Resource Location Games

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Abstract

In this paper, we introduce and analyze resource location games. We show core non-emptiness by providing a set of intuitive core allocations, called Resource-Profit allocations. In addition, we present a sufficient condition for which the core and the set of Resource-Profit allocations coincide. Finally, we provide an example showing that when the sufficient condition is not satisfied, the coincidence is not guaranteed.

Keywords: *cooperative game, core, resource-profit allocations, reallocation of resources*

1 Introduction

Consider a setting with several regions (e.g., villages, municipalities, or small districts), each inhabited by several residents. All these residents are interested in the realization of the same type of task (e.g., mowing the lawn, cleaning a rain gutter, or pruning the hedge). Such a task can be executed with a single resource, and each resident may or may not own such a single resource (e.g., a lawn mower, a gutter ladder, or a hedge trimmer). If a resident holds (and so has access to) such a resource, it generates a resident-specific profit (e.g., the profit or utility realized by mowing the lawn, cleaning the rain gutter, or pruning the hedge).

Residents amongst, but also within, regions can decide to collaborate. In such a collaboration, the participants decide in which regions to locate their resources. Each resource is then shared, and used, amongst all participants in the region where the resource is located, a so-called covered region. Such type of situations, in which a resource is shared and used amongst all participants in a covered region, is reasonable when, for instance, demand per participant is low (e.g., a hedge trimmer is only used a couple of hours, per year) or capacity of the resource is high. The aim of the collaborating residents is to (re)allocate the resources in such a way that total profit (i.e., the sum of the profits of the participating residents that belong to a covered region), is maximized. Typically, this results in additional profit (compared to the situation without any collaboration amongst the players) and thus the question arises about how to allocate this additional profit in a fair and efficient way amongst the collaborating participants. In this paper, we investigate this joint profit allocation aspect in a resource location (RL) situation. To tackle this aspect, we introduce a RL game wherein residents are represented by players that each may or may not own a single resource and each have an associated profit, indicating the worth of having access to a resource.

For these RL games, we study properties of the core (i.e., the set of all possible allocations for which no group of players has an incentive not to collaborate). We distinguish between the case with more resources than regions (i.e., oversupply) and the case with not more resources than regions (i.e., no oversupply). For both cases, we show that the core is non-empty. For the oversupply case, we provide a complete description of the core. For the no oversupply case, we provide a subset of the core. We do so by providing a set of intuitive core allocations, called Resource-Profit (RP) allocations. These RP allocations are based on a uniform price of owning a resource and the player-specific profit. In addition, for the no oversupply case, we present a sufficient condition for which the core and the set of RP allocations coincide. As a side result, we are able to identify how these RP allocations can be constructed via any core allocation. Finally, we provide an example showing that when the sufficient condition is not satisfied, the coincidence is not guaranteed, i.e., the set of RP allocations is a proper subset of the core.

RL games belong to the class of resource pooling games, in which resources are reallocated, or shared, amongst players to realize additional profit. In the last couple of years, there is an increasing interest in these games. Some examples are the pooling of technicians in the service industry (Anily and Haviv [1]), pooling of capacity in a production environment (Özen et al. [8], Anily and Haviv [2]), pooling of emergency vehicles in health care (Karsten et al. [6]), reallocation of inventory in a retail setting (Sošić [12]), pooling of spare parts in the capital intensive goods industry (Karsten et al. [5], Karsten and Basten [4], Guajardo and Rönnqvist [3]), and reallocation of spare parts and repair vans in a railway setting (Schlicher et al. [9, 10]). To the best of our knowledge, there are no resource pooling games in literature that consider our specific situation –the one in which players can share resources within a region and reallocate them amongst the regions. The only exception is the classical Böhm-Bawerk horse market (BBHM) game, which has been studied extensively in literature (see, e.g., Tejada and Núñez [14], Tejada [13], Núñez and Rafels [7] and Shapley and Shubik [11]). In BBHM games, there are sellers that each have one horse for sale and buyers that each wish to buy one such horse. These horses are all alike, while the sellers and buyers may have different valuations for such a horse. When collaborating, horses are sold towards those buyers that value horses most. Shapley and Shubik [11] showed that the core of these games coincide with a particular set of market allocations, which, per player, depends on its valuation and a uniform market price. Clearly, we study a generalization of BBHM games: when each region (of an RL game) inhabits exactly one player (with or without a resource), the players with a resource can be seen as potential sellers and the players without a resource can be seen as potential buyers. Hence, in the spirit of BBHM games, we contribute to the literature by generalizing this classical game and some of its corresponding results.

The outline of this paper is as follows. In Section 2, we introduce RL situations and describe the associated RL games. Then, in Section 3, we formally define the set of RP allocations and we analyze its relation with the core of RL games. We conclude this paper with a final remark about this relation. We want to emphasize that proofs of lemmas and theorems are relegated to the appendix. For the main results, which are presented in the form of theorems, we also give a sketch of proof in the main text.

2 RL situations and associated RL games

We start with introducing RL situations in Section 2.1. Thereafter, to tackle the allocation problem of the maximal joint profit increase in an RL situation, we describe the associated RL games in Section 2.2. We conclude this section with an example of such an RL game.

2.1 RL situation

An RL situation can be summarized by a tuple $\theta = (N, r, w, \mathscr{D}, D)$, where $N \subseteq \mathbb{N}$ is a finite set of players (e.g., residents). The parameter $r_i \in \{0, 1\}$ indicates whether player $i \in N$ owns a resource ($r_i = 1$) or not ($r_i = 0$). The vector $r \in \{0, 1\}^N$ summarizes these parameters. It is assumed that there is at least one player who owns a resource, i.e., $\sum_{i \in N} r_i \geq 1$. The player-specific profit $w_i \geq 0$ specifies the profit player $i \in N$ realizes in case he has access to a resource. A player has access to a resource if he owns a resource. When players collaborate, there is also another way of having access to a resource, which will be discussed later on in this section. The vector $w \in \mathbb{R}^N_{++}$ summarizes the player-specific profits. The set $\mathscr{D} \subseteq \mathbb{N}$ is a finite set of regions. Furthermore, $D_j \subseteq N$ represents the set of players that belong to region $j \in \mathscr{D}$. Every player belongs to exactly one region and thus the family of sets of players $D = \{D_j \mid j \in \mathscr{D}\}$ is a partition of N. The set of all RL situations is denoted by Θ.

It is assumed that the nature of the resources is such that the players within the same region can share resources with each other, i.e., all players within the same region can benefit from a single resource. As a consequence, if a region contains at least one player who owns a resource and all players in this region decide to cooperate, then this resource can be donated to this region and so every player in this region has access to a resource and thus receives its player-specific profit. Note that, due to the nature of the resources, it does not make a difference for a region whether it has one, or multiple players with a resource. Indeed, the remaining resources (if any) could be allocated to other regions –and this calls for collaboration amongst the regions as well. Doing this in an optimal way boils down to allocating the $\sum_{i \in N} r_i$ resources to the regions for which the regional profit, i.e., the sum of its player-specific profits, is the highest. These regions are called the covered regions and the remaining ones are called non-covered regons. We denote $\mathscr{D}^c_N \subseteq \mathscr{D}$ as the set of covered regions. Then, by assuming that initially there is no reallocation of resources and moreover resources are not shared amongst players in the same region (i.e., no collaboration within and amongst the regions), the maximal joint profit increase due to cooperation equals

$$\sum_{j \in \mathscr{D}_N^c} \sum_{i \in D_j} w_i - \sum_{i \in N} r_i w_i.$$

Here, the first part equals the sum of the regional profits of the covered regions, i.e., the total profit when there is full collaboration. The second part equals the sum of the player-specific profits of the players who initially own a resource, i.e., the total profit when there is no collaboration at all.

2.2 RL games

In this section, we introduce the associated RL game in order to tackle the allocation problem of the maximal joint profit increase when players decide to collaborate. For this, we start with a formal description of a cooperative game.

A cooperative game is a pair (N, v) where N denotes a non-empty, finite set of players and $v : 2^N \to \mathbb{R}$ assigns a monetary payoff to each coalition $S \subseteq N$, where 2^N denotes the collection of all subsets of N. The coalitional value $v(S)$ denotes the highest payoff the coalition S can jointly generate by means of optimal cooperation without help of players in $N \backslash S$. Coalition N is called the grand coalition. Furthermore, by convention, $v(\emptyset) = 0$.

In order to define a cooperative game associated with RL situations, we first need to introduce some notions and definitions. For each coalition $S \subseteq N$, $R(S)$ indicates the total number of resources in coalition S, i.e., $R(S) = \sum_{i \in S} r_i$. Additionally, for each region $j \in \mathscr{D}$, $D_j(S)$ identifies the players of coalition S that belong to region j, i.e., $D_j(S) = D_j \cap S$. The set $\mathscr{D}_S \subseteq \mathscr{D}$ contains the regions for which there exists a player of coalition S that belongs to this region, i.e., $\mathscr{D}_S = \{j \in \mathscr{D} \mid D_j(S) \neq \emptyset\}$. Moreover, we denote the sum of the player-specific profits of all players in coalition S that belong to region j by $W_j(S)$ and thus $W_j(S) = \sum_{i \in D_j(S)} w_i$. We call $W_j(S)$ the regional profit of region j for coalition S.

To tackle the allocation problem of the maximal joint profit increase in an RL situation $\theta = (N, r, w, \mathscr{D}, D)$, one can analyze an associated cooperative game (N, v^θ). Here, for a coalition $S \subseteq N \backslash \{\emptyset\}$, $v^\theta(S)$ reflects the maximal joint profit this coalition can make. For this, we assume that the players in S can only reallocate their own resources. Moreover, a player in coalition S cannot benefit from the resource of a player in the same region if he does not belong to coalition S. As a consequence, it is optimal for coalition S

to allocate his $R(S)$ resources to the $R(S)$ regions in $\mathscr{D}_S$ for which the regional profits for coalition S are the highest. In order to define $v^\theta(S)$ formally, we first introduce the bijection $\sigma_S : \{1, 2, \ldots, |\mathscr{D}_S|\} \to \mathscr{D}_S$. This bijection is uniquely defined and orders the regions in $\mathscr{D}_S$ in such a way that they are in non-increasing order with respect to regional profits for coalition S. Moreover, if there is a tie, then the region with the smallest index is chosen first. Formally,

$\sigma_S(1) = \min\{j \in \mathscr{D}_S \mid W_j(S) \geq W_k(S)\ \forall k \in \mathscr{D}_S\}$,

$\sigma_S(i) = \min\left\{j \in \mathscr{D}_S \backslash \{\sigma_S(1), ..., \sigma_S(i\text{-}1)\} \mid W_j(S) \geq W_k(S)\ \forall k \in \mathscr{D}_S \backslash \{\sigma_S(1), ..., \sigma_S(i\text{-}1)\}\right\}$,

for every $i \in \{2, 3, \ldots, |\mathscr{D}_S|\}$. As a result, coalition S allocates a resource to every region $j \in \mathscr{D}_S$ with $\sigma_S^{-1}(j) \leq R(S)$. We denote the set of covered regions for coalition S by $\mathscr{D}_S^c = \{j \in \mathscr{D}_S \mid \sigma_S^{-1}(j) \leq R(S)\}$ and the set of non-covered regions by $\mathscr{D}_S^{nc} = \{j \in \mathscr{D}_S \mid \sigma_S^{-1}(j) > R(S)\}$. The following definition provides the formal definition of an RL game.

Definition 1. *For every RL situation $\theta \in \Theta$, the associated RL game (N, v^θ) is defined by*

$$v^\theta(S) = \sum_{j \in \mathscr{D}_S^c} W_j(S),$$

for all $S \subseteq N \backslash \{\emptyset\}$ and $v^\theta(\emptyset) = 0$.

We conclude this section with an illustrative example.

Example 1. *Let $\theta \in \Theta$ with $N = \{1, 2, 3\}$, $r = (0, 0, 1)$, $w = (10, 6, 8)$, $\mathscr{D} = \{4, 5\}$, $D_4 = \{1, 2\}$, and $D_5 = \{3\}$. In Table 1, we present the coalitional values of (N, v^θ).*

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v^\theta(S)$	0	0	0	8	0	10	8	16

Table 1: The RL game (N, v^θ) of Example 1

Player 3 is the only player with a resource. When he cooperates with others, he can either keep it in his own region, or give it to another region. Since $w_1 > w_3 > w_2$, player 3 donates his resource to region 4 when cooperating with player 1, but keeps it in region 5 when cooperating with player 2. When all the players cooperate, it is best to allocate the resource to region 4. Then, both player 1 and player 2 use it, which results in a profit of $w_1 + w_2 = 10 + 6 = 16$. $\diamond$

3 The core of RL games

In this section, we study the core of RL games. We start with a formal description of the core, a lemma for core allocations and a lemma that provides upper bounds for the coalitional values in RL games. Then, we focus in Section 3.1 on the core of RL games that originate from RL situations with more resources than regions, i.e., with oversupply of resources. Finally, we focus in Section 3.2 on the core of RL games that originate from RL situations in which there are not more resources than regions, i.e., no oversupply of resources (so either undersupply or exactly enough resources).

The core $\mathscr{C}(N, v)$ of a cooperative game (N, v) is formally defined as the set of all allocations $x \in \mathbb{R}^N$ that are efficient ($\sum_{i \in N} x_i = v(N)$) and stable ($\sum_{i \in S} x_i \geq v(S)$ for all $S \subset N$). In Lemma 1 we present a result for core allocations that is frequently used throughout this paper. This lemma resembles that a coalition cannot claim too much from the value of the grand coalition, because this would not leave enough for the players outside the coalition. Recall that all proofs are relegated to the appendix.

Lemma 1. *Let (N, v) be a cooperative game and let $S \subset N$. For every $x \in \mathscr{C}(N, v)$ it holds that*

$$\sum_{i \in S} x_i \leq v(N) - v(N \backslash S).$$

In Lemma 2 we show that any coalition in an RL game can realize a coalitional value at most equal to the sum of the player-specific profits of all the players in that coalition. Moreover, in case there are enough resources for all regions of this coalition (i.e., no undersupply of resources for coalition S), all player-specific profits can be realized.

Lemma 2. *Let $\theta \in \Theta$ be an RL situation and let (N, v^θ) be the associated RL game. For any coalition $S \subseteq N$, the following holds:*

$$\begin{aligned} &\text{(i)}\ \ v^\theta(S) \leq \textstyle\sum_{i \in S} w_i \ \text{ if } \ R(S) < |\mathscr{D}_S|, \\ &\text{(ii)}\ \ v^\theta(S) = \textstyle\sum_{i \in S} w_i \ \text{ if } \ R(S) \geq |\mathscr{D}_S|. \end{aligned}$$

Note that in case of oversupply of resources for the grand coalition, it is also possible that there is no oversupply of resources for some coalitions. In other words, even though we consider in Section 3.1 the case $R(N) > |\mathscr{D}|$, it is still possible that there exists a coalition $S \subset N$ with $R(S) \leq |\mathscr{D}_S|$.

3.1 Oversupply of resources: $R(N) > |\mathscr{D}|$

In this section we describe (in Theorem 1) the core of RL games that originate from RL situations with oversupply of resources. More precisely, in this case, the core coincides with the vector of player-specific profits. We now provide a sketch of proof for this first result. We start the proof of this theorem with showing that each core element can be written as the vector of player-specific profits. We do so by proving that each player cannot claim more than its own profit, which follows by exploiting the results of Lemma 1 and Lemma 2. Subsequently, by exploiting the efficiency property (of a core allocation), we show that each player exactly claims its own profit. Finally, we prove that the vector of player-specific profits is a core element. We do so by showing that this vector is efficient and stable, which follows by exploiting the results of Lemma 2.

Theorem 1. *Let $\theta \in \Theta$ be an RL situation with $R(N) > |\mathscr{D}|$ and let (N, v^{θ}) be the associated RL game. It holds that*

$$\mathscr{C}(N, v^{\theta}) = \{w\}.$$

This theorem shows that, in case of oversupply of resources, the value that players assign to a resource reduces to zero and so all players obtain their player-specific profit.

3.2 No oversupply of resources: $R(N) \leq |\mathscr{D}|$

In this section we give (in Theorem 2) a partial description of the core of RL games that originate from RL situations with no oversupply of resources. More precisely, we introduce a (non-empty) set of intuitive core allocations. Moreover, we present (in Theorem 3) a sufficient condition for which this set of intuitive core allocations coincides with the core. Finally, we provide an example showing that when the sufficient condition is not satisfied, the coincidence is not guaranteed.

We start with introducing the intuitive core allocations, which per player $i \in N$, consists of two components. The first component is the resource component $\gamma \cdot r_i$ that compensates for owning a resource. The second components is the profit component α_i that compensates for the profit realized by a player. The allocation, which we call a Resoure-Profit (RP) allocation, is then formulated as

$$\gamma \cdot r_i + \alpha_i \text{ for all } i \in N.$$

We continue by formally defining these two components. First, we introduce the resource component, which depends on γ. This parameter is defined as follows:

$$\gamma \in \begin{cases} \left[W_{\sigma_N(R(N)+1)}(N), W_{\sigma_N(R(N))}(N)\right] & \text{if } R(N) < |\mathscr{D}|, \\ \left[0, W_{\sigma_N(R(N))}(N)\right] & \text{if } R(N) = |\mathscr{D}|. \end{cases} \tag{1}$$

The parameter γ resembles the principle of a market price. Firstly, because γ is at least equal to the regional profit of a region that has highest regional profit amongst all non-covered regions. Secondly, because γ is at most equal to the regional profit of a region that has lowest regional profit amongst all covered regions. Hence, any other price (than γ) would always give (some) players incentives to sell (or buy) a resource for a lower (or higher) price. The profit component is defined as follows:

$$\alpha_i \in \begin{cases} [0, w_i] & \text{for all } i \in D_j(N) \text{ with } j \in \mathscr{D}_N^c, \\ \{0\} & \text{for all } i \in D_j(N) \text{ with } j \in \mathscr{D}_N^{nc}, \end{cases} \tag{2}$$

with the additional condition that

$$\sum_{i \in D_j(N)} \alpha_i = W_j(N) - \gamma \text{ for all } j \in \mathscr{D}_N^c. \tag{3}$$

So, players that belong to a covered region can divide the regional profit, minus the price of the resource (that covers the region), freely, with the restriction that no one can demand more than their player-specific profit.

Next, for every RL situation $\theta \in \Theta$, we denote the set of RP allocations by

$$\Omega^\theta = \left\{ x \in \mathbb{R}^N \;\middle|\; x_i = \gamma \cdot r_i + \alpha_i \;\; \forall i \in N, \text{ (1), (2), (3)} \right\}.$$

We are now ready to give a partial description of the core of RL games, i.e., ready to show that RP allocations are core allocations. We prove this result (in Theorem 2) by showing that every RP allocation is efficient and stable. Efficiency follows by the construction of the resource and profit components in combination with the fact that there is no oversupply of resources. For stability, we use that the sum of resource and profit components of the players in a region exceeds the regional profit.

Theorem 2. *Let $\theta \in \Theta$ be an RL situation for which $R(N) \leq |\mathscr{D}|$ and let (N, v^θ) be the associated RL game. It holds that*

$$\Omega^\theta \subseteq \mathscr{C}(N, v^\theta).$$

An interesting follow-up question is under which conditions (if any) every core allocation can be described in terms of an RP allocation. In Theorem 3, we present a sufficient condition under which this is true, i.e., a sufficient condition under which the core coincides with the set of RP allocations. First, we introduce three relevant lemmas that illustrate properties of core allocations in RL games.

The following lemma shows that players who do not own a resource themselves, can claim only a limited share of the total profit.

Lemma 3. *Let $\theta \in \Theta$ be an RL situation and let (N, v^{θ}) be the associated RL game. Let $i \in N$ with $r_i = 0$. Then, for any $x \in \mathscr{C}(N, v^{\theta})$ it holds that*

$$x_i \in \begin{cases} \{0\} & \text{if } i \in D_j(N) \text{ for some } j \in \mathscr{D}_N^{nc}, \\ [0, w_i] & \text{if } i \in D_j(N) \text{ for some } j \in \mathscr{D}_N^{c}. \end{cases}$$

For a cooperative game (N, v), we define a coalition $S \subseteq N$ to be self-dual valued if

$$v(N) = v(S) + v(N \backslash S).$$

By Lemma 1, self-dual valued coalitions cannot claim more than their own coalitional value. Thus, by stability, they receive exactly their own coalitional value in every core allocation, i.e., $\sum_{i \in S} x_i = v(S)$ for every $x \in \mathscr{C}(N, v)$ and every self-dual valued coalition $S \subseteq N$. In the following two lemmas we provide two examples of self-dual valued coalitions in RL games.

Lemma 4. *Let $\theta \in \Theta$ be an RL situation and let (N, v^{θ}) be the associated RL game. Let $J \subseteq \mathscr{D}_N^{c}$ with $\sum_{j \in J} R(D_j(N)) = |J|$. For any $x \in \mathscr{C}(N, v^{\theta})$, it holds that*

$$\sum_{j \in J} \sum_{i \in D_j(N)} x_i = \sum_{j \in J} W_j(N).$$

Lemma 5. *Let $\theta \in \Theta$ be an RL situation and let (N, v^{θ}) be the associated RL game. Let $j \in \mathscr{D}_N^{c}$ with $R(D_j(N)) = 0$. Moreover, let $i \in D_l(N)$ for some $l \in \mathscr{D}_N^{nc}$ with $r_i = 1$. For any $x \in \mathscr{C}(N, v^{\theta})$, it holds that*

$$x_i + \sum_{k \in D_j(N)} x_k = W_j(N).$$

We are now ready to present a sufficient condition for which the core and the set of RP allocations coincide, namely the condition that each covered region has no more than two players who initially have a resource. We now provide a sketch of proof for this last result. We start the proof of this theorem by observing that, based on Theorem 2, it suffices to show that the core is a subset of the set of RP allocations. In particular, we do so by showing that every core allocation can be written as an RP allocation. For that, we distinguish between two cases: the situation in which each covered region has exactly one resource and the situation in which this is not the case. Then, per case, we construct a resource component (γ) and a vector of profit components ($(\alpha_i)_{i \in N}$) such that they form a core allocation. Finally, we show that these components do satisfy the properties of an RP allocation, i.e., the conditions in (1), (2) and (3).

Theorem 3. *Let $\theta \in \Theta$ be a resource location situation with $R(N) \leq |\mathscr{D}|$, $R(D_j(N)) \leq 2$ for all $j \in \mathscr{D}_N^c$ and let (N, v^θ) be the associated RL game. It holds that*

$$\Omega^\theta = \mathscr{C}(N, v^\theta).$$

The condition in Theorem 3 resembles the idea that a covered region should have limited bargaining power. We conclude this paper with an example, showing that when the sufficient condition (of Theorem 3) is not sastisfied, i.e., when a covered region has too much bargaining power, the set of RL allocations does not coincide with the core.

Example 2. *Let $\theta \in \Theta$ be an RL situation with $N = \{1,2,3,4,5\}$, $w = (1,2,3,4,5)$, $r = (1,1,1,0,0)$, $D = \{6,7,8\}$, $D_6 = \{1,2,3\}, D_7 = \{4\}$, and $D_8 = \{5\}$. In Table 2, we present the coalitional values of RL game (N, v^θ).*

S	$v^\theta(S)$	S	$v^\theta(S)$	S	$v^\theta(S)$	S	$v^\theta(S)$
$\emptyset$	0	$\{1,4\}$	4	$\{1,2,3\}$	6	$\{2,4,5\}$	5
$\{1\}$	1	$\{1,5\}$	5	$\{1,2,4\}$	7	$\{3,4,5\}$	5
$\{2\}$	2	$\{2,3\}$	5	$\{1,2,5\}$	8	$\{1,2,3,4\}$	10
$\{3\}$	3	$\{2,4\}$	4	$\{1,3,4\}$	8	$\{1,2,3,5\}$	11
$\{4\}$	0	$\{2,5\}$	5	$\{1,3,5\}$	9	$\{1,2,4,5\}$	9
$\{5\}$	0	$\{3,4\}$	4	$\{1,4,5\}$	5	$\{1,3,4,5\}$	9
$\{1,2\}$	3	$\{3,5\}$	5	$\{2,3,4\}$	9	$\{2,3,4,5\}$	10
$\{1,3\}$	4	$\{4,5\}$	0	$\{2,3,5\}$	10	$\{1,2,3,4,5\}$	15

Table 2: The RL game (N, v^θ) of Example 2

It can be checked that $x = (5,5,5,0,0) \in \mathscr{C}(N, v^\theta)$. Now, suppose that $x \in \Omega^\theta$. So, for each $i \in N$, we can write $x_i = \gamma \cdot r_i + \alpha_i$. For $i \in \{4,5\}$ this boils down to $\alpha_4 = x_4 = 0$ and $\alpha_5 = x_5 = 0$, because $r_4 = r_5 = 0$. Moreover, since $x \in \Omega^\theta$, it holds that $\gamma + \sum_{i \in D_j(N)} \alpha_i = W_j(N)$ for all $j \in \mathscr{D}_N^c$. So, for $j = 7$, this boils down to $\gamma + \alpha_4 = W_7(N) = 4$ and thus $\gamma = 4$. Now, observe that $\gamma + \alpha_5 = 4 \neq 5 = W_8(N)$, which is a contradiction. Hence, $x \notin \Omega^\theta$. $\diamond$

References

[1] Anily, S. and Haviv, M. Cooperation in service systems. *Operations Research*, 58(3): 660–673, 2010.

[2] Anily, S. and Haviv, M. Line balancing in parallel M/M/1 lines and loss systems as cooperative games. *Production and Operations Management*, 26(8):1568–1584, 2017.

[3] Guajardo, M. and Rönnqvist, M. Cost allocation in inventory pools of spare parts with service-differentiated demand classes. *International Journal of Production Research*, 53(1):220–237, 2015.

[4] Karsten, F. and Basten, R. Pooling of spare parts between multiple users: How to share the benefits? *European Journal of Operational Research*, 233(1):94–104, 2014.

[5] Karsten, F.; Slikker, M., and Van Houtum, G. Inventory pooling games for expensive, low-demand spare parts. *Naval Research Logistics*, 59(5):311–324, 2012.

[6] Karsten, F.; Slikker, M., and Van Houtum, G. Resource pooling and cost allocation among independent service providers. *Operations Research*, 63(2):476–488, 2015.

[7] Núñez, M. and Rafels, C. The Böhm-Bawerk horse market: a cooperative analysis. *International Journal of Game Theory*, 33(3):421–430, 2005.

[8] Özen, U.; Reiman, M., and Wang, Q. On the core of cooperative queueing games. *Operations Research Letters*, 39(5):385–389, 2011.

[9] Schlicher, L.; Slikker, M., and Van Houtum, G. A note on maximal covering location games. *Operations Research Letters*, 45(1):98–103, 2017.

[10] Schlicher, L.; Slikker, M., and Van Houtum, G. Probabilistic resource pooling games. *Naval Research Logistics*, 64(7):531–546, 2017.

[11] Shapley, L. and Shubik, M. The assignment game I: The core. *International Journal of game theory*, 1(1):111–130, 1971.

[12] Sošić, G. Transshipment of inventories among retailers: Myopic vs. farsighted stability. *Management science*, 52(10):1493–1508, 2006.

[13] Tejada, O. Analysis of the core of multisided Böhm-Bawerk assignment markets. *Top*, 21(1):189–205, 2013.

[14] Tejada, O. and Núñez, M. The nucleolus and the core-center of multi-sided Böhm-Bawerk assignment markets. *Mathematical methods of operations research*, 75(2):199–220, 2012.

Appendix

Lemma 1. *Let (N, v) be a cooperative game and let $S \subset N$. For every $x \in \mathscr{C}(N, v)$ it holds that*

$$\sum_{i \in S} x_i \leq v(N) - v(N \backslash S).$$

Proof **:** Let $x \in \mathscr{C}(N, v)$. Then,

$$\sum_{i \in S} x_i = \sum_{i \in N} x_i - \sum_{i \in N \backslash S} x_i = v(N) - \sum_{i \in N \backslash S} x_i \leq v(N) - v(N \backslash S),$$

where the second equality holds by efficiency and the inequality by stability. □

Lemma 2. *Let $\theta \in \Theta$ be an RL situation and let (N, v^θ) be the associated RL game. For any coalition $S \subseteq N$, the following holds:*

$$\begin{array}{rl} \textit{(i)} & v^\theta(S) \leq \sum_{i \in S} w_i \ \text{ if } \ R(S) < |\mathscr{D}_S|, \\ \textit{(ii)} & v^\theta(S) = \sum_{i \in S} w_i \ \text{ if } \ R(S) \geq |\mathscr{D}_S|. \end{array}$$

Proof **:** Let $S \subseteq N$. It holds that

$$v^\theta(S) = \sum_{j \in \mathscr{D}_S^c} W_j(S) \leq \sum_{j \in \mathscr{D}_S} W_j(S) = \sum_{i \in S} w_i,$$

where the inequality holds since $\mathscr{D}_S^c \subseteq \mathscr{D}_S$ and $W_j(S) \geq 0$ for all $j \in \mathscr{D}_S$ and all $S \subseteq N$. Moreover, if $R(S) \geq |\mathscr{D}_S|$, then coalition S has enough resources for covering all its corresponding regions, i.e, all regions containing at least one player from coalition S. As a consequence, $\mathscr{D}_S^c = \mathscr{D}_S$ and thus $v^\theta(S) = \sum_{i \in S} w_i$. □

Theorem 1. *Let $\theta \in \Theta$ be an RL situation with $R(N) > |\mathscr{D}|$ and let (N, v^θ) be the associated RL game. It holds that*

$$\mathscr{C}(N, v^\theta) = \{w\}.$$

Proof **:** $(\subseteq)$ Let $y \in \mathscr{C}(N, v^\theta)$ and let $i \in N$. Since $R(N) > |\mathscr{D}| = |\mathscr{D}_N|$, it follows from Lemma 2(ii) that $v^\theta(N) = \sum_{k \in N} w_k$. Moreover, since $r_i \in \{0, 1\}$, we have $R(N \backslash \{i\}) \geq R(N) - 1 \geq |\mathscr{D}| \geq |\mathscr{D}_{N \backslash \{i\}}|$. Hence, it also follows from Lemma 2(ii) that $v^\theta(N \backslash \{i\}) = \sum_{k \in N \backslash \{i\}} w_k = \sum_{k \in N} w_k - w_i = v^\theta(N) - w_i$. So, by Lemma 1, we have $y_i \leq v^\theta(N) - v^\theta(N \backslash \{i\}) = w_i$. In order to satisfy efficiency, it is necessary to have $y_i = w_i$ for all $i \in N$. Hence, $y = w$ and thus $\mathscr{C}(N, v^\theta) \subseteq \{w\}$.

($\supseteq$) Since $\sum_{i\in N} w_i = v^\theta(N)$, w is an efficient allocation. From Lemma 2 we also know $\sum_{i\in S} w_i \geq v^\theta(S)$ for every $S \subset N$. Hence, w is also a stable allocation and thus $w \in \mathscr{C}(N, v^\theta)$. $\square$

Theorem 2. *Let $\theta \in \Theta$ be an RL situation for which $R(N) \leq |\mathscr{D}|$ and let (N, v^θ) be the associated RL game. It holds that*

$$\Omega^\theta \subseteq \mathscr{C}(N, v^\theta).$$

Proof **:** Let $x \in \Omega^\theta$. We first show that x is an efficient allocation:

$$\begin{aligned}\sum_{i\in N} x_i &= \gamma \cdot R(N) + \sum_{j\in\mathscr{D}_N^c}\sum_{i\in D_j(N)} \alpha_i + \sum_{j\in\mathscr{D}_N^{nc}}\sum_{i\in D_j(N)} \alpha_i \\ &= \gamma \cdot R(N) + \sum_{j\in\mathscr{D}_N^c} (W_j(N) - \gamma) = \sum_{j\in\mathscr{D}_N^c} W_j(N) = v^\theta(N).\end{aligned}$$

The second equality holds due to condition (3) and because $\alpha_i = 0$ for all $i \in D_j(N)$ with $j \in \mathscr{D}_N^{nc}$ (condition (2)). In the third equality, it is explicitly used that there is no oversupply of resources and thus the number of covered regions equals the number of resources, i.e., $|\mathscr{D}_N^c| = R(N)$.

Secondly, we show that x is a stable allocation and thus let $S \subset N$. We first show that

$$\gamma + \sum_{i\in D_j(S)} \alpha_i \geq W_j(S) \text{ for all } j \in \mathscr{D}. \tag{4}$$

For this, we distinguish between two cases: $j \in \mathscr{D}_N^c$ and $j \in \mathscr{D}_N^{nc}$.

- **Case 1**: $j \in \mathscr{D}_N^c$.

  Then,

$$\begin{aligned}\gamma + \sum_{i\in D_j(S)} \alpha_i &= W_j(N) - \sum_{i\in D_j(N\backslash S)} \alpha_i = W_j(S) + W_j(N\backslash S) - \sum_{i\in D_j(N\backslash S)} \alpha_i \\ &= W_j(S) + \sum_{i\in D_j(N\backslash S)} (w_i - \alpha_i) \geq W_j(S).\end{aligned}$$

  The first equality holds by condition (3) and the last inequality holds as $\alpha_i \leq w_i$ for all $i \in N$ (condition (2)).

- **Case 2**: $j \in \mathscr{D}_N^{nc}$.

  Since there are non-covered regions, it holds that $R(N) < |\mathscr{D}|$. So,

$$\begin{aligned}\gamma + \sum_{i\in D_j(S)} \alpha_i &\geq W_{\sigma_N(R(N)+1)}(N) + \sum_{i\in D_j(S)} \alpha_i \geq W_{\sigma_N(R(N)+1)}(N) \geq W_j(N) \\ &= \sum_{i\in D_j(N)} w_i \geq \sum_{i\in D_j(S)} w_i = W_j(S).\end{aligned}$$

The first inequality holds because of condition (1) and the fact that $R(N) < |\mathscr{D}|$. The second inequality holds because $\alpha_i \geq 0$ for all $i \in N$ (condition (2)). The third inequality holds since every non-covered region has a regional profit at most equal to the regional profit of the non-covered region with highest regional profit. The last inequality holds since $D_j(S) \subseteq D_j(N)$ and $w_i \geq 0$ for all $i \in N$.

We have proven that (4) holds, both if $j \in \mathscr{D}_N^c$ and if $j \in \mathscr{D}_N^{nc}$. Now, observe that

$$\begin{aligned}\sum_{i \in S} x_i &= \gamma \cdot R(S) + \sum_{j \in \mathscr{D}} \sum_{i \in D_j(S)} \alpha_i \geq \gamma \cdot R(S) + \sum_{j \in \mathscr{D}_S^c} \sum_{i \in D_j(S)} \alpha_i \\ &\geq \gamma \cdot R(S) + \sum_{j \in \mathscr{D}_S^c} (W_j(S) - \gamma) = \sum_{j \in \mathscr{D}_S^c} W_j(S) = v^\theta(S).\end{aligned}$$

The first inequality holds as $\mathscr{D}_S^c \subseteq \mathscr{D}$ and $\alpha_i \geq 0$ for all $i \in N$ (condition (2)). The second inequality holds by applying (4). The penultimate equality holds since there is no oversupply of resources for coalition S (because $R(S) \leq R(N) \leq |\mathscr{D}|$) and thus the number of covered regions for coalition S equals the number of resources of coalition S, i.e., $|\mathscr{D}_S^c| = R(S)$.

Since x is both an efficient and stable allocation, we conclude that $x \in \mathscr{C}(N, v^\theta)$. $\square$

Lemma 3. *Let $\theta \in \Theta$ be an RL situation and let (N, v^θ) be the associated RL game. Let $i \in N$ with $r_i = 0$. Then, for any $x \in \mathscr{C}(N, v^\theta)$ it holds that*

$$x_i \in \begin{cases} \{0\} & \text{if } i \in D_j(N) \text{ for some } j \in \mathscr{D}_N^{nc}, \\ [0, w_i] & \text{if } i \in D_j(N) \text{ for some } j \in \mathscr{D}_N^{c}. \end{cases}$$

Proof **:** Let $x \in \mathscr{C}(N, v^\theta)$. Note that, since $r_i = 0$, we have

$$R(N \backslash \{i\}) = R(N) - r_i = R(N),$$

i.e., coalition $N \backslash \{i\}$ and coalition N have the same number of resources. From now on we distinguish between the two cases: $i \in D_j(N)$ for some $j \in \mathscr{D}_N^{nc}$ and $i \in D_j(N)$ for some $j \in \mathscr{D}_N^c$.

- **Case 1**: $i \in D_j(N)$ for some $j \in \mathscr{D}_N^{nc}$.

  Since $R(N \backslash \{i\}) = R(N)$ and because player i is not in a covered region in the optimal solution for the grand coalition, we know that the optimal allocation of resources for coalition $N \backslash \{i\}$ is the same as the optimal allocation of resources for the grand coalition. This means that also their coalitional values will be the same, i.e., $v^\theta(N \backslash \{i\}) = v^\theta(N)$. Hence, by Lemma 1, we have

  $$x_i \leq v^\theta(N) - v^\theta(N \backslash \{i\}) = 0.$$

Next, since $r_i = 0$, we have $v^\theta(\{i\}) = 0$. Hence, by stability, it holds that

$$x_i \geq v(\{i\}) = 0.$$

Consequently, we have $x_i = 0$.

- **Case 2**: $i \in D_j(N)$ for some $j \in \mathscr{D}_N^c$.
  Since $R(N\backslash\{i\}) = R(N)$, the optimal allocation of resources for the grand coalition is also a feasible allocation of resources for coalition $N\backslash\{i\}$. This allocation results in a profit of $v^\theta(N) - w_i$ for coalition $N\backslash\{i\}$. However, in contrast to case 1, this allocation is not necessarily optimal (because player i is in a covered region in the optimal solution for the grand coalition). Therefore, $v^\theta(N\backslash\{i\}) \geq v^\theta(N) - w_i$. Hence, by Lemma 1, we have

$$x_i \leq v^\theta(N) - v^\theta(N\backslash\{i\}) \leq w_i.$$

  Similar to case 1 we have due to stability $x_i \geq 0$ and thus $x_i \in [0, w_i]$. □

Lemma 4. *Let $\theta \in \Theta$ be an RL situation and let (N, v^θ) be the associated RL game. Let $J \subseteq \mathscr{D}_N^c$ with $\sum_{j\in J} R(D_j(N)) = |J|$. For any $x \in \mathscr{C}(N, v^\theta)$, it holds that*

$$\sum_{j\in J} \sum_{i\in D_j(N)} x_i = \sum_{j\in J} W_j(N).$$

Proof : We prove this lemma by first showing $v^\theta(\cup_{j\in J} D_j(N)) = \sum_{j\in J} W_j(N)$ and then showing that coalition $\cup_{j\in J} D_j(N)$ is self-dual valued. Firstly, since $R(\cup_{j\in J} D_j(N)) = \sum_{j\in J} R(D_j(N)) = |J|$, it follows from Lemma 2(ii) that $v^\theta(\cup_{j\in J} D_j(N)) = \sum_{j\in J} W_j(N)$. Secondly, note that coalition $N\backslash\left(\cup_{j\in J} D_j(N)\right)$ has $R(N) - |J|$ resources and thus it still has exactly enough resources to cover all regions in $\mathscr{D}_N^c\backslash J$ (because $J \subseteq \mathscr{D}_N^c$ and thus $|\mathscr{D}_N^c\backslash J| = |\mathscr{D}_N^c| - |J| = R(N) - |J|$). As a consequence,

$$\begin{aligned} v^\theta(N\backslash\left(\cup_{j\in J} D_j(N)\right)) &= \sum_{j\in \mathscr{D}_N^c\backslash J} W_j(N) = \sum_{j\in \mathscr{D}_N^c} W_j(N) - \sum_{j\in J} W_j(N) \\ &= v^\theta(N) - v^\theta(\cup_{j\in J} D_j(N)). \end{aligned}$$

Hence, coalition $\cup_{j\in J} D_j(N)$ is self-dual valued which finishes the proof. □

Lemma 5. *Let $\theta \in \Theta$ be an RL situation and let (N, v^θ) be the associated RL game. Let $j \in \mathscr{D}_N^c$ with $R(D_j(N)) = 0$. Moreover, let $i \in D_l(N)$ for some $l \in \mathscr{D}_N^{nc}$ with $r_i = 1$. For any $x \in \mathscr{C}(N, v^\theta)$, it holds that*

$$x_i + \sum_{k\in D_j(N)} x_k = W_j(N).$$

Proof : Similar to the proof of Lemma 4, we prove this lemma by first showing $v^\theta(D_j(N) \cup \{i\}) = W_j(N)$ and then showing that coalition $D_j(N) \cup \{i\}$ is self-dual valued. Firstly, note that coalition $D_j(N) \cup \{i\}$ has a single resource because $R(D_j(N)) = 0$ and $r_i = 1$. Moreover, since region j is a covered region and player i is in the non-covered region l, we know $W_j(N) \geq W_l(N) \geq w_i$. As a consequence, coalition $D_j(N) \cup \{i\}$ will cover region j with its single resource and thus $v^\theta(D_j(N) \cup \{i\}) = W_j(N)$. Secondly, note that coalition $N \backslash (D_j(N) \cup \{i\})$ has $R(N) - 1$ resources and thus it still has exactly enough resources to cover all regions in $\mathscr{D}_N^c \backslash \{j\}$ (because $j \in \mathscr{D}_N^c$ and thus $|\mathscr{D}_N^c \backslash \{j\}| = |\mathscr{D}_N^c| - 1 = R(N) - 1$). As a consequence,

$$v^\theta(N \backslash \left(D_j(N) \cup \{i\}\right)) = \sum_{k \in \mathscr{D}_N^c \backslash \{j\}} W_k(N) = \sum_{k \in \mathscr{D}_N^c} W_k(N) - W_j(N)$$
$$= v^\theta(N) - v^\theta(D_j(N) \cup \{i\}).$$

Hence, coalition $D_j(N) \cup \{i\}$ is self-dual valued which finishes the proof. □

Theorem 3. *Let $\theta \in \Theta$ be a resource location situation with $R(N) \leq |\mathscr{D}|$, $R(D_j(N)) \leq 2$ for all $j \in \mathscr{D}_N^c$ and let (N, v^θ) be the associated RL game. It holds that*

$$\Omega^\theta = \mathscr{C}(N, v^\theta).$$

Proof **:** From Theorem 2 we already know that $\Omega^\theta \subseteq \mathscr{C}(N, v^\theta)$, so we only need to prove $\mathscr{C}(N, v^\theta) \subseteq \Omega^\theta$, i.e., we need to prove that every core allocation can be written as an RP allocation. For this, let $x \in \mathscr{C}(N, v^\theta)$ and we will show that there exists a γ and a set $(\alpha_i)_{i \in N}$ for which the following five properties hold:

a) $x_i = \gamma \cdot r_i + \alpha_i$ for all $i \in N$,
b) $\alpha_i = 0$ for all $i \in D_j(N)$ and all $j \in \mathscr{D}_N^{nc}$,
c) $\alpha_i \in [0, w_i]$ for all $i \in D_j(N)$ and all $j \in \mathscr{D}_N^c$,
d) $\gamma + \sum_{i \in D_j(N)} \alpha_i = W_j(N)$ for all $j \in \mathscr{D}_N^c$,
e) $\gamma \in \begin{cases} \left[W_{\sigma_N(R(N)+1)}(N), W_{\sigma_N(R(N))}(N)\right] & \text{if } R(N) < |\mathscr{D}|, \\ \left[0, W_{\sigma_N(R(N))}(N)\right] & \text{if } R(N) = |\mathscr{D}|. \end{cases}$

Note that these five properties together imply that x belongs to the set of RP allocations. From now on we distinguish between two cases: in case 1 we assume that each covered region initially has exactly one resource and in case 2 we assume that this is not the case.

Case 1: $R(D_j(N)) = 1$ for all $j \in \mathscr{D}_N^c$.
Before proving that the five properties are satisfied, we will first show that every player

in a non-covered region initially has no resources. Since there is no oversupply of resources, the number of covered regions equals the number of resources, i.e., $|\mathscr{D}_N^c| = R(N)$. Therefore, since $R(D_j(N)) = 1$ for all $j \in \mathscr{D}_N^c$, it follows that $\sum_{j\in\mathscr{D}_N^c} R(D_j(N)) = |\mathscr{D}_N^c| = R(N)$. Using this relationship, it follows that $\sum_{j\in\mathscr{D}_N^{nc}} R(D_j(N)) = R(N) - \sum_{j\in\mathscr{D}_N^c} R(D_j(N)) = R(N) - R(N) = 0$. As a consequence, $R(D_j(N)) = 0$ for all $j \in \mathscr{D}_N^{nc}$, i.e., each non-covered region initially has no resources. Specifically, every player in a non-covered region initially has no resources, i.e., $r_i = 0$ for all $i \in D_j(N)$ with $j \in \mathscr{D}_N^{nc}$. Now, we set

$$\gamma = \min\{x_k \mid k \in N, r_k = 1\},$$

$$\alpha_i = x_i - \gamma \cdot r_i \text{ for all } i \in N,$$

and we will prove that the five properties are satisfied.

a) We need to prove $x_i = \gamma \cdot r_i + \alpha_i$ for all $i \in N$. This property follows automatically by construction of γ and $(\alpha_i)_{i\in N}$.

b) We need to prove $\alpha_i = 0$ for all $i \in D_j(N)$ and all $j \in \mathscr{D}_N^{nc}$. For this, let $j \in \mathscr{D}_N^{nc}$ and $i \in D_j(N)$. Since $r_i = 0$, it follows from Lemma 3 that $\alpha_i = x_i - \gamma \cdot r_i = x_i = 0$.

c) We need to prove $\alpha_i \in [0, w_i]$ for all $i \in D_j(N)$ and all $j \in \mathscr{D}_N^c$. For this, let $j \in \mathscr{D}_N^c$ and $i \in D_j(N)$. We distinguish between two cases: $r_i = 0$ and $r_i = 1$.

  - **Case 1.c.1**: $r_i = 0$.
    By Lemma 3, we have that $\alpha_i = x_i - \gamma \cdot r_i = x_i \in [0, w_i]$.
  - **Case 1.c.2**: $r_i = 1$.
    We distinguish between another two cases: $x_i = \gamma$ and $x_i \neq \gamma$.
    - **Case 1.c.2.1**: $x_i = \gamma$.
      It holds that $\alpha_i = x_i - \gamma = \gamma - \gamma = 0 \in [0, w_i]$.
    - **Case 1.c.2.2**: $x_i \neq \gamma$.
      Let $i^* \in \{k \in N \mid x_k = \gamma \text{ and } r_k = 1\}$, then $i \neq i^*$. Note that region $D_j(N)$ owns in total one resource (as $j \in \mathscr{D}_N^c$). Therefore, since $i \in D_j(N)$ with $r_i = 1$, there is no other player with a resource in this region. Hence, since $r_{i^*} = 1$, we know that $i^* \notin D_j(N)$. Now, it holds that

$$x_{i^*} - x_i + \sum_{k\in D_j(N)} x_k \geq v^\theta(D_j(N)\backslash\{i\} \cup \{i^*\}) \geq W_j(N) - w_i, \tag{5}$$

      where the first inequality holds by stability. The second inequality holds since coalition $D_j(N)\backslash\{i\} \cup \{i^*\}$ has one resource, which, as a possible allocation, can be positioned in region j. By Lemma 4 we have $\sum_{i\in D_j(N)} x_i = W_j(N)$ and thus from (5) it follows that $x_{i^*} - x_i \geq -w_i$. Hence,

$$\alpha_i = x_i - \gamma = x_i - x_{i^*} \leq w_i.$$

Moreover,

$$\alpha_i = x_i - \gamma = x_i - \min\{x_k \mid k \in N, r_k = 1\} \geq 0,$$

where the inequality holds as $i \in \{k \in N \mid r_k = 1\}$. Consequently, $\alpha_i \in [0, w_i]$.

d) We need to prove $\gamma + \sum_{i \in D_j(N)} \alpha_i = W_j(N)$ for all $j \in \mathscr{D}_N^c$. For this, let $j \in \mathscr{D}_N^c$. In addition, let $i^* \in D_j(N)$ for which $r_{i^*} = 1$. Note that i^* is unique since $R(D_j(N)) = 1$. From this, we can conclude that

$$\gamma + \sum_{i \in D_j(N)} \alpha_i = \gamma + \alpha_{i^*} + \sum_{i \in D_j(N) \setminus \{i^*\}} \alpha_i = x_{i^*} + \sum_{i \in D_j(N) \setminus \{i^*\}} x_i = \sum_{i \in D_j(N)} x_i = W_j(N),$$

where the last equality holds by Lemma 4, because $R(D_j(N)) = 1$.

e) We need to prove $\gamma \in \begin{cases} \left[W_{\sigma_N(R(N)+1)}(N), W_{\sigma_N(R(N))}(N)\right] & \text{if } R(N) < |\mathscr{D}|, \\ \left[0, W_{\sigma_N(R(N))}(N)\right] & \text{if } R(N) = |\mathscr{D}|. \end{cases}$

For this, we distinguish between two cases: $R(N) = |\mathscr{D}|$ and $R(N) < |\mathscr{D}|$.

- **Case 1.e.1**: $R(N) = |\mathscr{D}|$.

  From property d) we know that $\gamma + \sum_{i \in D_{\sigma_N(R(N))}(N)} \alpha_i = W_{\sigma_N(R(N))}(N)$. This implies that

$$\gamma \leq W_{\sigma_N(R(N))}(N),$$

  because $\alpha_i \geq 0$ for all $i \in D_{\sigma_N(R(N))}(N)$ by property c). Next, let $i^* \in \{k \in N \mid x_k = \gamma \text{ and } r_k = 1\}$. Note that i^* exists due to the definition of γ. Then, by stability and the fact that $r_{i^*} = 1$, we have

$$\gamma = x_{i^*} \geq v^\theta(\{i^*\}) = w_{i^*} \geq 0.$$

  Hence, $\gamma \in [0, W_{\sigma_N(R(N))}(N)]$.

- **Case 1.e.2**: $R(N) < |\mathscr{D}|$.

  Similar to case 1.e.1, it follows that

$$\gamma \leq W_{\sigma_N(R(N))}(N).$$

  Again, let $i^* \in \{k \in N | x_k = \gamma \text{ and } r_k = 1\}$. Observe that $i^* \notin D_{\sigma_N(R(N)+1)}(N)$ since $r_{i^*} = 1$ and $R(D_{\sigma_N(R(N)+1)}(N)) = 0$. Then,

$$x_{i^*} + \sum_{i \in D_{\sigma_N(R(N)+1)}(N)} x_i \geq v^\theta(D_{\sigma_N(R(N)+1)}(N) \cup \{i^*\}) \geq W_{\sigma_N(R(N)+1)}(N),$$

  where the first inequality holds by stability and the second inequality since allocating the single resource to region $\sigma_N(R(N)+1)$ is a possible allocation for coalition $D_{\sigma_N(R(N)+1)}(N) \cup \{i^*\}$. By Lemma 3 and because $r_i = 0$ for all $i \in D_{\sigma_N(R(N)+1)}(N)$, we have $x_i = 0$ for all $i \in D_{\sigma_N(R(N)+1)}(N)$. Hence,

$$\gamma = x_{i^*} \geq W_{\sigma_N(R(N)+1)}(N) - \sum_{i \in D_{\sigma_N(R(N)+1)}(N)} x_i = W_{\sigma_N(R(N)+1)}(N).$$

Hence, $\gamma \in [W_{\sigma_N(R(N)+1)}(N), W_{\sigma_N(R(N))}(N)]$.

Case 2: there exists a $j \in \mathscr{D}_N^c$ for which $R(D_j(N)) \neq 1$.
Before proving that the five properties are satisfied, we will first show that there exists a covered region that initially has no resources. For this, suppose for the sake of contradiction that this is not the case. Then, since there exists a $j \in \mathscr{D}_N^c$ for which $R(D_j(N)) \neq 1$, we know that this covered region initially has exactly two resources. Moreover, every other covered regions initially has at least one resource. Consequently, $R(N) \geq \sum_{j \in \mathscr{D}_N^c} R(D_j(N)) \geq |\mathscr{D}_N^c| + 1 = R(N) + 1 > R(N)$, which is a contradiction and thus there indeed exists a covered region that initially has no resources. Let j_0 denote such a region, i.e., let $j_0 \in \mathscr{D}_N^c$ with $R(D_{j_0}(N)) = 0$. Now, we set

$$\gamma = W_{j_0}(N) - \sum_{i \in D_{j_0}(N)} x_i,$$

$$\alpha_i = x_i - \gamma \cdot r_i \text{ for all } i \in N,$$

and we will prove that the five properties are satisfied.

a) Similar to property a) in case 1, we can conclude $x_i = \gamma \cdot r_i + \alpha_i$ for all $i \in N$.

b) We need to prove $\alpha_i = 0$ for all $i \in D_j(N)$ and all $j \in \mathscr{D}_N^{nc}$. For this, let $j \in \mathscr{D}_N^{nc}$ and $i \in D_j(N)$. We distinguish between two cases: $r_i = 0$ and $r_i = 1$.
   - **Case 2.b.1**: $r_i = 0$.
     Similar to property b) in case 1, we can conclude $\alpha_i = 0$.
   - **Case 2.b.2**: $r_i = 1$.
     By Lemma 5, we have $\alpha_i = x_i - \gamma = x_i + \sum_{i \in D_{j_0}(N)} x_i - W_{j_0}(N) = 0$.

c) We need to prove $\alpha_i \in [0, w_i]$ for all $i \in D_j(N)$ and all $j \in \mathscr{D}_N^c$. For this, let $j \in \mathscr{D}_N^c$ and $i \in D_j(N)$. We distinguish between two cases: $r_i = 0$ and $r_i = 1$.
   - **Case 2.c.1**: $r_i = 0$.
     Similar to property c) in case 1.c.1., we can conclude $\alpha_i \in [0, w_i]$.
   - **Case 2.c.2**: $r_i = 1$.
     Since $R(D_{j_0}(N)) = 0$ and thus every players in region j_0 initially has no resources, we have $i \notin D_{j_0}(N)$. Moreover, as $R(N \backslash (D_{j_0}(N) \cup \{i\})) = R(N) - 1$, we know that coalition $N \backslash (D_{j_0}(N) \cup \{i\})$ has exactly enough resources to cover all regions

in $\mathscr{D}_N^c \backslash \{j_0\}$ (because $j_0 \in \mathscr{D}_N^c$ and thus $|\mathscr{D}_N^c \backslash \{j_0\}| = |\mathscr{D}_N^c| - 1 = R(N) - 1$). This allocation results in a profit of $v^\theta(N) - W_{j_0}(N) - w_i$ for coalition $N \backslash (D_{j_0}(N) \cup \{i\})$. Note that this allocation is not necessarily optimal and thus

$$v^\theta(N \backslash (D_{j_0}(N) \cup \{i\})) \geq v^\theta(N) - W_{j_0}(N) - w_i.$$

As a consequence,

$$\alpha_i = x_i - \gamma = \sum_{k \in D_{j_0}(N) \cup \{i\}} x_k - W_{j_0}(N) \leq v^\theta(N) - v^\theta(N \backslash (D_{j_0}(N) \cup \{i\})) - W_{j_0}(N)$$
$$\leq v^\theta(N) - \left(v^\theta(N) - W_{j_0}(N) - w_i\right) - W_{j_0}(N) = w_i,$$

where the first inequality holds by applying Lemma 1. Next, observe that

$$\alpha_i = x_i - \gamma = \sum_{k \in D_{j_0}(N) \cup \{i\}} x_k - W_{j_0}(N) \geq v^\theta(D_{j_0}(N) \cup \{i\}) - W_{j_0}(N)$$
$$\geq W_{j_0}(N) - W_{j_0}(N) = 0,$$

where the first inequality holds by stability and the second inequality since allocating the single resource to region j_0 is a possible allocation for coalition $D_{j_0}(N) \cup \{i\}$. Consequently, $\alpha_i \in [0, w_i]$.

d) We need to prove $\gamma + \sum_{i \in D_j(N)} \alpha_i = W_j(N)$ for all $j \in \mathscr{D}_N^c$. For this, let $j \in \mathscr{D}_N^c$. We distinguish between three cases: $R(D_j(N)) = 0$, $R(D_j(N)) = 1$ and $R(D_j(N)) = 2$.

- **Case 2.d.1**: $R(D_j(N)) = 0$.

  We distinguish between another two cases: in the first case we assume that there is a covered region that initially has two resources and in the second case we assume that this is not the case.

  - **Case 2.d.1.1**: there exists a $j \in \mathscr{D}_N^c$ for which $R(D_j(N)) = 2$.

    Let $j_2 \in \mathscr{D}_N^c$ with $R(D_{j_2}(N)) = 2$. By Lemma 4, it holds that

$$\sum_{i \in D_{j_2}(N) \cup D_{j_0}(N)} x_i = W_{j_2}(N) + W_{j_0}(N),$$
$$\sum_{i \in D_{j_2}(N) \cup D_j(N)} x_i = W_{j_2}(N) + W_j(N).$$

    Subtracting the first equality from the second equality gives

$$\sum_{i \in D_{j_0}(N)} x_i - \sum_{i \in D_j(N)} x_i = W_{j_0}(N) - W_j(N). \tag{6}$$

    As a consequence, since $r_i = 0$ and thus $\alpha_i = x_i$ for all $i \in D_{j_0}(N)$, we have

$$\gamma + \sum_{i \in D_j(N)} \alpha_i = W_{j_0}(N) - \sum_{i \in D_{j_0}(N)} x_i + \sum_{i \in D_j(N)} x_i = W_j(N).$$

- **Case 2.d.1.2**: $R(D_j(N)) \neq 2$ for all $j \in \mathscr{D}_N^c$.

  In this case we have that every covered region initially has at most one resource. Moreover, since the covered region j_0 initially has no resources, we know that there must be a player in a non-covered region that initially has a resource. Let i^* denote such a player, i.e., let $i^* \in D_{j'}(N)$ for some $j' \in \mathscr{D}_N^{nc}$ with $r_{i^*} = 1$. Then, by Lemma 5, it holds that

$$x_{i^*} + \sum_{i \in D_{j_0}(N)} x_i = W_{j_0}(N),$$
$$x_{i^*} + \sum_{i \in D_j(N)} x_i = W_j(N).$$

  Subtracting the first equality from the second equality again gives (6). As a consequence, similar to case 2.d.1.1, we can conclude $\gamma + \sum_{i \in D_j(N)} \alpha_i = W_j(N)$.

- **Case 2.d.2**: $R(D_j(N)) = 1$.

  Similar to property d) in case 1, we can conclude $\gamma + \sum_{i \in D_j(N)} \alpha_i = W_j(N)$.

- **Case 2.d.3**: $R(D_j(N)) = 2$.

  Let $i_1, i_2 \in D_j(N)$ with $i_1 \neq i_2$ and $r_{i_1} = r_{i_2} = 1$. Then, for each $i \in D_j(N) \backslash \{i_1, i_2\}$, we have $r_i = 0$ and thus $\alpha_i = x_i$. As a consequence,

$$\begin{aligned}
\gamma + \sum_{i \in D_j(N)} \alpha_i &= \gamma + (x_{i_1} - \gamma) + (x_{i_2} - \gamma) + \sum_{i \in D_j(N) \backslash \{i_1, i_2\}} x_i = \sum_{i \in D_j(N)} x_i - \gamma \\
&= \sum_{i \in D_j(N)} x_i - \left(W_{j_0}(N) - \sum_{i \in D_{j_0}(N)} x_i \right) \\
&= \left(W_j(N) - W_{j_0}(N) - \sum_{i \in D_{j_0}(N)} x_i \right) - \left(W_{j_0}(N) - \sum_{i \in D_{j_0}(N)} x_i \right) \\
&= W_j(N),
\end{aligned}$$

  where the penultimate equality is a direct consequence of Lemma 4.

e) We need to prove $\gamma \in \begin{cases} \left[W_{\sigma_N(R(N)+1)}(N), W_{\sigma_N(R(N))}(N) \right] & \text{if } R(N) < |\mathscr{D}|, \\ \left[0, W_{\sigma_N(R(N))}(N) \right] & \text{if } R(N) = |\mathscr{D}|. \end{cases}$

For this, we distinguish between two cases: $R(N) = |\mathscr{D}|$ and $R(N) < |\mathscr{D}|$.

- **Case 2.e.1**: $R(N) = |\mathscr{D}|$.

  Similar to case 1.e.1, it follows that

$$\gamma \leq W_{\sigma_N(R(N))}(N).$$

  In addition, by Lemma 3, $x_i \leq w_i$ for all $i \in D_{j_0}(N)$ and thus

$$\gamma = W_{j_0}(N) - \sum_{i \in D_{j_0}(N)} x_i \geq W_{j_0}(N) - \sum_{i \in D_{j_0}(N)} w_i = W_{j_0}(N) - W_{j_0}(N) = 0.$$

Hence, $\gamma \in [0, W_{\sigma_N(R(N))}(N)]$.

- **Case 2.e.2**: $R(N) < |\mathscr{D}|$.

  Similar to case 1.e.1, it follows that

$$\gamma \leq W_{\sigma_N(R(N))}(N).$$

  Next, since $R(N \backslash D_{j_0}(N)) = R(N)$, we know that coalition $N \backslash D_{j_0}(N)$ has exactly enough resources to cover all regions in $\mathscr{D}_N^c \backslash \{j_0\} \cup \{\sigma_N(R(N)+1)\}$ (because $j_0 \in \mathscr{D}_N^c$ and $\sigma_N(R(N)+1) \notin \mathscr{D}_N^c$ and thus $|\mathscr{D}_N^c \backslash \{j_0\} \cup \{\sigma_N(R(N)+1)\}| = |\mathscr{D}_N^c| = R(N)$). This allocation results in a profit of $v^\theta(N) - W_{j_0}(N) + W_{\sigma_N(R(N)+1)}(N)$ for coalition $N \backslash D_{j_0}(N)$. Hence,

$$v^\theta(N \backslash D_{j_0}(N)) \geq v^\theta(N) - W_{j_0}(N) + W_{\sigma_N(R(N)+1)}(N).$$

  Consequently, by Lemma 1, we have

$$\begin{aligned}\gamma &= W_{j_0}(N) - \sum_{i \in D_{j_0}(N)} x_i \geq W_{j_0}(N) - v^\theta(N) + v^\theta(N \backslash D_{j_0}(N)) \\ &\geq W_{j_0}(N) - v^\theta(N) + \Big(v^\theta(N) - W_{j_0}(N) + W_{\sigma_N(R(N)+1)}(N)\Big) = W_{\sigma_N(R(N)+1)}(N).\end{aligned}$$

  To conclude, $\gamma \in [W_{\sigma_N(R(N)+1)}(N), W_{\sigma_N(R(N))}(N)]$.

We have proven that every core allocation can be written as an RP allocation, both if $R(D_j(N)) = 1$ for all $j \in \mathscr{D}_N^c$ (case 1) and if there exists a $j \in \mathscr{D}_N^c$ for which $R(D_j(N)) \neq 1$ (case 2). Hence, the set of RP allocations concides with the core. □