Homothetic Preferences Revealed*

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Abstract

We introduce a method to recover homothetic preferences from choice data with minor optimization or measurement errors. Our method allows for a more detailed graphical analysis to reveal subjects' preferences and to choose appropriate functional forms for parametric analysis. It has also various other uses such as improving applications of the money metric function or classifying choice data into different preference types. We illustrate some of those approaches using data from an experiment on social preferences.

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1 Introduction

Under homotheticity, everything there is to know about an agent's preference is implicit in a single indifference set. When there are only two goods, which is common in experimental economics, a preference can therefore be completely summarized graphically by showing a single indifference curve.

A finite set of choice data will, of course, only reveal a part of an agent's preference. This part can be recovered using nonparametric revealed preference techniques as suggested by Varian (1982), who shows how to construct bounds on indifference sets from a set of price and quantity data.[1] Knoblauch (1993) extends this approach by imposing homotheticity, which allows to construct substantially tighter bounds. However, she requires that data are perfectly consistent with homothetic utility maximization, which is very restrictive and almost never the case. Therefore, to the best of the authors' knowledge, the technique has never been applied. In this paper, we show how to implement Knoblauch's method when there are minor violations of homotheticity in the data. Our procedure adjusts the data using the homothetic efficiency approach as suggested by Heufer (2013) and Heufer and Hjertstrand (2017).[2]

Consider Figure 1, which uses choice data from a subject of an experiment conducted by Fisman et al. (2007). The hatched areas show the *revealed preferred* and *revealed worse* set for one of the subject's choices using Varian's (1982) technique. These two areas are the bounds on the indifference curve implied by the subject's choices. The indifference curve through the indicated point has to be somewhere in between the hatched parts.

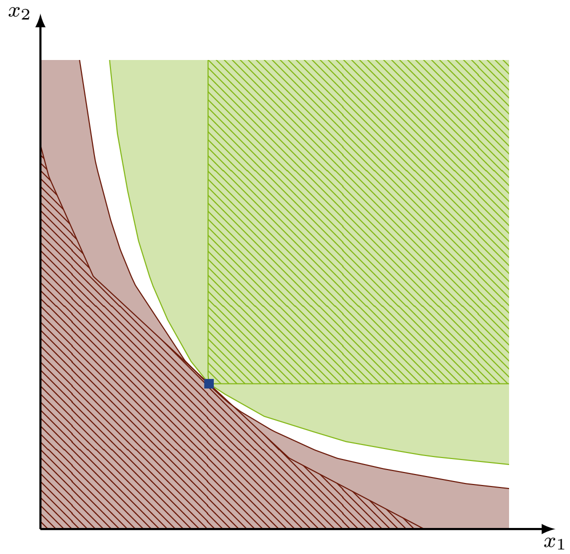

Figure 1: Recovered bounds on preferred and worse sets using data from Fisman et al. (2007): With (solid) and without (hatched) using homothetic recoverability (subject 14).

The choices contain some minor violations of homothetic utility maximization, so Knoblauch's (1993) technique cannot be applied directly. However, by adjusting the data, our version of Knoblauch's (1993) technique can provide bounds under homotheticity. The solidly shaded areas show the *homothetically*

[1] In a recent paper, Halevy et al. (2017) show that these bounds rely on the assumption of convexity of preferences and propose alternative bounds based only on the assumption of monotonicity.

[2] What constitutes a "minor" violation is subjective. Varian (1990) suggests to use an efficiency level of 95% but points out that the level should depend on the problem at hand, that is, the number of observations, the power of the test, and the model under consideration. Fisman et al. (2007), whose data we use in this paper, use an efficiency threshold of 80%. We do not take a direct stand on what constitutes a reasonable threshold but encourage readers to make up their own minds whether they consider a reported efficiency level to be small or large. One possible approach is to use Beatty and Crawford's (2011) and Heufer's (2008, 2012) application of a measure of predictive success to find a good tradeoff between test power and efficiency. See Heufer and Hjertstrand (2017) for an application and discussion of this approach. We also discuss the issue in Section 3 of this paper.

revealed preferred and worse sets for the same subject. Evidently, the bounds are much tighter, and the sets draw a much clearer picture of the kind of preference this subject exhibits.

This paper contributes to the possibilities of applying revealed preference methods to choice data, which too often is confined to simply pre-testing data for consistency with utility maximization before conducting further analysis. By showing how to practically recover preferences from choice data under homotheticity, the tighter bounds make applications based on recovered preferences more attractive. As seen in Figure 1, the approach also lends itself to a graphical analysis of the data, which can be used to summarize a subject's preference and use this to decide which functional form to be estimated.

Figure 1 provides a clear illustration of how useful the approach can be. The recovered sets make use of the *homothetically revealed preferred* relation, which extends the standard revealed preference relation. Every application that makes use of the recovered bounds on revealed preferred and worse sets can benefit from the tighter bounds as long as the assumption of homotheticity can be justified. Note, however, that our method can still suffer from problems of misspecification: if homothetic efficiency is low and the standard revealed preference relation already contains a lot of information, homothetic bounds can be less tight than the bounds calculated using Varian's technique. Our applications include a way to identify the cases where this caveat applies. For the experimental data we use, this is only the case for a few subjects.

We illustrate some of the possible applications using data from an experimental dictator game conducted by Fisman et al. (2007). In particular, we show graphical examples of homothetically revealed preferred and worse sets of four subjects with very different preferences, show how these sets can be used to systematically classify choice data into different preference types, and demonstrate that homothetic bounds on the money metric are typically much tighter.

The crucial part of our contribution is to show how to make Knoblauch's (1993) technique operational by using homothetic efficiency measures to adjust data which violate homotheticity because of minor optimization or measurement error. Of course, this approach suffers from the same drawbacks associated with imposing assumptions of functional forms of utility. However, this drawback is very limited compared to assuming a specific functional form (such as the popular CES function), as we only impose the assumption of homotheticity without any additional structure that comes with a particular utility function.

The rest of the paper is organized as follows. Section 2.1 introduces the notation and some basic definitions. Section 2.2 recalls the necessary tools to measure homothetic efficiency, which is required for the adjustment of data. Section 2.3 shows how the approach of Knoblauch (1993) can be made operational using the adjusted data. Section 3 proposes various applications and illustrates the approach using experimental data. Section 4 concludes.

2 Theory

2.1 Preliminaries

This section briefly summarizes basic definitions and results. A more extensive overview of the relevant concepts can be found in Sections 2 and 3 of Heufer and Hjertstrand (2017).

We use the following notation: For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^L$, $\mathbf{x} \geqq \mathbf{y}$ if $x_i \geq y_i$ for all $i = 1, \ldots, L$; $\mathbf{x} \geq \mathbf{y}$ if $\mathbf{x} \geqq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$; and $\mathbf{x} > \mathbf{y}$ if $x_i > y_i$ for all $i = 1, \ldots, L$. We denote $\mathbb{R}^L_+ = \{\mathbf{x} \in \mathbb{R}^L : \mathbf{x} \geqq (0, \ldots, 0)\}$ and $\mathbb{R}^L_{++} = \{\mathbf{x} \in \mathbb{R}^L : \mathbf{x} > (0, \ldots, 0)\}$. The commodity space is $\mathbb{R}^L_+$, and the price space is $\mathbb{R}^L_{++}$, where $L \geq 2$ is the number of different commodities. A budget set is defined as $B^i = B(\mathbf{p}^i) = \{\mathbf{x} \in \mathbb{R}^L_+ : \mathbf{p}^i\mathbf{x}^i \leq 1\}$, where $\mathbf{p}^i = (p^i_1, \ldots, p^i_L)' \in \mathbb{R}^L_{++}$ is the price vector, and income is normalized to 1. We observe $N \geq 1$ different budgets and the corresponding demand of a consumer which is assumed to be exhaustive (i.e., $\mathbf{p}^i\mathbf{x}^i = 1$). The entire set of observations on a consumer is denoted $\Omega = \{(\mathbf{x}^i, \mathbf{p}^i)\}^N_{i=1}$.

The bundle $\mathbf{x}^i$ is *directly revealed preferred* to a bundle $\mathbf{x}$, written $\mathbf{x}^i \,\mathrm{R}^0\, \mathbf{x}$, if $\mathbf{p}^i\mathbf{x}^i \geq \mathbf{p}^i\mathbf{x}$; it is *strictly directly revealed preferred* to $\mathbf{x}$, written $\mathbf{x}^i \,\mathrm{P}^0\, \mathbf{x}$, if $\mathbf{p}^i\mathbf{x}^i > \mathbf{p}^i\mathbf{x}$; it is *revealed preferred* to $\mathbf{x}$ if there exists a sequence $\mathbf{x}^j, \ldots, \mathbf{x}^k$, such that $\mathbf{x}^i \,\mathrm{R}^0\, \mathbf{x}^j \,\mathrm{R}^0\, \ldots\, \mathbf{x}^k \,\mathrm{R}^0\, \mathbf{x}$; this is written as $\mathbf{x}^i \,\mathrm{R}\, \mathbf{x}$, where R is the transitive closure of R^0. The bundle $\mathbf{x}^i$ is *strictly revealed preferred* to $\mathbf{x}$, written $\mathbf{x}^i \,\mathrm{P}\, \mathbf{x}$, if $\mathbf{x}^i \,\mathrm{R}\, \mathbf{x}^j \,\mathrm{P}^0\, \mathbf{x}^k \,\mathrm{R}\, \mathbf{x}$ for some $j, k = 1, \ldots, N$. We say that a utility function $u : \mathbb{R}^L_+ \to \mathbb{R}$ *rationalizes* a set of observations Ω if $u(\mathbf{x}^i) \geq u(\mathbf{y})$ whenever $\mathbf{x}^i \,\mathrm{R}^0\, \mathbf{y}$.

A set of observations Ω satisfies the *Generalized Axiom of Revealed Preference* (GARP) if for all $i, j = 1, \ldots, N$, it holds that [not $\mathbf{x}^i \,\mathrm{P}^0\, \mathbf{x}^j$] whenever $\mathbf{x}^j \,\mathrm{R}\, \mathbf{x}^i$. Varian (1982) shows that GARP is a necessary and sufficient condition for the existence of a monotonic and concave utility function that rationalizes the data. He also shows how to recover preferences implicit in the data. For some bundle $\mathbf{x}^0 \in \mathbb{R}^L_+$ which was not necessarily observed as a choice, the set of prices which *support* $\mathbf{x}^0$ is defined as

$$S(\mathbf{x}^0) = \big\{p^0 \in \mathbb{R}^L_{++} : \ \{(\mathbf{x}^i, \mathbf{p}^i)\}_{i=0}^N \text{ satisfies GARP and } \mathbf{p}^0\mathbf{x}^0 = 1\big\}. \tag{1}$$

The set $S(\mathbf{x}^0)$ is used to provide boundaries on the indifference set. The set of all bundles which are *revealed worse* than $\mathbf{x}^0$ is

$$RW(\mathbf{x}^0) = \big\{\mathbf{x} \in \mathbb{R}^L_+ : \text{ for all } \mathbf{p}^0 \in S(\mathbf{x}^0),\, \mathbf{x}^0 \,\mathrm{P}\, \mathbf{x}\big\} \tag{2}$$

and the set of all bundles which are *revealed preferred* to $\mathbf{x}^0$ is

$$RP(\mathbf{x}^0) = \big\{\mathbf{x} \in \mathbb{R}^L_+ : \text{ for all } \mathbf{p} \in S(\mathbf{x}),\, \mathbf{x} \,\mathrm{P}\, \mathbf{x}^0\big\}. \tag{3}$$

The *convex hull* of a set of points $Y = \big\{\mathbf{y}^i\big\}_{i=1}^M$ is defined as $\big\{\mathbf{x} \in \mathbb{R}^L_+ : \text{there exist } \alpha_i \in [0,1], \sum_{i=1}^M \alpha_i = 1 \text{ such that } \mathbf{x} = \sum_{i=1}^M \alpha_i \mathbf{y}^i\big\}$; the *convex monotonic hull* (*CMH*) of a set of points Y is $CMH(Y) =$ convex hull of $\big(\big\{\mathbf{x} \in \mathbb{R}^L_+ : \mathbf{x} \geq \mathbf{y}^i \text{ for some } i = 1, \ldots, M\big\}\big)$. Let $int\,CMH(Y)$ denote the interior of $CMH(Y)$. Varian (1982) and Knoblauch (1992) show that

$$int\,CMH(\{\mathbf{x}^i : \mathbf{x}^i \,\mathrm{R}\, \mathbf{x}^0\}) \subseteq RP(\mathbf{x}^0) \subseteq CMH(\{\mathbf{x}^i : \mathbf{x}^i \,\mathrm{R}\, \mathbf{x}^0\}), \tag{4}$$

which can be used to easily check if $\mathbf{x} \in RP(\mathbf{x}^0)$ and also if $\mathbf{x} \in RW(\mathbf{x}^0)$, because $\mathbf{x} \in RW(\mathbf{x}^0) \Leftrightarrow \mathbf{x}^0 \in RP(\mathbf{x})$ (see Varian 1982, Fact 3).

In practice, it is often the case that GARP is violated. That requires the use of efficiency measures to adjust the revealed preference relation in order to remove the violation. A particularly efficient way to do so is the *Varian Efficiency Vector* (VEV) described in Heufer and Hjertstrand (2017) and based on a suggestion by Varian (1993). They define $\mathbf{x}^i \,\mathrm{R}^0(v_i)\, \mathbf{x}$ if $v_i\mathbf{p}^i\mathbf{x}^i \geq \mathbf{p}^i\mathbf{x}$, with $\mathrm{R}(v_i)$ being the transitive closure and $\mathrm{P}^0(v_i)$ defined analogously. Then $\mathbf{v} = (v_1, \ldots, v_N) \in [0,1]^N$ is a VEV if it satisfies GARP($\mathbf{v}$) (i.e., [not $\mathbf{x}^i \,\mathrm{P}^0(v_i)\, \mathbf{x}^j$] whenever $\mathbf{x}^j \,\mathrm{R}(v_j)\, \mathbf{x}^i$) and there does not exist an $\mathbf{v}' \geq \mathbf{v}$ which satisfies GARP($\mathbf{v}'$). The sets RP and RW can then be based on $\mathrm{R}(v_i)$ instead of R. See Heufer and Hjertstrand (2017) for an implementation and justification of this approach.

2.2 Testing for Homotheticity and Homothetic Efficiency

This section briefly introduces the concept of measuring homothetic efficiency. For more details, see Heufer and Hjertstrand (2017).

A utility function is *homothetic* if it is a positive monotonic transformation of a linearly homogeneous utility function; that is, if $u(\mathbf{x}) > u(\mathbf{y})$ then $u(\lambda\mathbf{x}) > u(\lambda\mathbf{y})$ for all $\lambda > 0$. Varian (1983) introduces a homothetic analogue to GARP and shows that it is necessary and sufficient for homothetic utility maximization:

Definition 1 (Varian 1983)**.** *A set of observations Ω satisfies the* Homothetic Axiom of Revealed Preference *(HARP) if for all distinct choices of indices $i, j, \ldots, \ell$, it holds that* $(\mathbf{p}^i\mathbf{x}^j)(\mathbf{p}^j\mathbf{x}^k)\cdots(\mathbf{p}^\ell\mathbf{x}^i) \geq 1$.

Theorem 1 (Varian 1983)**.** *There exists a homothetic, monotonic, continuous, and concave utility function which rationalizes the set of observations Ω if and only if Ω satisfies HARP.*

Because data rarely satisfies HARP, Heufer (2013) and Heufer and Hjertstrand (2017) develop approaches to measure *homothetic efficiency.* Similar to the Afriat Efficiency Index (Afriat 1972, also known as the Critical Cost Efficiency Index), their *Homothetic Efficiency Index* (HEI) is a lower bound that can be interpreted as wasted expenditure. A *Homothetic Efficiency Vector* (HEV) is a more disaggregated measure of efficiency that provides information about inefficiency for each observation rather than a lower bound for the entire data set. To define it, we first need the definition of HARP($\mathbf{h}$).

Definition 2 (Heufer and Hjertstrand 2017)**.** *A set of observations Ω satisfies HARP($\mathbf{h}$) for some $\mathbf{h} = (h_1, \ldots, h_N) \in (0,1]^N$ if for all $i, j = 1, \ldots, N$, it holds that*

$$\left(\frac{\mathbf{p}^i\mathbf{x}^j}{h_i}\right)\left(\frac{\mathbf{p}^j\mathbf{x}^k}{h_j}\right)\cdots\left(\frac{\mathbf{p}^\ell\mathbf{x}^i}{h_\ell}\right) \geq 1. \tag{5}$$

An HEV is a vector $\mathbf{h}$ such that HARP($\mathbf{h}$) is satisfied and there does not exist a vector $\mathbf{h}' \geq \mathbf{h}$ such that HARP($\mathbf{h}'$) is also satisfied. In our approach to homothetic recoverability in Section 2.3, we only make use of the HEV, but the approach could also be based on the HEI.

Heufer and Hjertstrand (2017) propose a linear programming procedure to calculate a first-order approximation of the HEV for observed data. This procedure provides a generally applicable and computationally efficient method for large data sets. However, for the data we use here, we can exploit the fact that there are only two goods available ($L = 2$), which means that we can apply a pairwise version of HARP, which Heufer (2013) has shown to be equivalent to HARP. In particular, for $L = 2$, $(\mathbf{p}^i\mathbf{x}^j/h_i)(\mathbf{p}^j\mathbf{x}^i/h_j) \geq 1$ for all $i, j = 1, \ldots, N$ is equivalent to HARP($\mathbf{h}$). A simple optimization problem that maximizes the sum of all h_i subject to the N^2 pairwise constraints is still computationally feasible for $N = 50$. This allows to find an HEV that minimizes the distance to the unit vector in the L1-norm.

Note that the HEV is not necessarily unique, and that therefore optimal solution values may give different adjusted data to work with. For example, different norms will lead to different HEV. As no specific norm is generally superior to another, the choice of norm is ultimately arbitrary. We note, however, that differences between HEV will be minor if violations of homotheticity are minor. Major differences are to be expected if homothetic efficiency is low, but we do not recommend to apply methods that assume homotheticity for those cases.[3]

Using the HEV for empirical work is well justified, as one can show that if HARP($\mathbf{h}$) is satisfied, there exists a homothetic utility function that comes close to representing the data. For that, we say that a utility function u $\mathbf{h}$*-rationalizes* a set of observations Ω if $u(\mathbf{x}^i) \geq u(\mathbf{y})$ whenever $\mathbf{x}^i \, \mathrm{R}^0(h_i) \, \mathbf{y}$. Theorem 2 shows the relationship between HARP($\mathbf{h}$) and $\mathbf{h}$-rationalization.

Theorem 2 (Heufer and Hjertstrand 2017)**.** *For any $\mathbf{h} = (h_1, \ldots, h_N) \in (0,1]^N$, there exists a homothetic, monotonic, continuous, and concave utility function that $\mathbf{h}$-rationalizes the set of observations Ω if and only if Ω satisfies HARP($\mathbf{h}$).*

[3]Regarding the question of what level of homothetic efficiency is too low, see Footnote 2 in Section 1 and the references therein.

2.3 Homothetic Recoverability

Following Knoblauch (1993), define for a set of observations which satisfies HARP a scalar

$$t_{i,m} = \min \left\{ (\mathbf{p}^i \mathbf{x}^j)(\mathbf{p}^j \mathbf{x}^k) \cdots (\mathbf{p}^\ell \mathbf{x}^m) \right\}, \tag{6}$$

where the minimum is over all finite sequences $i, j, \ldots, \ell$ between 1 and N inclusive, and $t_{i,i} = 1$. Note that we can also have $m = 0$ for a bundle $\mathbf{x}^0$ that was not observed as a choice. We can compute $t_{i,0}$ as we do not need a price vector $\mathbf{p}^0$. We say that $t_{i,m}\mathbf{x}^i$ is *homothetically revealed preferred* to $\mathbf{x}^m$, written $t_{i,m}\mathbf{x}^i \,\mathrm{H}\, \mathbf{x}^m$. The scalar $t = t_{i,m}$ is the smallest value such that $t\,\mathbf{x}^i \,\mathrm{H}\, \mathbf{x}^m$.

Knoblauch (1993) also shows how to recover homothetic preferences implicit in a set of observations which satisfies HARP. Define the set of bundles which are *homothetically revealed preferred* to $\mathbf{x}^m$ as

$$\mathit{HRP}(\mathbf{x}^m) = \mathit{int}\,\mathit{CMH}\left(\mathbf{x}^m \cup \bigcup_{i=1}^{N} t_{i,m}\mathbf{x}^i \right). \tag{7}$$

The set $\mathit{HRP}(\mathbf{x}^m)$ is very useful indeed, as Theorem 3 below shows that it describes the set of bundles which any rationalizing homothetic utility function must rank higher than $\mathbf{x}^m$. Define the set of bundles which are *homothetically revealed worse* to $\mathbf{x}^m$ as

$$\mathit{HRW}(\mathbf{x}^m) = \{\mathbf{x} \in \mathbb{R}^L_+ : \mathbf{x}^m \in \mathit{HRP}(\mathbf{x})\}. \tag{8}$$

As HRP can be easily computed as the convex hull of a finite number of points, it is also easy to test for any bundle if $\mathbf{x} \in \mathit{HRW}(\mathbf{x}^m)$.

Theorem 3 (Knoblauch 1993)**.** *If Ω satisfies HARP, then $\mathbf{x} \in \mathit{HRP}(\mathbf{x}^m)$ if and only if every homothetic, monotonic, continuous, and concave utility function which rationalizes Ω satisfies $u(\mathbf{x}) > u(\mathbf{x}^m)$.*

These definitions and results rely on the data satisfying HARP. If the data violate HARP, then there exist observations such that $(\mathbf{p}^i\mathbf{x}^j)(\mathbf{p}^j\mathbf{x}^k)\cdots(\mathbf{p}^\ell\mathbf{x}^i) < 1$. These can again be multiplied with each other, ad infinitum. Therefore, the scalars in Eq. 6 are not well defined, and the infimum is 0. Alternatively, one could define the scalars over the shortest simple path, that is, by using every observation only once. However, that would be computationally inefficient (to the point of being completely infeasible for larger data sets), and we would not be able to obtain a result that justifies the approach like the one in Theorem 4 below. We therefore propose an approach that relies on adjusting the data by their homothetic efficiency.

Let $\mathbf{h}$ be an HEV of a set of observations Ω. Define

$$\tilde{t}_{i,m} = \min \left\{ \left(\frac{\mathbf{p}^i \mathbf{x}^j}{h_i h_j} \right) \left(\frac{\mathbf{p}^j \mathbf{x}^k}{h_j h_k} \right) \cdots \left(\frac{\mathbf{p}^\ell \mathbf{x}^m}{h_\ell h_m} \right) \right\}. \tag{9}$$

where, if $m = 0$ for a bundle $\mathbf{x}^0$, we set $h_0 = 1$. The reason for this definition is that if we adjust the data, we can only confidently state that $(\mathbf{p}^i\mathbf{x}^j/h_i)\mathbf{x}^i$ is homothetically revealed preferred to $h_j\mathbf{x}^j$ but not necessarily to $\mathbf{x}^j$. Note that $\lambda\,\mathbf{x}\,\mathrm{H}\,\mathbf{y}$ implies $\mathbf{x}\,\mathrm{H}\,\mathbf{y}/\lambda$, so we have $(\mathbf{p}^i\mathbf{x}^j/[h_ih_j])\mathbf{x}^i \,\mathrm{H}\, \mathbf{x}^j$.

We say that $\tilde{t}_{i,m}\mathbf{x}^i$ is *homothetically revealed preferred at efficiency* $\mathbf{h}$ to $\mathbf{x}^m$, written $\tilde{t}_{i,m}\mathbf{x}^i \,\tilde{\mathrm{H}}\, \mathbf{x}^m$. To motivate this definition and show that it is useful, we first need to define the set of bundles which are homothetically revealed preferred at efficiency vector $\mathbf{h}$ to $\mathbf{x}^m$ as

$$\widetilde{\mathit{HRP}}_{\mathbf{h}}(\mathbf{x}^m) = \mathit{int}\,\mathit{CMH}\left(\mathbf{x}^m \cup \bigcup_{i=1}^{N} \tilde{t}_{i,m}\mathbf{x}^i \right), \tag{10}$$

and

$$\widetilde{HRW}_{\mathbf{h}}(\mathbf{x}^m) = \{\mathbf{x} \in \mathbb{R}^L_+ : \mathbf{x}^m \in \widetilde{HRP}_{\mathbf{h}}(\mathbf{x})\}. \tag{11}$$

Theorem 4 below justifies the use of $\widetilde{HRP}_{\mathbf{h}}$ and $\widetilde{HRW}_{\mathbf{h}}$: It shows that every homothetic utility function which **h**-rationalizes the data will not contradict our construction of the homothetically revealed preferred and worse sets.

Theorem 4. *If Ω satisfies HARP($\mathbf{h}$), then for every homothetic, monotonic, continuous, and concave u which $\mathbf{h}$-rationalizes Ω, it holds that $\widetilde{HRP}_{\mathbf{h}}(\mathbf{x}^m) \subseteq \{\mathbf{x} \in \mathbb{R}^L_+ : u(\mathbf{x}) \geq u(\mathbf{x}^m)\}$ and $\widetilde{HRW}_{\mathbf{h}}(\mathbf{x}^m) \subseteq \{\mathbf{x} \in \mathbb{R}^L_+ : u(\mathbf{x}) \leq u(\mathbf{x}^m)\}$.*

The proof can be found in the appendix. Figure 2 shows examples to illustrate the construction. Figure 2.(a) shows two observations; possible HEV are $\mathbf{h} = (\sqrt{5/6}, \sqrt{5/6})$, $\mathbf{h} = (5/6, 1)$, and $\mathbf{h} = (1, 5/6)$. Note that the two scalar factors $\tilde{t}^{1,2}$ and $\tilde{t}^{2,1}$ would be adjusted by dividing them by the product of h_1 and h_2, so the choice of $\mathbf{h}$ does not matter. Without adjustment, $\mathbf{x}^1$ would be scaled up by $(\mathbf{p}^1\mathbf{x}^2)$, but $\mathbf{x}^2$ is strictly revealed preferred over $(\mathbf{p}^1\mathbf{x}^2)\mathbf{x}^1$. Scaling $\mathbf{x}^1$ by $(\mathbf{p}^1\mathbf{x}^2)/(5/6)$ instead resolves this conflict. In Figure 2.(b) the choice $\mathbf{x}^3 = (8, 16)$ with prices $\mathbf{x}^3 = (1/24, 1/24)$ is added. For homotheticity, there is no conflict between $\mathbf{x}^1$ and $\mathbf{x}^3$. Possible HEV include $\mathbf{h} = (\sqrt{5/6}, \sqrt{5/6}, \sqrt{5/6})$ and $\mathbf{h} = (1, 5/6, 1)$. In this example, the choice of $\mathbf{h}$ matters. Setting $h_i = \sqrt{5/6}$ for $i = 1, 2, 3$ results in an unnecessary adjustment between $\mathbf{x}^1$ and $\mathbf{x}^3$. By setting h_1 and h_3 to 1, we can scale up $\mathbf{x}^1$ by $(\mathbf{p}^1\mathbf{x}^3)$ instead of $(\mathbf{p}^1\mathbf{x}^3)/(5/6)$. The scalar factor between $\mathbf{x}^1$ and $\mathbf{x}^2$ is unaffected, as the adjustment would be by $\sqrt{5/6}\sqrt{5/6} = 5/6$ for both proposed choices of $\mathbf{h}$.

Figure 2.(c) shows RP (hatched) and $\widetilde{HRP}$ (solid) for $\mathbf{x}^3$. Note that the homothetically revealed preferred set is contained in the standard set, that is, the bounds are tighter without homotheticity. This is because $\mathbf{x}^2$ is already directly revealed preferred to $\mathbf{x}^3$ but has to be scaled up in order to adjust for violations of homotheticity. Figure 2.(d) replaces $\mathbf{x}^2$ with $\hat{\mathbf{x}}^2$, which is not revealed preferred to $\mathbf{x}^3$. This does not change homothetic efficiency as $\hat{\mathbf{x}}^2$ is on the same ray as $\mathbf{x}^2$ and chosen on a budget parallel to the one on which $\mathbf{x}^2$ was chosen. Hence, $\widetilde{HRP}$ remains unchanged while RP shrinks. This example also illustrates that whether or not homothetic bounds are tighter depends not only on homothetic efficiency, but also on how much information is already contained in the standard revealed preference relation.

3 Applications

In this section we illustrate some of the possible applications of our approach using data from Fisman et al. (2007). The data comes from an experimental dictator game in which a subject (the dictator) had to divide money between himself and some other subject (the recipient). The payoffs for the two subjects can be interpreted as two distinct goods. The transfer rate between the dictator and recipient varied, such that for every unit given up by the dictator, the recipient could receive more or less than one unit, which implies different prices for the two goods. Thus, subjects chose from budget sets as defined in Section 2.1. Fisman et al. (2007) estimate CES utility functions for their data. As this function is homothetic, applying the assumption of homotheticity without a particular functional form as we do here is still less restrictive. In terms of robustness (i.e., drawing conclusions not justified by the data), our approach lies between using only the revealed preference relation to recover bounds and estimating parameters of a homothetic utility function. The greater the homothetic efficiency, the closer the robustness of our approach is to that of the former.

The data set is well suited for our illustration because there are only two goods. This facilitates a graphical illustration, and the recovered preferences are easy to interpret in terms of social preferences.

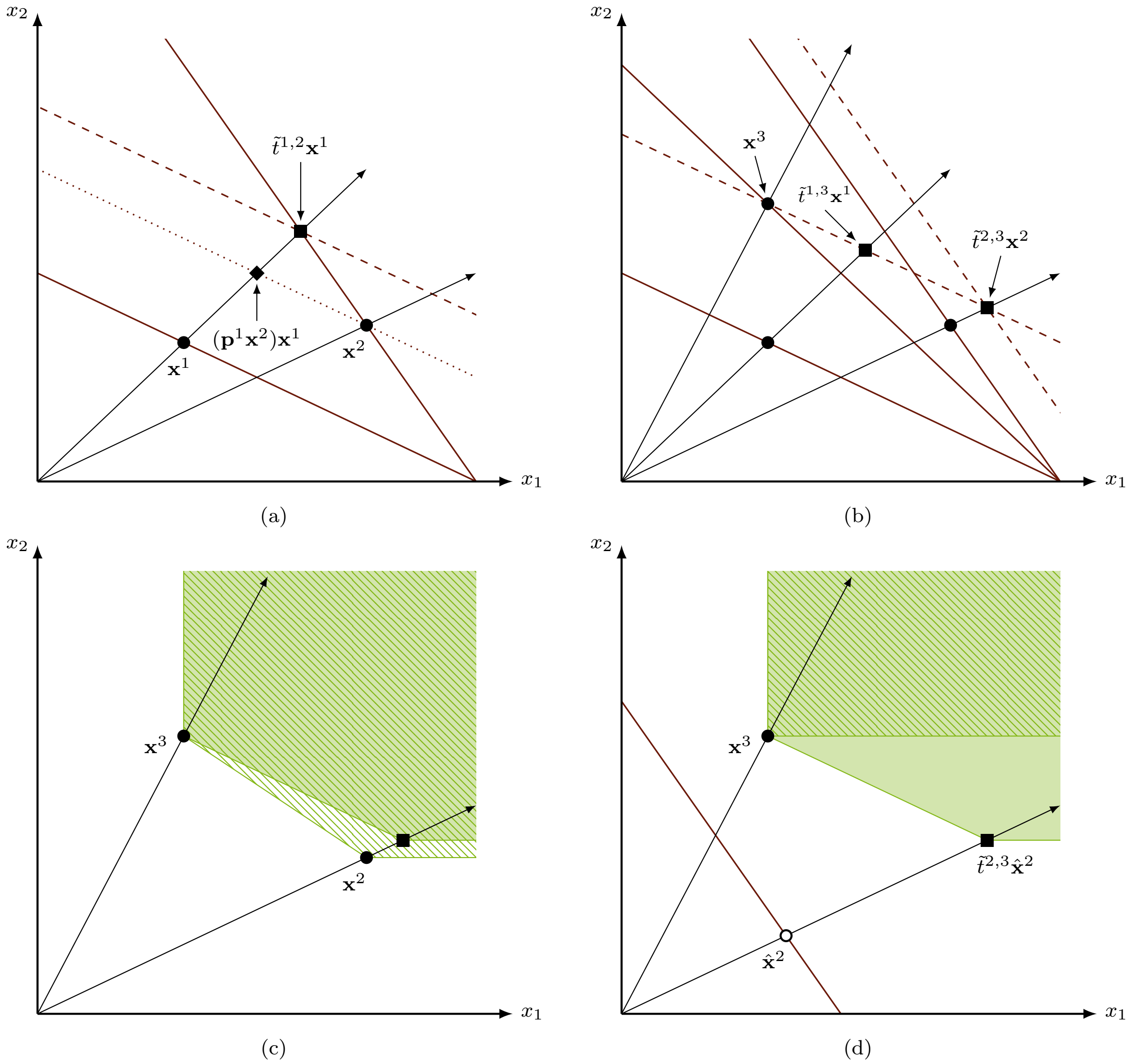

Figure 2: (a) Example with choices $\mathbf{x}^1 = (8, 8)$, $\mathbf{x}^2 = (18, 9)$ and prices $\mathbf{p}^1 = (1/24, 1/12)$, $\mathbf{p}^2 = (1/24, 1/36)$. (b) The choice $\mathbf{x}^3 = (8, 16)$ with prices $\mathbf{x}^3 = (1/24, 1/24)$ is added to the previous example. (c) Revealed preferred (hatched) and homothetically revealed preferred set (solid) of $\mathbf{x}^3$. (d) Choice $\mathbf{x}^2$ with prices $\mathbf{p}^2$ is replaced by $\hat{\mathbf{x}}^2 = (9, 4.5)$ with prices $\hat{\mathbf{p}}^2 = (2/18, 2/9)$, which changes the revealed preferred set of $\mathbf{x}^3$.

Furthermore, the subjects chose from 50 different budgets, which provides a rich data set. Note that given the large number of observations, there is already a lot to recover about preferences without homotheticity, so we compare our approach with an already quite informative benchmark. For simplicity, we will refer to the non-homothetic benchmarks as the "standard" case.

Subjects in Fisman et al. (2007) were not required to spend the entire endowment; instead, they could click with the mouse on or below a line representing the budget. Most subjects picked bundles on or very close to the budget line, but a few made unreasonable choices far below the line. Fisman et al. (2007) also address this issue and exclude 12 (15.79%) of the 76 subjects from their analysis because their choices were too wasteful; we do the same here and only present the result for the remaining 64 subjects. We use the mean of each subject's VEV and HEV to characterize their level of efficiency; the mean (median) over the 64 subjects is 0.989 (0.999) for the VEV, and 0.964 (0.978) for the HEV. Heufer and Hjertstrand (2017) define a misspecification index based on vector efficiency (MSIV), which is the average normalized difference between the values of the VEV and HEV. It measures by how much more data which have already been adjusted to fit the utility maximization model need to be adjusted to also fit the homothetic model. The mean (median) of the MSIV over all 64 subjects is 0.020 (0.023).

3.1 Graphical Illustrations

Choi et al. (2007b) already provide a short illustrative graphical analysis of revealed preferred and worse sets for data on risk preferences collected by Choi et al. (2007a). We extend this approach using our homothetically revealed preferred and worse sets. Figure 3 shows the recovered standard and homothetically revealed preferred and worse sets of four subjects of Fisman et al. (2007) with different prototypical preferences. As in Figure 1, the hatched areas show the revealed preferred set $RP(\mathbf{x})$ and the revealed worse set $RW(\mathbf{x})$ using the revealed preference relation adjusted by the VEV. The solidly filled areas show the corresponding homothetically revealed preferred and worse sets, $\widetilde{HRP}_{\mathbf{h}}(\mathbf{x})$ and $\widetilde{HRW}_{\mathbf{h}}(\mathbf{x})$, respectively. The bundles for which Figure 3 shows the bounds on preferred and worse sets are indicated in the figure; they are actually chosen bundles that are representative of the respective subjects' choices. Note how much narrower the space between the homothetically revealed preferred and worse sets is compared to the standard sets. For all subjects presented in Figure 3, homothetic efficiency is high and misspecification is low.[4] We therefore have high confidence in the validity of the recovered bounds.

For Figure 3, we have chosen four subjects whose preferences are diverse, ranging from strong inequality aversion to a strong preference for maximizing the sum of the two payoffs, and another one who is very selfish. Nonetheless, they all come close to preferences that can be represented by the CES utility function $u(x_1, x_2) = (\alpha x_1^\rho + (1-\alpha)x_2^\rho)^{1/\rho}$ with $\alpha \in [0, 1]$ and $\rho \leq 1$. The four prototypical utility functions defined below are all nested within the CES function (Mas-Colell et al. 1995, p. 97).

The subject in Figure 3.(a) clearly exhibits almost perfectly selfish preferences, that is, $u(x_1, x_2) = x_1$. This function is obtained from the CES function by setting $\alpha = 1$ (with arbitrary ρ). Figure 3.(b) shows the preferences of a subject who considers own payoff and payoff for somebody else to be almost perfect substitutes and is therefore very concerned about the total amount, that is, $u(x_1, x_2) = x_1 + x_2$. This function corresponds to $\alpha = 1/2$ and $\rho = 1$ in the CES function. In sharp contrast, the subject in Figure 3.(c) exhibits preferences that come close to maximizing the minimum payout for the dictator and recipient, that is, they come close to maximizing the Rawls (or Leontief) function $u(x_1, x_2) = \min\{x_1, x_2\}$ (Rawls 1971). This function is obtained as $\rho \to -\infty$ with $\alpha = 1/2$. Figure 3.(d) shows preferences of a subject who comes close to maximizing a Cobb-Douglas function $x_1^\alpha x_2^{1-\alpha}$ with α close to 1/2. This subject therefore comes close to maximizing the Nash product, that is, $u(x_1, x_2) = x_1 x_2$ (Nash 1950), which strikes a balance between inequality and the total amount that is paid out. This function is obtained as $\rho \to 0$ with $\alpha = 1/2$.

This section demonstrates the potential of graphical inspection and illustrates how much there is to learn from homothetic recoverability. A more systematic analysis is conducted in Section 3.3, where we use several ways to classify subjects based on which of the four preferences their choices come closest to.

3.2 Money Metric

Varian (1982) considers the revealed preference implications for Samuelson's (1974) "money metric" cardinalization of utility. The money metric utility of a bundle $\mathbf{x}^0$ at prices $\mathbf{p}^0$ is the amount a consumer must spend to obtain the utility associated with $\mathbf{x}^0$ at prices $\mathbf{p}^0$. Varian provides an approximation of the upper bound on the true money metric function; Knoblauch (1992) shows that this bound is already

[4] The mean (median) of the HEV values are: Subject 8: 0.995 (0.996); subject 27: 0.987 (0.990); subject 16: 0.954 (0.972); subject 35: 0.992 (0.993); subject 14 (Figure 1): 0.989 (0.990). The MSIV values are: Subject 8: 0.004; subject 27: 0.009; subject 16: 0.028; subject 35: 0.007; subject 14: 0.011.

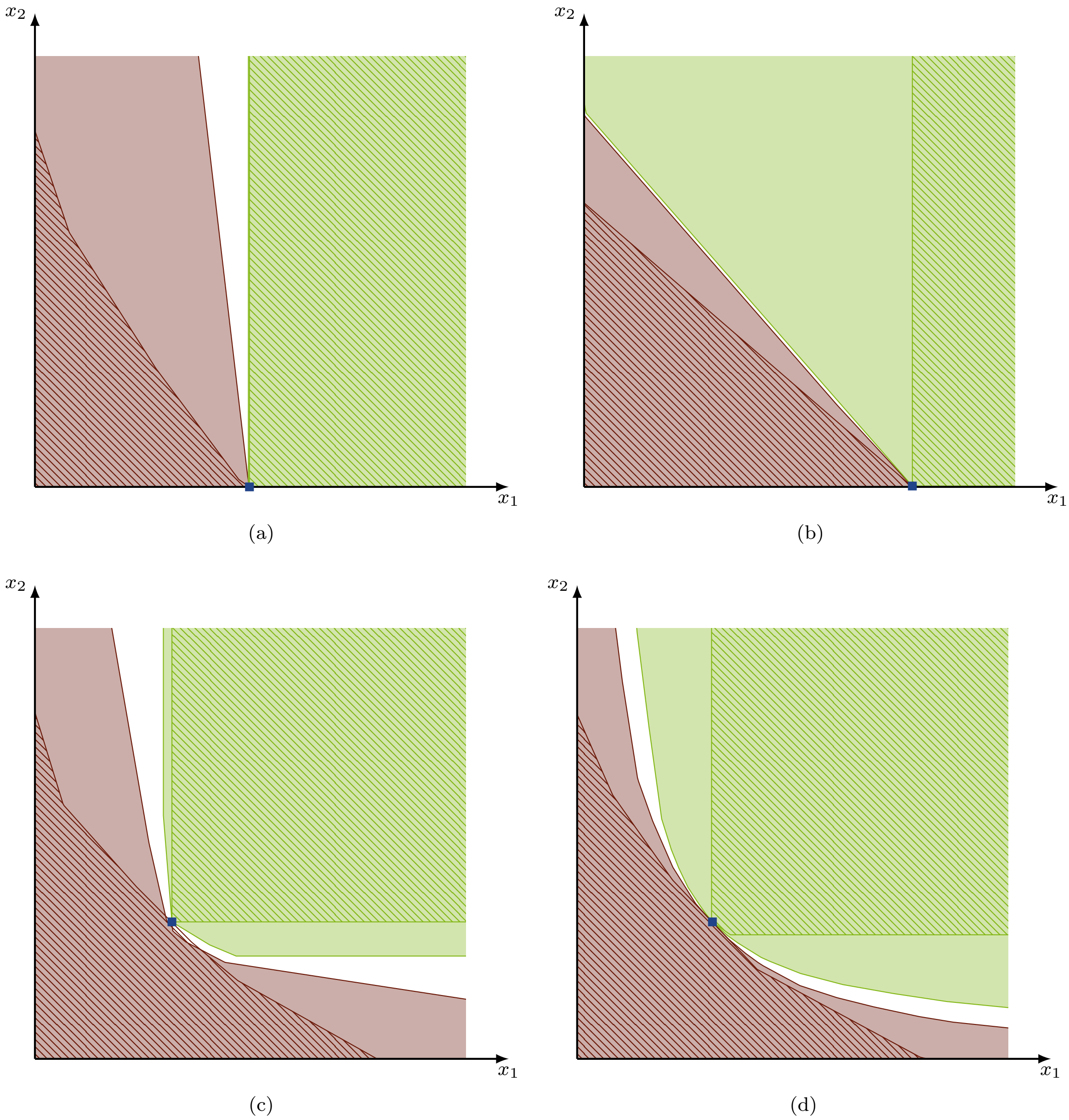

Figure 3: Standard and homothetically revealed preferred and worse sets for four subjects with prototypical preferences: (a) Selfish preferences, subject 8; (b) perfect substitutes, subject 27; (c) Rawls or Leontief preferences, subject 16; (d) Nash preferences, subject 35. Data from Fisman et al. (2007).

the best possible bound given the observed data. In Knoblauch (1993) she suggests to base the upper bound not on the revealed preferred but the homothetically revealed preferred set. If the assumption of homotheticity can be justified, this can provide a considerably tighter bound on the money metric.

In particular, if homothetic efficiency is the same as the standard efficiency (i.e., the data requires no further adjustment in addition to what is necessary to fit the utility maximization model), then the homothetic bounds will be tighter. If instead homothetic efficiency is considerably lower and large additional adjustments are needed, then the homothetic bounds are not necessarily tighter. In that sense, the ratio of the homothetic and standard bounds can be interpreted as a measure for how useful it is to adjust the data to fit the homothetic model; see also the discussion of the results in Table 1 below.

The difference between the standard money metric and the homothetic money metric quantifies the differences of the extents to which preferences can be recovered from the data. Revisiting the idea of the money metric is also very useful because of the recent application of the money metric function to parametric recoverability by Halevy et al. (2018). When their approach is used to recover preferences of a homothetic utility function, imposing homotheticity on the money metric function does not add further restrictions and optimally exploits the information contained in the data. Blackorby and Donaldson (1988) show that money metric utility functions for arbitrary reference prices are generally not concave unless preferences are homothetic, which also adds to the attractiveness of our approach.

Varian (1982) defines the upper bound of the *money metric function* as $m^+(\mathbf{p}^0, \mathbf{x}^0) = \inf_{\mathbf{x} \in RP(\mathbf{x}^0)} \mathbf{p}^0\mathbf{x}$. Knoblauch (1992) clarifies some issues and shows that the following more practically useful definition is equivalent:

$$m^+(\mathbf{p}^0, \mathbf{x}^0) = \min \mathbf{p}^0\mathbf{x}^i \quad \text{such that} \quad \mathbf{x}^i \,\mathrm{R}\, \mathbf{x}^0. \tag{12}$$

Knoblauch (1993) introduces the *homothetic* money metric function based on $HRP(\mathbf{x}^0)$:

$$\begin{aligned} hm^+(\mathbf{p}^0, \mathbf{x}^0) &= \min_{\mathbf{x} \in HRP(\mathbf{x}^0)} \mathbf{p}^0\mathbf{x} \\ &= \min_{i \in \{0,\dots,N\}} \{\mathbf{p}^0\, t_{i,0}\mathbf{x}^i\}. \end{aligned} \tag{13}$$

Figure 4 illustrates the idea. It shows the bounds of the standard and homothetically revealed preferred sets of an arbitrary bundle $\mathbf{x}^0$. The $\mathbf{x}$ that minimizes $\mathbf{p}^0\mathbf{x}$ such that $\mathbf{x} \in RP(\mathbf{x}^0)$ is $\mathbf{x}^i$; thus, $m^+(\mathbf{p}^0, \mathbf{x}^0) = \mathbf{p}^0\mathbf{x}^i = \lambda_1$. Then the budget line with $\mathbf{p}^0$ divided by λ_1 (or equivalently, the budget line with expenditure λ_1) passes through $\mathbf{x}^i$. The bound for the homothetic money metric function is $hm^+(\mathbf{p}^0, \mathbf{x}^0) = \mathbf{p}^0\, t_{i,0}\mathbf{x}^i = \lambda_2$, which is a considerably tighter upper bound than λ_1.

A particularly useful application of the money metric is to evaluate bundles that have not been observed as a choice at prices not faced by decision makers and to predict the utility an individual would experience from it. To illustrate the differences between money metric and homothetic money metric, we therefore evaluate the same bundle (which has never been chosen) at the same price vector (which subjects never faced) for every subject. We use the mean demand of all subjects, $\bar{\mathbf{x}} = (41.161, 10.074)$, as the bundle and evaluate it at prices $\mathbf{q}$ with $q_1 = q_2$ such that $\mathbf{q}\bar{\mathbf{x}} = 1$; this also means that if there is no relevant information in the data at all, both $m^+(\mathbf{q}, \bar{\mathbf{x}})$ and $hm^+(\mathbf{q}, \bar{\mathbf{x}})$ will be equal to 1. We also repeated the analysis with price ratios of 1/2 and 2 and obtained qualitatively similar results, which we will not report here. If assuming homotheticity does indeed allow to construct tighter upper bounds, $hm^+(\mathbf{q}, \bar{\mathbf{x}})$ should be smaller than $m^+(\mathbf{q}, \bar{\mathbf{x}})$. We note that the difference between the two values could also be helpful for deciding whether or not it is justified to assume homotheticity; for a subject with tighter bounds without homotheticity, it might be better not to.

Table 1 shows the mean values over all subjects and various percentiles. It also shows the means and

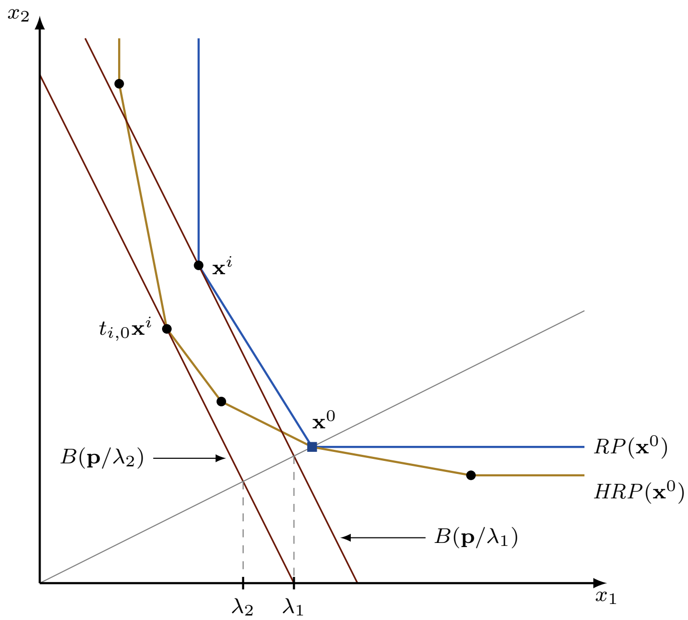

Figure 4: A tighter upper bound for the money metric utility function. The revealed preferred set $RP(\mathbf{x}^0)$ is contained in the homothetically revealed preferred set $HRP(\mathbf{x}^0)$; the figure only shows their boundaries. The upper money metric bound $m^+(\mathbf{p}, \mathbf{x}^0)$ is equal to λ_1, as $\mathbf{px}$ is minimized at $\mathbf{px}^i$, while the homothetic upper money metric bound $hm^+(\mathbf{p}, \mathbf{x}^0)$ is equal to λ_2.

percentiles of the individual ratios of the homothetic and standard money metric as well as the differences. The mean of the lower bounds for the standard money metric is 0.978, which is noticeably greater than the mean of the lower bounds for the homothetic money metric at 0.91; this general finding is confirmed by the ratios and the differences.

Money metric evaluations of mean demand with $q_1/q_2 = 1$						
	mean	p5	p25	p50	p75	p95
$m^+(\mathbf{q}, \bar{\mathbf{x}})$ (standard)	0.978	0.906	0.964	0.995	1.000	1.000
$hm^+(\mathbf{q}, \bar{\mathbf{x}})$ (homothetic)	0.910	0.823	0.845	0.916	0.991	1.000
$m^+(\mathbf{q}, \bar{\mathbf{x}})/hm^+(\mathbf{q}, \bar{\mathbf{x}})$ (ratios)	1.081	0.988	1.000	1.052	1.142	1.206
$m^+(\mathbf{q}, \bar{\mathbf{x}}) - hm^+(\mathbf{q}, \bar{\mathbf{x}})$ (differences)	0.067	-0.012	0.000	0.048	0.119	0.168

Table 1: Upper bounds of the standard and homothetic money metric utility function for a price ratio of one with normalized price vectors for the mean demand ($\bar{\mathbf{x}} = (41.161, 10.074)$), for the 64 participants without excessively wasteful choices. Data from Fisman et al. (2007).

We consider this to be an economically significant improvement on the tightness of the upper bound of the money metric. However, it is important to note that the improvement is not universal, as we can already see by the result for the 5th percentile. Figure 5 shows an example of a subject for which the bound is tighter without homotheticity; it is caused by the necessity to adjust a small number of choices considerably more to achieve consistency with HARP than with GARP.

Figure 6 shows a scatter plot of the results. Points below the 45° line show the subjects whose choices, due to low homothetic efficiency compared to standard efficiency, require so much adjustment that the bound on the money metric is less tight under homotheticity. There are 5 (7.81%) subjects for whom this is the case. In total, there are 19 (29.69%) for whom the difference is less than 0.01 and 32 (50.00%) for whom the difference is less than 0.05. As expected, the benefit depends on the level of homothetic efficiency and, in particular, on the extent of misspecification. The correlation coefficient between the ratios and (i) the mean HEV is 0.362 and (ii) the MSIV is -0.438. The correlation coefficient between the

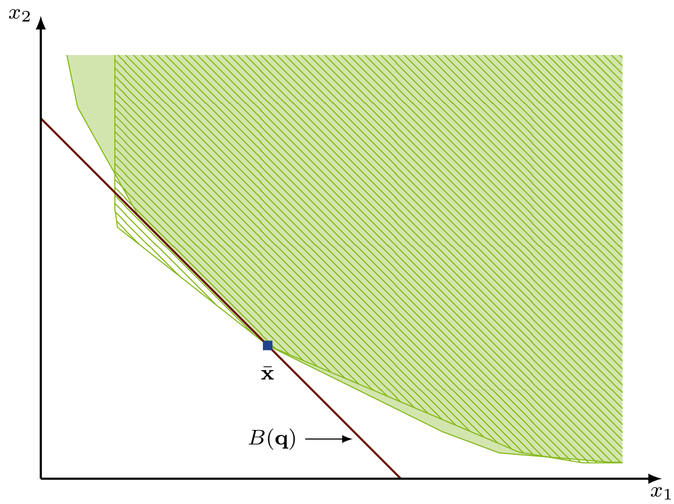

Figure 5: Revealed preferred (hatched) and homothetically revealed preferred (solid) sets of subject 74 at $\bar{\mathbf{x}} = (41.161, 10.074)$ and a budget line through $\bar{\mathbf{x}}$ with a price vector $\mathbf{q} = (q_1, q_2)$ such that $q_1 = q_2$ and $\mathbf{q}\bar{\mathbf{x}} = 1$. Even though the mean HEV for this subject is 0.933, two choices have to be adjusted by an h_i of 0.772 and 0.779 respectively, while the minimum of the VEV is 0.939. Accordingly, this subject has a moderately high MSIV of 0.055. As a result, the RP set is not always completely contained in HRP. In this case, the bound for the standard money metric is tighter than for the homothetic money metric, that is, $m^+(\mathbf{q}, \bar{\mathbf{x}}) < hm^+(\mathbf{q}, \bar{\mathbf{x}})$.

differences and (i) the mean HEV is 0.401 and (ii) the MSIV is -0.476.

We conclude that assuming homotheticity allows us to construct significantly tighter bounds for at least half of the subjects while it is rare that the assumption leads to worse results. This demonstrates that in most cases, assuming homotheticity allows the researcher to draw more precise conclusions. Nonetheless, the findings highlight the importance of carefully considering the level of homothetic efficiency and testing if assuming homotheticity really does add to the analysis in a meaningful way. The differences in the bounds on the money metric can guide the researcher in interpreting the practical effects of choosing the homothetic model, as even a high level of homothetic efficiency does not necessarily lead to considerably tighter bounds.

3.3 Classification

Classifying subjects into preference types can be helpful to obtain an overview of the data and the extent of heterogeneity between subjects. In the context of dictator games, Andreoni and Miller (2002) and Porter and Adams (2016) classify their subjects depending on how close their choices come to being generated by certain prototypical utility functions. For example, Andreoni and Miller (2002) use three of the four utility functions that we define in Section 3.1 (selfish, perfect substitutes, Rawls) and classify subjects based on which one of the utility functions generates choices that are closest in terms of Euclidean distance to the choices made by a subject. The four examples of our analysis in Section 3.1 (see Figure 3) indicate that it should be possible to use the tight homothetic bounds on preferred and worse sets to classify choices. Alternatively, we can use the homothetic money metric to evaluate "distance" between choices in economic terms rather than Euclidean. We use both approaches to determine which of the four utility functions defined in Section 3.1 come closest to the recovered preferences of subjects in Fisman et al. (2007). Note that neither Andreoni and Miller (2002) nor Porter and Adams (2016) use the Nash utility function. We use it here because it is an intermediate case between the two extreme cases of Rawls and perfect substitutes preferences. It also appears to be an empirically relevant case as several subjects do exhibit preferences like those shown in Figure 3.(d).

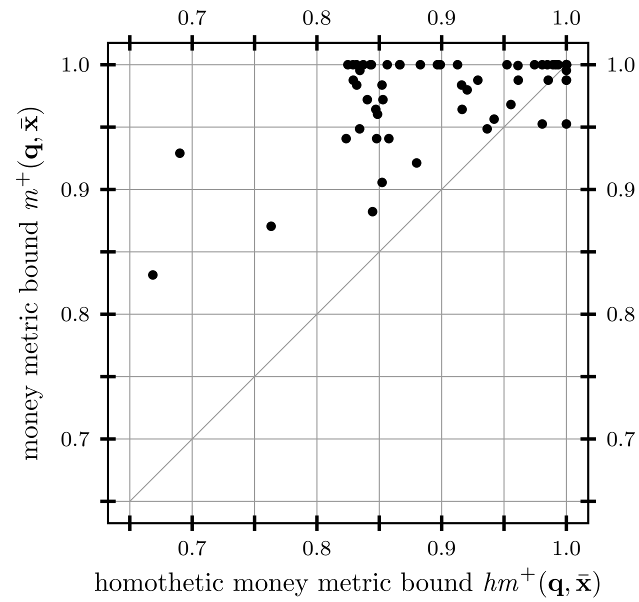

Figure 6: Scatter plot of the bounds on the homothetic and standard money metric for each subject at the mean demand $\bar{\mathbf{x}}$. Data from Fisman et al. (2007).

To use the bounds on preferred and worse sets, we proceed as follows. For each subject, we generate choices on each budget by maximizing the prototypical utility functions. For each prototypical choice on each budget, we construct the homothetically revealed preferred set for that choice based on the actual choices of that subject. We then count how many of the other prototypical choices are in that set. If, say, the choice implied by Rawls preferences is in the homothetically revealed preferred set of the choice implied by Nash preferences, we can conclude that the subject prefers the Rawls choice over the Nash choice. We then classify subjects according to which set of prototypical choices is most frequently preferred.

For each subject, let $i \in \{1, \ldots, 50\}$ be the index of the 50 budgets. Let $j, k \in \{$*selfish*, *perfect substitutes*, *Rawls*, *Nash*$\}$ be indices for choices generated by the four prototypical utility functions. Let $\mathbf{y}_j^i$ be the bundle that maximizes utility function j on budget i. Note that $\widetilde{HRP}_{\mathbf{h}}(\mathbf{y}_j^i)$ is the homothetically revealed preferred set of $\mathbf{y}_j^i$ based on the subject's choices. The bundle chosen on budget i by the subject, $\mathbf{x}^i$, will be different from $\mathbf{y}_j^i$ unless the subject happened to choose exactly the bundle that maximizes utility function j on that budget. Therefore, budget i will often intersect $\widetilde{HRP}_{\mathbf{h}}(\mathbf{y}_j^i)$; the question is if one of the other prototypical choices $\mathbf{y}_k^i$, $k \neq j$, are included in it. Let c_k be the score for utility function k; it counts for how many budgets the subject prefers the bundle that maximizes utility function k over the bundle that maximizes one of the other three utility functions. The following pseudo-code describes the procedure.

for $k \in \{$*selfish*, *perfect substitutes*, *Rawls*, *Nash*$\}$ **do**
  set $c_k = 0$;
  for $j \in \{$*selfish*, *perfect substitutes*, *Rawls*, *Nash*$\}$ **do**
    for $i \in \{1, \ldots, 50\}$ **do**
      if $\mathbf{y}_k^i \in \widetilde{HRP}_{\mathbf{h}}(\mathbf{y}_j^i)$ **then**
        set $c_k = c_k + 1$;

Each c_k will be between 50 and 200. Each subject will be classified as being closest to the prototypical utility function associated with the greatest c_k.

This way of classifying choices leads to an interesting observation. Suppose that a subject in a dictator game always gives away a very small amount to the recipient (x_2) and keeps the rest (x_1). The choices will be near the x_1-axis but never on it. By the construction of *HRP*, the homothetically revealed preferred sets of these choices will never contain a point on the x_1-axis. A choice $\mathbf{y}$ generated by maximizing the perfectly selfish prototypical utility function, however, will always be on the x_1-axis. This $\mathbf{y}$ will therefore never be in any $HRP(\mathbf{x}^0)$ unless $\mathbf{x}^0$ is itself on the x_1-axis, and the subject will not be classified as selfish. This is even more relevant when using *RP* instead. In the homothetic case, a single choice on the x_1-axis can be scaled and makes it possible for all *HRP* sets to contain parts of the x_1-axis, which is not the case for *RP*.

We consider this to be neither better nor worse than other ways of classification, but rather as a complementary view. An individual with the utility function $u(x_1, x_2) = x_1^{1-\varepsilon} x_2^{\varepsilon}$ will make choices near the x_1-axis for small enough ε, and not classifying this individual as being closest to selfishness may seem odd. But it is indeed the case that for all $\varepsilon > 0$ this individual will prefer any other bundle over the perfectly selfish one, with the exception of the opposite extreme on the x_2-axis.

For the second approach, we evaluate each of the prototypical choices $\mathbf{y}_j^i$ using the homothetic money metric function, that is, we compute $hm^+(\mathbf{p}^i, \mathbf{y}_j^i)$ for all $i \in \{1, \ldots, 50\}$ and all $j \in \{$*selfish*, *perfect substitutes*, *Rawls*, *Nash*$\}$ and classify each subject as being closest to the prototypical utility function associated with the j with the greatest mean homothetic money metric.

The results for these two approaches are shown in Table 2. The table also shows the results based on the same approaches without homotheticity, that is, based on *RP* instead of *HRP*, and on m^+ instead of hm^+. As expected, classification based on homothetic revealed preferred sets leads to considerably fewer subjects being classified as selfish, and the difference is even greater for standard revealed preferred sets. Classification based on *HRP* leads to more Rawls and fewer Nash types than classification based on *RP*. Figure 3.(c) and (d) might help to explain this. The part of the preferred and worse sets that are recovered without homotheticity are quite similar for the two subjects, while the recovered parts under homotheticity allow to draw a clearer distinction between the two. Classification results based on the money metric appear to be somewhat between those based on Euclidean distance and revealed preferred sets, with only minor differences between hm^+ and m^+. As discussed in Section 3.2, part of the advantage of hm^+ over m^+ comes into play when evaluating bundles at prices not faced by subjects. It is possible that assuming homotheticity here does not make much of a difference since the prices used for the money metric evaluations are those that subjects actually were presented with.

	Preference types			
Method	Selfish	Perfect substitutes	Rawls	Nash
Euclidean distance	43 (67.19%)	5 (7.81%)	3 (4.69%)	13 (20.31%)
Money metric	39 (60.94%)	2 (3.12%)	1 (1.56%)	22 (34.38%)
Homothetic money metric	37 (57.81%)	5 (7.81%)	2 (3.12%)	20 (31.25%)
Revealed preferred	27 (42.19%)	4 (6.25%)	2 (3.12%)	31 (48.44%)
Homothetically revealed preferred	31 (48.44%)	3 (4.69%)	10 (15.62%)	20 (31.25%)

Table 2: Classification of the 64 participants without excessively wasteful choices. Participants are assigned to one of four prototypical preference types using various methods. Data from Fisman et al. (2007).

Experimental data as well as large household demand panel or cross-section data sets can also be analyzed in terms of how many preference types there are, either by finding the minimal partition of the data such that each part is perfectly rationalizable (see Crawford and Pendakur 2013) or by finding the number of types depending on a threshold for a measure of difference between them (see Castillo and Freer 2018). The approaches depend on consistency with GARP and GARP-based efficiency measures, but could also be conducted in terms of HARP. As HARP is a more stringent condition, one would naturally expect to need a greater number of types to represent the data. For example, two households with very different incomes will choose from very different budgets which are likely to not intersect. It may then be possible to represent both by the same utility function, even if it is a utility function that leads to very different budget allocations at the two different income levels, while it may not be possible to represent the households by the same *homothetic* utility function. Whether this is a desirable result or not depends among other things on whether or not it is reasonable to assume homotheticity in the given context.

4 Discussion and Conclusion

In this paper we provide methods to recover homothetic preferences from choice data and illustrate how this can be used when data do not perfectly satisfy homotheticity but come close to it. The illustrations are based on experimental data because the methods in this paper were motivated by and are particularly well suited for the kind of data encountered in experimental economics. Recovering much of a preference on an individual level requires choices under various different relative prices, and as experimenters are free to choose the budgets offered to subjects, that data is easy to observe in the lab. Experiments are also often conducted with research questions on individual preferences and preference heterogeneity among individuals in mind. Those are questions for which the methods provided here are particularly helpful.

There are many other applications possible which we did not consider in detail here. In general, every application that relies on revealed preferred and worse sets can potentially be improved with our method. For example, Heufer (2014) shows how to use revealed preference to operationalize Yaari's (1969) approach to comparative risk aversion. He compares the risk aversion implicit in choices over lotteries between different individuals based on their revealed preferred and worse sets. He finds that there are pairs of subjects whose choices do not reveal enough information in order to distinguish their level of risk aversion. Given how much more we can recover under homotheticity, the approach can benefit from using homothetically revealed preferred and worse set rather than the standard ones whenever the assumption of homotheticity is justified. As a further example, consider Halevy et al. (2018) method to estimate parameters of a utility function by minimizing a loss function corresponding to the money metric index proposed by Varian (1990). As the bounds on the money metric function can be tightened using homothetic revealed preferences, our approach can be used to estimate parameters by optimally exploiting the information contained in choice data under homotheticity.

Heufer and Hjertstrand (2017) also compute homothetic efficiency of demand for consumption goods by households using a large panel data set. The methods introduced in this paper build on that concept of homothetic efficiency, but applications to household demand data requires to address some additional issues.

First, recent literature has shown that household behavior can potentially be better explained by a collective household model (Cherchye et al. 2007, 2011a, Cherchye et al. 2011b, Aguiar and Kashaev 2017). Our approach can be modified to recover homothetic preferences in the collective household model based on the revealed preference characterization by Cherchye et al. (2011a). In this context, the assumption of homotheticity may not only give tighter bounds on indifference curves but it may also allow for a more efficient procedure to test the collective household model (see also Shananin and Tarasov (2017) on computational issues under the assumption of homotheticity).

Second, we need to compare our approach to the related work by Blundell et al. (2003, 2007, 2008) and Blundell et al. (2015). These authors use empirical expansion paths in combination with revealed preference theory to conduct welfare analysis in a population of heterogeneous households. Their approach is largely based on the assumption that there exists demand data for (different) cross-sections of households drawn at consecutive time periods. Our approach of preference recoverability is not suitable for some of this kind of data, as we require choices on the level of an individual or a single household. On the other hand, while it would be possible to use non- or semi-parametric methods to estimate individual Engel curves in experimental data, the amount of data required is usually not available on an individual level.

The approach of Blundell et al. (2003, 2007, 2008) and Blundell et al. (2015) is applicable to aggregated time series data to analyze the preferences of a representative consumer or to each individual household in household survey panel data; however, the latter case requires a sufficient amount of data so that non-parametric estimation of individual household Engel curves is possible. In contrast, our approach can be meaningfully applied to data sets with only a small number of observations. This also illustrates the difference in how the two approaches deal with heterogeneity. While the approach of Blundell et al. (2003, 2007, 2008) and Blundell et al. (2015) assumes that observable characteristics can capture much of the heterogeneity between households, we can apply our approach to each individual household or experimental subjects and therefore avoid any assumptions about preference homogeneity across individuals.

We believe that revealed preference methods have more to offer than only testing data for consistency with a particular decision model. In this paper we have demonstrated that using efficiency measures to make small adjustments to data that are close to consistency with homotheticity allows to learn much about preferences implicit in choice data and can open up a range of applications. We hope that our paper thereby contributes to the growing literature aiming to establish nonparametric revealed preference analysis as a robust complement to parametric analysis.

A Appendix

Proof of Theorem 4. By induction. Suppose u is homothetic, monotonic, continuous, and concave, and $\mathbf{h}$-rationalizes the data. By definition, we can assume without loss of generality that u is homogeneous of degree 1. Because u is concave, we only need to consider the vertices of the closure of $\widetilde{HRP}_{\mathbf{h}}$, that is, we only need to check if $u(\tilde{t}_{i,m}\mathbf{x}^i) < u(\mathbf{x}^m)$ is possible.

Step 1 By homogeneity of degree 1, $u([\mathbf{p}^i\mathbf{x}^m/(h_i h_m)]\mathbf{x}^i) = [\mathbf{p}^i\mathbf{x}^m/(h_i h_m)]u(\mathbf{x}^i)$. Let $\mathbf{y} = [h^i\mathbf{x}^m]/[\mathbf{p}^i\mathbf{x}^m]$; then $\mathbf{p}^i\mathbf{y} = h_i$, and therefore $\mathbf{x}^i \, \mathrm{R}^0(h_i) \, \mathbf{y}$. Suppose $u([\mathbf{p}^i\mathbf{x}^m/(h_i h_m)]\mathbf{x}^i) < u(\mathbf{x}^m)$. Then $u(\mathbf{x}^i) < (h_i h_m/[\mathbf{p}^i\mathbf{x}^m])u(\mathbf{x}^m)$. But with $h_m \leq 1$, $(h_i h_m/[\mathbf{p}^i\mathbf{x}^m])u(\mathbf{x}^m) \leq (h_i/[\mathbf{p}^i\mathbf{x}^m])u(\mathbf{x}^m) = u(\mathbf{y})$. Then $u(\mathbf{x}^i) < u(\mathbf{y})$, but $\mathbf{x}^i \, \mathrm{R}^0(h_i) \, \mathbf{y}$, so u cannot $\mathbf{h}$-rationalize Ω. Thus, $([\mathbf{p}^i\mathbf{x}^m]/h_i h_m)u(\mathbf{x}^i) \geq u(\mathbf{x}^m)$.

Step 2 Assume without loss of generality that

$$\tilde{t}_{1,n} = \frac{\mathbf{p}^1\mathbf{x}^2}{h_1 h_2}\frac{\mathbf{p}^2\mathbf{x}^3}{h_2 h_3}\cdots\frac{\mathbf{p}^{n-1}\mathbf{x}^n}{h_{n-1}h_n}$$

and that $\tilde{t}_{1,m} = \tilde{t}_{1,n}[\mathbf{p}^n\mathbf{x}^m]/(h_n h_m)$. Suppose $\tilde{t}_{1,n}u(\mathbf{x}^1) \geq u(\mathbf{x}^n)$. Then

$$\tilde{t}_{1,m}u(\mathbf{x}^1) = \tilde{t}_{1,n}u(\mathbf{x}^1)\frac{\mathbf{p}^n\mathbf{x}^m}{h_n h_m} \geq u(\mathbf{x}^n)\frac{\mathbf{p}^n\mathbf{x}^m}{h_n h_m} \geq u(\mathbf{x}^m),$$

where the last inequality follows from Step 1. Thus, $\tilde{t}_{1,m}u(\mathbf{x}^1) \geq u(\mathbf{x}^m)$. By induction, Steps 1 and 2 show that $u(\tilde{t}_{i,m}\mathbf{x}^i) \geq u(\mathbf{x}^m)$ for all i and all homothetic u which $\mathbf{h}$-rationalize Ω. That concludes the proof for $\widetilde{HRP}$. The second part of the theorem then follows from the definition of $\widetilde{HRW}$. $\square$

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