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Discrete Optimization

Min-ordering and max-ordering scalarization methods for multi-objective robust optimization

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ABSTRACT

Several robustness concepts for multi-objective uncertain optimization have been developed during the last years, but not many solution methods. In this paper we introduce two methods to find min-max robust efficient solutions based on scalarizations: the min-ordering and the max-ordering method. We show that all point-based min–max robust weakly efficient solutions can be found with the max-ordering method and that the min-ordering method finds set-based min–max robust weakly efficient solutions, some of which cannot be found with formerly developed scalarization based methods. We then show how the scalarized problems may be approached for multi-objective uncertain combinatorial optimization problems with special uncertainty sets. We develop compact mixed-integer linear programming formulations for multi-objective extensions of bounded uncertainty (also known as budgeted or Γ-uncertainty). For interval uncertainty, we show that the resulting problems reduce to well-known single-objective problems.

1. Introduction

When applying optimization techniques to real-world problems, one often encounters the difficulties, that several objectives need to be optimized at the same time and that not all parameters are known exactly in advance. In multi-objective optimization several objectives are optimized simultaneously by choosing a *(Pareto) efficient* solution that cannot be improved in one objective without worsening it in another objective. Robust optimization is a way to handle uncertainties, without having to assume any information on probability distributions, hedging against (all) possible outcomes. During the last years, concepts of those fields have been combined to *multi-objective robust optimization*.

Several concepts on how to define robust solutions in multi-objective optimization have been developed. The common (single-objective) concept of *min–max robustness* aims to find a solution that minimizes the objective function in the worst case. One generalization to multi-objective optimization, which we call *point-based min–max robust efficiency*, was first introduced by Kuroiwa and Lee (2012). They consider the worst case in each objective independently, which results in a deterministic multi-objective problem with bottleneck objective functions, called the robust counterpart. However, the resulting worst case point for a solution can differ significantly from the possible outcomes. Therefore, a second generalization of min–max robustness for multiple objectives has been developed by Ehrgott, Ide, and Schöbel (2014). They look at the outcome set of a solution under every scenario and compare these sets to each other to find so-called *set-based min–max robust efficient* solutions. A comparison of these two and other concepts for robust efficiency can be found in Ide and Schöbel (2016) and Wiecek and Dranichak (2016).

Common methods to find efficient solutions in the deterministic case, i.e. without uncertainty, are so-called *scalarization methods*, where the multi-objective problem is transformed to a family of single-objective problems, whose solutions are (weakly) efficient for the original problem. By solving the resulting problems, several different (and possibly all) efficient solutions are found. For an overview on scalarization methods see, e.g., Ehrgott (2006).

In the uncertain case, several methods to find min–max robust efficient solutions have been developed, which are based on

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scalarizations: on the weighted sum and ϵ-constraint scalarization (Ehrgott et al., 2014), on the augmented weighted Chebyshev scalarization (Ide (2014)) and on p-norm scalarizations (Bokrantz & Fredriksson, 2017). Point-based min–max robust efficient solutions can also be found by applying deterministic scalarization methods to the robust counterpart (see, e.g., Fliege & Werner, 2014; Hassanzadeh, Nemati, & Sun, 2013; Kuroiwa & Lee, 2012).

In this paper we introduce two new methods to find min–max robust efficient solutions based on scalarizations: the *max-ordering* and *min-ordering* method, resulting in problems of the form min–max–max respectively min–max–min. The min-ordering problem can therefore be interpreted as a so-called *adjustable robust problem* (Ben-Tal, Goryashko, Guslitzer, & Nemirovski, 2004), where only part of the decisions have to be made before the realization of the uncertain parameters.

In robust optimization, the considered uncertainty set, i.e., the possible values the uncertain parameters can attain, plays an important role w.r.t. solvability and complexity of the resulting robust problems. In this paper we investigate the min-ordering and max-ordering optimization problems for multi-objective min–max robust combinatorial optimization problems with specific uncertainty sets: One popular assumption is that each parameter attains a value in a given interval independently of the realization of the other parameters (*interval uncertainty*). Based on this, Bertsimas and Sim (2003) introduced the (single-objective) concept of *bounded uncertainty*, assuming that the parameters vary in intervals, but the worst case is not attained for all parameters simultaneously. The concept has been studied extensively in single-objective robust optimization also under the names of *budgeted uncertainty* or Γ-*uncertainty*. Uncertainty sets for multi-objective optimization based on bounded uncertainty have been considered in Doolittle, Kerivin, and Wiecek (2012) and Wang, Li, Ding, Sun, and Wang (2017) (only considering uncertainty in the constraints) and in Hassanzadeh et al. (2013) and Raith, Schmidt, Schöbel, and Thom (2018b) (resulting in an objective-wise uncertainty set). We introduce an extension of bounded uncertainty to multi-objective optimization for the case that the uncertainties in the objectives are not independent of each other.

Solution approaches for multi-objective min–max robust combinatorial problems with objective-wise bounded uncertainty have been developed in Raith et al. (2018b). Kuhn, Raith, Schmidt, and Schöbel (2016) consider bi-objective robust combinatorial problems with finite and polyhedral uncertainty sets for several robustness concepts. The multi-objective robust version of the shortest path problem with finite uncertainty set is considered in Raith, Schmidt, Schöbel, and Thom (2018a), where labeling algorithms are extended in order to find robust efficient solutions.

This paper is structured as follows: First, we give a short introduction to multi-objective robust optimization. In Section 3 we introduce the min-ordering and max-ordering optimization problems and show their general properties. In Section 4 we consider combinatorial multi-objective optimization problems with particular uncertainty sets and investigate the complexity and solvability of the resulting min-ordering and max-ordering problems.

2. Preliminaries

In this section we introduce some general notation and give a short introduction to multi-objective optimization and multi-objective robust optimization.

Throughout this paper, we use the symbols $<$ (strictly less than) and $\leqq$ (less than or equal to) to compare values in $\mathbb{R}$. Further, ∂M denotes the boundary of a set $M \subseteq \mathbb{R}^k$ and we use $i \in [k]$ as an abbreviation for $i \in \{1, \dots, k\}$.

To shorten the text we use a [./.] notation, e.g., instead of "x is smaller than y if $x < y$ and x is smaller than or equal to y if $x \leqq y$" we write "x is smaller than [·/or equal to] y if $x[</\leqq]y$".

2.1. Multi-objective robust optimization

Definition 1. Given a set $\mathcal{X}$ of feasible solutions and $k \in \mathbb{N}$ objective functions $z_1, \dots, z_k : \mathcal{X} \to \mathbb{R}$, we call

$$\min_{x \in \mathcal{X}} z(x) = \begin{pmatrix} z_1(x) \\ \vdots \\ z_k(x) \end{pmatrix}$$

a *multi-objective optimization problem (MOP).*

If $k = 1$ we say that the problem is a *single-objective* problem. For $k \geqq 2$, a solution that minimizes all objectives simultaneously does usually not exist. Therefore, we use the concept of *efficient solutions*.

Definition 2. For two vectors $y^1, y^2 \in \mathbb{R}^k$ we use the notation

$$y^1 < y^2 \Leftrightarrow y_i^1 < y_i^2 \text{ for } i \in [k],$$
$$y^1 \leq y^2 \Leftrightarrow y_i^1 \leqq y_i^2 \text{ for } i \in [k] \text{ and } y^1 \neq y^2,$$
$$y^1 \leqq y^2 \Leftrightarrow y_i^1 \leqq y_i^2 \text{ for } i \in [k].$$

We also define $\mathbb{R}^k_{[>/\geq/\geqq]} := \{y \in \mathbb{R}^k : 0[</\leq/\leqq]y\}$.

Definition 3. A solution $x \in \mathcal{X}$ is a *[weakly/·/strictly] efficient solution* for (MOP), if there is no $x' \in \mathcal{X}$ such that $z(x')[</\leq/\leqq]z(x)$.

Note that a solution $x \in \mathcal{X}$ is [weakly/·/strictly] efficient if and only if there is no $x' \in X$ with

$$z(x') \in z(x) - (\mathbb{R}^k_{[>/\geq/\geqq]}).$$

We now assume that the input data is uncertain, i.e., not all parameters are exactly known in advance. Instead, they depend on a scenario, which will only be revealed after one has chosen a solution. The set $\mathcal{U}$ of all possible scenarios is called the *uncertainty set*.

Definition 4. Given a feasible set of solutions $\mathcal{X}$, an uncertainty set $\mathcal{U}$, and a multi-objective function $z : \mathcal{X} \times \mathcal{U} \to \mathbb{R}^k$, the family of multi-objective optimization problems

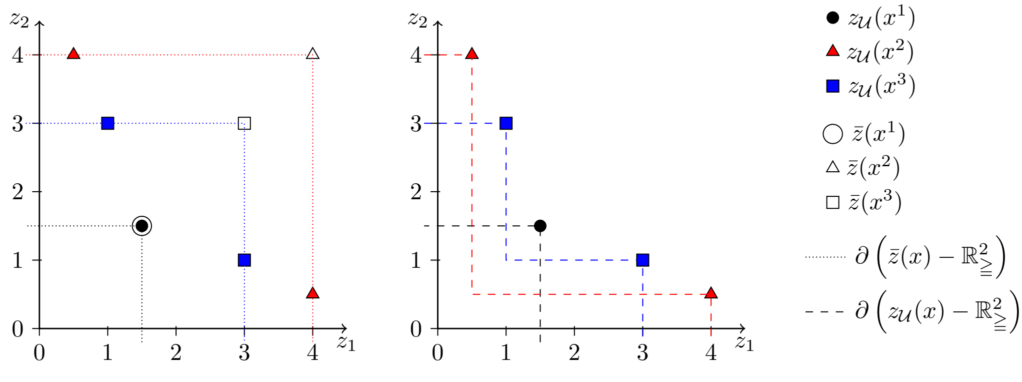
$$\left(\min_{x \in \mathcal{X}} z(x, \xi), \xi \in \mathcal{U} \right)$$

is called a *multi-objective uncertain optimization problem* (MOUP).

In the following we assume $\mathcal{X}$ and $\mathcal{U}$ to be compact and nonempty and the z_i to be continuous in x and ξ. If a problem or part of a problem is not subject to uncertainty, we say that it is *deterministic*, e.g., this is the case for a (MOUP) with $|\mathcal{U}| = 1$.

Note that the formulation in Definition 4 only considers uncertainty in the objective function. If the constraints, i.e., the set of feasible solutions, are subject to uncertainty, we aim to find solutions which are feasible in all scenarios (as proposed in the seminal works on robustness, see, e.g., Ben-Tal & Nemirovski, 1998; Soyster, 1973). For this purpose, the sets of feasible solutions under all scenarios can be intersected in advance to obtain a (deterministic) set of *robust feasible solutions*. Hence, in the following, we assume the feasible set $\mathcal{X}$ to be deterministic.

To decide what is a good solution for a multi-objective uncertain problem is not trivial. In single-objective robust optimization one looks for so-called robust optimal solutions. Often these are defined as solutions, which have a minimal worst case value, i.e., one solves $\min_{x \in \mathcal{X}} \max_{\xi \in \mathcal{U}} z(x, \xi)$ (see, e.g., Ben-Tal, El Ghaoui, & Nemirovski, 2009). This concept has been generalized to robust efficiency for multi-objective problems in various ways



(a) Only x^1 is pointMR efficient. (b) All solutions are setMR efficient.

Fig. 1. Determining pointMR efficient solutions and setMR efficient solutions for the instance in Example 6.

(e.g., Ehrgott et al., 2014; Kuroiwa & Lee, 2012), since the notion of *worst case* is not clear in the multi-objective case.

We present the two most common concepts for min–max robust efficiency: *point-based min–max robust efficiency* and *set-based min–max robust efficiency*. For point-based min–max robust efficiency, we determine the worst case for each solution x and objective i individually, and compare the solutions w.r.t. the resulting point $\bar{z}(x)$. For set-based min–max robust efficiency, we check whether there exists a solution $x' \in \mathcal{X}$ with $\{z(x', \xi) : \xi \in \mathcal{U}\} \subseteq \{z(x, \xi) : \xi \in \mathcal{U}\} - \mathbb{R}^k_{\geq}$ (analogous to determining efficiency in the deterministic case by checking whether a solution $x' \in \mathcal{X}$ with $z(x') \in z(x) - \mathbb{R}^k_{\geq}$ exists).

Definition 5 (Ehrgott et al., 2014; Kuroiwa & Lee, 2012)**.** Given a multi-objective uncertain optimization problem, we define

$$\bar{z}(x) := \begin{pmatrix} \max_{\xi \in \mathcal{U}} z_1(x, \xi) \\ \vdots \\ \max_{\xi \in \mathcal{U}} z_k(x, \xi) \end{pmatrix}.$$

A solution $x \in \mathcal{X}$ is *point-based min–max robust [weakly/·/strictly] efficient* for (MOUP) (abbreviated: pointMR [weakly/·/strictly] efficient), if it is a [weakly/·/strictly] efficient solution for the *robust counterpart* $\min_{x \in \mathcal{X}} \bar{z}(x)$, i.e., if there is no $x' \in \mathcal{X}$ with

$$\bar{z}(x') \in \bar{z}(x) - \mathbb{R}^k_{[>/\geq/\geqq]}.$$

Defining

$$z_{\mathcal{U}}(x) := \{z(x, \xi) : \xi \in \mathcal{U}\},$$

a solution $x \in \mathcal{X}$ is *set-based min–max robust [weakly/·/strictly] efficient* for (MOUP) (abbreviated: setMR [weakly/·/strictly] efficient), if there exists no $x' \in \mathcal{X}$ with

$$z_{\mathcal{U}}(x') \subseteq z_{\mathcal{U}}(x) - \mathbb{R}^k_{[>/\geq/\geqq]}.$$

Both concepts reduce to min–max robustness for $k = 1$, i.e., the pointMR efficient solutions and setMR efficient solutions are then identical to the solutions of $\min_{x \in \mathcal{X}} \max_{\xi \in \mathcal{U}} z_1(x, \xi)$. Note that every pointMR [weakly/strictly] efficient solution is also setMR [weakly/strictly] efficient and that the two concepts coincide, if (MOUP) is *objective-wise uncertain*, i.e., if $\mathcal{U} = \mathcal{U}_1 \times \cdots \times \mathcal{U}_k$ and $z_i(x, \xi) = z_i(x, \xi_i), \xi_i \in \mathcal{U}_i \ \forall \ i \in [k]$.

Example 6. Let a multi-objective uncertain optimization problem be given with $\mathcal{X} := \{x^1, x^2, x^3\}$, $\mathcal{U} := \{\xi^1, \xi^2\}$ and

$$z(x^1, \xi^1) = z(x^1, \xi^2) = (1.5, 1.5)$$
$$z(x^2, \xi^1) = (0.5, 4), z(x^2, \xi^2) = (4, 0.5)$$
$$z(x^3, \xi^1) = (1, 3), z(x^3, \xi^2) = (3, 1).$$

Fig. 1(a) shows $\bar{z}(x)$ and $\partial(\bar{z}(x) - \mathbb{R}^k_{\geq})$ and Fig. 1(b) shows $z_{\mathcal{U}}(x)$ and $\partial(z_{\mathcal{U}}(x) - \mathbb{R}^k_{\geq})$ for $x \in \mathcal{X}$. All three solutions are setMR efficient, whereas only x^1 is pointMR efficient.

The following lemma characterizes setMR efficient solutions.

Lemma 7 (Ehrgott et al., 2014)**.** *Given a multi-objective uncertain optimization problem (MOUP). For all* $x, x' \in \mathcal{X}$,

$$z_{\mathcal{U}}(x') \subseteq z_{\mathcal{U}}(x) - \mathbb{R}^k_{[>/\geq/\geqq]} \Leftrightarrow$$
$$\forall \ \xi \in \mathcal{U} \ \exists \ \eta \in \mathcal{U} : z(x', \xi)[< / \leq / \leqq]z(x, \eta).$$

2.2. Methods to find robust efficient solutions based on scalarizations

In (deterministic) multi-objective optimization it is common to find a set of efficient solutions with a *scalarization method*, i.e., by solving a family of single-objective, so-called *scalarized*, problems (see, e.g., Ehrgott, 2006). For finding pointMR efficient solutions, these methods can directly be applied to the robust counterpart $\min_{x \in \mathcal{X}} \bar{z}(x)$. In case of set-based min–max robust efficiency, the extension of scalarization methods is not as straightforward, because the robust counterpart is a set-valued problem. The following methods to find setMR efficient solutions based on scalarizations have been developed.

Ehrgott et al. (2014) introduce two methods based on scalarizations: The *weighted sum scalarization method* and the ϵ*-constraint method*, which are extensions of the corresponding methods for the deterministic case. They show that both methods find setMR weakly efficient solutions. The solutions for the weighted sum scalarized problems are even setMR efficient, if the weights are chosen strictly greater than zero. The solutions found with the ϵ*-constraint method* are always pointMR weakly efficient. The authors show that the two methods do not always find the same solutions and that there can exist setMR efficient solutions, which cannot be found by either of these methods.

Ide (2014) introduces a method based on the (augmented) weighted Chebyshev scalarization with reference point 0. Ide (2014) shows that all solutions found with this *(augmented) weighted Chebyshev method* are setMR weakly efficient. In case of objective-wise uncertainty, the scalarized problem in Ide (2014) is identical to the scalarized problem in Hassanzadeh et al. (2013) (if the robust utopian point in Hassanzadeh et al. (2013) can be chosen as 0), where the deterministic augmented weighted Chebychev method is applied to the robust counterpart $\min_{x \in \mathcal{X}} \bar{z}(x)$ to find pointMR efficient solutions.

Bokrantz and Fredriksson (2017) consider order-preserving scalarizing functions $s : \mathbb{R}^k \to \mathbb{R}$ and the resulting scalarized problems $\min_{x \in \mathcal{X}} \max_{\xi \in \mathcal{U}} s(z(x))$. They show that for so-called *strongly increasing* scalarizing functions the solutions for the scalarized problem are setMR efficient. In an application they consider weighted p-norms as scalarizing functions, resulting in the *p-norm scalarization method* (e.g., the weighted sum scalarization method for $p = 1$).

3. Min-ordering and max-ordering method for multi-objective uncertain problems

Max-ordering problems have been of interest in multi-objective optimization since Bowman (1976). The max-ordering approach has been used, e.g., for multi-objective location problems in Ehrgott, Nickel, and Hamacher (1999) and for biobjective combinatorial problems in Ehrgott and Skriver (2003). We also refer to Ehrgott (2005). In this section we make use of the max-ordering and of the min-ordering scalarization to identify robust efficient solutions.

Definition 8. Let

$$(\text{P}) \ \left(\min_{x\in\mathcal{X}} z(x,\xi), \xi \in \mathcal{U}\right)$$

be a multi-objective uncertain optimization problem. For a given weight vector $\lambda \in \mathbb{R}^k_>$ and reference point $r \in \mathbb{R}^k$ we define the corresponding *min-ordering optimization problem* as

$$(\text{P-min}(r,\lambda)) \ \min_{x\in\mathcal{X}} \max_{\xi\in\mathcal{U}} \min_{i\in[k]} \lambda_i(z_i(x,\xi) - r_i)$$

and the corresponding *max-ordering optimization problem* as

$$(\text{P-max}(r,\lambda)) \ \min_{x\in\mathcal{X}} \max_{\xi\in\mathcal{U}} \max_{i\in[k]} \lambda_i(z_i(x,\xi) - r_i).$$

We further denote the objective value for a given $x \in \mathcal{X}$ by

$$\alpha^{\min}(x,r,\lambda) := \max_{\xi\in\mathcal{U}} \min_{i\in[k]} \lambda_i(z_i(x,\xi) - r_i) \qquad \text{for (P-min}(r,\lambda)),$$

$$\alpha^{\max}(x,r,\lambda) := \max_{\xi\in\mathcal{U}} \max_{i\in[k]} \lambda_i(z_i(x,\xi) - r_i) \qquad \text{for (P-max}(r,\lambda)).$$

Note that $\alpha^{\min}(x, r, \lambda)$ and $\alpha^{\max}(x, r, \lambda)$ exist for all $x \in \mathcal{X}$ because $\mathcal{U}$ is compact and nonempty and the finitely many functions $z_i(x,\cdot) : \mathcal{U} \to \mathbb{R}$ are continuous. The values $\alpha^{\min}(x, r, \lambda)$ and $\alpha^{\max}(x, r, \lambda)$ also have a geometric interpretation, which we detail in Section 3.1.

In Sections 3.2 and 3.3, we show that optimal solutions for (P-min(r, λ)) and (P-max(r, λ)) are setMR weakly efficient and solutions for (P-max(r, λ)) even pointMR weakly efficient. Similar to the existing methods discussed in Section 2.2, we obtain a *min-ordering* resp. *max-ordering scalarization method* to find a set of robust efficient solutions by varying the parameters r, λ and solving the resulting problems (P-min(r, λ)) resp. (P-max(r, λ)). The max-ordering scalarization method is similar to the weighted Chebyshev method for multi-objective robust problems given in Ide (2014) (and for objective-wise uncertainty in Hassanzadeh et al. (2013)), but with arbitrary reference point.

Before investigating properties of the solutions for (P-min(r, λ)) and (P-max(r, λ)), we provide a brief example to give an intuition on their meaning for the original problem: Consider a student organization who wants to offer cheap lunch for students in several university towns and has to decide on a dish $x \in \mathcal{X}$ in advance. They can price the dish differently in each town and because of a very small profit margin the price depends on the prices of the ingredients in the supermarket in town. They aim to minimize the lunch prices in all towns simultaneously, i.e., $z_i(x, \xi)$ is the price of dish x in town i, where the uncertainty in the price development is modeled by $\xi \in \mathcal{U}$. Solving (P-max(r, λ)) with $r = (0, \dots, 0)^T$, $\lambda = (1, \dots, 1)^T$ means then to minimize the highest price any student in any town has to pay for their meal in the worst case. Solving (P-min(r, λ)) with the same r, λ means to minimize the best price the organization can offer in some university, assuming the worst price development. I.e., this is the price p they can legitimately use in their advertisement “Cheap student lunch - starting from p!”, because in some town the price will not be higher than p.

The remainder of this section is structured as follows: We first give a geometric interpretation of the problems (P-min(r, λ)) and (P-max(r, λ)) and a characterization of their solutions in Section 3.1. We then investigate properties of the solutions found with the max-ordering method in Section 3.2 and with the min-ordering method in Section 3.3.

In Section 4 we show how (P-min(r, λ)) and (P-max(r, λ)) can be solved for multi-objective uncertain combinatorial problems with particular uncertainty sets and investigate their complexity. For this, we use the following reformulations of (P-min(r, λ)) and (P-max(r, λ)) in case of a single scenario.

Remark 9. If the uncertainty set $\mathcal{U}$ contains only one scenario ξ, i.e., (MOUP) is a deterministic problem.

(P-min(r, λ)) then reduces to $\min_{x\in\mathcal{X}, i\in[k]} \lambda_i(z_i(x,\xi) - r_i)$. This can be solved by solving the k single-objective deterministic problems

$$(\text{P}_i) \quad \min_{x\in\mathcal{X}} \lambda_i(z_i(x,\xi) - r_i)$$

and choosing the best of the obtained solutions.

(P-max(r, λ)) then reduces to $\min_{x\in\mathcal{X}} \max_{i\in[k]} \lambda_i(z_i(x,\xi) - r_i)$. By interpreting $[k]$ as an uncertainty set whose scenarios $i \in [k]$ determine which objective function $\lambda_i(z_i(x,\xi) - r_i)$ to use, this can be interpreted as a single-objective min–max robust problem with a discrete uncertainty set.

3.1. Geometric interpretation of (P-max(r, λ)) and (P-min(r, λ))

The sublevel set of the function $\max_{i\in[k]} \lambda_i(z_i - r_i)$ for level $\alpha \in \mathbb{R}$ is

$$\begin{aligned} L^{\max,r,\lambda}_{\leqq}(\alpha) &= \left\{ z \in \mathbb{R}^k : \max_{i\in[k]} \lambda_i(z_i - r_i) \leqq \alpha \right\} \\ &= \left\{ z \in \mathbb{R}^k : z_i \leqq \frac{\alpha}{\lambda_i} + r_i \ \forall i \ \in [k] \right\} \\ &= \left\{ z \in \mathbb{R}^k : z \leqq \alpha \left(\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_k} \right)^T + r \right\}. \end{aligned}$$

and that of the function $\min_{i\in[k]} \lambda_i(z_i - r_i)$ is

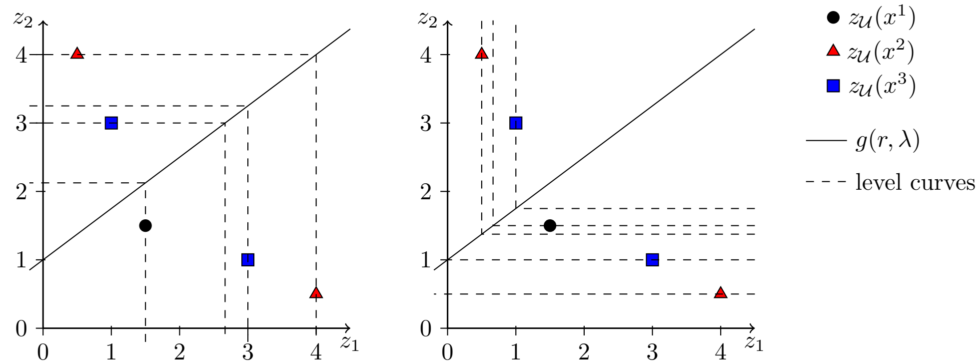
$$\begin{aligned} L^{\min,r,\lambda}_{\leqq}(\alpha) &= \left\{ z \in \mathbb{R}^k : \min_{i\in[k]} \lambda_i(z_i - r_i) \leqq \alpha \right\} \\ &= \left\{ z \in \mathbb{R}^k : \exists i \in [k] \text{ with } z_i \leqq \frac{\alpha}{\lambda_i} + r_i \right\} \\ &= \left\{ z \in \mathbb{R}^k : z \not> \alpha \left(\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_k} \right)^T + r \right\}. \end{aligned}$$

Therefore, every sublevel set of $\max_{i\in[k]} \lambda_i(z_i - r_i)$ or $\min_{i\in[k]} \lambda_i(z_i - r_i)$ can be uniquely identified with a point on the line

$$g(r,\lambda) := \left\{ y(\alpha) := r + \alpha \left(\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_k} \right)^T : \alpha \in \mathbb{R} \right\}.$$

For two points $y(\alpha)$, $y(\alpha') \in g(r, \lambda)$ we have $y(\alpha) \leq y(\alpha') \Leftrightarrow y(\alpha) < y(\alpha') \Leftrightarrow \alpha < \alpha'$, because of $\lambda_i > 0 \, \forall i \in [k]$. Fig. 2 shows the level curves of $\max_{i\in[k]} \lambda_i(z_i - r_i)$ and $\min_{i\in[k]} \lambda_i(z_i - r_i)$ for $r = (0, 1)^T$ and $\lambda = (3, 4)^T$ that contain $z(x, \xi)$ for some $x \in \mathcal{X}$ and $\xi \in \mathcal{U}$ from Example 6.

Recall the definitions of $\bar{z}(x)$ and $z_{\mathcal{U}}(x)$, used in the definition of pointMR efficiency and setMR efficiency (Definition 5). The following theorem shows that the optimal solutions for (P-max(r, λ)) can be identified by comparing the intersection points of $g(r, \lambda)$ with



(a) $\max_{i\in[k]} \lambda_i(z_i - r_i)$ (b) $\min_{i\in[k]} \lambda_i(z_i - r_i)$

Fig. 2. Level curves of the functions $\max_{i\in[k]} \lambda_i(z_i - r_i)$ and $\min_{i\in[k]} \lambda_i(z_i - r_i)$ with $r = (0, 1)^T$, $\lambda = (3, 4)^T$, which contain any $z(x, \xi)$ from Example 6.

$\partial(\bar{z}(x) + \mathbb{R}^k_{\geqq})$ for all $x \in \mathcal{X}$. Similarly, the optimal solutions of (P-min(r, λ)) can be identified by comparing the intersection points of $g(r, \lambda)$ with $\partial(z_{\mathcal{U}}(x) - \mathbb{R}^k_{\geqq})$ for all $x \in \mathcal{X}$.

Theorem 10. *Let $r \in \mathbb{R}^k, \lambda \in \mathbb{R}^k_{>}$ be given. A feasible solution $x^* \in \mathcal{X}$ is optimal for* (P-max(r, λ)) *if and only if there exists $y^* \in \mathbb{R}^k$ such that (x^*, y^*) is an efficient solution for*

$$(\text{G-max}(r,\lambda)) \qquad \min\ y \quad \text{s.t. } y \in g(r,\lambda) \cap \partial(\bar{z}(x) + \mathbb{R}^k_{\geqq}),\ x \in \mathcal{X}.$$

A feasible solution $x^* \in \mathcal{X}$ is optimal for (P-min(r, λ)) *if and only if there exists $y^* \in \mathbb{R}^k$ such that (x^*, y^*) is an efficient solution for*

$$(\text{G-min}(r,\lambda)) \qquad \min\ y \quad \text{s.t. } y \in g(r,\lambda) \cap \partial(z_{\mathcal{U}}(x) - \mathbb{R}^k_{\geqq}),\ x \in \mathcal{X}.$$

Proof. We first show

$$g(r,\lambda) \cap \partial(\bar{z}(x) + \mathbb{R}^k_{\geqq}) = \left\{ r + \alpha^{max}(x,r,\lambda)\left(\frac{1}{\lambda_1},\dots,\frac{1}{\lambda_k}\right)^T \right\}$$

for every $x \in \mathcal{X}, r \in \mathbb{R}^k, \lambda \in \mathbb{R}^k_{>}$. For every $\alpha \in \mathbb{R}$ with $\alpha > \alpha^{\max}(x, r, \lambda)$ we have

$$\begin{aligned}
\alpha > \alpha^{\max}(x,r,\lambda) &= \max_{\xi\in\mathcal{U}} \max_{i\in[k]} \lambda_i(z_i(x,\xi) - r_i) \\
&\geqq \lambda_i(\max_{\xi\in\mathcal{U}} z_i(x,\xi) - r_i) \qquad \forall\, i \in [k]
\end{aligned}$$

$$\Rightarrow\ r_i + \alpha \cdot \frac{1}{\lambda_i} > \max_{\xi\in\mathcal{U}} z_i(x,\xi) = \bar{z}_i(x) \qquad \forall\, i \in [k]$$

$$\Rightarrow\ r + \alpha\left(\frac{1}{\lambda_1},\dots,\frac{1}{\lambda_k}\right)^T \in \bar{z}(x) + \mathbb{R}^k_{>} = (\bar{z}(x) + \mathbb{R}^k_{\geqq}) \setminus \partial(\bar{z}(x) + \mathbb{R}^k_{\geqq}).$$

Further, for every $\alpha \in \mathbb{R}$ with $\alpha < \alpha^{\max}(x, r, \lambda)$,

$$\alpha < \alpha^{\max}(x,r,\lambda) = \max_{\xi\in\mathcal{U}} \max_{i\in[k]} \lambda_i(z_i(x,\xi) - r_i)$$

$$\Rightarrow\ r_i + \alpha \cdot \frac{1}{\lambda_i} < \max_{\xi\in\mathcal{U}} z_i(x,\xi) = \bar{z}_i(x) \qquad \text{for at least one } i \in [k]$$

$$\Rightarrow\ r + \alpha\left(\frac{1}{\lambda_1},\dots,\frac{1}{\lambda_k}\right)^T \notin \bar{z}(x) + \mathbb{R}^k_{\geqq}$$

$$\Rightarrow\ r + \alpha\left(\frac{1}{\lambda_1},\dots,\frac{1}{\lambda_k}\right)^T \notin \partial(\bar{z}(x) + \mathbb{R}^k_{\geqq}) \qquad \text{since } \bar{z}(x) + \mathbb{R}^k_{\geqq} \text{ is closed.}$$

It follows that $y(\alpha^{max}(x, r, \lambda))$ is the unique intersection point of $g(r, \lambda)$ with $\partial(\bar{z}(x) + \mathbb{R}^k_{\geqq})$. Hence, the only $y \in \mathbb{R}^k$, such that (x, y) is feasible for (G-max(r, λ)), is $y(\alpha^{max}(x, r, \lambda))$. It follows that

$$\begin{aligned}
&x^* \text{ is optimal for}(\text{P-max}(r,\lambda)) \\
&\Leftrightarrow \nexists\ x \in \mathcal{X} : \alpha^{max}(x,r,\lambda) < \alpha^{max}(x^*,r,\lambda) \\
&\Leftrightarrow \nexists\ x \in \mathcal{X} : y(\alpha^{max}(x,r,\lambda)) \le y(\alpha^{max}(x^*,r,\lambda)) \\
&\Leftrightarrow (x^*, y(\alpha^{max}(x^*,r,\lambda))) \text{ is an efficient solution for } (\text{G-max}(r,\lambda)).
\end{aligned}$$

Similarly, we show

$$g(r,\lambda) \cap \partial(z_{\mathcal{U}}(x) - \mathbb{R}^k_{\geqq}) = \left\{ r + \alpha^{max}(x,r,\lambda)\left(\frac{1}{\lambda_1},\dots,\frac{1}{\lambda_k}\right)^T \right\}$$

for every $x \in \mathcal{X}, r \in \mathbb{R}^k, \lambda \in \mathbb{R}^k_{>}$. For every $\alpha \in \mathbb{R}$ with $\alpha > \alpha^{\min}(x, r, \lambda)$ we have

$$\alpha > \alpha^{\min}(x,r,\lambda) \geqq \min_{i\in[k]} \lambda_i(z_i(x,\xi) - r_i) \qquad \forall\, \xi \in \mathcal{U}$$

$$\Rightarrow\ \forall\, \xi \in \mathcal{U}\ \exists\, i \in [k] : r_i + \alpha \cdot \frac{1}{\lambda_i} > z_i(x,\xi)$$

$$\Rightarrow\ r + \alpha\left(\frac{1}{\lambda_1},\dots,\frac{1}{\lambda_k}\right)^T \notin z(x,\xi) - \mathbb{R}^k_{\geqq} \qquad \forall \xi \in \mathcal{U}$$

$$\Rightarrow\ r + \alpha\left(\frac{1}{\lambda_1},\dots,\frac{1}{\lambda_k}\right)^T \notin z_{\mathcal{U}}(x) - \mathbb{R}^k_{\geqq}$$

$$\Rightarrow\ r + \alpha\left(\frac{1}{\lambda_1},\dots,\frac{1}{\lambda_k}\right)^T \notin \partial(z_{\mathcal{U}}(x) - \mathbb{R}^k_{\geqq}), \qquad \text{since } z_{\mathcal{U}}(x) - \mathbb{R}^k_{\geqq} \text{ is closed,}$$

and for every $\alpha \in \mathbb{R}$ with $\alpha < \alpha^{\min}(x, r, \lambda)$,

$$\alpha < \alpha^{\min}(x,r,\lambda) = \min_{i\in[k]} \lambda_i(z_i(x,\xi) - r_i) \qquad \text{for at least one } \xi \in \mathcal{U}$$

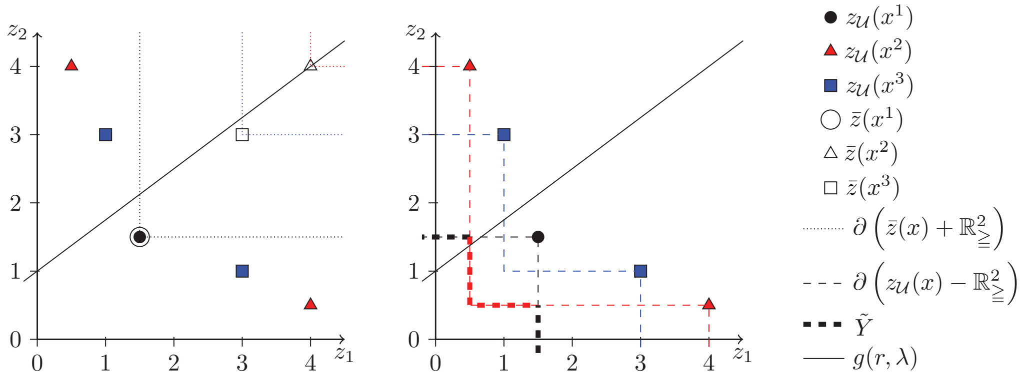
$$\Rightarrow\ \exists \xi \in \mathcal{U} \text{ such that } \forall i \in [k] : r_i + \alpha \cdot \frac{1}{\lambda_i} < z_i(x,\xi)$$

$$\Rightarrow\ \exists \xi \in \mathcal{U} : r + \alpha\left(\frac{1}{\lambda_1},\dots,\frac{1}{\lambda_k}\right)^T \in z(x,\xi) - \mathbb{R}^k_{>}$$

$$\begin{aligned}
\Rightarrow\ r + \alpha\left(\frac{1}{\lambda_1},\dots,\frac{1}{\lambda_k}\right)^T &\in z_{\mathcal{U}}(x) - \mathbb{R}^k_{>} \\
&= (z_{\mathcal{U}}(x) - \mathbb{R}^k_{\geqq}) \setminus \partial(z_{\mathcal{U}}(x) - \mathbb{R}^k_{\geqq}).
\end{aligned}$$

Hence, for all $x \in \mathcal{X}$, $y(\alpha^{min}(x, r, \lambda))$ is the unique intersection point of $g(r, \lambda)$ with $\partial(z_{\mathcal{U}}(x) - \mathbb{R}^k_{\geqq})$. Therefore, x^* is optimal for (P-min(r, λ)) if and only if $(x^*, y(\alpha^{min}(x^*, r, \lambda)))$ is an efficient solution for (G-min(r, λ)). □

Note that it follows from the proof of Theorem 10 that for (G-max(r, λ)) and (G-min(r, λ)) every weakly efficient solution is also efficient, because we have $y(\alpha) \le y(\alpha') \Leftrightarrow y(\alpha) < y(\alpha')$ for two



(a) Finding solutions for (P-max(r,λ)) (b) Finding solutions for (P-min(r,λ))

Fig. 3. Determining the intersection point of $g(r, \lambda)$ with $\partial(\bar{z}(x)+\mathbb{R}^2_{\geqq})$ (a) and $\partial(z_{\mathcal{U}}(x)-\mathbb{R}^2_{\geqq})$ (b) for the solutions in Example 6. As an example, $g(r, \lambda)$ is shown for $r=(0,1)^T, \lambda=(3,4)^T$.

points $y(\alpha),\ y(\alpha')\in g(r,\ \lambda)$. Theorem 10 implies, that a solution $x\in\mathcal{X}$ can be found with the [max-ordering/min-ordering] method if and only if there exist $\lambda\in\mathbb{R}^k_>, r\in\mathbb{R}^k, y\in\mathbb{R}^k$, such that $(x,\ y)$ is (weakly) efficient for [(G-max$(r,\ \lambda)$)/(G-min$(r,\ \lambda)$)]. Fig. 3 illustrates $g(r,\ \lambda)$, $\partial(\bar{z}(x)+\mathbb{R}^k_{\geqq})$ and $\partial(z_{\mathcal{U}}(x)-\mathbb{R}^k_{\geqq})$ for the feasible solutions in Example 6. It is easy to see in Fig. 3(a) that for each choice of r, λ the intersection point of $g(r,\ \lambda)$ with $\partial(\bar{z}(x^1)+\mathbb{R}^k_{\geqq})$ has smaller coordinates than the intersection point of $g(r,\ \lambda)$ with $\partial(\bar{z}(x^2)+\mathbb{R}^k_{\geqq})$ or $\partial(\bar{z}(x^3)+\mathbb{R}^k_{\geqq})$, hence x^1 is the unique optimal solution for (P-max$(r,\ \lambda)$).

Let us now consider the sets

$$Y := \bigcup_{x\in\mathcal{X}} \partial(z_{\mathcal{U}}(x)-\mathbb{R}^k_{\geqq}) \text{ and } \tilde{Y} := \{y\in Y : \nexists y'\in Y : y' < y\}.$$

For each $y\in\tilde{Y}$ there exists $r\in\mathbb{R}^k, \lambda\in\mathbb{R}^k_>, x\in\mathcal{X}$ such that $(x,\ y)$ is efficient for (G-min$(r,\ \lambda)$): choose $r=y$, then $y\in g(r,\ \lambda)$, hence there exists x such that $(x,\ y)$ is feasible for (G-min$(r,\ \lambda)$), because $y\in Y$. Further there is no feasible $(x',\ y')$ with $y'<y$, because $y\in\tilde{Y}$, hence $(x,\ y)$ is (weakly) efficient for (G-min$(r,\ \lambda)$).

Fig. 3(b) shows $\partial(z_{\mathcal{U}}(x)-\mathbb{R}^k_{\geqq})$ for all $x\in\mathcal{X}$ in Example 6 as dashed lines and $\tilde{Y}$ as thick dashed line. Since $\partial(z_{\mathcal{U}}(x^1)-\mathbb{R}^k_{\geqq})\cap\tilde{Y}$ and $\partial(z_{\mathcal{U}}(x^2)-\mathbb{R}^k_{\geqq})\cap\tilde{Y}$ are not empty, x^1 and x^2 can be found with the min-ordering method. On the other hand, it is easy to see that for every $r\in\mathbb{R}^k, \lambda\in\mathbb{R}^k_>$ there exists a point $\tilde{y}\in\tilde{Y}\cap g(r,\lambda)$ such that $\tilde{y}\leq y$ for $y\in g(r,\lambda)\cap\partial(z_{\mathcal{U}}(x^3)-\mathbb{R}^k_{\geqq})$. Therefore, x^3 is not optimal for (P-min$(r,\ \lambda)$).

3.2. Solutions found with the max-ordering method

Ide (2014) shows that (for fixed reference point 0) every [·/unique] solution of (P-max$(r,\ \lambda)$) is setMR [weakly/strictly] efficient. We show that for every reference point $r\in\mathbb{R}^k$ every [·/unique] solution of the max-ordering optimization problem is even pointMR [weakly/strictly] efficient and that for a small enough r all pointMR weakly efficient solutions can be found by choosing an appropriate λ.

Theorem 11. *Let $r\in\mathbb{R}^k, \lambda\in\mathbb{R}^k_>$ be given and let x be an optimal solution for (P-max$(r,\ \lambda)$). Then*

1. *x is pointMR weakly efficient for* (P) *and*
2. *if x is the unique optimal solution for (P-max$(r,\ \lambda)$), then x is pointMR strictly efficient.*

Proof. Let x be [an/the unique] optimal solution for (P-max$(r,\ \lambda)$). Assume that x is not pointMR [weakly/strictly] efficient. Then there exists a solution $x'\in\mathcal{X}$ with

$$\max_{\xi\in\mathcal{U}} z_i(x',\xi)\ [<\ /\ \leqq]\ \max_{\xi\in\mathcal{U}} z_i(x,\xi) \qquad \forall i\in[k]$$

$$\Leftrightarrow \max_{\xi\in\mathcal{U}} \lambda_i\big(z_i(x',\xi)-r_i\big)\ [<\ /\ \leqq]\ \max_{\xi\in\mathcal{U}} \lambda_i(z_i(x,\xi)-r_i) \qquad \forall i\in[k]$$

$$\Rightarrow \max_{i\in[k]}\max_{\xi\in\mathcal{U}} \lambda_i\big(z_i(x',\xi)-r_i\big)\ [<\ /\ \leqq]\ \max_{i\in[k]}\max_{\xi\in\mathcal{U}} \lambda_i(z_i(x,\xi)-r_i)$$

$$\Rightarrow \max_{\xi\in\mathcal{U}}\max_{i\in[k]} \lambda_i\big(z_i(x',\xi)-r_i\big)\ [<\ /\ \leqq]\ \max_{\xi\in\mathcal{U}}\max_{i\in[k]} \lambda_i(z_i(x,\xi)-r_i)$$

1. If x is not pointMR weakly efficient, i.e., $<$ holds, this is a contradiction to x being an optimal solution for (P-max$(r,\ \lambda)$).
2. If x is not pointMR strictly efficient, i.e., $\leqq$ holds, then x is not optimal for (P-max$(r,\ \lambda)$) or x' is optimal as well. This contradicts x being the unique optimal solution. □

Theorem 11 implies that not all setMR weakly efficient solutions can be found with the max-ordering method, because a setMR weakly efficient solution is not necessarily pointMR weakly efficient. However, the following theorem shows that for a suitable choice of r all pointMR weakly efficient solutions can be found by varying λ.

Theorem 12. *Let x be a pointMR weakly efficient solution and let a reference point $r\in\mathbb{R}^k$ with $r_i<\max_{\xi\in\mathcal{U}} z_i(x,\xi)\ \forall i\in[k]$ be given. Then there exists a weight vector $\lambda\in\mathbb{R}^k_>$ such that x is an optimal solution for (P-max$(r,\ \lambda)$).*

Proof. Because of $r_i<\max_{\xi\in\mathcal{U}} z_i(x,\xi)$ we obtain well-defined and positive weights by setting

$$\lambda_i := \frac{1}{\max_{\xi\in\mathcal{U}} z_i(x,\xi)-r_i}\ \forall\ i\ =1,\ldots,k.$$

It follows that $\max_{\xi\in\mathcal{U}}\lambda_i(z_i(x,\xi)-r_i)=\lambda_i\big(\max_{\xi\in\mathcal{U}} z_i(x,\xi)-r_i\big)=1\ \forall\ i\in[k]$.

Let $x'\in X$ be any feasible solution. Since x is weakly pointMR efficient, there exists at least one index $j\in\{1,\ldots,k\}$ with $\max_{\xi\in\mathcal{U}} z_j(x,\xi)\leqq\max_{\xi\in\mathcal{U}} z_j(x',\xi)$. It follows that

$$\max_{i\in[k]}\max_{\xi\in\mathcal{U}}\lambda_i(z_i(x,\xi)-r_i)=1$$

$$=\max_{\xi\in\mathcal{U}}\lambda_j\big(z_j(x,\xi)-r_j\big)=\lambda_j\left(\max_{\xi\in\mathcal{U}} z_j(x,\xi)-r_j\right)$$

$$\leqq\lambda_j\left(\max_{\xi\in\mathcal{U}} z_j(x',\xi)-r_j\right)=\max_{\xi\in\mathcal{U}}\lambda_j\big(z_j(x',\xi)-r_j\big)$$

$$\leqq\max_{i\in[k]}\max_{\xi\in\mathcal{U}}\lambda_i\big(z_i(x',\xi)-r_i\big),$$

hence, x is optimal for $(Pmax(r,\ \lambda))$. □

The results from Section 3.1 provide a geometric interpretation of the proof of Theorem 12: For given r, x and the λ constructed in the proof of Theorem 12, $g(r, \lambda)$ is the line through r and $\bar{z}(x)$. Then, $\bar{z}(x) = y(\alpha^{max}(x, r, \lambda))$ and $y(\alpha) \in \bar{z}(x) - \mathbb{R}^k_{\geqq}$ for all $\alpha < (\alpha^{max}(x, r, \lambda))$. If x is pointMR efficient, $\bar{z}(x) - \mathbb{R}^k_{\geq} \cap \partial(\bar{z}(x') + \mathbb{R}^k_{\geq})$ is empty for all $x' \in \mathcal{X}$, hence $(x, y(\alpha^{max}(x, r, \lambda))$ is an efficient solution for (G-max(r, λ)).

3.3. Solutions found with the min-ordering method

For (P-min(r, λ)) we show that every [·/unique] solution is set-based robust [weakly/strictly] efficient, i.e., the min-ordering scalarization method is suitable for finding setMR (weakly) efficient solutions. Moreover, we show that with this method we can find setMR efficient solutions that cannot be found with the other known scalarization methods presented in Section 2.2, including the weighted sum, ϵ-constraint and augmented weighted Chebychev method. This also implies that solutions for (P-min(r, λ)) are not necessarily pointMR efficient.

Theorem 13. *Let $r \in \mathbb{R}^k, \lambda \in \mathbb{R}^k_{>}$ be given and let x be an optimal solution for (P-min(r, λ)). Then*

1. *x is setMR weakly efficient for* (P) *and*
2. *if x is the unique optimal solution for* (P-min(r, λ))*, then x is setMR strictly efficient.*

Proof. Let x be [an/the unique] optimal solution for (P-min(r, λ)). Assume that x is not setMR [weakly/strictly] efficient. From Lemma 7 it follows that there exists a feasible solution x' with $\forall\ \xi \in \mathcal{U}\ \ \exists\ \ \eta \in \mathcal{U} : z(x', \xi)[< / \leqq]z(x, \eta)$. Let $\xi' \in \operatorname{argmax}_{\xi \in \mathcal{U}} \min_{i \in [k]} \lambda_i(z_i(x', \xi) - r_i)$ be a worst case scenario of x' w.r.t. (P-min(r, λ)) Then there exists $\eta' \in \mathcal{U}$ with

$$z_i(x', \xi')[< / \leqq]z_i(x, \eta') \qquad \forall\ i \in [k]$$
$$\Leftrightarrow \lambda_i\big(z_i(x', \xi') - r_i\big)[< / \leqq]\lambda_i\big(z_i(x, \eta') - r_i\big) \qquad \forall\ i \in [k]$$

We hence conclude that

$$\begin{aligned}\max_{\xi \in \mathcal{U}} \min_{i \in [k]} \lambda_i\big(z_i(x', \xi) - r_i\big) &= \min_{i \in [k]} \lambda_i\big(z_i(x', \xi') - r_i\big)\\ &[< / \leqq] \min_{i \in [k]} \lambda_i\big(z_i(x, \eta') - r_i\big)\\ &\leqq \max_{\xi \in \mathcal{U}} \min_{i \in [k]} \lambda_i(z_i(x, \xi) - r_i).\end{aligned}$$

1. If x is not setMR weakly efficient, i.e., $<$ holds, this is a contradiction to x being an optimal solution for (P-min(r, λ)).
2. If x is not setMR strictly efficient, i.e., $\leqq$ holds, then x is not optimal for (P-min(r, λ)) or x' is optimal as well. This contradicts x being the unique optimal solution of (P-min(r, λ)). □

In Example 14 we illustrate that the min-ordering method can serve to find some (but not all) setMR efficient solutions to multi-objective uncertain optimization problems which cannot be found with other known scalarization methods presented in Section 2.2, as

- the ϵ-constraint method,
- the p-norm scalarization method,
- or solving any scalarized problem of the form

$$\min_{x \in \mathcal{X}} \left(\rho_1 \max_{\xi \in \mathcal{U}} \max_{i \in [k]} \nu_i(z_i(x, \xi) - r_i) + \rho_2 \max_{\xi \in \mathcal{U}} \sum_{i \in [k]} \mu_i(z_i(x, \xi) - r_i) \right) \tag{1}$$

with $\rho \in \mathbb{R}^2_{\geq}, \nu, \mu \in \mathbb{R}^k_{>}, r_i \in \mathbb{R}^k$.

Example 14. Consider the multi-objective uncertain optimization problem given in Example 6. Recall that all three solutions are setMR efficient. Because of

$$\begin{aligned}&\left(\max_{\xi \in \mathcal{U}} z_1(x^1, \xi), \max_{\xi \in \mathcal{U}} z_2(x^1, \xi)\right) = (1.5, 1.5)\\ &< \left(\max_{\xi \in \mathcal{U}} z_1(x^3, \xi), \max_{\xi \in \mathcal{U}} z_2(x^3, \xi)\right) = (3, 3)\\ &< \left(\max_{\xi \in \mathcal{U}} z_1(x^2, \xi), \max_{\xi \in \mathcal{U}} z_2(x^2, \xi)\right) = (4, 4),\end{aligned}$$

x^1 is the only pointMR weakly efficient solution, hence only x^1 can be found with the ϵ-constraint method (Ehrgott et al., 2014).

Bokrantz and Fredriksson (2017) show that a solution $x \in \mathcal{X}$ can only be found with the p-norm scalarization method if

$$\nexists x' \in \mathcal{X} : z_{\mathcal{U}}(x') \in \operatorname{Conv}(z_{\mathcal{U}}(x)) - \mathbb{R}^k_{>},$$

where $\operatorname{Conv}(z_{\mathcal{U}}(x))$ denotes the convex hull of $z_{\mathcal{U}}(x)$. Since $(1.5, 1.5) \in \operatorname{Conv}(\{(1, 3), (3, 1)\}) - \mathbb{R}^k_{>}$ and $(1.5, 1.5) \in \operatorname{Conv}(\{(0.5, 4), (4, 0.5)\}) - \mathbb{R}^k_{>}$, x^1 is the only solution that can be found with the p-norm scalarization method.

Let now $\rho \in \mathbb{R}^2_{\geq}, \nu, \mu \in \mathbb{R}^k_{>}, r_i \in \mathbb{R}^k$ be given and consider the scalarized problem (1). We define

$$f(x) := \max_{\xi \in \mathcal{U}} \max_{i \in [k]} \nu_i(z_i(x, \xi) - r_i) \text{ and}$$
$$h(x) := \max_{\xi \in \mathcal{U}} \sum_{i \in [k]} \mu_i(z_i(x, \xi) - r_i)$$

From Theorem 11 it follows that only x^1 can be optimal for $\min_{x \in \mathcal{X}} f(x)$, because it is the only pointMR weakly efficient solution. In the following we show that x^1 is also the only optimal solution for $\min_{x \in \mathcal{X}} h(x)$. Let $\mu \in \mathbb{R}^2_{>}, \mu_i \geqq \mu_j, \{i, j\} = \{1, 2\}$. Then

$$\begin{aligned}h(x^1) &= 1.5\mu_1 + 1.5\mu_2 - \mu_1 r_1 - \mu_2 r_2 \qquad \leqq 3\mu_i - \mu_1 r_1 - \mu_2 r_2\\ h(x^3) &= \max\{3\mu_1 + \mu_2, \mu_1 + 3\mu_2\} - \mu_1 r_1 - \mu_2 r_2\\ &\qquad > 3\mu_i - \mu_1 r_1 - \mu_2 r_2\\ h(x^2) &= \max\{4\mu_1 + 0.5\mu_2, 0.5\mu_1 + 4\mu_2\} - \mu_1 r_1 - \mu_2 r_2\\ &\qquad > 3\mu_i - \mu_1 r_1 - \mu_2 r_2\end{aligned}$$

It follows that x^1 is the unique optimal solution for $\min_{x \in \mathcal{X}} h(x)$. Since it is also uniquely optimal for $\min_{x \in \mathcal{X}} f(x)$, x^1 is the unique optimal solution for (1) for every $\rho \in \mathbb{R}^2_{\geq}$.

We conclude that the setMR efficient solutions x^2 and x^3 cannot be found with any of the methods listed in the statement.

In Fig. 3(b) it is easy to see that there exists no $r \in \mathbb{R}^k$ and $\lambda \in \mathbb{R}^k_{>}$, such that the minimal intersection point of $g(r, \lambda)$ with $\bigcup_{x \in \mathcal{X}} \partial(z_{\mathcal{U}}(x) - \mathbb{R}^k_{\geqq})$ is in $\partial(z_{\mathcal{U}}(x^3) - \mathbb{R}^k_{\geqq})$. With Theorem 10 it follows that x^3 cannot be found with the min-ordering scalarization method either.

However, x^2 is optimal for (P-min(r, λ)) with $r = (0, 0)^T, \lambda = (1, 1)^T$, because

$$\begin{aligned}\max_{\xi \in \mathcal{U}} \min_{i=1,2} z_i(x^2, \xi) &= 0.5 < \max_{\xi \in \mathcal{U}} \min_{i=1,2} z_i(x^3, \xi)\\ &= 1 < \max_{\xi \in \mathcal{U}} \min_{i=1,2} z_i(x^1, \xi) = 1.5.\end{aligned}$$

Our findings also hold for non-discrete uncertainty sets. The next example is a modification of Example 6 to show how such sets look like.

Example 15. We consider the same instance as in Example 6 with $\mathcal{X} := \{x^1, x^2, x^3\}$, but with a compact uncertainty set $\mathcal{U} := [\xi^1, \xi^2]$. We illustrate two ways to describe $z(x, \xi)$:

a) In Fig. 4(a) we just extend the outcome set of Example 6 linearly,

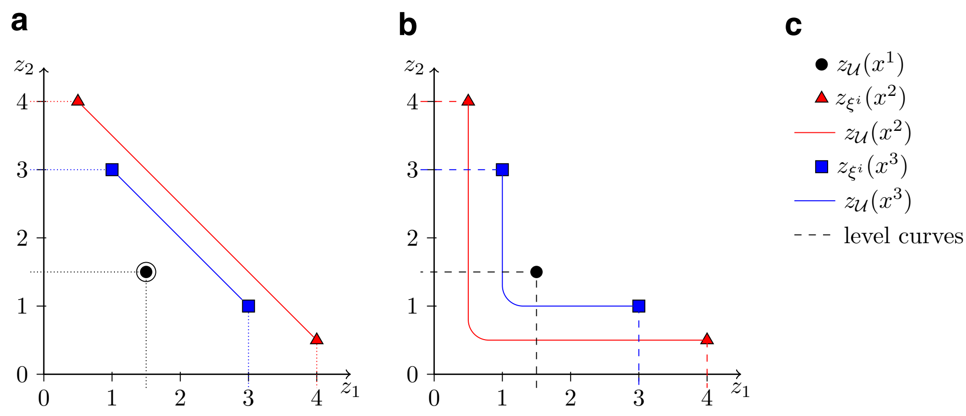


Fig. 4. Solutions and level curves for the two examples with compact uncertainty set described in Example 15.

b) while in Fig. 4(b) it is extended by curves approximating the situation of Fig. 1(b).

In case a), x^1 is the only setMR efficient solution and the only solution which is found by the min-ordering or the max-ordering scalarization methods. In case b) the situation is the same as in the discrete case, i.e., we again obtain the results of Example 14.

4. Min-ordering and max-ordering optimization problem for multi-objective uncertain combinatorial problems

In Section 3 we have shown that all pointMR efficient solutions can be found with the max-ordering method, and the min-ordering method finds setMR efficient solutions, some of which are not found with any of the formerly developed scalarization based methods (see Section 2.2). On an example, we have shown the meaning of the particular solutions obtained with the min-ordering and the max-ordering method. Now, we investigate how the problems (P-min(r, λ)) and (P-max(r, λ)) can be solved for combinatorial problems. We show that in case of interval uncertainty the uncertainty set can be reduced to one scenario, resulting in problems which have already been considered in the literature. For a multi-objective extension of the so-called *bounded uncertainty set* we develop compact mixed-integer linear programming (MILP) formulations, i.e., formulations without nested minimum and maximum functions.

We consider *multi-objective combinatorial problems with uncertain costs* (*MOUCO*): Let a finite set of elements $E = \{e_1, \dots, e_n\}$ and a feasible set $\mathcal{X} \subseteq \{0,1\}^n$ be given. Each feasible solution $x \in \mathcal{X}$ represents a subset of E, which contains element e_j if and only if $x_j = 1$. Further, a cost matrix $c \in \mathbb{R}^{k\times n}$ is given, assigning a cost $c_{i,j}$ to element e_j in the ith objective function for $i \in [k]$. The costs are uncertain, i.e., $c \in \mathcal{U} \subseteq \mathbb{R}^{k\times n}$. The k objective functions $z_i(x, c)$ hence depend on x and on the realization of the costs and are given as

$$z_i(x,c) := \sum_{j\in[n]} c_{i,j}x_j \ \forall x \in \mathcal{X}, c \in \mathcal{U}.$$

For a solution $x \in \mathcal{X}$ we write $|x| := \Sigma_{j\in[n]}x_j$.

4.1. Interval uncertainty

We use a straight-forward extension of the often used single-objective concept of interval uncertainty, where each uncertain parameter takes any value in a given interval, independent of the realization of the other parameters.

Definition 16. Let lower bounds $\hat{c} \in \mathbb{R}^{k\times n}$ and interval lengths $\delta \in \mathbb{R}^{k\times n}_{\geqq}$ be given. We define the *interval uncertainty set*

$$\mathcal{U}^I := \left\{c \in \mathbb{R}^{k\times n} : c_{i,j} = \hat{c}_{i,j} + \beta_{i,j}\delta_{i,j},\ \beta_{i,j} \in [0,1]\ \forall i \in [k],\ j \in [n]\right\}.$$

The following theorem shows that in case of interval uncertainty it is sufficient to consider the upper bounds of the intervals, i.e., the uncertainty set can be reduced to a single scenario. Therefore, (P-min(r, λ)) can be solved by solving k single-objective deterministic combinatorial problems and (P-max(r, λ)) can be interpreted as a single-objective min–max robust combinatorial problem with discrete uncertainty set (see Remark 9).

Theorem 17. *Let (P) be a MOUCO with uncertainty set* $\mathcal{U} := \mathcal{U}^I$. *We define* $\bar{c}_{i,j} := \hat{c}_{i,j} + \delta_{i,j}$ *for all* $j \in [n]$, $i \in [k]$. *Then (P-min(r, λ)) is equivalent to*

$$\min_{x\in\mathcal{X}, i\in[k]} \lambda_i(z_i(x,\bar{c}) - r_i)$$

and (P-max(r, λ)) is equivalent to

$$\min_{x\in\mathcal{X}} \max_{i\in[k]} \lambda_i(z_i(x,\bar{c}) - r_i).$$

Proof. From $\bar{c} \in \mathcal{U}^I$ and $c_{i,j} \leqq \bar{c}_{i,j}\ \forall\ c \in \mathcal{U}^I, j \in [n], i \in [k]$ we conclude

$$\begin{aligned}
&\lambda_i(z_i(x,c) - r_i) \leqq \lambda_i(z_i(x,\bar{c}) - r_i) && \forall\ x \in \mathcal{X}, c \in \mathcal{U}^I, i \in [k]\\
\Rightarrow\ &\min_{i\in[k]} \lambda_i(z_i(x,c) - r_i) \leqq \min_{i\in[k]} \lambda_i(z_i(x,\bar{c}) - r_i) && \forall\ x \in \mathcal{X}, c \in \mathcal{U}^I\\
\overset{\bar{c}\in\mathcal{U}^I}{\Rightarrow}\ &\max_{c\in\mathcal{U}^I}\min_{i\in[k]} \lambda_i(z_i(x,c) - r_i) = \min_{i\in[k]} \lambda_i(z_i(x,\bar{c}) - r_i) && \forall\ x \in \mathcal{X}\\
\Rightarrow\ &\min_{x\in\mathcal{X}}\max_{c\in\mathcal{U}^I}\min_{i\in[k]} \lambda_i(z_i(x,c) - r_i) = \min_{x\in\mathcal{X}}\min_{i\in[k]} \lambda_i(z_i(x,\bar{c}) - r_i),
\end{aligned}$$

where all minima and maxima exist due to the finiteness of $\mathcal{X}$, the compactness of $\mathcal{U}^I$ and the continuity of $z(x,\cdot) : \mathcal{U}^I \to \mathbb{R}$. For (P-max($r$, λ)) we analogously obtain $\min_{x\in\mathcal{X}}\max_{c\in\mathcal{U}^I}\max_{i\in[k]}\lambda_i(z_i(x,c) - r_i) = \min_{x\in\mathcal{X}}\max_{i\in[k]}\lambda_i(z_i(x,\bar{c}) - r_i)$. □

It follows that (P-min(r, λ)) with interval uncertainty set $\mathcal{U}^I$ is polynomially solvable if the single-objective deterministic problem can be solved in polynomial time. However, (P-max(r, λ)) is as complex as a single-objective min–max or min–max robust problem with discrete uncertainty set. This has been shown to be NP-hard for several combinatorial problems, which can be solved in polynomial time in the single-objective deterministic case, e.g., the shortest path, minimum spanning tree and assignment problem, see Murthy and Her (1992) and Kouvelis and Yu (1997).

4.2. Bounded uncertainty

The concept of *bounded uncertainty*, also called Γ-uncertainty or cardinality constrained uncertainty, was introduced for single-objective optimization by Bertsimas and Sim (2003). Its idea is that it is unlikely that all uncertain parameters, which vary in intervals, attain their worst case value simultaneously. Therefore, the authors assume that not more than Γ parameters differ from their so-called *nominal* value. We extend this idea to multi-objective uncertain combinatorial optimization by assuming that at most a given number Γ of all cost parameters can deviate from their minimal value. For the sake of simplicity we assume here that the nominal value of a cost parameter is its minimal value.

Definition 18. Let $\hat{c} \in \mathbb{R}^{k\times n}$, $\delta \in \mathbb{R}^{k\times n}_{\geqq}$ and $\Gamma \in \mathbb{Z}$ with $0 \leqq \Gamma \leqq (n \cdot k)$ be given. We define the *discretely bounded uncertainty set* as

$$\mathcal{U}^d := \left\{ c \in \mathbb{R}^{k\times n} : c_{i,j} = \hat{c}_{i,j} + \beta_{i,j}\delta_{i,j}, \beta_{i,j} \in \{0,1\}\ \forall i \in [k], j \in [n], \sum_{i\in[k], j\in[n]} \beta_{i,j} \leqq \Gamma \right\}$$

Bertsimas and Sim (2003) also allow more than Γ parameters to deviate from their minimal value if not all attain their maximal value, but deviate to a lesser extent. In the single-objective robust optimization case treated in Bertsimas and Sim (2003), restricting to what extent the parameters may deviate in total leads to the same objective value as restricting the number of deviating parameters. However, Example 22 shows that this does not hold for (P-min(r, λ)). Therefore, we also consider the *continuously bounded uncertainty set*:

Definition 19. Let $\hat{c} \in \mathbb{R}^{k\times n}$, $\delta \in \mathbb{R}^{k\times n}_{\geqq}$ and $\Gamma \in \mathbb{Z}$ with $0 \leqq \Gamma \leqq (n \cdot k)$ be given. We define the *continuously bounded uncertainty set*

$$\mathcal{U}^c := \left\{ c \in \mathbb{R}^{k\times n} : c_{i,j} = \hat{c}_{i,j} + \beta_{i,j}\delta_{i,j}, \beta_{i,j} \in [0,1]\ \forall i \in [k], j \in [n], \sum_{i\in[k], j\in[n]} \beta_{i,j} \leqq \Gamma \right\}$$

If we can assume that the uncertainties in the objectives are independent of each other, another possibility to extend the idea of bounded uncertainty to multi-objective optimization is to restrict the deviation of the parameters for each objective independently. This has been done in Hassanzadeh et al. (2013) and Raith et al. (2018b). They use an objective-wise extension of the concept of bounded uncertainty, which we will refer to as *objective-wise bounded uncertainty*.

Definition 20. Let $\hat{c} \in \mathbb{R}^{k\times n}$, $\delta \in \mathbb{R}^{k\times n}_{\geqq}$ and $\Gamma_i \in \mathbb{Z}$ with $0 \leqq \Gamma_i \leqq n\ \forall i \in [k]$ be given. We define the *objective-wise bounded uncertainty set*

$$\mathcal{U}^{\text{owb}} := \left\{ c \in \mathbb{R}^{k\times n} : c_{i,j} = \hat{c}_{i,j} + \beta_{i,j}\delta_{i,j}, \beta_{i,j} \in [0,1]\ \forall i \in [k], j \in [n], \sum_{j\in[n]} \beta_{i,j} \leqq \Gamma_i\ \forall i \in [k] \right\}.$$

In the following, we focus on discretely and continuously bounded uncertainty.

4.2.1. MILP-formulation for (P-max(r, λ)) with bounded uncertainty

In this section we introduce a MILP-formulation for (P-max(r, λ)) with discretely or continuously bounded uncertainty set. We show that we can apply the same approach that Hassanzadeh et al. (2013) use to develop an augmented weighted Chebyshev method for multi-objective uncertain linear problems with objective-wise bounded uncertainty set.

In the following we show that for (P-max(r, λ)) we do not need to distinguish between the uncertainty sets $\mathcal{U}^d$ and $\mathcal{U}^c$. Moreover, even using $\mathcal{U}^{\text{owb}}$ results in an equivalent problem, if the bound Γ_i is the same for all objectives.

Lemma 21. *For given $\mathcal{X} \subseteq \{0,1\}^n$, $\lambda \in \mathbb{R}^k_>$, $r \in \mathbb{R}^k$, $\hat{c}, \delta \in \mathbb{R}^{k\times n}$, $\Gamma = \Gamma_1 = \cdots = \Gamma_k \in \mathbb{Z}_{\geqq}$:*

$$\begin{aligned}\min_{x\in\mathcal{X}} \max_{c\in\mathcal{U}^d} \max_{i\in[k]} \lambda_i(z_i(x,c) - r_i) &= \min_{x\in\mathcal{X}} \max_{c\in\mathcal{U}^c} \max_{i\in[k]} \lambda_i(z_i(x,c) - r_i) \\ &= \min_{x\in\mathcal{X}} \max_{\xi\in\mathcal{U}^{\text{owb}}} \max_{i\in[k]} \lambda_i(z_i(x,c) - r_i).\end{aligned}$$

Proof. Let $x \in \mathcal{X}, i \in [k]$ be given. Let $\pi\colon [n] \to [n]$ be a permutation such that $\delta_{i,\pi(1)}x_{\pi(1)} \geqq \delta_{i,\pi(2)}x_{\pi(2)} \geqq \cdots \geqq \delta_{i,\pi(n)}x_{\pi(n)}$. We construct the scenario c^* by setting $c^*_{i',j} := \hat{c}_{i',j} + \beta^*_{i',j}\delta_{i',j}$ for all $i' \in [k]$ and $j \in [n]$ with

$$\beta^*_{i',j} := \begin{cases} 1 & \text{for } i = i', j = \pi(l), 1 \leqq l \leqq \Gamma \\ 0 & \text{else.} \end{cases}$$

Then $\sum_{i'\in[k], j\in[n]} \beta^*_{i',j} = \Gamma$, hence $c^* \in \mathcal{U}^d$. Further, for any $\beta \in [0,\ 1]^{k\times n}$ with $\sum_{j\in[n]}\beta_{i,j} \leqq \Gamma$ we have $\sum_{j\in[n]} \beta_{i,j}\delta_{i,j}x_j \leqq \sum_{j\in[n]} \beta^*_{i,j}\delta_{i,j}x_j$, because $\delta_{i,\pi(l)}x_{\pi(l)} \leqq \delta_{i,\pi(l')}x_{\pi(l')}$ for $l \geqq l'$. Consequently,

$$\begin{aligned} z_i(x,c) &= \sum_{j\in[n]} (\hat{c}_{i,j} + \beta_{i,j}\delta_{i,j})x_j \leqq \sum_{j\in[n]} (\hat{c}_{i,j} + \beta^*_{i,j}\delta_{i,j})x_j \\ &= z_i(x,c^*) \leqq \max_{c'\in\mathcal{U}^d} z_i(x,c')\ \forall\ c \in \mathcal{U}^{\text{owb}} \end{aligned}$$

and therefore

$$\max_{c\in\mathcal{U}^{\text{owb}}} \lambda_i(z_i(x,c) - r_i) \leqq \max_{c\in\mathcal{U}^d} \lambda_i(z_i(x,c) - r_i).$$

Further,

$$\max_{c\in\mathcal{U}^d} \lambda_i(z_i(x,c) - r_i) \leqq \max_{c\in\mathcal{U}^c} \lambda_i(z_i(x,c) - r_i) \leqq \max_{c\in\mathcal{U}^{\text{owb}}} \lambda_i(z_i(x,c) - r_i),$$

because of $\mathcal{U}^d \subseteq \mathcal{U}^c \subseteq \mathcal{U}^{\text{owb}}$. Since these results hold for all $x \in \mathcal{X}, i \in [k]$, we get

$$\begin{aligned} \max_{c\in\mathcal{U}^d} \lambda_i(z_i(x,c) - r_i) &= \max_{c\in\mathcal{U}^c} \lambda_i(z_i(x,c) - r_i) \\ &= \max_{c\in\mathcal{U}^{\text{owb}}} \lambda_i(z_i(x,c) - r_i) \qquad \forall\ i \in [k], x \in \mathcal{X} \\ \Rightarrow \max_{i\in[k]} \max_{c\in\mathcal{U}^d} \lambda_i(z_i(x,c) - r_i) &= \max_{i\in[k]} \max_{c\in\mathcal{U}^c} \lambda_i(z_i(x,c) - r_i) \\ &= \max_{i\in[k]} \max_{c\in\mathcal{U}^{\text{owb}}} \lambda_i(z_i(x,c) - r_i) \quad \forall\ x \in \mathcal{X} \\ \Rightarrow \min_{x\in\mathcal{X}} \max_{i\in[k]} \max_{c\in\mathcal{U}^d} \lambda_i(z_i(x,c) - r_i) &= \min_{x\in\mathcal{X}} \max_{i\in[k]} \max_{c\in\mathcal{U}^c} \lambda_i(z_i(x,c) - r_i) \\ &= \min_{x\in\mathcal{X}} \max_{i\in[k]} \max_{c\in\mathcal{U}^{\text{owb}}} \lambda_i(z_i(x,c) - r_i), \end{aligned}$$

where, again, all minima and maxima exist due to the finiteness of $\mathcal{X}$, the compactness of $\mathcal{U}^d, \mathcal{U}^c, \mathcal{U}^{\text{owb}}$ and the continuity of $z(x,\cdot) : [\mathcal{U}^d/\mathcal{U}^c/\mathcal{U}^{\text{owb}}] \to \mathbb{R}$. □

Because of this identity we can use the approach given in Hassanzadeh et al. (2013) also for the uncertainty sets $\mathcal{U}^d$ or $\mathcal{U}^c$:

$$\begin{aligned} &\min_{x\in\mathcal{X}} \max_{c\in[\mathcal{U}^d/\mathcal{U}^c/\mathcal{U}^{\text{owb}}]} \max_{i\in[k]} \lambda_i(z_i(x,c) - r_i) \\ &= \min_{x\in\mathcal{X}} \max_{c\in\mathcal{U}^{\text{owb}}} \max_{i\in[k]} \lambda_i(z_i(x,c) - r_i) \\ &= \min_{x\in\mathcal{X}} \max_{i\in[k]} \lambda_i\left(\max_{c\in\mathcal{U}^{\text{owb}}} z_i(x,c) - r_i \right) \end{aligned}$$

is equivalent to

$$\begin{aligned} \min\ & y \\ \text{s.t. } & y \geqq \lambda_i(\tilde{z}_i - r_i) && \forall i \in [k] \\ & \tilde{z}_i \geqq \max_{c \in \mathcal{U}^{\text{owb}}} z_i(x, c) && \forall i \in [k] \\ & x \in \mathcal{X}. \end{aligned}$$

As shown by Bertsimas and Sim (2003), for every $x \in \mathcal{X}$, the dual of the single-objective problem $\max_{c \in \mathcal{U}^{\text{owb}}} z_i(x, c)$ is equivalent to the linear program

$$\begin{aligned} \min\ & \sum_{j \in [n]} \hat{c}_{i,j} x_j + \theta \Gamma + \sum_{j \in [n]} \rho_j \\ \text{s.t. } & \rho_j + \theta \geqq \delta_{i,j} x_j && \forall\ j \in [n] \\ & \theta \geqq 0 \\ & \rho_j \geqq 0 && \forall\ j \in [n]. \end{aligned}$$

Similar to Hassanzadeh et al. (2013), we conclude that (P-max(r, λ)) is equivalent to

$$\begin{aligned} \min\ & y \\ \text{s.t. } & y \geqq \lambda_i(\tilde{z}_i - r_i) && \forall\ i \in [k] \\ & \tilde{z}_i - \sum_{j \in [n]} \hat{c}_{i,j} x_j - \theta_i \Gamma - \sum_{j \in [n]} \rho_{i,j} \geqq 0 && \forall\ i \in [k] \\ & \rho_{i,j} + \theta_i - \delta_{i,j} x_j \geqq 0 && \forall\ j \in [n], i \in [k] \\ & \rho_{i,j}, \theta_i \geqq 0 && \forall j \in [n], i \in [k] \\ & x \in \mathcal{X}. \end{aligned}$$

4.2.2. MILP-formulation for (P-min(r, λ)) with continuously bounded uncertainty

For a fixed x we can reformulate $\max_{c \in \mathcal{U}^c} \min_{i \in [k]} \lambda_i(z_i(x, c) - r_i)$ as follows:

$$\begin{aligned} (M(x))\quad & \max_{c \in \mathcal{U}^c} \min_{i \in [k]} \lambda_i(z_i(x, c) - r_i) \\ & \Leftrightarrow \\ & \max\ z \\ & \text{s.t. } z \leqq \lambda_i\Big(\sum_{j \in [n]} \hat{c}_{i,j} x_j + \sum_{j \in [n]} \beta_{i,j} \delta_{i,j} x_j - r_i\Big) \quad \forall\ i \in [k] \\ & \sum_{j \in [n], i \in [k]} \beta_{i,j} \leqq \Gamma \\ & \beta_{i,j} \in [0, 1] \quad \forall\ j \in [n], i \in [k] \end{aligned}$$

Since $\beta_{i,j}$ only contributes to the objective function if $x_j \neq 0$ and 0 is the only lower bound on $\beta_{i,j}$, there is always an optimal solution with $x_j = 0 \Rightarrow \beta_{i,j} = 0\ \forall j \in [n], i \in [k]$. Hence, we can replace $\beta_{i,j}$ with $\tilde{\beta}_{i,j} := \beta_{i,j} x_j$. Further, $\tilde{\beta}_{i,j} x_j = \tilde{\beta}_{i,j}$, hence we obtain the equivalent problem

$$\begin{aligned} \max\ & z \\ \text{s.t. } & z \leqq \lambda_i\Big(\sum_{j \in [n]} \hat{c}_{i,j} x_j + \sum_{j \in [n]} \tilde{\beta}_{i,j} \delta_{i,j} - r_i\Big) && \forall\ i \in [k] \\ & \sum_{j \in [n], i \in [k]} \tilde{\beta}_{i,j} \leqq \Gamma \\ & \tilde{\beta}_{i,j} \leqq x_j && \forall\ j \in [n], i \in [k] \\ & \tilde{\beta}_{i,j} \geqq 0 && \forall\ j \in [n], i \in [k] \end{aligned}$$

and its dual

$$\begin{aligned} (D(x))\quad \min\ & \sum_{i \in [k], j \in [n]} \lambda_i \hat{c}_{i,j} x_j \tau_i - \lambda_i r_i \tau_i + \Gamma \pi + \sum_{j \in [n], i \in [k]} x_j \nu_{i,j} \\ \text{s.t. } & \sum_{i \in [k]} \tau_i = 1 \\ & -\lambda_i \delta_{i,j} \tau_i + \pi + \nu_{i,j} \geqq 0 && \forall\ j \in [n], i \in [k] \\ & \tau_i, \pi, \nu_{i,j}, \geqq 0 && \forall\ j \in [n], i \in [k] \end{aligned}$$

In order to use ($D(x)$) instead of ($M(x)$) as inner problem of (P-min(r, λ)), we replace $x_j \tau_i$ by the new variable $\tilde{\tau}_{i,j}$ and $x_j \nu_{i,j}$ by $\tilde{\nu}_{i,j}$. Since $x_j \in \{0, 1\}$, $\tau_i \geq 0$ and $\sum_{i \in [k]} \tau_i = 1 \Rightarrow \tau_i \leq 1$, we can ensure $\tilde{\tau}_{i,j} = x_j \tau_i$ by adding the constraints

$$\begin{aligned} & \tilde{\tau}_{i,j} \leqq \tau_i \\ & \tilde{\tau}_{i,j} \leqq x_j \\ & \tilde{\tau}_{i,j} \geqq \tau_i - (1 - x_j) \\ & \tilde{\tau}_{i,j} \geqq 0. \end{aligned}$$

Further, consider a feasible solution for ($D(x)$) with $\nu_{i,j} > \lambda_i \delta_{i,j}$. Since $\nu_{i,j}$ occurs in only one constraint, which requires

$$\nu_{i,j} \geqq \lambda_i \delta_{i,j} \tau_i - \pi,$$

we can choose $\nu_{i,j} = \lambda_i \delta_{i,j}$ instead and obtain a still feasible solution. Its objective value is not worse, since $\nu_{i,j}$ contributes with nonnegative factor to the objective function. Hence, we can restrict the feasible space of ($D(x)$) by adding the constraint $\nu_{i,j} \leqq \lambda_i \delta_{i,j}$. Then, the following constraints ensure that $\tilde{\nu}_{i,j} = x_j \nu_{i,j}$:

$$\begin{aligned} & \tilde{\nu}_{i,j} \leqq \nu_{i,j} \\ & \tilde{\nu}_{i,j} \leqq x_j \lambda_i \delta_{i,j} \\ & \tilde{\nu}_{i,j} \geqq \nu_{i,j} - \lambda_i \delta_{i,j}(1 - x_j) \\ & \tilde{\nu}_{i,j} \geqq 0. \end{aligned}$$

We obtain the following MILP-formulation for (P-min(r, λ)) with uncertainty set $\mathcal{U}^c$:

$$\begin{aligned} \min\ & \sum_{i \in [k], j \in [n]} \lambda_i \hat{c}_{i,j} \tilde{\tau}_i - \lambda_i r_i \tau_i + \Gamma \pi + \sum_{j \in [n], i \in [k]} \tilde{\nu}_{i,j} \\ \text{s.t. } & \sum_{i \in [k]} \tau_i = 1 \\ & -\lambda_i \delta_{i,j} \tau_i + \pi + \nu_{i,j} \geqq 0 && \forall\ j \in [n], i \in [k] \\ & \tilde{\tau}_{i,j} - \tau_i \leqq 0 && \forall\ j \in [n], i \in [k] \\ & \tilde{\tau}_{i,j} - x_j \leqq 0 && \forall\ j \in [n], i \in [k] \\ & \tilde{\tau}_{i,j} - \tau_i - x_j \geqq -1 && \forall\ j \in [n], i \in [k] \\ & \tilde{\nu}_{i,j} - \nu_{i,j} \leqq 0 && \forall\ j \in [n], i \in [k] \\ & \tilde{\nu}_{i,j} - x_j \lambda_i \delta_{i,j} \leqq 0 && \forall\ j \in [n], i \in [k] \\ & \tilde{\nu}_{i,j} - \nu_{i,j} - \lambda_i \delta_{i,j} x_j \geqq -\lambda_i \delta_{i,j} && \forall\ j \in [n], i \in [k] \\ & \tau_i, \tau_0, \nu_{i,j}, \tilde{\tau}_{i,j}, \tilde{\nu}_{i,j} \geqq 0 && \forall\ j \in [n], i \in [k] \\ & x \in \mathcal{X} \end{aligned}$$

4.2.3. MILP formulation for (P-min(r, λ)) with discretely bounded uncertainty

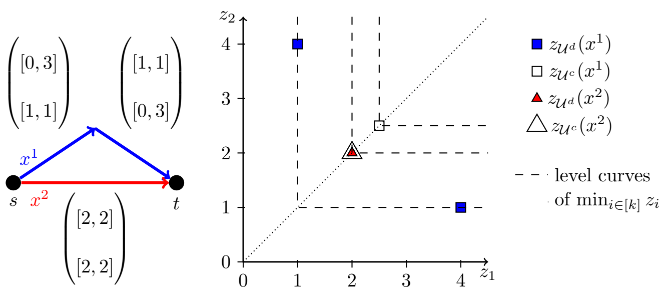
In contrast to (P-max(r, λ)), the solutions for (P-min(r, λ)) with discretely bounded uncertainty can differ from the solution for (P-min(r, λ)) with continuously bounded uncertainty, as the following example shows.

Example 22. Consider an instance of (P-min(r, λ)) with weights $\lambda = (1, 1)^T$, reference point $r = (0, 0)^T$, feasible set $\mathcal{X} = \{x^1 = (1, 1, 0), x^2 = (0, 0, 1)\}$ and discretely bounded uncertainty set $\mathcal{U}^d$ with $\Gamma = 1$. Our nominal costs are given by $\hat{c}$ and the interval lengths are given by δ as specified below:

$$\hat{c} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix}, \delta = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}.$$

The instance can for example be interpreted as an instance of the multi-objective robust shortest path problem in the graph shown in Fig. 5(a).

Since only one cost value can deviate from its lower bound, we either have $z_1(x^1, c) = 0 + 1$ or $z_2(x^1, c) = 1 + 0$. Hence, $\max_{c \in \mathcal{U}^d} \min_{i \in [k]} z_i(x^1, c) = 1$.



(a) Graph in Example 22 (b) $z_{\mathcal{U}^d}(x)$, $z_{\mathcal{U}^c}(x)$ and level curves of $\min_{i\in[k]} z_i$

Fig. 5. Example 22 shows that (P-min(r, λ)) with $\mathcal{U}^d$ is not equivalent to (P-min(r, λ)) with $\mathcal{U}^c$.

However, if we consider the continuous bounded uncertainty set $\mathcal{U}^c$ with the same $\Gamma, \hat{c}, \delta$ instead of $\mathcal{U}^d$, by setting

$$\beta' = \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{pmatrix},$$

we obtain the cost matrix

$$c' = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 1.5 & 0 & 0 \\ 0 & 1.5 & 0 \end{pmatrix} = \begin{pmatrix} 1.5 & 1 & 2 \\ 1 & 1.5 & 2 \end{pmatrix}$$

Therefore, $\max_{c\in\mathcal{U}^c} \min_{i\in[k]} z_i(x^1, c) \geq 2.5$. On the other hand we have $\max_{c\in\mathcal{U}^d} \min_{i\in[k]} z_i(x^2, c) = \max_{c\in\mathcal{U}^c} \min_{i\in[k]} z_i(x^2, c) = 2$. It follows that x^1 is the only optimal solution for (P-min(r, λ)) with uncertainty set $\mathcal{U}^d$, but x^2 is the only optimal solution for (P-min(r, λ)) with uncertainty set $\mathcal{U}^c$. The objective vectors $z(x, \xi)$ and the corresponding level curves are shown in Fig. 5(b).

Therefore, the derived MILP-formulation for (P-min(r, λ)) with continuously bounded uncertainty is not valid for (P-min(r, λ)) with discretely bounded uncertainty. The example shows also, that the inner maximization problem of (P-min(r, λ)) is not equivalent to its linear relaxation. Hence, we cannot use the approach to dualize the linearly relaxed inner problem here. However, with help of the identity we prove in Theorem 27 we can nevertheless find a minimization problem which is equivalent to the inner maximization problem and derive a MILP formulation for (P-min(r, λ)) with discretely bounded uncertainty set.

Definition 23. Let δ be a vector in $\mathbb{R}^n$ or a matrix in $\mathbb{R}^{k\times l}$ and let an index set $I\subseteq[n]$ resp. $I\subseteq[k]\times[l]$ be given. We denote the j-smallest of all entries δ_i with $i\in I$ as $j\text{-}\min_I \delta$ and the j-greatest as $j\text{-}\max_I \delta$. For $j = 0$ or $j > |I|$ we set $j\text{-}\min_I \delta = j\text{-}\max_I \delta = 0$.

Notation 24. For a binary vector $x\in\{0, 1\}^n$ we write $I(x) := \{j \in [n] : x_j = 1\}$.

Definition 25. Let $r\in\mathbb{R}^k, \lambda\in\mathbb{R}^k_>$ and $x\in\mathcal{X}$ be given. We define $M\in\mathbb{R}^{k\times(\Gamma+1)}$ by its entries

$$m_{i,l} := \lambda_i\left(-r_i + \sum_{j\in I(x)} \hat{c}_{i,j} + \sum_{h=1}^{l-1} h\text{-}\max_{I(x)}\ \delta_{(i,\cdot)}\right),$$

i.e., $\left(\frac{m_{i,l}}{\lambda_i} + r_i\right)$ is the sum of the nominal cost of x in the ith objective and the l highest interval lengths $\delta_{i,j}$ among those with $x_j = 1$ w.r.t. the ith objective.

Example 26. Consider an instance of (P-min(r, λ)) with $r = (0, 0, 0)^T, \lambda = (1, 3, 1)^T$ and uncertainty set $\mathcal{U}^d$ with $\Gamma = 6$. Let a feasible solution x be given with $|x| = 6$ and

$$\sum_{j\in I(x)} \hat{c}_{(\cdot,j)} = \begin{pmatrix} 10 \\ 4 \\ 14 \end{pmatrix},$$

$$\left\{\delta_{(\cdot,j)} : j\in I(x)\right\} = \left\{\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}\right\}.$$

We exemplarily compute $m_{1,3}$. For the first objective, the highest interval length $\delta_{1,j}$ among those with $j\in I(x)$ is 5, the second highest 4 and the nominal cost 10. Hence, we obtain

$$m_{1,3} = \lambda_1\left(-r_1 + \sum_{j\in I(x)} \hat{c}_{1,j} + \sum_{h=1}^{2} h\text{-}\max_{I(x)}\ \delta_{(1,\cdot)}\right)$$
$$= 1(-0 + 10 + 5 + 4) = 19.$$

The complete matrix for this example is

$$M = \begin{pmatrix} 10 & 15 & 19 & 22 & 24 & 25 & 25 \\ 12 & 15 & 18 & 21 & 24 & 27 & 30 \\ 14 & 20 & 25 & 29 & 32 & 34 & 35 \end{pmatrix}.$$

Theorem 27. *Given $x\in\mathcal{X}$ and the corresponding matrix M, the optimal objective value z^* of the inner maximization problem of (P-min(r, λ)) equals the $(\Gamma+1)$-smallest entry in M, i.e.,*

$$z^* := \max_{c\in\mathcal{U}} \min_{i\in[k]} \lambda_i(z_i(x, c) - r_i) = (\Gamma+1)\text{-}\min_{[k]\times[\Gamma+1]} M =: m^*.$$

Proof. We show first, that $z^* \leqq m^*$. Let c^* with $c^*_{i,j} = \hat{c}_{i,j} + \beta^*_{i,j}\delta_{i,j}$ be an optimal solution of the inner maximization problem $\max_{c\in\mathcal{U}} \min_{i\in[k]} \lambda_i(z_i(x, c) - r_i)$ with objective value z^*. Let us now look at the structure of the cost matrix c^*, or, more precisely, at each row $c^*_{(i,\cdot)}$ of this matrix, representing the costs under objective i in scenario c^*. Let $l_i := \sum_{j\in[n]} \beta^*_{i,j}$ be the number of entries in this row which deviate from their nominal value. Since we maximize the costs we can w.l.o.g. assume that among all $i\in I(x)$ the l_i indices with highest entries in $\delta_{(i,\cdot)}$ are chosen to deviate from the nominal value.

Due to the construction of the matrix M, it follows that the objective value of x in scenario c^* with respect to objective i is equal to the (l_i+1)st entry of line $m_{i,\cdot}$:

$$\lambda_i(z_i(x, c^*) - r_i) = \lambda_i\left(-r_i + \sum_{j\in I(x)} \hat{c}_{i,j} + \sum_{h=1}^{l_i} h\text{-}\max_{I(x)}\ \delta_{(i,\cdot)}\right)$$
$$= m_{i,(l_i+1)}.$$

M is constructed such that in each row i we have $m_{i,l} \leqq m_{i,l'}\ \forall\ l \leqq l'$. Hence, in row i there are at most l_i matrix entries smaller than m_{i,l_i+1} and in total there are at most $\sum_{i\in[k]} l_i = \sum_{i\in[k]}\sum_{j\in[n]} \beta^*_{i,j} \leqq \Gamma$ matrix entries smaller than $\min_{i\in[k]} m_{i,l_i+1}$. This implies

$$z^* = \min_{i\in[k]} m_{i,(l_i+1)} \leqq (\Gamma+1)\text{-}\min_{i\in[k], j\in[\Gamma+1]} M = m^*.$$

To show $z^* \geqq m^*$, we construct a scenario $\tilde{c}\in\mathcal{U}$ with objective value m^*. For each $i\in[k]$ we define

$$\hat{l}_i := \max\{l : m_{i,l} < m^*\}.$$

Because of $m_{i,l} \leqq m_{i,l'}\ \forall\ l \leqq l'$, we have $m_{i,l} < m^*\ \forall\ l \leqq \hat{l}_i$ and $m^* \leqq m_{i,(\hat{l}_i+1)} \leqq m_{i,l'}\ \forall\ l' > \hat{l}_i$. Thus we conclude

$$\sum_{i=1}^{k} \hat{l}_i \leqq \Gamma \text{ and } m^* = \min_{i\in[k]} m_{i,(\hat{l}_i+1)}.$$

We construct a $\tilde{\beta}$ such that the solution $\tilde{c}$ with $\tilde{c}_{i,j} = \hat{c}_{i,j} + \tilde{\beta}_{i,j}\delta_{i,j}$ is feasible and has objective value m^*: For each $i\in[k]$ we choose a set $\hat{J}_i \subseteq I(x)$ of $\hat{l}_i$ indices with largest interval lengths, i.e., such that $|\hat{J}_i| = \hat{l}_i$ and $\delta_{i,j} \geqq \delta_{i,j'}\ \forall\ j\in\hat{J}_i, j'\in I(x)\setminus\hat{J}_i$. We set

$$\tilde{\beta}_{i,j} := \begin{cases} 1 & \text{for } j\in\hat{J}_i \\ 0 & \text{else} \end{cases} \qquad \text{and} \qquad \tilde{c}_{i,j} := \hat{c}_{i,j} + \tilde{\beta}_{i,j}\delta_{i,j}.$$

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Then $\sum_{i\in[k], j\in[n]} \tilde{\beta}_{i,j} = \sum_{i\in[n]} \hat{l}_i \leqq \Gamma$, hence, $\tilde{c} \in \mathcal{U}^d$. Further,

$$\begin{aligned} z^* &\geqq \min_{i\in[k]} \lambda_i (z_i(x, \tilde{c}) - r_i) = \min_{i\in[k]} \lambda_i \left(\sum_{j\in[n]} (\hat{c}_{i,j} x_j + \tilde{\beta}_{i,j} \delta_{i,j} x_j) - r_i \right) \\ &= \min_{i\in[k]} \lambda_i \left(-r_i + \sum_{j\in[n]} \hat{c}_{i,j} x_j + \sum_{j\in \hat{J}_i} \delta_{i,j} \right) \\ &= \min_{i\in[k]} \lambda_i \left(-r_i + \sum_{j\in[n]} \hat{c}_{i,j} x_j + \sum_{h=1}^{\hat{l}_i} h\text{-}\max_{I(x)} \ \delta_{(i,\cdot)} \right) \\ &= \min_{i\in[n]} m_{i,(\hat{l}_i+1)} = m^*. \end{aligned}$$

□

Example 28. Consider the instance in Example 26 and the feasible solution x. We have $\Gamma + 1 = 7$ and the 7th smallest entry in M is 19. It follows that $\max_{c\in\mathcal{U}^d} \min_{i\in[k]} \lambda_i (z_i(x, c) - r_i) = 19$.

With help of this equality we derive a MILP formulation for (P-min(r, λ)). In a preprocessing step, for each $i \in [k]$ we sort the entries of the vector $\delta_{(i,\cdot)}$ non-increasingly and set

$$y_{i,j,j'} := \begin{cases} 1 & \text{if } \delta_{i,j} \text{ before } \delta_{i,j'} \text{ w.r.t. this sorting} \\ 0 & \text{else} \end{cases}$$

Then, for a given x, we can formulate $\max_{\xi\in\mathcal{U}} \min_{i\in[k]} \lambda_i (z_i(x, c) - r_i)$ as a minimization problem with the variables

z	being the objective value
$m_{i,l}$	representing $m_{i,l}$ as given in Definition 25
$w_{i,l}$	indicating if $m_{i,l}$ is one of the $\Gamma + 1$ smallest entries of M
$u_{i,j,l}$	indicating if $\delta_{i,j}$ is one of the summands *in* $m_{i,l}$
q_l	indicating the number of elements in x (if $q_l = 0$, x contains $l-1$ elements or more, if $q_l = 1$, x contains $l-1$ elements or less)

and the constants

$$N_i := \sum_{j\in[n]} (\hat{c}_{i,j} + \delta_{i,j}) \ \forall i \in [k].$$

If x is known, many of the values can be precomputed. However, when using the problem as inner problem for (P-min(r, λ)), they are variables. We construct the following MILP formulation for $\max_{\xi\in\mathcal{U}} \min_{i\in[k]} \lambda_i (z_i(x, c) - r_i)$:

$$\min \ z$$

$$\text{s.t.} \ z \geqq m_{i,l} - (1 - w_{i,l}) N_i \qquad \forall\ i \in [k], l \in [\Gamma + 1] \quad (1)$$

$$\sum_{\substack{i\in[k] \\ l\in[\Gamma+1]}} w_{i,l} = \Gamma + 1 \quad (2)$$

$$m_{i,l} = \lambda_i \left(\sum_{j\in[n]} \hat{c}_{i,j} x_j + \sum_{j\in[n]} u_{i,j,l} \delta_{i,j} - r_i \right) \qquad \forall\ i \in [k], l \in [\Gamma + 1] \quad (3)$$

$$\sum_{j\in[n]} u_{i,j,l} \geqq (l-1) - \Gamma q_l \qquad \forall\ i \in [k], l \in [\Gamma + 1] \quad (4)$$

$$\sum_{j\in[n]} u_{i,j,l} \geqq \sum_{j\in[n]} x_j - |E|(1 - q_l) \qquad \forall\ i \in [k], l \in [\Gamma + 1] \quad (5)$$

$$u_{i,j,l} \leqq x_j \qquad \forall\ j \in [n], i \in [k], l \in [\Gamma + 1] \quad (6)$$

$$u_{i,j',l} - u_{i,j,l} \leqq 1 - y_{i,j,j'} x_j \qquad \forall\ j, j' \in [n], i \in [k], l \in [\Gamma + 1] \quad (7)$$

$$u_{i,j,l}, w_{i,l}, q_l \in \{0, 1\} \qquad \forall\ j \in [n], i \in [k], l \in [\Gamma + 1] \quad (8)$$

The first two constraints ensure that z, when minimized, is set to the $(\Gamma + 1)$-smallest of the variables $m_{i,l}$. Because of Constraints (4) and (5), for each i and l at least $\min\{|x|, l-1\}$ of the $u_{i,j,l}$ are set to 1. Hence, at least $\min\{|x|, l-1\}$ of the $\delta_{i,j}$ are summed up in Constraint (3). Constraints (6) and (7) ensure, that these are the largest $\delta_{i,j}$ among those with $x_j = 1$. Since every $u_{i,j,l} = 1$ increases the value of $m_{i,l}$ (Constraint (3)) and thus potentially the value of our minimization objective z (Constraint (1)), we can assume without loss of generality that exactly $\min\{|x|, l-1\}$ of the $u_{i,j,l}$ are set to 1 and thus exactly $\min\{|x|, l-1\}$ of the $\delta_{i,j}$ are summed up in Constraint (3). We obtain

$$\sum_{h=1}^{l-1} h\text{-}\max_{I(x)} \ \delta_{(i,\cdot)} = \sum_{j\in[n]} u_{i,j,l} \delta_{i,j} \ \forall\ l \in [\Gamma + 1], i \in [k],$$

Thus, $m_{i,l}$ take exactly the values given in Definition 25 (Constraint (3)). We conclude that (P-min(r, λ)) with uncertainty set $\mathcal{U}^d$ can be formulated as

$$\begin{array}{lll} (\text{P-min}(r, \lambda)) & \min & z \\ & \text{s.t.} & (1) - (8) \\ & & x \in \mathcal{X}. \end{array}$$

*4.2.4. Complexity of (P-min(*r*, λ)) and (P-max(*r*, λ)) with bounded uncertainty*

For $\Gamma = 0$, the uncertainty sets $\mathcal{U}^d$ and $\mathcal{U}^c$ only contain one scenario. From Remark 9 it hence follows, analogous to the case of interval uncertainty, that (P-min(r, λ)) is polynomially solvable, if the single-objective deterministic problem is polynomially solvable, whereas (P-max(r, λ)) is NP-hard for several combinatorial problems, e.g., the shortest path, minimum spanning tree and assignment problem.

The following Theorem shows that (P-min(r, λ)) with uncertainty set $\mathcal{U}^d$ is NP-hard for the shortest path and minimum spanning tree problem, if $\Gamma = 1$.

Theorem 29. *(P-min(r, λ)) with uncertainty set $\mathcal{U}^d$ and $\Gamma = 1$ is NP-hard for the shortest path problem and the minimum spanning tree problem, even for two objectives, $\lambda = (1, 1)^T$ and $r = (0, 0)^T$.*

Proof. We consider the single-objective min–max robust shortest path resp. minimum spanning tree problem with a discrete scenario set consisting of two scenarios. This has been proven to be NP-hard for both problems (see Kouvelis & Yu, 1997). We reduce it to (P-min(r, λ)) with two objectives and discretely bounded uncertainty set with $\Gamma = 1$.

Let an instance I of the single-objective min–max robust problem be given. In case of the shortest path problem, we have given a graph G with edge set $E = \{e_1, \dots, e_n\}$, and a start node s and end node t in G. The set of feasible solutions $\mathcal{X} \subseteq \{0, 1\}^n$ contains all vectors that represent a simple path from s to t. In case of the minimum spanning tree problem, E is again the edge set of a graph G and the feasible solutions represent the spanning trees in G. Further, we have given two scenarios ξ^1, ξ^2 and edge costs $b \in \mathbb{R}^{2\times n}$, assigning cost $b_{i,j}$ to edge e_j under scenario ξ^i. We construct an instance I' of (P-min(r, λ)) as following:

- We start with the graph G from I and construct edge costs for the discretely bounded uncertainty set: $\hat{c}_{i,j} := b_{i,j}, \delta_{i,j} = 0 \ \forall j \in [n], i \in [2]$.
- We then add one new node s' and one new edge e_{n+1}: For the minimum spanning tree problem, e_{n+1} connects s' to any of the other nodes. For the shortest path problem, the edge e_{n+1} leads from s' to the original start node s.

- We construct cost intervals for the new edge: For some upper bound $B \geq \max_{i=1,2} \sum_{j\in[n]} \hat{c}_{i,j}$ we define $\hat{c}_{(\cdot,n+1)} := (0,0)^T, \delta_{(\cdot,n+1)} := (B,B)^T$.
- We define the new feasible set $\mathcal{X}' := \left\{ \begin{pmatrix} x \\ 1 \end{pmatrix} : x \in \mathcal{X} \right\}$.

Note, that in case of the spanning tree problem, $\mathcal{X}'$ represents the set of all spanning trees in the new graph, since the only edge connecting s' to the old graph is e_{n+1}. In case of the robust shortest path problem, $\mathcal{X}'$ represents the set of all paths from the new node s' to the original destination node t in the new graph, because s' has exactly one outgoing edge e_{n+1}, which ends in the original start node s.

Constructed like this, for every $x \in \mathcal{X}$ the solution $x' := (x, 1)^T$ is feasible for I' and for every $x' \in \mathcal{X}'$, the solution $x := (x'_1, \ldots, x'_n)^T$ is feasible for I. Hence, every feasible solution x for I corresponds to a feasible solution x' for I' and vice versa.

Since for every $x' \in \mathcal{X}'$ we have $x'_{(n+1)} = 1$, its worst case scenario is either

$$c^1 : c^1_{1,(n+1)} = B, c^1_{2,(n+1)} = 0, c^1_{i,j} = \hat{c}_{i,j} \ \forall\ j \neq n+1 \text{ or}$$
$$c^2 : c^1_{1,(n+1)} = 0, c^2_{2,(n+1)} = B, c^2_{i,j} = \hat{c}_{i,j} \ \forall\ j \neq n+1,$$

because all other feasible scenarios are equivalent to just considering the nominal edge lengths (since $\Gamma = 1$). The choice of B ensures $z_1(x', c^1) \geqq z_2(x', c^1)$ and $z_2(x', c^2) \geqq z_1(x', c^2)$ for all $x' \in \mathcal{X}'$. It follows that for every $x' \in \mathcal{X}'$

$$\begin{aligned} \max_{c\in\mathcal{U}} \min_{i=1,2} z_i(x', c) &= \max \left\{ \min\{z_1(x', c^1), z_2(x', c^1)\}, \right. \\ &\quad \left. \min\{z_1(x', c^2), z_2(x', c^2)\} \right\} \\ &= \max\left\{ z_2(x', c^1), z_1(x', c^2) \right\} \\ &= \max\left\{ \sum_{j\in[n]} \hat{c}_{2,j} x'_j, \sum_{j\in[n]} \hat{c}_{1,j} x'_j \right\} \\ &= \max_{i=1,2} \sum_{j\in[n]} \hat{c}_{i,j} x'_j = \max_{i=1,2} \sum_{j\in[n]} b_{i,j} x'_j. \end{aligned}$$

We conclude that an optimal solution for I' corresponds to an optimal solution for I and vice versa. □

5. Conclusion

In this paper we introduced two methods to find min–max robust efficient solutions based on scalarizations: the min-ordering and the max-ordering method. We have shown that the max-ordering method finds (all) point-based min–max robust weakly efficient solutions. The min-ordering solution finds set-based min–max robust weakly efficient solutions, which cannot necessarily be found with scalarization based methods for multi-objective robust optimization from the literature.

We investigated the resulting scalarized problems (P-min(r, λ)) and (P-max(r, λ)) for multi-objective combinatorial problems with particular uncertainty sets. For interval uncertainty we could show that only one scenario needs to be considered. Then, (P-max(r, λ)) reduces to a single-objective min–max robust problem with discrete uncertainty set, whereas a solution to (P-min(r, λ)) can be found by solving several single-objective deterministic problems with the same feasible set. We further extended the single-objective concept of bounded uncertainty to the multi-objective case. We developed MILP-formulations for both (P-min(r, λ)) and (P-max(r, λ)) with bounded uncertainty and investigated the complexity of the resulting problems.

The first question in mind for further investigations is, how to solve (P-min(r, λ)) and (P-max(r, λ)) in case of multi-objective robust combinatorial problems with other uncertainty sets, e.g., discrete scenarios sets or polyhedral or ellipsoidal uncertainty. Also, the complexity of (P-min(r, λ)) with uncertainty set $\mathcal{U}^c$ remains an open question.

Further research could be done on specialized solution approaches for particular combinatorial problems, for example the shortest path or minimal spanning tree problem. It is also interesting to check if solutions to other robustness concepts, e.g., hull-based min–max robust efficiency (Bokrantz & Fredriksson, 2017), multi-scenario efficiency (Botte & Schöbel, 2019), or lightly robust efficiency (Ide & Schöbel, 2016) can be found with the min-ordering or max-ordering method.

A variant of the max-ordering or min-ordering optimization problem is to look for the second/third/... highest or smallest objective instead of the maximum or minimum. Moreover, we have shown that the solutions of (P-min(r, λ)) and (P-max(r, λ)) have quite different properties and characterizations. It would therefore also be of interest to consider a combination of both by choosing any ordered median function as scalarizing function and analyze the resulting problems.

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