(r), 0) = A^{-1}(q, 1) = (r, 1)$ .

So we have proved that for all  $r \in (0, 1) : \phi(T^2(r), 0) = (r, 1)$ .  $\square$

LEMMA 2.5. Let  $\phi$  be an IIA-solution. Then

- (i) if  $\phi(T^1(q), 0) = (1, q)$  for some  $q \in (0, 1)$ , then  $\phi = W^1$ ,
- (ii) if  $\phi(T^2(q), 0) = (q, 1)$  for some  $q \in (0, 1)$ , then  $\phi = W^0$ .

PROOF. We only prove (ii). Suppose  $\phi(T^2(q), 0) = (q, 1)$ . We

want to prove that  $\phi(S, d) = \bar{w}(S, d)$  for each  $(S, d) \in \underline{B}$ .

Take  $(S, d) \in \underline{B}$ . Let  $v := (u_1(S_d) + 1, u_2(S_d))$  and put

$\hat{S} := \text{conv}(S_d \cup \{v\})$ . Then  $S_d \subset \hat{S}$ . Hence, if we can show that

$\phi(\hat{S}, d) = \bar{w}(S, d)$  then the IIA-property guarantees that

$\phi(S, d) = \phi(S_d, d) = \phi(\hat{S}, d) = \bar{w}(S, d)$ .

Since  $W(\hat{S}) = \{(s_1, s_2) \in \mathbb{R}^2 ; \bar{w}(S, d) \leq s_1 \leq u_1(S_d) + 1, s_2 = u_2(S_d)\}$ ,

we can conclude by lemma 2.3, that  $\phi(\hat{S}, d) = \bar{w}(S, d)$  if

$\phi(\hat{S}, d) \neq v$ . There is an  $\epsilon > 0$ , such that  $t := (v_1 - \epsilon, v_2) \in W(\hat{S})$ .

Let  $D := \text{conv}(d, v, t)$ . Then there is an  $r \in (0, 1)$  and an af-

fine map  $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  as in (P.3) such that  $A(0) = d$ ,

$A(r,1) = t$  and  $A(1,1) = v$ . Now  $\phi(T^2(r),0) = (r,1)$  by lemma 2.4. So by (P.3):  $\phi(D,d) = \phi(A(T^2(r)),A(0)) = A(\phi(T^2(r),0)) = A(r,1) = t \neq v$ . Since  $D \subset \hat{S}$ , by the IIA-property we have:  $\phi(\hat{S},d) \neq v$ . So  $\phi(\hat{S},d) = \bar{w}(S,d)$ ,  $\phi(S,d) = \bar{w}(S,d)$ ,  $\phi = W^0$ .  $\square$ .

LEMMA 2.6. Let  $\phi$  be an IIA-solution.

- (i) Let  $(T,e) \in \underline{V}^1$  and  $\phi(T,e) = \underline{w}(T,e) \neq \underline{p}(T,e)$ . Then  $\phi = W^1$ .
- (ii) Let  $(T,e) \in \underline{V}^2$  and  $\phi(T,e) = \bar{w}(T,e) \neq \bar{p}(T,e)$ . Then  $\phi = W^0$ .

PROOF. We only prove (ii). Let  $E := \text{conv}(e, \bar{w}(T,e), \bar{p}(T,e))$ .

Then there is an  $A : \mathbb{R}^2 \times \mathbb{R}^2$  as in (P.3) and a  $T^2(r)$  ( $0 < r < 1$ ) such that  $A(E) = T^2(r)$ . Then, by the IIA-property and (P.3):  $\phi(T^2(r),0) = A(\phi(E,e)) = A(\bar{w}(T,e)) = (r,1)$ . By lemma 2.5 we conclude that  $\phi = W^0$ .  $\square$

LEMMA 2.7. Let  $\phi$  be an IIA-solution.

- (i) If  $\phi(T^1(0),0) = (1,0)$  and  $\phi \neq W^1$ , then  $\phi = M^1$ .
- (ii) If  $\phi(T^2(0),0) = (0,1)$  and  $\phi \neq W^0$ , then  $\phi = M^0$ .

PROOF. We only prove (ii). So we suppose that

$\phi(T^2(0),0) = (0,1)$  and  $\phi \neq W^0$ . (a) Take  $(S,d) \in \underline{V}^2$ . Let  $v := (u_1(S_d)+1, u_2(S_d))$  and  $\tilde{S} := \text{conv}(S_d \cup \{v\})$ . Then by lemma 2.3,  $\phi(\tilde{S},d) \in \{\bar{w}(S,d), v\}$ . If  $\phi(\tilde{S},d) = v$ , then by IIA also  $\phi(\text{conv}(d, \bar{w}(S,d), v), d) = v$ . But then, by (P.3),  $\phi(T^2(0),0) = (1,1)$ , and that is impossible. Hence,  $\phi(\tilde{S},d) = \bar{w}(S,d)$ . But then, by IIA,

$$\phi(S,d) = \phi(\tilde{S},d) = \bar{w}(S,d) = M^0(S,d).$$

(b) Take  $(S,d) \in \underline{V}^2$ . Let  $z := (u_1(S_d)+1, \frac{1}{2}(d_1+u_2(S_d)))$  and  $\tilde{S} := \text{conv}(S_d \cup \{z\})$ . Then, by lemma 2.3,  $\phi(\tilde{S},d) \in \{\bar{w}(S,d), v\}$ .  $\phi(\tilde{S},d) = \bar{w}(S,d)$  would imply that also  $\phi(\text{conv}(d, \bar{w}(S,d), \bar{p}(S,d)), d) = \bar{w}(S,d)$ . But then, by (P.3),  $\phi(T^2(q),0) = (q,1)$  for some  $q \in (0,1)$ . This implies, by lemma 2.5, that  $\phi = W^0$  and that is not true. So

$\phi(\tilde{S},d) \neq \bar{w}(S,d)$ . Let  $\hat{p} \in P(\tilde{S})$ ,  $\hat{p} \neq \bar{p}(\tilde{S},d) = \bar{p}(S,d)$ . Define  $S^* := \text{conv}(\tilde{S} \cup \{(d_1, \hat{p}_2)\})$ . By (P.3) and  $\phi(T^2(0),0) = (0,1)$ , we have  $\phi(\text{conv}(d, (d_1, \hat{p}_2), \hat{p}), d) = (d_1, \hat{p}_2)$ . Then IIA implies that  $\hat{p} \neq \phi(S^*, d) = \phi(\tilde{S}, d)$ . But then  $\phi(\tilde{S}, d) = \bar{p}(S, d)$ . Hence, by IIA:  $\phi(S, d) = \phi(\tilde{S}, d) = \bar{p}(S, d) = M^0(S, d)$ .  $\square$

PROOF OF THEOREM 2.2. Let  $\phi$  be an IIA-solution. We have to prove that  $\phi \in \{F^t ; 0 \leq t \leq 1\} \cup \{W^0, W^1, M^0, M^1\}$ .

(i) Suppose first that  $\phi : \mathbb{B} \rightarrow \mathbb{R}^2$  is a strong solution. Then, by theorem 2.1,  $\phi \in \{F^t; 0 \leq t \leq 1\}$ .

(ii) Suppose now that  $\phi$  is not strong. Then there is an  $(S, d) \in \mathbb{B}$  such that  $\phi(S, d) = \bar{w}(S, d) \neq \bar{p}(S, d)$  (or  $\phi(S, d) = \underline{w}(S, d) \neq \underline{p}(S, d)$ ). If  $(S, d) \in \mathbb{V}^2$ , then  $\phi = W^0$  by lemma 2.6 (ii). Suppose  $(S, d) \in \mathbb{U}^2$ . If  $\phi = W^0$ , then there is nothing left to prove. So suppose  $\phi \neq W^0$ . Since

$\phi(\text{conv}\{\bar{d}, \bar{w}(S, d), \bar{p}(S, d)\}, d) = \bar{w}(S, d)$ , it follows from (P.3) that  $\phi(T^2(0), 0) = (0, 1)$ . By lemma 2.7 (ii), we have  $\phi = M^0$ .

(In case  $\phi(S, d) = \underline{w}(S, d) \neq \underline{p}(S, d)$  it follows, similarly, that  $\phi = M^1$  or  $\phi = W^1$ ). Hence  $\phi \in \{F^t; 0 \leq t \leq 1\} \cup \{W^0, W^1, M^0, M^1\}$ .  $\square$

### 3. THE MULTISOLUTIONS WITH THE IIA-PROPERTY.

We define the multisolutions  $\mu^i, v^i, \sigma^i$  ( $i=1, 2$ ) by

$$\mu^0(S, d) = v^0(S, d) = \sigma^0(S, d) := \text{conv}(\bar{w}(S, d), \bar{p}(S, d)) \text{ if } (S, d) \in \mathbb{V}^2,$$

$$v^0(S, d) := \text{conv}(\bar{w}(S, d), \bar{p}(S, d)), v^0(S, d) := \{\bar{w}(S, d)\} \text{ and}$$

$$\sigma^0(S, d) := \{\bar{p}(S, d)\} \cup (\mu^0(S, d) \setminus v^0(S, d)) \text{ if } (S, d) \in \mathbb{U}^2.$$

$$\mu^1(S, d) = v^1(S, d) = \sigma^1(S, d) := \text{conv}(\underline{w}(S, d), \underline{p}(S, d)) \text{ if } (S, d) \in \mathbb{V}^1,$$

$$v^1(S, d) := \text{conv}(\underline{w}(S, d), \underline{p}(S, d)), v^1(S, d) := \{\underline{w}(S, d)\} \text{ and}$$

$$\sigma^1(S, d) := \{\underline{p}(S, d)\} \cup (\mu^1(S, d) \setminus v^1(S, d)) \text{ if } (S, d) \in \mathbb{U}^1.$$

The easy proof of the following theorem is left to the reader.

THEOREM 3.1.  $\mu^0, \mu^1, v^0, v^1, \sigma^0, \sigma^1$  are IIA-multisolutions.

For the characterization of all IIA-multisolutions we need

LEMMA 3.2. Let  $\phi$  be an IIA-multisolution. Let  $a, b \in \phi(S, d)$  and  $a \neq b$ . Then  $a_1 = b_1$  or  $a_2 = b_2$ .

PROOF. Suppose that  $a_1 \neq b_1$  and  $a_2 \neq b_2$ . Then w.l.o.g. we suppose that  $d = 0$ ,  $a_1 < b_1$  and  $a_2 > b_2$ . We distinguish two cases: (i)  $0 = a_1 < b_1$ , (ii)  $0 < a_1 < b_1$ .

In case (i) consider the map  $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  with

$A(x_1, x_2) = (\frac{1}{2}x_1, x_2)$ . Let  $T := \text{conv}\{0, b, a\}$ . Then  $A(T) \subset T$  and  $a \in A(T) \cap \phi(T, 0) \neq \emptyset$ . So  $\phi(A(T), 0) = A(T) \cap \phi(T, 0)$  by IIA.

By (P.3),  $A(b) = (\frac{1}{2}b_1, b_2) \in \phi(A(T), 0)$ . But then

$(\frac{1}{2}b_1, b_2) \in \phi(T, 0)$ , which is in contradiction with (P.2). So in case (i):  $a_1 = b_1$  or  $a_2 = b_2$ .

For case (ii) we take  $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  of the form as in (P.3), with  $A(0) = 0$  and  $A(\frac{1}{2}(a+b)) = a$ . Let  $T$  be as above and  $E := \text{conv}\{0, b, \frac{1}{2}(a+b)\}$ . Then a simple calculation shows that

$A(E) \subset T$  and  $A(b) \notin W(T)$ . Since  $a \in A(E) \cap \phi(T, 0)$ , by IIA:  $\phi(A(E), 0) = \phi(T, 0) \cap A(E)$ . By (P.3),  $A(b) \in \phi(A(E), 0)$  because  $b \in \phi(E, 0)$ . Then  $A(b) \in \phi(T, 0)$ , but that is, by (P.2), in contradiction with  $A(b) \notin W(T)$ . Hence, also in case (ii):

$$a_1 = b_1 \text{ or } a_2 = b_2. \quad \square$$

The main result of this section is

THEOREM 3.3. Let  $\phi : \underline{B} \rightarrow \mathbb{R}^2$  be an IIA-multisolution. Then  $\phi \in \{F^t; 0 \leq t \leq 1\} \cup \{W^0, W^1, M^0, M^1\} \cup \{\mu^0, \mu^1, \nu^0, \nu^1, \sigma^0, \sigma^1\}$ .

PROOF SKETCH. (i) If  $\phi$  is a solution, then by theorem 2.2:  $\phi \in \{F^t; 0 \leq t \leq 1\} \cup \{W^0, W^1, M^0, M^1\}$ .

(ii) Suppose there is an  $(S, d) \in \underline{B}$  and  $a, b \in \mathbb{R}^2$ ,  $a \neq b$  such that  $a \in \phi(S, d)$  and  $b \in \phi(S, d)$ . Then, by lemma 3.2:

$$\phi(S, d) \subset \text{conv}(\bar{w}(S, d), \bar{p}(S, d)) \text{ or } \phi(S, d) \subset \text{conv}(w(S, d), p(S, d)).$$

Suppose that the first inclusion holds. We distinguish two cases.

(a) Let  $(S, d) \in V^2$ .

Suppose there is a  $z \in \text{conv}(\bar{w}(S, d), \bar{p}(S, d))$  with  $z \notin \phi(S, d)$ .

Let  $V := \text{conv}(d, a, z)$ , then by arguments similar to those in the proof of lemma 2.4, we obtain  $z \in \phi(V, d)$ . But IIA and  $z \notin \phi(S, d)$  imply  $z \notin \phi(V, d) = \phi(S, d) \cap V$ , a contradiction.

So, by an argument as in lemma 2.6,  $\phi(S, d) = \text{conv}(\bar{w}(S, d), \bar{p}(S, d))$  for each  $(S, d) \in V^2$ .

(b) Let  $(S, d) \in \underline{U}^2$ . If  $\phi = \mu^0$ , then there is nothing to prove. If  $\phi \neq \mu^0$ , then we can prove that  $\phi = \nu^0$  or  $\phi = \sigma^0$ . The proof of this fact is rather elaborate, but follows similar lines of reasoning as the proof of theorem 2.2 (with the aid of analogous lemmas). We leave this to the reader.  $\square$

#### 4. RESTRICTEDLY MONOTONIC MULTISOLUTIONS

In this section we want to characterize all closed multisolutions with the RM-property. We start with two propositions about RM-solutions. The first proposition shows that each RM-multisolution is a strong multisolution.

PROPOSITION 4.1. Let  $\phi : \underline{B} \rightarrow \mathbb{R}^2$  be an RM-multisolution.

Then  $\phi(T, d) \subset P(T)$  for all  $(T, d) \in \underline{B}$ .

PROOF. Let  $(T, d) \in \underline{B}$  and suppose that  $a \in W(T) \setminus P(T)$ . We want to show that  $a \notin \phi(T, d)$ . Let  $S := \text{conv}(d, \bar{p}(S, d), p(S, d))$ .

Since  $S \subset T$  and  $u(S_d) = u(T_d)$ , the RM-property guarantees that  $\phi(T, d) \subset \phi(S, d) + \mathbb{R}_+^2$ . Now  $a \notin \phi(S, d) + \mathbb{R}_+^2$ , so  $a \notin \phi(T, d)$ .  $\square$

PROPOSITION 4.2. Let  $\phi : \mathbb{B} \times \mathbb{R}^2$  be an RM-multisolution. Let  $(S, d) \in \mathbb{B}$  and  $(T, d) \in \mathbb{B}$  with  $u(S_d) = u(T_d)$ . Then

$$\phi(S, d) \cap P(T) \subset \phi(T, d).$$

PROOF. Let  $S^* := \{x \in \mathbb{R}^2; x \geq d, \exists_{y \in S} [x \leq y]\}$ , and  $T^* := \{x \in \mathbb{R}^2; x \geq d, \exists_{y \in T} [x \leq y]\}$ . Then by the RM-property and proposition 4.1:  $\phi(S, d) = \phi(S_d, d) = \phi(S^*, d)$  and  $\phi(T, d) = \phi(T^*, d)$ . Let  $D := S^* \cap T^*$ . Then  $u(D_d) = u(S_d^*) = u(T_d^*)$ . Now take  $y \in \phi(S, d) \cap P(T)$ . Then  $y \in P(D)$ . By the RM-property there is an  $x \in \phi(D, d)$  with  $x \leq y$ . Since  $x, y \in P(D)$  by proposition 4.1, we have  $y = x \in \phi(D, d)$ . Again it follows from the RM-property that there is a  $z \in \phi(T^*, d)$  with  $y \leq z$ . Since  $y, z \in P(T)$  we have  $y = z \in \phi(T^*, d)$ ,  $y \in \phi(T, d)$ . Hence,

$$\phi(S, d) \cap P(T) \subset \phi(T, d).$$

$\square$

Let  $\Delta := \text{conv}\{(1, 0), (0, 1), (1, 1)\}$ . We look at multifunctions  $\lambda : [1, 2] \rightarrow \Delta$  with the following properties:

(C.1) For all  $t \in [1, 2]$ ,  $\lambda(t)$  is a non-empty closed subset of  $\{x \in \Delta; x_1 + x_2 = t\}$ .

(C.2) For all  $s, t \in [1, 2]$  with  $s \leq t$ :  $\lambda(t) \subset \lambda(s) + \mathbb{R}_+^2$ ,  
 $\lambda(s) \subset \lambda(t) - \mathbb{R}_+^2$ .

The family of multifunctions with these properties is denoted by  $\Lambda$  and the elements of  $\Lambda$  are called monotonic multicurves.

For  $\lambda \in \Lambda$ ,  $D(\lambda) := \bigcup_{t \in [1, 2]} \lambda(t)$ . In the following we characterize the closed RM-multisolutions with the aid of monotonic multicurves. We need some lemmas. The proof of lemma 4.3 is straightforward and left to the reader.

LEMMA 4.3. Let  $\lambda \in \Lambda$ . Then  $\lambda$  is an upper semicontinuous and lower semicontinuous multifunction and  $D(\lambda)$  is a closed subset of  $\Delta$ .

LEMMA 4.4. Let  $(S, 0) \in \mathbb{B}$ ,  $u(S_0) = (1, 1)$  and  $\lambda \in \Lambda$ .

- (i) If  $a \in D(\lambda)$  and  $(a - \mathbb{R}_+^2) \cap P(S) \neq \emptyset$ , then  $(a - \mathbb{R}_+^2) \cap P(S) \cap D(\lambda) \neq \emptyset$ .
- (ii) If  $b \in D(\lambda)$  and  $(b + \mathbb{R}_+^2) \cap P(S) \neq \emptyset$ , then  $(b + \mathbb{R}_+^2) \cap P(S) \cap D(\lambda) \neq \emptyset$ .

PROOF. We only prove (ii). If  $b \in P(S)$  or  $(1, 1) \in P(S)$ , then there is nothing to prove. So, suppose  $b \notin P(S)$  and  $(1, 1) \notin P(S)$ . Let  $K := \{x \in \Delta; b \leq x \leq (1, 1)\}$  and let

$\beta := b_1 + b_2$ . Let  $\bar{\lambda} : [\beta, 2] \rightarrow \Delta$  be the multifunction with  $\bar{\lambda}(s) = \lambda(s) \cap K$  for all  $s \in [\beta, 2]$ . In view of (C.2),  $\bar{\lambda}(s) \neq \emptyset$  for each  $s \in [\beta, 2]$  and  $\bar{\lambda}$  is upper and lower semicontinuous in view of lemma 4.3. Now let  $V := \{x \in K; x \notin P(S), (x - \mathbb{R}_+^2) \cap P(S) \neq \emptyset\}$ ,  $I_1 := \{t \in [\beta, 2]; \bar{\lambda}(t) \subset V\}$ ,  $W := \{x \in K; x \notin P(S), (x + \mathbb{R}_+^2) \cap P(S) \neq \emptyset\}$ ,  $I_2 := \{t \in [\beta, 2]; \bar{\lambda}(t) \cap W \neq \emptyset\}$ . Note that  $2 \in I_1$ , that  $\beta \in I_2$  because  $b \in \bar{\lambda}(\beta)$  and that  $I_1 \cap I_2 = \emptyset$ . Since  $V$  and  $W$  are open subsets of  $K$  (in the relative topology) it follows from the continuity of the multifunction  $\bar{\lambda}$ , that  $I_1$  and  $I_2$  are open subsets of  $[\beta, 2]$ . Now  $I_1 \cup I_2 = [\beta, 2]$  if  $(b + \mathbb{R}_+^2) \cap P(S) \cap D(\lambda) = \emptyset$  and that is in contradiction with the connectedness of  $[\beta, 2]$ . Hence,  $(b + \mathbb{R}_+^2) \cap D(\lambda) \cap P(S) \neq \emptyset$ .  $\square$

Now we associate with each  $\lambda \in \Lambda$  a bargaining multisolution  $\Pi^\lambda$ . Let  $(S, d) \in \underline{B}$ . If  $d = (0, 0)$  and  $u(S_d) = (1, 1)$ , then put  $\Pi^\lambda(S, d) := D(\lambda) \cap P(S)$ . If  $d \neq (0, 0)$  or  $u(S_d) \neq (1, 1)$ , then construct a map  $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  as in (P.3) such that  $A(d) = (0, 0)$  and  $u(A(S)_0) = (1, 1)$  and put  $\Pi^\lambda(S, d) := A^{-1}\Pi^\lambda(A(S), A(d))$ .

**THEOREM 4.5.** Let  $\lambda \in \Lambda$ . Then  $\Pi^\lambda$  is a closed RM-multisolution.

**PROOF.** By lemma 4.4,  $\Pi^\lambda(S, d) \neq \emptyset$  for all  $(S, d)$ . Since  $D(\lambda)$  is closed in view of lemma 4.3 and also  $P(S)$  is closed, we have  $\Pi^\lambda(S, d)$  is closed for all  $(S, d) \in \underline{B}$ . Furthermore, it is obvious that  $\Pi^\lambda$  satisfies (P.1), (P.2) and (P.3). Hence,  $\Pi^\lambda$  is a closed multisolution. To prove that  $\Pi^\lambda$  satisfies the RM-property, let  $(S, 0)$  and  $(T, 0)$  be bargaining games with  $u(S_0) = u(T_0) = (1, 1)$  and  $S \subset T$ . Take  $a \in \Pi^\lambda(T, d)$ . Then  $(a - \mathbb{R}_+^2) \cap P(S) \neq \emptyset$ . By lemma 4.4 (i):  $\emptyset \neq (a - \mathbb{R}_+^2) \cap P(S) \cap D(\lambda) = (a - \mathbb{R}_+^2) \cap \Pi^\lambda(S)$ . This implies that  $\Pi^\lambda(T, d) \subset \Pi^\lambda(S, d) + \mathbb{R}_+^2$ . Analogously, it follows with lemma 4.4 (ii) that  $\Pi^\lambda(S, d) \subset \Pi^\lambda(T, d) - \mathbb{R}_+^2$ . But then  $\Pi^\lambda$  is a closed RM-multisolution.  $\square$

The main result of this section is that each closed RM-multisolution corresponds to a monotonic multicurve.

**THEOREM 4.6.** Let  $\phi : \underline{B} \rightarrow \mathbb{R}^2$  be a closed RM-multisolution. Then there exists a  $\lambda \in \Lambda$  such that  $\phi = \Pi^\lambda$ .

PROOF. Let  $V(t) := \text{conv}\{(0,0), (1,0), (1,t-1), (t-1,1), (0,1)\}$  for each  $t \in [1,2]$ . Define the multifunction  $\lambda : [1,2] \rightarrow \mathbb{R}^2$  by  $\lambda(t) := \phi(V(t), 0)$  for all  $t \in [1,2]$ . Then  $\lambda(t)$  is a non-empty closed subset of  $P(V(t)) = \{x \in \Delta; x_1+x_2 = t\}$  and for  $1 \leq s \leq t \leq 2$  we have in view of the RM-property of  $\phi$ :

$$\lambda(t) = \phi(V(t), 0) \subset \phi(V(s), 0) + \mathbb{R}_+^2 = \lambda(s) + \mathbb{R}_+^2,$$

$$\lambda(s) = \phi(V(s), 0) \subset \phi(V(t), 0) - \mathbb{R}_+^2 = \lambda(t) - \mathbb{R}_+^2.$$

Hence,  $\lambda \in \Lambda$ . We want to prove that  $\phi = \pi^\lambda$ . In view of (P.3) it is sufficient to show that

$$\phi(S, 0) = \pi^\lambda(S, 0) \text{ if } (S, 0) \in \underline{B} \text{ and } u(S_0) = (1, 1).$$

Note that  $\phi(V(t), 0) = \pi^\lambda(V(t), 0)$  for all  $t \in [1,2]$ . Take  $x \in \pi^\lambda(S, 0)$ . Let  $s := x_1+x_2$ . Then, by applying proposition 4.2 we obtain:

$$x \in \pi^\lambda(S, 0) \cap P(V(s)) \Rightarrow x \in \pi^\lambda(V(s), 0) = \phi(V(s), 0),$$

$$x \in \phi(V(s), 0) \cap P(S) \Rightarrow x \in \phi(S, 0).$$

Hence,  $\pi^\lambda(S, 0) \subset \phi(S, 0)$ . For the converse, take an  $y \in \phi(S, 0)$  and let  $t := y_1+y_2$ . Then, in view of proposition 4.2:

$$y \in \phi(S, 0) \cap P(V(t)) \Rightarrow y \in \phi(V(t), 0) = \pi^\lambda(V(t), 0),$$

$$y \in \pi^\lambda(V(t), 0) \cap P(S) \Rightarrow y \in \pi^\lambda(S, 0).$$

So,  $\phi(S, 0) \subset \pi^\lambda(S, 0)$ . We have proved that  $\pi^\lambda(S, 0) = \phi(S, 0)$ .  $\square$

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