THE BEHAVIOR OF THE RENEWAL SEQUENCE IN CASE THE TAIL OF THE WAITING-TIME DISTRIBUTION IS REGULARLY VARYING WITH INDEX −1

J. B. G. FRENK,* Erasmus University, Rotterdam

Abstract

A second-order asymptotic result for the probability of occurrence of a persistent and aperiodic recurrent event is given if the tail of the distribution of the waiting time for this event is regularly varying with index −1.

RENEWAL THEORY; REGULAR VARIATION

1. Introduction and results

Suppose ε is a persistent and aperiodic recurrent event (for definition see [6], p. 308) and define:

\[ f_n \triangleq \Pr \{ \varepsilon \text{ occurs for the first time at the } n\text{th trial} \}, \]
\[ u_n \triangleq \Pr \{ \varepsilon \text{ occurs at the } n\text{th trial} \}, \]
\[ u_0 \triangleq 1, \quad f_0 \triangleq 0. \]

\([\{u_n\}_{n \in \mathbb{N}}\) is the so-called renewal sequence.]

Using the probabilistic interpretation this yields

\[ u_n = \sum_{k=1}^{\infty} f_k u_{n-k} \quad \text{for } n \geq 1. \]


\[ \lim_{n \to \infty} u_n = \frac{1}{\mu} \text{ for } \mu < \infty \]
\[ = 0 \quad \text{for } \mu = \infty \]

with \( \mu = \sum_{n=1}^{\infty} n f_n \). Garsia and Lamperti [8] obtained a stronger result when \( \mu = \infty \) for a certain class of lattice probability distributions \( F \). They proved with

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* Postal address: Erasmus Universiteit Rotterdam, Faculteit der Economische Wetenschappen, Postbus 1738, 3000 DR Rotterdam, The Netherlands.

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\[ F(n) \triangleq \sum_{m=0}^n f_m \text{ and } m(n) \triangleq \sum_{m=0}^n 1 - F(m) \quad (n \in \mathbb{N}) \]

\[ \begin{align*}
(a) \quad \lim_{n \to \infty} m(n)u_n &= \frac{\sin \pi \alpha}{\pi (1 - \alpha)} \quad \text{for } \frac{1}{2} < \alpha < 1 \\
&\quad \text{if } 1 - F(n) \in \text{RVS}_{-\alpha}, \\
(b) \quad \liminf_{n \to \infty} m(n)u_n &= \frac{\sin \pi \alpha}{\pi (1 - \alpha)} \quad \text{for } 0 < \alpha \leq \frac{1}{2} \\
&\quad \text{if } 1 - F(n) \in \text{RVS}_\alpha.
\end{align*} \tag{1.2} \]

(For the definition of \text{RVS}_\alpha the reader is referred to the next section.)

Erickson [5] considered the case \( \alpha = 1 \) and proved

\[ \lim_{n \to \infty} m(n)u_n = 1 \quad \text{if } 1 - F(n) \in \text{RVS}_{-1}. \tag{1.3} \]

In this paper we are going to prove among some other results the following statement which is stronger than Erickson’s.

**Theorem.**

\[ 1 - F(n) \in \text{RVS}_{-1} \iff -u_{\lfloor t \rfloor} \in \Pi^\infty. \tag{1.4} \]

([\lfloor t \rfloor] \triangleq \text{integral part of } t: \text{ for the definition of } \Pi^\infty \text{ the reader is referred to the next section.})

Both relations imply

\[ \lim_{n \to \infty} \frac{u_n - \frac{1}{m(n)}}{n(1 - F(n))m^{-2}(n)} = 0. \]

2. **Proofs**

Using the theory of Banach algebras we provide a proof of the theorem in the case \( \mu < \infty \). This method of proving the theorem for this special case will be given because of its brevity. It is not possible to use the same method when \( \mu \) is infinite and we therefore give a proof of this case using the Fourier representation of \( \mu_n \). (This proof also applies to the case \( \mu < \infty \).) However, before starting we need some definitions and lemmas.

**Definition 1.** A sequence of eventually positive numbers \( \{c(n)\}_{n \in \mathbb{N}} \) is called a regularly varying sequence of index \( \rho \) if

\[ \lim_{n \to \infty} \frac{c([\lambda n])}{c(n)} = \lambda^\rho \quad \forall \lambda > 0 \quad (\triangleq: c(n) \in \text{RVS}_\rho). \]
An ultimately positive function \( R \) on \((0, \infty)\) is called regularly varying with index \( p \) if \( \lim_{t \to \infty} \left( R(At)/R(t) \right) = \lambda^p \ \forall \lambda > 0 \) (\( \Delta: R(t) \in \text{RVF}^p \)).

The following lemma shows that the theory of regularly varying functions also applies to regularly varying sequences.

**Lemma 1.** If \( \{c(n)\}_{n \in \mathbb{N}} \) is a regularly varying sequence of index \( p \), the function \( R \) defined on \([0, c)\) by \( R(t) \triangleq c([t]) \) is a regularly varying function of index \( p \).

**Proof.** See [14].

**Definition 2.** A sequence \( \{c(n)\}_{n \in \mathbb{N}} \) belongs to the class \( \text{IIS}^\infty \) if there exists a sequence \( L(n) \in \text{RVSo}^\infty \) such that \( \lim_{n \to \infty} \frac{c([nx]) - c(n)}{L(n)} = \log x \ \forall x > 0 \) (\( \Delta: c(n) \in \text{IIS}^\infty \)). A function \( R \) on \((0, \infty)\) belongs to the class \( \text{H}^\infty \) if there exists a function \( L(t) \) such that \( \lim_{t \to \infty} \frac{R(tx) - R(t)}{L(t)} = \log x \ \forall x > 0 \) (\( \Delta: R(t) \in \text{H}^\infty \)). \( L(t) \) is then automatically in \( \text{RVF}_\infty \).

**Lemma 2.** If \( \{c(n)\}_{n \in \mathbb{N}} \in \text{IIS}^\infty \) the function \( R \) defined on \((0, \infty)\) by \( R(t) \triangleq c([t]) \) is in \( \text{HI}^\infty \).

**Proof.** Using the definition of \( \text{IIS}^\infty \) we obtain

\[
\lim_{n \to \infty} \frac{c([nx]) - c(n)}{L(n)} = \lim_{n \to \infty} \frac{c([nx]) - c(n)}{L([nx])} \cdot \frac{L([nx])}{L(n)} = \log x + \log x \quad \forall x > 0.
\]

This implies (take \( x = \frac{1}{2}; \ z = 2; \ n = 2k + 1 \ (k \in \mathbb{N}) \) \( \lim_{n \to \infty} \frac{c(n+1) - c(n)}{L(n)} = 0 \). Hence for all \( x > 0 \)

\[
\lim_{t \to \infty} \frac{c([tx]) - c([t])}{L([t])} = \lim_{t \to \infty} \frac{c([tx]) - c([t]x)}{L([tx])} \cdot \frac{L([tx])}{L([t])} + \lim_{t \to \infty} \frac{c([t]x) - c([t])}{L([t])} = \log x \text{ since } [tx] - [t]x \text{ is bounded.}
\]

**Case \( \mu < \infty \).** Define \( \hat{U}(s) \triangleq \sum_{n=0}^{\infty} u_n s^n \) and \( \hat{F}(s) \triangleq \sum_{n=0}^{\infty} f_n s^n \) for \( |s| < 1 \). Before proving the theorem we recall the following result.

**Lemma 3.** If \( \hat{F}(e^{it}) = 1 \) for some \( t_0 \neq 0 \) and \( \hat{F}(e^{it}) \) is the characteristic function of \( F, F \) should be a lattice distribution with the point spectrum contained in the set \( \{2k\pi/t_0; \ k = 0, 1, \cdots \} \).

**Proof.** [9], p. 94.

**Proof of the theorem** (in the case \( \mu < \infty \)). We have \( \hat{U}(s) = 1/(1 - \hat{F}(s)) \) for
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\[ \vert s \vert < 1 \ (\text{[6], p. 311}). \]

Define

\[ \hat{M}(s) = \frac{1 - \hat{F}(s)}{1 - s} = \sum_{n=0}^{\infty} (1 - F(n)) s^n \quad \text{for} \quad \vert s \vert \leq 1. \]

Since \( \mu \triangleq \sum_{k=1}^{\infty} k f_k < \infty \) we obtain using the monotone convergence theorem

\[ \lim_{t \to 1} \hat{M}(s) = \mu > 0. \]

Since \( s < 1 \) and hence using a theorem of Wiener ([13], p. 665) we obtain

\[ 1/\hat{M}(s) = \sum_{n=0}^{\infty} \lambda_n s^n \quad \text{with} \quad \sum_{n=0}^{\infty} \lambda_n < \infty \quad \text{and} \quad \vert s \vert \leq 1. \]

This implies together with \( 1 - F(n) \in \text{RV}_S \) (\[1\], p. 258)

\[ (2.1) \]

\[ \lambda_n \sim -\frac{1}{\mu^2} (1 - F(n)) \quad (n \to \infty) \]

(take \( d_n = 1 - F(n) \) and \( \Lambda(x) = 1/x \)).

Since \( \hat{U}(s) = 1/(1-s)\hat{M}(s) \) we get \( u_n = \sum_{p=0}^{\infty} \lambda_p \) for all \( n \geq 0 \). Consequently \( \sum_{n=0}^{p} u_n - (p + 1) u_p = -\sum_{n=1}^{p} n \lambda_n \). Using (2.1) and Lemma 1 we get

\[ \lim_{p \to \infty} \frac{\sum_{n=0}^{p} u_n - (p + 1) u_p}{p^2(1 - F(p))} = -\lim_{p \to \infty} \frac{\sum_{n=1}^{p} n \lambda_n}{p^2(1 - F(p))} = \frac{1}{\mu^2}. \]

Combining this with

\[ \lim_{p \to \infty} \frac{\sum_{n=0}^{p} u_n - \frac{1}{m(p)}}{p(1 - F(p))} = \frac{1}{\mu^2} \]

([7], Theorem 3) we get

\[ \lim_{p \to \infty} \frac{u_p - \frac{1}{m(p)}}{p(1 - F(p))} = 0. \]

Clearly this implies \( -u_{\lceil t \rceil} \in \Pi^\infty \) since \( -1/m(\lceil t \rceil) \in \Pi^\infty \). The converse statement \( (\Pi^\infty \Rightarrow 1 - F(n) \in \text{RV}_S) \) will be proved at the end of this section.

We remark that the renewal theorem of Kolmogorov (for the case \( \mu < \infty \)) can be proved easily using Wiener’s theorem. Using the same method we can also prove a second-order asymptotic result for the case \( \alpha > 1 \).

**Lemma 4.**

\[ 1 - F(n) \in \text{RV}_S^{\infty}(\alpha > 1) \Leftrightarrow u_n - \frac{1}{\mu} \in \text{RV}_S^{\infty}. \]
Both imply

$$\lim_{n \to \infty} \frac{u_n - (1/\mu)}{n(1 - F(n))} = \frac{1}{\mu^2(1 - \alpha)}.$$ 

**Proof.** Since

$$u_n - \frac{1}{\mu} = \sum_{k=0}^{n} \lambda_k - \sum_{k=0}^{\infty} \lambda_k = -\sum_{k=n+1}^{\infty} \lambda_k;$$

$$\lambda_n \sim (-1/\mu^2)(1 - F(n))$$ if \(1 - F(n) \in \text{RVS}_-^\infty\) ([1], p. 258) and Lemma 1 we obtain the desired result.

To prove the converse statement we consider the following cases.

(a) \(1 < \alpha < 2 \Rightarrow \sum_{p=0}^{n} u_p - (n/\mu) \in \text{RIS}_-^\infty\) and applying [11], Theorem A, yields \(1 - F(n) \in \text{RVS}_-^\infty\).

(b) \(\alpha = 2 \Rightarrow \sum_{p=0}^{n} u_p - (n/\mu) \in \text{IIS}_-^\infty\) and applying [7], Theorem 2, yields \(1 - F(n) \in \text{RVS}_-^\infty\).

(c) \(\alpha > 2 \Rightarrow \sum_{p=0}^{n} (u_p - (1/\mu)) \in \text{RVS}_-^\infty\) and applying [7], Theorem 1, yields \(1 - F(n) \in \text{RVS}_-^\infty\).

The case \(\mu = \infty\). Define \(\phi(\theta) \triangleq \int_0^\infty e^{i\theta x} dF(x)\) with \(F\) some probability distribution on \((0, \infty)\). Hence in our case \(\phi(\theta) = \sum_{n=1}^{\infty} e^{i\theta n} f_n\).

Before we start with the proof of the theorem we state the following lemma.

**Lemma 5.** If the tail of the distribution \(F\) is regularly varying with index \(-1\) and \(0 < \varepsilon < 1\) is some chosen number we can find \(A_1, A_2, A_3 > 0\) such that

$$\forall n \geq A_1 \forall \theta \in [A_2, \varepsilon n] \left| \frac{\text{Re} \left( \frac{\theta - \pi}{n} \right) - \text{Re} \left( \frac{\theta}{n} \right)}{1 - F(n)} \right| \leq A_3 . \theta^\gamma \quad \text{with} \quad \gamma < 1.$$ 

**Proof.** The definition of \(\phi\) yields

$$\text{Re} \left( \frac{\theta - \pi}{n} \right) - \text{Re} \left( \frac{\theta}{n} \right) = -\int_0^\infty \int_0^{\theta x/n} \sin z \, dz \, dF(x)$$

$$= \int_0^\infty \left( F \left( \frac{nz}{\theta} \right) - F \left( \frac{nz}{\theta - \pi} \right) \right) \sin z \, dz.$$ 

Hence

\[
\frac{1}{\theta} \left( \text{Re} \left( \frac{\theta - \pi}{n} \right) - \text{Re} \left( \frac{\theta}{n} \right) \right) = \int_0^\infty \left( F(nw) - F\left( nw \left( \frac{\theta}{\theta - \pi} \right) \right) \right) \sin \theta w \, dw.
\]

Divide \(\int_0^\infty (F(nw) - F\left( nw \left( \frac{\theta}{\theta - \pi} \right) \right)) \sin \theta w \, dw\) into two parts, the first part

$$I_1(\theta, n, \eta) \triangleq \int_0^{\eta} \left( F(nw) - F\left( nw \left( \frac{\theta}{\theta - \pi} \right) \right) \right) \sin \theta w \, dw$$
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and the second

$$I_2(\theta, n, \eta) \triangleq \int_{\theta - \pi}^{\theta - \pi} \left( F(nw) - F \left( nw \left( \frac{\theta}{\theta - \pi} \right) \right) \right) \sin \theta w \, dw$$

with $\eta \in (0, \frac{1}{2}(\sqrt{5} - 1))$.

Consider $I_1(\theta, n, \eta)$. Using Fubini's theorem we get

$$I_1(\theta, n, \eta) = \theta \int_0^{\theta - \pi} \int_p^{\theta - \pi} \left( F \left( nw \left( 1 + \frac{\pi}{\theta - \pi} \right) \right) - F(nw) \right) \, dw \cos \theta p \, dp.$$

Since

$$\int_p^{\theta - \pi} 1 - F \left( nw \left( 1 + \frac{\pi}{\theta - \pi} \right) \right) \, dw = \frac{\theta - \pi}{\theta} \int_p^{\theta - \pi} \frac{(1 - F(nw)) \, dw}{(1 + \pi/(\theta - \pi))}$$

we obtain

$$\int_p^{\theta - \pi} F \left( nw \left( 1 + \frac{\pi}{\theta - \pi} \right) \right) - F(nw) \, dw$$

$$= \frac{\pi}{\theta} \int_p^{\theta - \pi} \frac{(1 - F(nw)) \, dw}{(1 + \pi/(\theta - \pi))} + \int_p^{\theta - \pi} (1 - F(nw)) \, dw$$

$$- \int_p^{\theta - \pi} (1 - F(nw)) \, dw.$$

Combining these relations we have

$$I_1(\theta, n, \eta) = \theta I_{11}(\theta, n, \eta) + \theta I_{12}(\theta, n, \eta) - \theta I_{13}(\theta, n, \eta)$$

with

$$I_{11}(\theta, n, \eta) \triangleq \frac{\pi}{\theta} \int_0^{\theta - \pi} \int_p^{\theta - \pi} \frac{(1 - F(nw)) \, dw \cos \theta p \, dp}{(1 + \pi/(\theta - \pi))}$$

$$I_{12}(\theta, n, \eta) \triangleq \int_0^{\theta - \pi} \int_p^{\theta - \pi} (1 - F(nw)) \, dw \cos \theta p \, dp$$

$$I_{13}(\theta, n, \eta) \triangleq \int_0^{\theta - \pi} \int_p^{\theta - \pi} (1 - F(nw)) \, dw \cos \theta p \, dp.$$

Using Lemma 1 we can apply the result stated by Pitman [12], Lemma 2;

$$\forall h, c > 0 \exists A_1(c, h) 0 \leq \frac{1 - F(nw)}{1 - F(n)} \leq \frac{A}{w^{1-h}}$$

for all $n \geq A_1(c, h)$, $0 < w < c$ and $A$ some constant if $1 - F \in RVF_{-1}$ and $t^{1+h}(1 - F(t))$ is bounded in the neighbourhood of 0. (The condition $t^{1+h}(1 - F(t))$ is bounded in the neighbourhood of 0 is omitted in Pitman's lemma. One can easily construct a counterexample if the condition is not fulfilled.) Using
this inequality we find for \( h = \eta \) and \( \theta \) sufficiently large that for all \( n \geq A_1(3, \eta) \)

\[
\begin{align*}
(a) \quad \left| \frac{I_{11}(\theta, n, \eta)}{1 - F(n)} \right| & \leq A \int_0^{\eta} \int_0^{\eta - (1 + \pi/(\theta - \pi))} w^{-1 - n} \, dw \, dp \leq \frac{C_1}{\theta} \cdot \theta^{n+2 - 2n} \\

and \( C_1 \) some constant.

(b) \quad \left| \frac{I_{12}(\theta, n, \eta)}{1 - F(n)} \right| & \leq A \int_0^{\eta} \int_0^{\eta - (1 + \pi/(\theta - \pi))} w^{-1 - n} \, dw \, dp \\
& = \frac{A \cdot \pi}{(\theta - \pi) \eta} \left( 1 - \frac{1}{\frac{\pi}{\theta - \pi}} \right)^{\eta-n} \int_0^{\eta} p^{-n} \, dp \leq \frac{C_2}{\theta} \cdot \theta^{n+2 - 2n}

and \( C_2 \) some constant.

(c) \quad \left| \frac{I_{13}(\theta, n, \eta)}{1 - F(n)} \right| & \leq A \int_0^{\eta} \int_0^{\eta - (1 + \pi/(\theta - \pi))} w^{-1 - n} \, dw \left| \int_0^{\eta} \cos \theta p \, dp \right| \\
& = \frac{A \cdot \pi}{\eta(\theta - \pi)} \left( 1 - \frac{1}{\frac{\pi}{\theta - \pi}} \right)^{\eta-n} \frac{1}{\theta} \left| \int_0^{\eta} \cos z \, dz \right| \\
& \leq \frac{C_3}{\theta^2} \cdot \theta^{n+2 - 2n}

and \( C_3 \) some constant.

Hence using these inequalities we get \( \forall n \geq A_1(3, \eta) \).

\[
\begin{align*}
(2.3) \quad \left| \frac{I_1(\theta, n, \eta)}{1 - F(n)} \right| & \leq C_1 \cdot \theta^{2 - n} + C_2 \cdot \theta^{n+2 - n} + C_3 \cdot \theta^{n+2 - 1} \leq C_4 \cdot \theta^{n+2 - n}

with \( C_4 \) some constant and \( \theta \) sufficiently large.

Consider \( I_2(\theta, n, \eta) \). Since

\[
I_2(\theta, n, \eta) \triangleq \int_{-\infty}^{\infty} \left( 1 - F(nw \left( \frac{\theta}{\theta - \pi} \right) ) \right) - (1 - F(nw)) \sin \theta w \, dw

and \( 1 - F \) a positive non-increasing function we can apply Bonnet's form of the second mean-value theorem ([15], p. 17) to get

\[
\left| I_2(\theta, n, \eta) \right| \leq \frac{2}{\theta} \left( 1 - F(n\theta^{-n} \left( \frac{\theta}{\theta - \pi} \right) ) \right) + \frac{2}{\theta} (1 - F(n\theta^{-n})).

Hence using Pitman's lemma we have

\[
(2.4) \quad \left| \frac{I_2(\theta, n, \eta)}{1 - F(n)} \right| \leq \frac{C_5}{\theta} \left( \theta^{-n} - 1 - n \right) = \frac{C_5}{\theta} \cdot \theta^{n+2 - n}.
\]
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Combining (2.2), (2.3), (2.4) we obtain

$$\left| \text{Re} \left( \frac{\theta - \pi}{n} \right) - \text{Re} \left( \frac{\theta}{n} \right) \right| \leq C_4 \cdot \theta^{n^2-n+1} + C_5 \cdot \theta^{n^2+n} \leq C_6 \cdot \theta^\gamma$$

with $\gamma \triangleq \max(\eta^2-\eta+1, \eta^2+\eta)$ for all $n \geq A_4(3, \eta)$ and $\theta \in [A_2, \varepsilon n]$. Since $0 < \eta < \frac{1}{2}(\sqrt{5} - 1)$ we have $\gamma < 1$.

**Proof of the theorem.** For $u_n$ the following representation is well known ([5], p. 266 or [8], p. 226):

$$u_n = \frac{2}{\pi} \int_0^\pi W(\theta) \cos n\theta \, d\theta \quad \text{and} \quad W(\theta) \triangleq \text{Re} \left( \frac{1}{1 - \phi(\theta)} \right).$$

Hence for all $p > 1$

$$\frac{\pi}{2} (u_n - u_{[np]}) = \left( \int_0^{B/n} \cos n\theta W(\theta) \, d\theta - \int_0^{B/[np]} \cos ([np]\theta) W(\theta) \, d\theta \right)$$

$$\quad + \left( \int_e^\pi \cos n\theta W(\theta) \, d\theta - \int_e^{B/n} \cos ([np]\theta) W(\theta) \, d\theta \right)$$

$$\quad + \int_e^\pi (\cos n\theta - \cos ([np]\theta)) W(\theta) \, d\theta.$$

We shall consider these three parts separately and prove

(a) \( \lim_{n \to \infty} \frac{2}{\pi} \left( \int_0^{B/n} \cos n\theta W(\theta) \, d\theta - \int_0^{B/[np]} \cos ([np]\theta) W(\theta) \, d\theta \right) \frac{n}{(1 - F(n)) m^{-2}(n)} = \log p; \)

(b) \( \limsup_{n \to \infty} \left| \frac{2}{\pi} \int_e^\pi (\cos n\theta - \cos ([np]\theta)) W(\theta) \, d\theta \right| \frac{n}{(1 - F(n)) m^{-2}(n)} = 0; \)

(c) \( \limsup_{n \to \infty} \left| \frac{2}{\pi} \int_{B/n}^e \cos n\theta W(\theta) \, d\theta - \int_{B/[np]}^e \cos ([np]\theta) W(\theta) \, d\theta \right| \frac{n}{(1 - F(n)) m^{-2}(n)} = O(B^{1(\gamma - 1)}). \)

In order to prove (c) and (b) it is sufficient to prove

(c') \( \limsup_{n \to \infty} \left| \frac{2}{\pi} \int_e^{B/n} \cos n\theta W(\theta) \, d\theta \right| \frac{n}{(1 - F(n)) m^{-2}(n)} = O(B^{1(\gamma - 1)}); \)

(b') \( \limsup_{n \to \infty} \left| \frac{2}{\pi} \int_e^\pi \cos n\theta W(\theta) \, d\theta \right| \frac{n}{(1 - F(n)) m^{-2}(n)} = 0. \)
We shall first provide the proof of (a) and (b') since the proof of (c') is lengthy and rather technical.

Proof of (a). Using partial integration we obtain for every $p \geq 1$ and $B > 0$

$$\int_0^{B/(np)} \cos ([np] \theta) W(\theta) \, d\theta = \cos B \int_0^{B/(np)} W(\theta) \, d\theta + \frac{1}{n} \int_0^{B/(np)} \sin ([np] \theta) \int_0^\theta W(z) \, dz \, d\theta.$$ 

Hence

$$\int_0^{B/n} \cos n\theta W(\theta) \, d\theta - \int_0^{B/(np)} \cos ([np] \theta) W(\theta) \, d\theta = \cos B \int_0^B \frac{W(\theta)}{n} \, d\theta + \frac{1}{n} \int_0^B \sin \theta \int_{\theta/n}^\theta \frac{W(z)}{n} \, ds \, d\theta.$$

Since $1 - F(n) \in \text{RVS}^{-1}$ we also have using Lemma 1 ([15], p. 271)

$$W\left(\frac{1}{n}\right) \in \text{RVF}^{-1} \quad \text{and} \quad W\left(\frac{1}{n}\right) - \frac{\pi}{2} \cdot \frac{n^2(1 - F(n))}{m^2(n)}.$$

Combining the last two results it is easy to deduce

$$\lim_{n \to \infty} \frac{2}{\pi} \frac{\int_0^{B/n} \cos n\theta W(\theta) \, d\theta - \int_0^{B/(np)} \cos ([np] \theta) W(\theta) \, d\theta}{n(1 - F(n))m^{-2}(n)} = \log p.$$ 

Proof of (b'). Since $\cos n\theta = -\cos n(\theta + \pi/n)$ we obtain

$$2 \int_0^\pi \cos (n\theta) W(\theta) \, d\theta = \int_0^\pi \cos n\theta \left(W(\theta) - W\left(\theta - \frac{\pi}{n}\right)\right) \, d\theta + \int_0^{\pi/(n)} \cos n\theta W\left(\theta - \frac{\pi}{n}\right) \, d\theta$$

$$- \int_0^{\pi/(n)} \cos n\theta W\left(\theta - \frac{\pi}{n}\right) \, d\theta.$$

Because $W(\theta)$ is bounded on $[A, B]$ with $0 < A < B \leq \pi$ we get

$$\limsup_{n \to \infty} \left| \int_0^{\pi/(n)} \cos n\theta W\left(\theta - \frac{\pi}{n}\right) \, d\theta \right| \leq \limsup_{n \to \infty} C_n \cdot \frac{\pi}{W\left(\frac{1}{n}\right)} = 0.$$

$$\limsup_{n \to \infty} \left| \int_{\pi/(n)}^{\pi + (\pi/n)} \cos n\theta W\left(\theta - \frac{\pi}{n}\right) \, d\theta \right| \leq \limsup_{n \to \infty} C_n \cdot \frac{\pi}{W\left(\frac{1}{n}\right)} = 0.$$
Erickson proved ([5], Lemma 5)

\[ |\phi(\theta_1) - \phi(\theta_2)| \leq \frac{2}{|\theta_1 - \theta_2|} \cdot m\left(\frac{1}{|\theta_1 - \theta_2|}\right) \quad \forall \theta_1 \neq \theta_2 \]

and thus using the definition of \( W(\theta) \)

\[ \left| W(\theta) - W\left(\theta - \frac{\pi}{n}\right) \right| \leq \frac{\pi \cdot m\left(\frac{n}{\pi}\right)}{|1 - \phi(\theta)| \left|1 - \phi\left(\theta - \frac{\pi}{n}\right)\right|} \leq \frac{\pi}{n} C_{\epsilon,\pi} m\left(\frac{n}{\pi}\right) \]

for \( \theta \in [\epsilon, \pi] \) and

\[ C_{\epsilon,\pi} = \max_{\theta \in [\epsilon, \pi]} \left( \frac{1}{|1 - \phi(\theta)| \left|1 - \phi\left(\theta - \frac{\pi}{n}\right)\right|} \right). \]

(This is possible since \( \epsilon \) is not periodic.) Hence

\[ (f) \quad \limsup_{n \to \infty} \left| \frac{1}{n} \cos n \theta (W(\theta) - W(\theta - \pi/n)) \right| = 0. \]

Using (d), (e), (f) and

\[ \frac{1}{n} W\left(\frac{1}{n}\right) \sim \frac{\pi}{2} n(1 - F(n)) \]

we thus get

\[ \limsup_{n \to \infty} \frac{1}{n(1 - F(n))} m^{-2}(n) \to 0. \]

Proof of \((c')\). We write

\[ 2 \int_{B/n}^{e} \cos n \theta W(\theta) d\theta = \frac{1}{n} \int_{B}^{B+\pi} \cos \theta W\left(\frac{\theta - \pi}{n}\right) d\theta - \frac{1}{n} \int_{e/n}^{e+n+\pi} \cos \theta W\left(\frac{\theta - \pi}{n}\right) d\theta + \frac{1}{n} \int_{B}^{e} \cos \theta \left(W\left(\frac{\theta}{n}\right) - W\left(\frac{\theta - \pi}{n}\right)\right) d\theta \triangleq \frac{1}{n} (I_1(n) - I_2(n) + I_3(n)). \]

Obviously

\[ \lim_{n \to \infty} \frac{1}{n} W\left(\frac{1}{n}\right) = \int_{B}^{B+\pi} \cos \theta W\left(\frac{\theta - \pi}{n}\right) d\theta \]
and hence
\[ \lim_{n \to \infty} \sup_{n} \frac{1}{n} |I_1(n)| \leq \delta \text{ for } B \text{ sufficiently large and } \delta \text{ sufficiently small.} \]

Since \( W(1/n) \to \infty(n \to \infty) \) and \( W((\theta - \pi)/n) \) bounded on \( \theta \in [\varepsilon n, \varepsilon n + \pi] \) we find
\[ \lim_{n \to \infty} \sup_{n} \frac{1}{n} |I_2(n)| = 0. \]

Finally we have to consider \( 1/n \cdot I_3(n) \). Using the definition of \( W(\theta) \) we get
\[ W(\frac{\theta}{n}) - W(\frac{\theta - \pi}{n}) = \frac{\text{Re} \phi(\frac{\theta - \pi}{n}) - \text{Re} \phi(\frac{\theta}{n})}{|1 - \phi(\frac{\theta}{n})|^2} \]
\[ + \left(1 - \text{Re} \phi(\frac{\theta - \pi}{n})\right) \left(\frac{1}{|1 - \phi(\frac{\theta}{n})|^2} - \frac{1}{|1 - \phi(\frac{\theta - \pi}{n})|^2}\right) \]
\[ \Delta \text{Integrand}_{31} (\theta, n) + \text{Integrand}_{32} (\theta, n). \]

Hence
\[ \frac{1}{n} I_3(n) = \frac{1}{n} \int_{B}^{\varepsilon n} \cos \theta \cdot \text{Integrand}_{31} (\theta, n) \, d\theta \]
\[ + \frac{1}{n} \int_{B}^{\varepsilon n} \cos \theta \cdot \text{Integrand}_{32} (\theta, n) \, d\theta. \]

We first consider \( 1/n \int_{B}^{\varepsilon n} \cos \theta \cdot \text{Integrand}_{31} (\theta, n) \, d\theta \). Erickson proved ([5], Lemma 5)
\[ \theta m(\frac{1}{\theta}) \leq k |1 - \phi(\theta)| \text{ for all } \theta \in (0, 2\pi) \text{ and } k \text{ some constant.} \]

Using this inequality, Pitman’s result and Lemma 5 we find for \( h = (1 - \gamma)/4 > 0 \)
\[ \left| \frac{1}{n} \int_{B}^{\varepsilon n} \cos \theta \cdot \text{Integrand}_{31} (\theta, n) \, d\theta \right| \geq A_3 \cdot k^2 \int_{B}^{\varepsilon n} \theta \gamma m^2(n) \, d\theta \]
\[ \geq O\left(\int_{B}^{\varepsilon n} \theta^{\gamma+2h-2} \, d\theta\right) \leq O(B^{-1(\gamma-1)}). \]
The behavior of the renewal sequence

Consider the second part of \((1/n) \cdot I_3(n)\). Since

\[
|1 - \phi(\theta)|^2 = (1 - \text{Re} \phi(\theta))^2 + \text{Im}^2 \phi(\theta)
\]

and

\[
\frac{1}{1 - \phi(\frac{\theta}{n})^2} \left| \frac{1}{1 - \phi(\frac{\theta - \pi}{n})^2} \right| = \frac{|1 - \phi(\frac{\theta - \pi}{n})|^2 - |1 - \phi(\frac{\theta}{n})|^2}{|1 - \phi(\frac{\theta}{n})|^2 |1 - \phi(\frac{\theta - \pi}{n})|^2}
\]

we get the following relation:

\[
\text{Integrand}_{32}(\theta, n) = \left(1 - \text{Re} \phi(\frac{\theta - \pi}{n})\right) \left| \frac{\text{Re} \phi(\frac{\theta}{n}) - \text{Re} \phi(\frac{\theta - \pi}{n})}{1 - \phi(\frac{\theta}{n})^2} \left| \frac{1 - \phi(\frac{\theta - \pi}{n})}{1 - \phi(\frac{\theta}{n})^2} \right| \right|
\]

\[+ \left(1 - \text{Re} \phi(\frac{\theta - \pi}{n})\right) \left| \frac{\text{Im} \phi(\frac{\theta - \pi}{n}) - \text{Im} \phi(\frac{\theta}{n})}{1 - \phi(\frac{\theta}{n})^2} \left| \frac{\text{Im} \phi(\frac{\theta - \pi}{n}) + \text{Im} \phi(\frac{\theta}{n})}{1 - \phi(\frac{\theta}{n})^2} \right| \right|
\]

In a similar way as Erickson provides the proof of [5], Lemma 5, we get

\[
\left| \text{Im} \phi(\frac{\theta - \pi}{n}) - \text{Im} \phi(\frac{\theta}{n}) \right| \leq \frac{\pi}{n} m\left(\frac{n}{\pi}\right)
\]

and

\[
\left| \text{Re} \phi(\frac{\theta - \pi}{n}) - \text{Re} \phi(\frac{\theta}{n}) \right| \leq \frac{\pi}{n} m\left(\frac{n}{\pi}\right) \text{ for all } \theta \in [B, \epsilon n].
\]

Using the above relations and the mentioned inequality for \(|1 - \phi(\theta)|\) we find

\[
\text{Integrand}_{32}(\theta, n) \leq \frac{k^4 n^3 m\left(\frac{n}{\pi}\right) \pi \left(1 - \text{Re} \phi(\frac{\theta - \pi}{n})\right) \left(2 - \text{Re} \phi(\frac{\theta}{n}) - \text{Re} \phi(\frac{\theta - \pi}{n})\right)}{m^2 \left(\frac{n}{\theta - \pi}\right) m^2 \left(\frac{n}{\theta}\right) \theta^2 (\theta - \pi)^2}
\]

\[
+ \frac{k^4 n^3 \pi m\left(\frac{n}{\pi}\right) \left(1 - \text{Re} \phi(\frac{\theta - \pi}{n})\right) \left| \text{Im} \phi(\frac{\theta}{n}) + \text{Im} \phi(\frac{\theta - \pi}{n}) \right|}{m^2 \left(\frac{n}{\theta - \pi}\right) m^2 \left(\frac{n}{\theta}\right) \theta^2 (\theta - \pi)^2}
\]

\[\Delta \text{Integrand}_{321}(\theta, n) + \text{Integrand}_{322}(\theta, n).
\]
First we consider $\text{Integrand}_{321}(\theta, n)$. Since $1 - \text{Re} \phi(1/n) \sim \frac{1}{2} \pi (1 - F(n)) (n \to \infty)$ and $w^{1+\eta}(1 - \text{Re} \phi(1/w))$ is bounded in the neighbourhood of 0 we can apply Pitman’s lemma ([12], Lemma 2) to the following integral and find for $0 < \eta < \frac{1}{6}$ and $n$ sufficiently large

$$\left| \frac{1}{n} \int_{B}^{e \pi} \cos \theta \cdot \text{Integrand}_{321}(\theta, n) \, d\theta \right| \leq C \int_{B}^{e \pi} \frac{(\theta - \pi)^{1+\eta} + (\theta - \pi)^{1+\eta} \theta^{2n} (\theta - \pi)^{2n}}{\theta^2 (\theta - \pi)^2} \, d\theta \leq O(B^{-(1-6\eta)}).$$

Hence

$$\limsup_{n \to \infty} \left| \frac{1}{n} \int_{B}^{e \pi} \cos \theta \cdot \text{Integrand}_{321}(\theta, n) \, d\theta \right| \leq O(B^{-(1-6\eta)}) \limsup_{n \to \infty} \frac{n(1 - F(n))}{m(n)} = 0.$$ 

Since $\text{Im} \phi(1/n) \sim m(n)/n$ and $\eta$ sufficiently small we find analogously

$$\limsup_{n \to \infty} \left| \frac{1}{n} \int_{B}^{e \pi} \cos \theta \cdot \text{Integrand}_{322}(\theta, n) \, d\theta \right| = O(B^{-1})$$

Combination of the above results yields

$$\limsup_{n \to \infty} \frac{|I_{3}(n)|}{n^2 \cdot (1 - F(n)) m^{-2}(n)} = O(B^{k(\gamma - 1)})$$

and hence

$$\limsup_{n \to \infty} \left| \frac{2}{\pi} \int_{B/n}^{e} \cos n\theta W(\theta) \, d\theta \right| = O(B^{k(\gamma - 1)}).$$

The proof of (c') is now completed and we obtain by combination of (a), (b), (c)

$$\lim_{n \to \infty} \frac{u_{n} - u_{[np]}}{n(1 - F(n)) m^{-2}(n)} = \log p \quad \forall p > 0.$$ 

This implies by Lemma 2, [3], Proposition 2, and [2], p. 41,

$$\lim_{n \to \infty} \frac{\sum_{p=0}^{n} u_{p} - nu_{n}}{n^2(1 - F(n)) m^{-2}(n)} = 1.$$
The behavior of the renewal sequence

On the other hand we proved in [7], Theorem 3,

\[ \lim_{n \to \infty} \frac{\sum_{p=0}^{n} u_p - n}{m(n)} \frac{1}{n^2(1 - F(n))m^{-2}(n)} = 1. \]

Combination of both relations yields

\[ \lim_{n \to \infty} \frac{u_n - 1}{m(n)} \frac{1}{n(1 - F(n))m^{-2}(n)} = 0. \]

We now prove the converse statement of the theorem. This statement is obvious since \(-u_{[t]} \in \Pi^\infty\) implies \(-1/\lfloor t \rfloor \sum_{p=0}^{[t]} u_p \in \Pi^\infty\) and [7], Theorem 1, then yields \(1 - F(n) \in \text{RVS}^{-1}_{\alpha} \).

As an application of the foregoing we can sharpen the result concerning the limit distribution of the residual waiting time. Following Example (b) of [6], p. 332, and the above results it is easy to prove

(i) \(1 - F(n) \in \text{RVS}^{-\alpha}_{\alpha} (\alpha > 1) \Rightarrow \lim_{n \to \infty} \frac{W_n(r) - \frac{1}{\mu}(1 - F(r-1))}{\mu} = \frac{1 - F(r-1)}{\mu^2(\alpha - 1)} \)

(ii) \(1 - F(n) \in \text{RVS}^{0}_{\alpha} \Rightarrow \lim_{n \to \infty} \frac{W_n(r) - 1 - F(r-1)}{m(n)} \frac{1}{n(1 - F(n))m^{-2}(n)} = 0 \)

with \(W_n(r) \triangleq \text{Pr}\{\text{first occurrence of } e \text{ after the } n\text{th trial takes place at the } (n + r)\text{th trial}\} \).

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References