Fair allocations for cooperation problems in vaccination

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Abstract

Vaccination is a very effective measure to fight an outbreak of an infectious disease, but it often suffers from delayed deliveries and limited stockpiles. To use these limited amounts of vaccines effectively, health agencies can decide to cooperate and share their vaccines. In this paper, we analyze this type of cooperation. Typically cooperation leads to an increased total return, but cooperation is only plausible when this total return can be distributed among the agents in a fair way. Using cooperative game theory, we derive theoretical sufficient conditions under which cooperation is plausible and we show that the resources can be traded for a market price in those cases. We perform numerical analyses to generalize these findings and we derive analytical expressions for market prices that can be used in general for distributing the total return in a fair way. Our results demonstrate that cooperation is a delicate matter. Cooperation is most likely to be plausible when the total amount of resources is limited or very large. In those cases, trading resources for a market price often results in a fair allocation of the total return. We confirm these findings with a case study on the redistribution of influenza vaccines.

Keywords: cooperative game theory, market allocations, core, S-shaped return functions, vaccination

1 Introduction

Vaccination is a powerful preventive measure to avoid a large outbreak of an infectious disease. However, often there are insufficient vaccines available to vaccinate the entire population. Various parties, such as health agencies, may have their own stockpiles of vaccines. To use these limited amounts of vaccines effectively, these

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parties can decide to cooperate and share their vaccines. The Centers for Disease Control and Prevention (CDC) in the United States allows parties to redistribute or sell their vaccines to others in case of delayed delivery or another emergency (Centers for Disease Control and Prevention, 2009b). By cooperating, the limited resources can be used in a smarter way which might lead to increased health benefits. Despite the fact that cooperation often leads to increased benefits, there can be various political, organizational or financial reasons why parties are not willing to cooperate. In this paper, we study under which conditions cooperation is plausible. We consider cooperation to be plausible when there exists an allocation of the total health benefits among all parties that is advantageous for everyone.

In literature, there are several papers that study cooperation in the context of vaccination. Most of these papers consider a central planner that coordinates the cooperation and assume that parties are willing to cooperate if there is individual rationality (i.e., if the benefits of cooperation are such that every individual party receives at least as much as it could obtain on its own). Sun, Yang, and De Véricourt (2009) and Wang, de Véricourt, and Sun (2009) model the behavior of countries that can decide to keep their vaccines for themselves or to donate some vaccines to others. Both papers compare the decentralized solution (i.e., the situation without cooperation) to the solution of a central planner, such as the World Health Organisation (WHO). Mamani, Chick, and Simchi-Levi (2013) also consider a central planner, but they do not enforce cooperation by imposing a solution. Instead, they propose a contract that coordinates the behavior of countries via subsidies. Such a contract indirectly stimulates countries to cooperate. In contrast to a central planner who directly or indirectly enforces cooperation, parties can also decide to cooperate themselves. The fact that cooperation typically leads to increased health benefits provides a motivation for parties to cooperate.

In our analysis, we study under which conditions parties are willing to cooperate without central coordination. Thereto, we make use of cooperative game theory. This field deals with the modeling and analysis of situations in which parties, also called 'players', can benefit from coordinating their actions. In this paper, we introduce and analyze a specific type of cooperative game, a so-called 'resource pooling game', in which players redistribute their resources in an optimal way in order to achieve together a higher total return. A natural question arises about the allocation among the players of this additional return compared to the situation without cooperation. We use the concept of the core to find a fair allocation of the total return. The core is defined as the set of allocations that divide the total return in such a way that no individual player nor any group of players is worse off. These allocations are therefore an extension to allocations that only consider individual rationality (Mamani et al., 2013; Sun et al., 2009; Wang et al., 2009).

We model the benefits that players can obtain from a certain amount of resources with a return function. The nonlinearities in vaccination give rise to a typical pattern in the return function, which is characterized by increasing returns to scale in case of limited resources and decreasing returns to scale in case of many resources (L. E. Duijzer, van Jaarsveld, Wallinga, & Dekker, 2018; Mamani et al., 2013; Wu, Riley, & Leung, 2007). Such type of return functions is also known as S-shaped return functions. For these type of return functions, we show that cooperation is a delicate matter. Even though cooperation typically leads to an overall increase of the total return, there is not always a fair way to allocate this total return among the cooperating players. In addition, we present a number of interesting cases for which cooperation is plausible. For those cases we present a fair and intuitive allocation with a uniform market price for trading resources. We numerically study situations in which it is difficult to determine the total return that players can achieve through cooperation, because of the complexity of the underlying decision problem (i.e., the problem of redistributing the resources in an optimal way). We analyze whether comparable market prices can be used in those situations. We conclude that when cooperation is plausible, trading resources for a market price often results in a fair allocation of the total return and we provide analytical expressions for potential market prices. We illustrate our findings in a case study on influenza vaccination.

With our analysis we contribute to the literature in two ways. Firstly, we contribute to the literature on cooperation in vaccination. This literature mainly considers cooperation that is organized via a central planner. Our results show that under certain conditions, a central planner is not needed to enforce cooperation but that players can organize the cooperation themselves. Secondly, we contribute to the cooperative game theory literature by being the first to analyze resource pooling games with S-shaped return functions. Next to applications in vaccination, these type of functions have also been used for returns from investing into a market (Zschocke, Mantin, & Jewkes, 2013), for sales response in marketing (Abedi, 2017) and for fill rates in exchangeable-item repair systems (Dreyfuss & Giat, 2017), among others.

The remainder of this paper is structured as follows. We start with a literature review in Section 2. In Section 3, we formulate the game and introduce the core. We introduce our 'market allocations' and discuss their relation to the core in Section 4. In Section 5, we derive sufficient conditions under which cooperation is always plausible and thus and a fair allocation of the total return exists. We generalize these findings in Section 6. In Section 7, we apply our results to a case study on vaccine distribution. We close with a discussion and conclusion in Section 8.

2 Literature

This paper considers a cooperation problem in which multiple parties together decide how a limited amount of vaccines has to be distributed among multiple groups of individuals. Many papers on vaccine allocation consider one central decision maker who decides how the available vaccines have to be allocated among the various regions, age groups or risk groups (L. E. Duijzer, van Jaarsveld, & Dekker, 2018; Keeling & Rohani, 2011, and references therein). In this paper, however, we consider multiple decision makers who each have an amount of vaccines available. We use cooperative game theory to analyze the cooperation between these decision makers.

With our cooperative perspective we contribute to the literature on cooperation and coordination in vaccination. We discuss this literature in Section 2.1. The context of vaccination asks for a cooperative game formulation that has not been studied before. In Section 2.2, we briefly discuss the related literature on cooperative game theory. We close with a discussion of the literature on S-shaped return functions in Section 2.3.

2.1 Cooperation and coordination in vaccination

In literature, many studies of coordination in vaccination focus on the production of vaccines. The various parties involved in vaccine production often have conflicting objectives. Governments and public health agencies strive for high vaccine stockpiles. But vaccine producers might not be willing to produce large amounts, because of the various supply uncertainties that play a role in the production of vaccines. Several studies use game theory to analyze coordination on the vaccine market via contracts or subsidies (e.g., Adida, Dey, & Mamani, 2013; Arifoğlu, Deo, & Iravani, 2012; Chick, Hasija, & Nasiry, 2017; Chick, Mamani, & Simchi-Levi, 2008; Dai, Cho, & Zhang, 2016).

There are also some studies that apply game theoretical approaches to vaccine allocation problems. These studies analyze independent agents that decide themselves on the amount of vaccines allocated to their population. Sun et al. (2009) study an epidemic that starts in a source country and spreads both within and across countries. Each country has its own stockpile of vaccines and the authors analyze when countries are willing to give up part of their stockpile or whether they act selfishly. The authors show that when the transmission from one country to another is small enough, countries either give all their vaccines to the source country or do not give away anything. Under certain conditions this decentralized solution can be improved by a central planner who decides how to allocate all resources. Wang et al. (2009) perform a comparable analysis. They restrict themselves to two countries, but analyze the outbreak with a more extensive epidemiological model and for a longer time horizon. They show that the decentralized solution is equal to the centralized solution when countries are either altruistic and all vaccines are given to one of the two countries or when every country acts selfishly and keeps his own stockpile. Any other solution results in more infections for at least one of the countries. Mamani et al. (2013) do not focus on the allocation of a given amount of vaccines, but on the decision how many vaccines to order. Ordering more vaccines brings higher purchasing costs, but reduces the costs related to infections. Their model incorporates characteristics of the models of both Sun et al. (2009) and Wang et al. (2009). Mamani et al. (2013) propose a coordinating contract in which every country pays a subsidy to the source country where the epidemic started. This coordinating contract aligns the objectives of the countries and reduces the overall costs for infections. These papers use non-cooperative game theory and enforce cooperation via contracts. In contrast, we analyze whether players are willing to cooperate without enforcement and we therefore use cooperative game theory for our analysis. Although enforced cooperation might be easier to arrange than self-organized cooperation, other studies have shown that digital tools can help to facilitate self-organized cooperation (Ergun, Gui, Heier Stamm, Keskinocak, & Swann, 2014).

2.2 Cooperative game theory

Cooperative game theory primarily deals with the modeling and analysis of situations in which groups of players can benefit from coordinating their actions. In particular, we focus only on a specific class of cooperative games, namely those in which binding agreements are made between players and side payments are allowed, i.e., transferable utility (TU) games. For such a cooperative game, one lists for every possible group of players a single number, representing, for instance, the health benefits for this group of players when they coordinate their actions. In the theory of cooperative games, an important question is how to allocate this associated amount when all players decide to cooperate. An allocation rule identifies how to divide this amount among the participating players for a class of cooperative games. Typically, one strives for allocation rules with appealing fairness properties. Three, well-known and accepted, fairness properties are efficiency (i.e., the amount should be allocated completely), individual rationality (i.e., everyone player gets at least what he would get while acting alone) and coalitional stability (i.e., every group of players gets at least what they would get while acting together). The set of allocations that does satisfy these properties together is better known as the core (Gillies, 1959).

The game that we study in this paper belongs to the class of operations research (OR) games, a stream of literature that studies TU games, arising from underlying situations in which a group of collaborating players faces a joint optimization problem (see, e.g., Borm, Hamers, & Hendrickx, 2001, for a review on OR games). In particular, within this class of OR games, our game can be recognized as a resource pooling game. In such a game, resources are reallocated, or shared among players to realize additional profit (or reduce costs). In the last couple of years, there is an increasing interest in these games, especially with a focus on logistics. Some examples are the reallocation of inventory in a retail setting (Sošić, 2006), pooling of emergency vehicles in health care (Karsten, Slikker, & Van Houtum, 2015), pooling of technicians in the service industry (Anily & Haviv, 2010), pooling of capacity in a production environment (Anily & Haviv, 2017; Özen, Reiman, & Wang, 2011), pooling of spare parts in the capital intensive goods industry (Guajardo & Rönnqvist, 2015; Karsten & Basten, 2014; Karsten, Slikker, & van Houtum, 2012), and reallocation of repair vans, tamping machines, and spare parts in a railway setting (Schlicher, Slikker, & van Houtum, 2017a, 2017b, 2018). To the best of our knowledge, we are the first to focus on a resource pooling game with an application in vaccination.

In a broader perspective, our game can be recognized as a slightly modified version of market games (Shapley & Shubik, 1969). In these games, which are studied intensively in literature (see e.g., Osborne & Rubinstein, 1994), each player is associated with a set of resources and a convex utility function, identifying the amount of profit realized for the given set of resources. Players can cooperate by reallocating resources to maximize the sum of the convex utility functions. Shapley and Shubik (1969) show that the core of these market games is always non-empty, by providing an intuitive market allocation. Debreu and Scarf (1963) show that core non-emptiness of market games is no longer guaranteed when utility functions are non-convex. We study a modified version of market games, since we consider the utility function (per player) to be S-shaped (i.e., convex-concave). To the best of our knowledge, there are no market games nor resource pooling games in literature that consider the specific individual utility function with a convex-concave form.

2.3 S-shaped return functions

The decision problem underlying our cooperative game, is a resource allocation problem with S-shaped return functions to measure the return obtained from a certain number of resources. The S-shape establishes convex returns for limited amounts of resources and concave returns in case of many resources. S-shaped return functions are used to express the relation between the number of distributed vaccines and the health benefits/costs in a population (Chick et al., 2017; L. E. Duijzer, van Jaarsveld, Wallinga, & Dekker, 2018; Mamani et al., 2013), but also in marketing (e.g., Abedi, 2017; Ağralı & Geunes, 2009). Although S-shaped return functions have various applications, decision problems involving these functions are in general difficult.

Ağralı and Geunes (2009) even show that a resource allocation problem involving such return functions is NP-hard. Several methods have been proposed to find solutions for this problem. Ginsberg (1974) was the first to consider this problem and he derived conditions under which the optimal solution can be described analytically. Based on these analytical solutions, L. E. Duijzer, van Jaarsveld, Wallinga, and Dekker (2018) developed a heuristic which works well for vaccine allocation problems. Ağralı and Geunes (2009) and Srivastava and Bullo (2014) approach the problem theoretically and develop approximation algorithms with theoretical performance guarantees and polynomial time complexity. Although the computation time of these approaches is polynomial, the computation time can be quite large for large instances or when a high precision is required. Abedi (2017) analyze a more general version of the problem in which the return functions are correlated. They study an application in marketing and develop a branch and cut algorithm.

In this paper, we need to solve a resource allocation problem with S-shaped return functions for every possible group of players in order to determine whether cooperation is plausible. This implies that the number of NP-hard problems we need to solve is exponential in the number of players. We therefore prefer a solution approach that is very fast. We use the heuristic of L. E. Duijzer, van Jaarsveld, Wallinga, and Dekker (2018), which is shown to work well in the context of vaccination. If possible (i.e., for small instances), we will use complete enumeration to determine the optimal solution of the NP-hard resource allocation problem.

3 Problem

The cooperation problem in vaccination that we study in this paper is an application of a general cooperation problem in resource allocation. We formulate our problem and our results in terms of general resources and return to the application of vaccination in the case study in Section 7. This way of formulating emphasizes the generality of our problem and our contribution to the literature on cooperative game theory.

We formulate our problem and the corresponding cooperative game in Section 3.1. In Section 3.2, we discuss the type of allocations that we are interested in. In Section 3.3, we show that these desirable allocations do not always exists.

3.1 Cooperative game formulation

We consider a finite set of players N and every player $i \in N$ initially has some resources $r_i \geq 0$. The value that player $i \in N$ obtains from a certain amount of resources is determined by the S-shaped return function $F_i(\cdot)$. We assume that $F_i(\cdot)$ only depends on the amount of resources of player i, and not on the resources of the other players. For a discussion of this assumption we refer to Section 8. We assume that the return function for every player $i \in N$ satisfies the following three conditions.

Assumption 1. *Consider a return function $F_i(\cdot)$, then:*

1. *$F_i(\cdot)$ is continuous, non-negative and non-decreasing,*
2. *$F_i(\cdot)$ is strictly convex on the interval $[0, c_i)$ and strictly concave on $(c_i, +\infty)$ for some $c_i \geq 0$,*
3. *$F_i(af) < aF_i(f)$ on the interval $f \in [d_i, +\infty)$ for some $d_i \geq c_i$ and for any $a > 1$.*

The S-shaped function captures the structure of increasing returns to scale when a player has few resources, but decreasing returns to scale when he has many resources. Condition (3) of Assumption 1 guarantees decreasing returns to scale. Functions that satisfy the conditions in Assumption 1 are referred to as *nicely convex-concave* by Ginsberg (1974).

We discuss in more detail how we measure the return of a player. Let $M_i > 0$ denote the size of player i. For example, when a player corresponds to a geographic region, this size can represent the number of inhabitants in that region. W.l.o.g. we assume that the return function measures the fractional return for a player, i.e., the actual return for player i is equal to $M_i F_i(\cdot)$. Furthermore, we assume that the return depends on the ratio of the amount of resources and the size of the player. For example, player i obtains a value $M_i F_i(r/M_i)$ from r resources. If the same amount of resources were given to player j, then he would obtain a value of $M_j F_j(r/M_j)$. This type of return function is common in vaccination (E. Duijzer, van Jaarsveld, Wallinga, & Dekker, 2016; L. E. Duijzer, van Jaarsveld, Wallinga, & Dekker, 2018; Mamani et al., 2013), where the players represent populations and M_i denotes the number of individuals in population i. When there are r vaccines available for population i, then r/M_i represents the fraction of the population that can be vaccinated. Then, $F_i(r/M_i)$ measures per individual the monetary health benefits that are the result of distributing r vaccines in population i. We note that this modeling choice is not restrictive, because any return function that satisfies Assumption 1 can be rewritten to model the fractional return. To illustrate, for any return function $G(\cdot)$ that satisfies Assumption 1 and measures the total return that a player of size M obtains from a number of resources, we can construct a function $F(\cdot)$ that also satisfies Assumption 1 but measures the fractional return, by setting $F(r/M) = G(r)/M$ for all $r \geq 0$.

We introduce our game as a pair (N, v), where $N \subseteq \mathbb{N}$ represents the set of players and $v : 2^N \to \mathbb{R}$ denotes the value function which is introduced in this section. We use the term 'coalition' to refer to a subset of players $S \subseteq N$. The total set of players N is referred to as the grand coalition. The value function

$v(S)$ measures the maximum return that a coalition of players $S \subseteq N$ can achieve by redistributing their resources without the help of the players in $N \backslash S$. This maximum return for coalition $S \subseteq N$ is equal to the value of the following optimization problem:

$$\begin{aligned} v(S) = \max \quad & \sum_{i \in S} M_i F_i\left(f_i\right) \\ \text{s.t.} \quad & \sum_{i \in S} f_i M_i \leq \sum_{i \in S} r_i \\ & f_i \geq 0 \qquad \forall i \in S, \end{aligned} \tag{1}$$

In Problem (1), the decision variable f_i for all $i \in S$ represents the fraction of resources player i receives relative to its size M_i, when player i is cooperating with all other players in S. In literature, the above problem per coalition, is referred to as a knapsack problem with S-shaped return functions (e.g., Ağralı & Geunes, 2009; Ginsberg, 1974; Srivastava & Bullo, 2014). Ağralı and Geunes (2009) show that this problem is NP-hard.

We illustrate our game with the following example. Because of the NP-hardness of Problem (1) it is difficult to determine the value function $v(\cdot)$. We therefore use discretized enumeration with step size 10^{-4} to approximate the value function and we round the numerical values to four decimal places. This implies that the actual value function can deviate from the reported numbers. However, because the return function is continuous and non-decreasing (Assumption 1) this deviation is small and it does not affect the message of the example.

Example 1: Consider a situation with three identical players. Let $F_i(f) = \frac{1}{1+\exp\{-45f+25\}}$ for $i = 1, 2, 3$. Furthermore, $\mathbf{M} = [1, 1, 1]$ and $\mathbf{r} = [0.2, 0.2, 0.2]$. It can be verified that

$$v(\{i\}) = \frac{1}{1 + \exp\{-45(r_i/M_i) + 25\}} = \frac{1}{1 + \exp\{16\}} = 0 \text{ for } i = 1, 2, 3.$$

When the three players cooperate, they can achieve a higher return by giving all resources to one player. This results in

$$v(N) = \frac{1}{1 + \exp\{-45 \cdot (3/5) + 25\}} + \frac{1}{1 + \exp\{-45 \cdot 0 + 25\}} + \frac{1}{1 + \exp\{-45 \cdot 0 + 25\}} = 0.8808.$$

$\triangle$

Example 1 demonstrates that cooperation can increase the total return. When all players keep their resources to themselves, the total return is approximately equal to zero. But combining the resources results in a total return of almost 0.9. In this case, the high return could be achieved because one player benefited from the willingness of the other players to give away their resources. However, a player is only willing to give away (part of) his resources if he can also benefit from the increased return of the other player. In the next section, we therefore discuss how the total return, achieved through cooperation, can be allocated among the players.

3.2 The core

To find a fair allocation of the total return, we use the concept of the core (Gillies, 1959) from cooperative game theory. The core of a game is formally defined as the set of all allocations $\mathbf{x} \in \mathbb{R}^N$ that satisfy the following conditions:

$$\begin{aligned} &\textbf{Efficiency} \quad && \sum_{i \in N} x_i = v(N) \\ &\textbf{Stability} \quad && \sum_{i \in S} x_i \geq v(S) \quad \forall S \subset N \end{aligned} \tag{2}$$

The efficiency condition guarantees that the total return is divided among all players. By the (coalitional) stability conditions, this division is done in such a way that no coalition of players can improve their return by leaving the grand coalition. Stability is thus stronger than individual rationality, which would only require that no individual player is willing to leave the grand coalition ($x_i \geq v(\{i\})$ for all $i \in N$). Core allocations are desirable, because they provide a fair way to divide the total return.

We illustrate the concept of the core in the following example. Recall that all numerical values are rounded to four decimal places.

Example 1 (continued): The return function of the three identical players is illustrated in Figure 1.

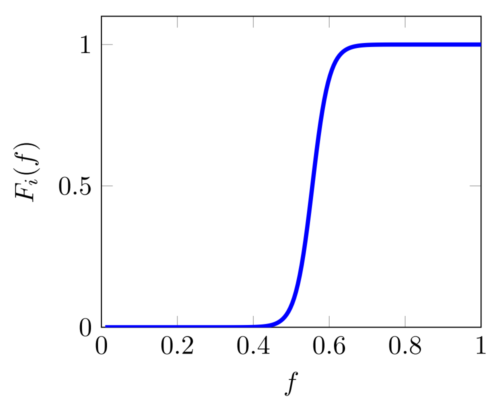

Figure 1: Graphical representation of the return function: $F_i(f) = \frac{1}{1+\exp\{-45f+25\}}$.

Given that $\mathbf{r} = [0.2, 0.2, 0.2]$, one can derive from Figure 1 that every coalition of players maximizes its return by giving all resources to one player. This results in the following value function:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	0.0009	0.0009	0.0009	0.8808

One can verify that $\mathbf{x} = \left[\frac{0.8808}{3}, \frac{0.8808}{3}, \frac{0.8808}{3}\right]$ satisfies the efficiency and stability conditions, which implies that $\mathbf{x}$ is a core allocation. $\triangle$

In Example 1, resources are scarce with respect to $F_i(\cdot)$ and M_i for all $i \in N$. The return that a single player obtains from his resources is negligible and the same holds when two players cooperate. Only when three players cooperate, they can obtain a high return. This implies that the fair allocation which divides the total return equally among all players is in the core. Moreover, practically any allocation in which each player receives a strictly positive share is in the core, because a player can almost never obtain a higher return in any subcoalition. We note that, in order to satisfy the stability conditions, every subcoalition consisting of two players should receive at least 0.0009. Therefore, the allocation in which all return is given to a single player is not in the core, even though this allocation is efficient and individually rational.

The simple setting of Example 1 illustrates the existence of core allocations. However, it is possible that the core is empty, meaning that no allocation exists that satisfies the efficiency and stability conditions in (2). In this paper, we argue that the existence of a core allocation is necessary for cooperation. However, the existence of a core allocation does not automatically imply that all players are also willing to cooperate with each other. There might still be other (political, organizational, financial) reasons not to do so. We therefore say that cooperation is *plausible* when the core is non-empty.

3.3 An empty core

With the following example we illustrate that the core can be empty. We again determine the value function via discretized enumeration with step size 10^{-4} and round the numerical values to four decimal places.

Example 2: Consider a situation with three identical players: $\mathbf{M} = [1, 1, 1]$ and $\mathbf{r} = \left[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right]$. Let $F_i(f) = \frac{1}{1+\exp\{-45f+25\}}$, for $i = 1, 2, 3$. One can conclude from Figure 1 that every coalition of players maximizes its return by giving all resources to one player. This results in the following value function:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	0.9933	0.9933	0.9933	1.0000

We will show that the core of this game is empty. Suppose for the sake of contradiction there exists an allocation $\mathbf{x} \in \mathbb{R}^3$ that satisfies the efficiency and stability conditions. Then,

$$\begin{aligned} x_1 &= v(N) - x_2 - x_3 && \text{by the efficiency condition} \\ &\leq 1.0000 - 0.9933 && \text{by the stability condition: } x_2 + x_3 \geq v(\{2,3\}) \\ &= 0.0067 \end{aligned}$$

Analogously, we can derive that $x_2 \leq 0.0067$ and $x_3 \leq 0.0067$. This implies that $x_1 + x_2 + x_3 \leq 0.0201 < 1.0000 = v(N)$. Hence, there is no allocation that satisfies both the efficiency and the stability conditions. And thus, the core is empty. △

The intuition behind an empty core is related to the convex-concave shape of the return function. This shape establishes the existence of a *sweet spot* that strikes the right balance between the increasing and decreasing returns to scale. In Figure 1, this sweet spot is somewhere around $f = 0.65$. Having more resources does hardly increase the return, but having less resources will result in a big loss. Resources are deployed in the most effective way around this sweet spot. If the total amount of resources is such that this sweet spot is not reached in the grand coalition, then $v(N)$ suffers from loss in effectiveness. This is what we see in Example 2 where some resources are not used in the most effective way in the grand coalition. In those cases, it is likely that a smaller coalition of players can use their own resources more effectively. In Example 2, this applies to any coalition of two players. These coalitions of players will only join the grand coalition, if they are compensated for their loss in effectiveness. However, there might be no players willing to pay for this compensation, such as in the above example. This leads to an empty core.

4 Market allocations

Core allocations satisfy a set of clear conditions, but their actual interpretation can be difficult. We therefore propose another type of allocations, so-called *market allocations*, that have a clear and intuitive structure. In Section 4.1, we introduce our market allocations. We discuss the relation between market allocations and the core in Section 4.2. In Section 4.3, we present a theoretical result on games with two players by showing that for two player games the core is always non-empty and that all core allocations are also market allocations.

4.1 Introduction market allocations

We introduce a particular type of allocation, namely those with a market price. Let $\mathbf{f}^* = [f_i^*]_{i\in N}$ denote an optimal solution to Problem (1) for the grand coalition N. Then, any allocation $\mathbf{y} \in \mathbb{R}^N$ can be written as

$$y_i = M_i F_i\left(f_i^*\right) + p_i\left(r_i - f_i^* M_i\right) \quad \forall i \in N. \tag{3}$$

Above allocation can be interpreted as follows. All players cooperate and determine the best possible division of all resources, i.e., $\mathbf{f}^*$. Each of them obtains a certain return from the resources that he gets in this division. This return corresponds to $M_i F(f_i^*)$ for every player $i \in N$. Some players end up with less resources than they initially had and others with more. To compensate for the loss of resources, players receive some money. At the same time, players that have received more resources have to pay for the extra resources. From the allocation $\mathbf{y}$ we can determine a price p_i per resource bought/sold for every player i. We call an allocation $\mathbf{y}$ a *market allocation* if $p_i = p_j$ for all $i, j \in N$. That implies that there exists a single price p, called the *market price*. All players either sell or buy resources for this market price. Such a market price is likely to enhance cooperation in practice, because it prevents having dissatisfied players who found out that other players have bought (sold) resources for a lower (higher) price.

4.2 Market allocations and the core

Although market allocations have a nice and clear interpretation, their relation to the core is not always clear. For special types of games, so-called 'market games' the core is always non-empty and there is always a market allocation in the core (see our discussion in Section 2), but this does not hold for the game that we consider in this paper. For our game, the core can be empty. Nevertheless, market allocations can be constructed for any situation and thus market allocations always exist, even if the core is empty. Conversely, it is also possible to have a non-empty core that does not contain a market allocation. This is illustrated with the following example. Again, we determine the value function via discretized enumeration and present the numerical values rounded to four decimal places.

Example 3: Consider a situation with three players: $M_i = 1$ for $i = 1, 2, 3$ and $\mathbf{r} = [0.1, 1, 0.65]$. Let $F_i(f) = \frac{1}{1+\exp\{-45f+25\}}$, for $i = 1, 2, 3$. The corresponding value function is as follows:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	1.0000	0.9859	1.0000	0.9998	2.0000	2.3319

One can easily verify that following allocation $\mathbf{x} = [0.1319, 1.1, 1.1]$ is in the core and thus the core is non-empty. Although the core is non-empty, we will show that there is no market allocation that belongs to the core. By contradiction, suppose there exists a market allocation $\mathbf{y}$ with market price p that belongs to the core. Via enumeration we can determine that the optimal distribution of resources for the grand coalition is as follows: $f_i^* = \frac{1.75}{3} = 0.5833$, with $F_i(f_i^*) = 0.7773$ for $i = 1, 2, 3$. To move towards the optimal distribution $\mathbf{f}^*$, player 1 has to buy some resources and player 2 and 3 have to sell some of their resources. By stability of allocation $\mathbf{y}$, the following can be derived for the market price that players 1 and 3 are willing to pay/receive:

$$y_1 = M_1 F_1(f_1^*) + p(r_1 - f_1^* M_1) \geq v(\{1\}) \quad \Leftrightarrow \quad p \leq \frac{0.7773 - 0}{0.5833 - 0.1} = 1.6083$$
$$y_3 = M_3 F_3(f_3^*) + p(r_3 - f_3^* M_1) \geq v(\{3\}) \quad \Leftrightarrow \quad p \geq \frac{0.9859 - 0.7773}{0.65 - 0.5833} = 3.1274$$

Hence, player 1 is willing to pay a price per resource at most equal to 1.6083. However, player 3 wants to receive at least 3.1274 per resource. This implies that there is no market price that satisfies the individual rationality condition of both player 1 and 3. Hence, there is no market allocation $\mathbf{y}$ that is in the core, even though the core is non-empty. $\triangle$

The fact that there is no market price in Example 3 is again caused by the convex-concave shape of the return function. Player 3 is initially at the sweet spot and by selling some resources he looses a lot in effectiveness. Since player 1 is the only player that buys resources, he needs to pay a high price to compensate player 3. However, in case of a market price, player 1 must also pay the same high price for the resources he buys from player 2. Player 1 is not willing to do so and therefore there is no market allocation in the core.

The fact that the core is non-empty can be explained as follows. We can use equation (3) to determine the prices p_1, p_2, p_3 that follow from the core allocation $\mathbf{x} = [0.1319, 1.1, 1.1]$. We find that $p_1 = 1.3354$, $p_2 = 0.7744$ and $p_3 = 4.8381$. Note that player 2 and 3 are sellers, so they sell their resources at different prices. Player 1 is the only buyer: he buys 1-0.5833=0.4167 resources from player 2 for a price p_2 per resource and he buys 0.65-0.5833 =0.0667 resources from player 3 for a price p_3 per resource. On average he pays the price p_1 per resource. We see that player 1 buys a few resources for a very high price and most of the resources for a low price, resulting in an acceptable average price for this player. In this way, all players are willing to cooperate with each other.

In general, Example 3 shows that the requirement of a single market price can be quite restrictive. There can be many allocations belonging to the core that correspond to individual prices for the players.

4.3 Two players

In the previous sections, we have seen that the core can be empty and that a market allocation that belongs to the core does not necessarily exist. In this section, we provide an interesting result for games with two players. These games have the elegant structure that one player gives (part of) his resources to the other player. Thus, there is one 'buyer' (denoted with the letter b) and one 'seller' (denoted with the letter s). The following theorem shows that in case of two players the core is non-empty. Moreover, all core allocations are also market allocations. The proof of this theorem can be found in Appendix A.

Theorem 1. *In the case of two players (i.e., $|N| = 2$) the core is equal to the set of all market allocations with a market price*

$$p \in \left[\frac{M_s \left[F_s\left(\frac{r_s}{M_s}\right) - F_s(f_s^*) \right]}{r_s - M_s f_s^*}, \frac{M_b \left[F_b(f_b^*) - F_b\left(\frac{r_b}{M_b}\right) \right]}{M_b f_b^* - r_b} \right].$$

Theorem 1 can intuitively be explained as follows. The market price compensates the seller for selling his resources. Therefore, this price must be high enough for the seller and at the same time not too high for the buyer. Such a price exists, because by redistributing the resources the players can achieve at least the same total return as on their own. For two players, all core allocations are market price allocations, because one player buys resources from the other. There are no other players involved, and hence the compensation that the seller receives for giving away resources is completely paid by the buyer.

5 Identical return functions

In the previous section, we have seen that the core is always non-empty for games with two players, but that for more than two players the core can be empty. In this section, we study conditions for games with $|N|$ players under which the core is non-empty and a market allocation is in the core. To do so, we consider a class of games where all players have the same return function: i.e., $F_i(\cdot) = F(\cdot)$ for all $i \in N$.

Identical return functions can be motivated as follows. Consider the example of vaccine allocation, then every player i corresponds to a population consisting of a certain number of individuals M_i. The return function $F_i(f)$ then measures per individual the monetary health benefits of vaccinating a fraction f of the population. If all regions face an outbreak of the same infectious disease that spreads through the population in a similar way, then we can assume that these regions respond similarly to vaccination and that their return functions are identical (L. E. Duijzer, van Jaarsveld, Wallinga, & Dekker, 2018; Keeling & Shattock, 2012).

We start in Section 5.1 with analyzing the characteristics of the value function. Based on these characteristics, we derive sufficient conditions for the core to be non-empty in Section 5.2. We do so by providing a market allocation that is in the core.

5.1 Analysis of the value function

To analyze this game, we first analyze the value function. The following lemma shows that any coalition of players can always use all the resources they have. This lemma follows directly from the fact that the functions $F_i(\cdot)$ are non-negative and non-decreasing (see Assumption 1).

Lemma 1. *For every coalition $S \subseteq N$, there always exists an optimal solution $\mathbf{f}^* \in \mathbb{R}^S$ to Problem (1) for which $\sum_{i \in S} f_i^* M_i = \sum_{i \in S} r_i$.*

To investigate in what way the players will divide their resources when they cooperate, we introduce the following concept. Let the function $D_F(\cdot)$ measure the additional return per resource for a player with return function $F(\cdot)$ (cf., L. E. Duijzer, van Jaarsveld, Wallinga, & Dekker, 2018). The additional return is the return that is obtained from the resources compared to having no resources at all. In many applications having no resources also implies no return (i.e., $F(0) = 0$), but this need not always be the case. For example, in case nobody is vaccinated during an epidemic there might still be health benefits related to individuals escaping infection the by chance. The function $D_F(\cdot)$ is defined as follows:

$$D_F(f) = \frac{1}{f}\left[F(f) - F(0)\right] \qquad \text{for all } f > 0 \tag{4}$$

Note that $D_F(f)$ can also be interpreted as the average slope of the return function $F(\cdot)$ on the interval $[0, f]$. We derive the following result, which follows from the characteristics of the function $F(\cdot)$ in Assumption 1.

Corollary 1. *The function $D_F(\cdot)$ is maximized by the unique fraction of resources $\tilde{f}_F$ for which $F'(\tilde{f}_F) = D_F(\tilde{f}_F)$. Moreover, $D_F(\cdot)$ is increasing on the interval $[0, \tilde{f}_F)$ and decreasing on the interval $(\tilde{f}_F, \infty)$.*

Corollary 1 is illustrated in Figure 2. This figure shows that there is a unique fraction of resources, $\tilde{f}_F$, for which the additional return per resource is the highest. This fraction is unique because of the nicely convex-concave shape of the function $F(\cdot)$. The resource fraction $\tilde{f}_F$ can be interpreted as the sweet spot that strikes the right balance between the convex and concave part of the return function.

The fraction $\tilde{f}_F$ plays an important role in the optimal solution to Problem (1). This is illustrated in the following theorem, which is based on Theorem 5 of L. E. Duijzer, van Jaarsveld, Wallinga, and Dekker (2018). From now on, we denote the total amount of resources for a coalition of players $S \subseteq N$ by $R(S) = \sum_{i \in S} r_i$.

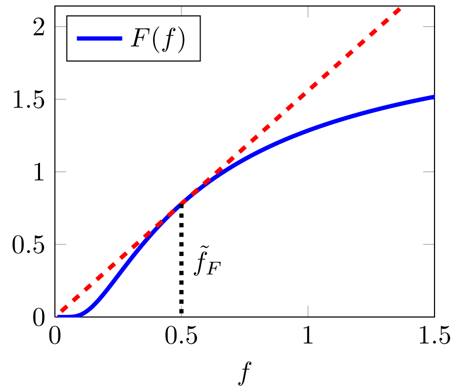

Figure 2: This figure illustrates the existence and uniqueness of $\tilde{f}_F$ in the following S-shaped return function: $F(f) = \exp\{0.75 - 0.5/f\}$.

Theorem 2. *In the case of identical return functions ($F_i(\cdot) = F(\cdot)$ for all $i \in N$) the optimal solution $\mathbf{f}^* \in \mathbb{R}^S$ for every coalition $S \subseteq N$ to Problem* (1) *for particular cases is as follows:*

(a). If $R(S) < \tilde{f}_F \min_{i\in S}\{M_i\}$, then it is best to give all resources to the player with the smallest size. Let $k \in \arg\min_{i\in S}\{M_i\}$, then the optimal solution is given by $f_k^* = \frac{R(S)}{M_k}$ and $f_j^* = 0$ for $j \in S\backslash\{k\}$.

(b). If $R(S) = \sum_{j\in K} \tilde{f}_F M_j$ for some $K \subseteq S$, then it is best to divide the resources pro rata among the players in K. The optimal solution is given by $f_j^* = \frac{R(S)}{\sum_{i\in K} M_i} = \tilde{f}_F$ for $j \in K$ and $f_j^* = 0$ for $j \in S\backslash\{K\}$.

(c). If $R(S) > \sum_{j\in S} \tilde{f}_F M_j$, then it is best to divide all resources pro rata over all the players. The optimal solution is given by $f_j^* = \frac{R(S)}{\sum_{i\in S} M_i}$ for all $j \in S$.

Above theorem shows that in order to make the best possible use of the return functions, the optimal solution tries to be close to $\tilde{f}_F$ for (a subset of) the players. If it is possible to give a subset of players exactly the resource fraction $\tilde{f}_F$, then this is optimal (case (b)). If the fraction $\tilde{f}_F$ cannot be attained for any of the players, then it is best to give all resources to the player with the smallest size (case (a)). In that case, there are very few resources and one can benefit the most from the convex part of the return function by setting the resource fraction as high as possible. This can be achieved by prioritizing the smallest player, because the resource fraction $f_i = R(N)/M_i$ is higher for a player i with a small size M_i. When there are many resources, it is best to divide them pro rata over all players (case (c)).

5.2 Core allocations

In this section, we derive sufficient conditions under which the core is non-empty. We show that in those cases there is a market allocation in the core. That means that in those cases, cooperation with all players is plausible and the total return is divided in such a way that all players buy and sell resources for the same price. We provide analytical expressions for these market prices and compare these prices to a simple price that could be used as a rule of thumb. This simple price is based on the intuition that the price of a resource is equal to the total additional return divided by the total amount of resources. To formalize this intuition, we introduce $A(N)$ as the average additional return per resource when all players in N cooperate.

$$A(N) = \frac{v(N) - \sum_{i \in N} M_i F_i(0)}{R(N)} \tag{5}$$

Observe that the numerator of above expression represents the total additional return that is gained compared to having no resources at all. This total additional return is divided by the total amount of resources.

Let us now analyze the cases for which we can show that a market allocation is in the core. To do so, we use the three cases of Theorem 2 for which the value function can easily be determined. The proofs of the theorems in this section can be found in Appendix A. The following theorem considers the case of scarcity, i.e, when there are very few resources in total (case (a) of Theorem 2).

Theorem 3. *Consider identical return functions ($F_i(\cdot) = F(\cdot)$ for all $i \in N$) and let $k \in \arg\min_{i \in N}\{M_i\}$. If $R(N) < \tilde{f}_F M_k$, then the market allocation with price $D_F\left(\frac{R(N)}{M_k}\right)$ is in the core. This implies that the core is non-empty.*

Theorem 3 can intuitively be explained as follows: when resources are scarce, the highest possible return is achieved when all resources are given to the player with the smallest size. This results in an additional return of $D_F\left(\frac{R(N)}{M_k}\right)$ per resource. Every coalition of players can obtain no more than this, because of the limited amount of resources and the convex part of the return function. Therefore, every coalition of players is happy if they can receive an additional return of $D_F\left(\frac{R(N)}{M_k}\right)$ per resource. This corresponds to the market price of the proposed allocation. Moreover, one can verify that $D_F\left(\frac{R(N)}{M_k}\right) = A(N)$ for the situation described in Theorem 3.

The core is not only non-empty in case of scarcity, but also when the total amount of resources is such that we can exactly provide a subset of players with the most efficient fraction of resources $\tilde{f}_F$ (case (b) of Theorem 2).

Theorem 4. *Consider identical return functions ($F_i(\cdot) = F(\cdot)$ for all $i \in N$). If $R(N) = \tilde{f}_F \sum_{i \in K} M_i$ for some $K \subseteq N$, then the market allocation with price $D_F\left(\tilde{f}_F\right)$ is in the core. This implies that the core is non-empty.*

To prove this theorem, we make use of duality for convex non-linear optimization problems. Deriving core allocations using duality theory is often done for linear optimization problems (Deng, Ibaraki, & Nagamochi, 1999; Owen, 1975), we extend this approach to non-linear problems.

To interpret Theorem 4, recall that the additional gain per resource, $D_F(\cdot)$, is maximized for $\tilde{f}_F$. Hence, any subset of players can never obtain an additional gain per resource higher than $D_F\left(\tilde{f}_F\right)$. The proposed allocation gives every player $i \in N$ exactly $D_F\left(\tilde{f}_F\right)$ per resource for his initial amount of resources r_i. This implies that no player can be better off on his own or in a coalition. Therefore, the market allocation of Theorem 4 is in the core.

This can be also seen in Figure 2. In this figure, the dashed line can be interpreted as the relation between the initial amount of resources r_i/M_i (on the horizontal axis) for player i which has $f_i^* = \tilde{f}_F$ and the market allocation with a market price $D_F\left(\tilde{f}_F\right)$ (on the vertical axis). The difference between the dashed line and the S-curve can be interpreted as the willingness of player i to pay/receive the price $D_F\left(\tilde{f}_F\right)$. Since, the dashed line lies above the S-curve, this player is always willing to sell/buy resources for this price p until he has $\tilde{f}_F M_i$ resources.

Similar to the case of scarcity, we can show that for any situation satisfying the conditions of Theorem 4 it holds that $D_F\left(\tilde{f}_F\right) = A(N)$.

Finally, we can also show non-emptiness of the core when there is an oversupply of resources (case (c) of Theorem 2). The following theorem presents a core-allocation:

Theorem 5. *Consider identical return functions ($F_i(\cdot) = F(\cdot)$ for all $i \in N$). If $R(N) \geq \tilde{f}_F \sum_{i \in N} M_i$, then the allocation with the market price $F'\left(\frac{R(N)}{\sum_{i \in N} M_i}\right)$ is in the core. This implies that the core is non-empty.*

We can intuitively explain Theorem 5 as follows. By Theorem 2, we know that all players receive a fraction of resources equal to $f^* = \frac{R(N)}{\sum_{i \in N} M_i}$ when the total amount of resources is high enough. This implies that the market price of Theorem 5 is equal to $F'(f^*)$. When the price is such, no player is willing to deviate from the optimal resource distribution $\mathbf{f}^*$ by buying or selling resources. If the price would be higher than $F'(f^*)$, players would rather sell some more resources. The money they would get by selling some resources, is higher than the return they can obtain by keeping them. Similarly, with a price lower than $F'(f^*)$ players would rather buy some more resources. Hence, when the price is exactly equal to $F'(f^*)$ no player is willing to deviate from the optimal resource distribution.

When there are many resources, the value of a resource will go down. Players with many resources are therefore willing to sell some of their resources for a relatively low price. Rewriting the market price of Theorem 5 gives the following:

$$F'\left(\frac{R(N)}{\sum_{j\in N} M_j}\right) \leq D_F\left(\frac{R(N)}{\sum_{j\in N} M_j}\right) = A(N)$$

The inequality follows from the fact that $D_F(f) \geq F'(f)$ for $f \geq \tilde{f}_F$ (Corollary 1). Above analysis shows that in this case the market price is smaller than the average additional return per resource $A(N)$. This contradicts the result of Theorems 3 and 4, where the market price is equal to the $A(N)$. To explain this difference, we recall that the function $D_F(f)$ measures the additional return per resource for a player. This additional return is increasing for $f \leq \tilde{f}_F$ and decreasing for $f > \tilde{f}_F$.

Both in Theorem 3 and 4 the total amount of resources is such that in the optimal division of resources, all players will receive an amount of resources in the region of increasing additional returns. This implies that any player who buys resources, will obtain a higher return per resource from the resources he bought than from the resources he initially had. Since both the initial resources and the bought resources contribute to $A(N)$, this player is willing to pay the price $A(N)$ for the resources he buys.

In contrast, in Theorem 5 there is such an abundance of resources, that in the optimal division of resources all players receive an amount of resources in the region of decreasing additional returns. This implies that a player who buys resources will potentially have a higher return per resource for his initial resources, than for the resources he buys. The return per resource for the bought resources might be even lower than the average return $A(N)$. Therefore, this player might only be willing to pay a price less than $A(N)$ for the resources he buys.

6 General case

In the previous section, we have identified situations for which we showed non-emptiness of the core by providing a market allocation that belongs to the core. That implies that for those specific situations, cooperation is plausible and the benefits achieved through cooperation can be allocated in a fair way via buying and selling all resources for a market price. In this section, we study the general case without assuming identical return functions or restrictions on the total amount of resources $R(N)$. We investigate for this general case when cooperation is plausible by analyzing the non-emptiness of the core in Section 6.1. In Section 6.2, we derive expressions for market price candidates of which we expect that the corresponding market allocation is in the core. To evaluate our proposed market allocations, we compare them to a number of benchmark allocations. The numerical results on the performance of the different allocations can be found in Section 6.3.

6.1 Non-emptiness of the core

In this section, we study the possibilities of cooperation. Cooperation is plausible when the core is non-empty, but it becomes complicated in case of an empty core. The conditions that describe the core depend on the value function of our cooperative game. Therefore, in order to be able to determine whether the core is empty or not we first need to determine the value function for every coalition. In many cases, however, the value function is difficult to determine, because of the NP-hardness of Problem (1). This is particularly true when the players do not have identical return functions, because then the results of Theorem 2 no longer hold. To overcome this problem, we propose to approximate the value function by means of an approximation for Problem (1). In literature, there are more studies that approximate the value function when this function is difficult to determine (cf., Goemans & Skutella, 2004; Zeng, Zhang, Cai, & Li, 2018).

In contract to the small examples that we discussed before, it is no longer possible to approximate the value function with complete enumeration. Therefore, from now on we use the heuristic proposed by L. E. Duijzer, van Jaarsveld, Wallinga, and Dekker (2018), which is shown to perform close to optimal for a similar optimization problem. We refer to Appendix B.1 for a complete description of our approximation. We use $\bar{v}(S)$ to denote the approximation of the value function for every coalition $S \subseteq N$.

When we talk about the core, we technically mean the core of the game with the approximated value function. Moreover, because we approximate the value function, we have to be careful with numerical precision. Therefore, we introduce a slack parameter ϵ to account for small deviations. We set $\epsilon = 10^{-5}$ and we consider the ϵ-core: all allocations $\mathbf{x}$ that satisfy the efficiency condition ($\sum_{i \in N} x_i = \bar{v}(N)$) and the stability conditions ($\sum_{i \in S} x_i \geq \bar{v}(S) - \epsilon$ for all $S \subset N$) determined with the approximated value functions $\bar{v}(\cdot)$. For the reader's convenience, we will simply refer to the core instead of the ϵ-core.

To analyze the non-emptiness of the core, we randomly generate instances with five players. We analyze identical return functions (the same function for all players) and individual return functions (functions may differ between the players). We use the following two types of return functions that are commonly used to model a convex-concave shape and that both satisfy Assumption 1.

(i) $F_i(f) = \exp\{a_i - b_i/f\}$,

(ii) $F_i(f) = \frac{1}{1+\exp\{-a_i f + b_i\}}$.

Type (i) functions are also used by Ginsberg (1974) and type (ii) is the well-known logistic function. For return functions of type (i) we generate a_i and b_i randomly on the interval $[0,1]$ for all $i \in N$. For these functions, $\tilde{f}_{F_i} = b_i$. For type (ii) functions we randomly generate $a_i \in [25, 100]$ and $b_i \in [0, a_i]$ for all $i \in N$. This ensures that $\tilde{f}_{F_i}$ also lies in the interval [0,1] for almost all generated instances. The population sizes M_i are randomly generated on the interval [10,100]. The initial amount of resources for player i is randomly generated on the interval $\left[0, d\tilde{f}_{F_i} M_i\right]$, where we let $d \in \{0.05, 0.5, 1, 1.5, ..., 10\}$. The case that $d = 0$ is not interesting, since this would imply that none of the players have resources and thus $R(N) = 0$. A low value of d results in a game where players have relatively few resources. For high values of d there are many resources available.

Figure 3 shows how often the core is non-empty for different values of d. We can easily determine whether the core is empty or not by analyzing whether there is a feasible solution that satisfies the core conditions (see Appendix B.2 for a full description).

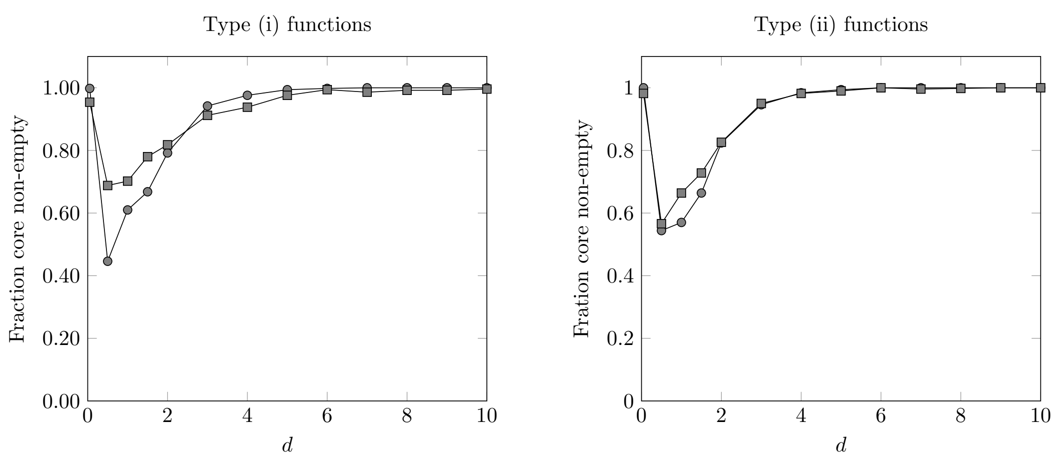

Figure 3: This figure presents the fraction of instances for which the core is non-empty. We take the average over 500 generated instances with five players and we analyze both identical return functions (—●—) and individual return functions (—■—).

Figure 3 shows that the core is non-empty in most of the cases. Only for moderate amounts of resource ($d \in [0.5, 3]$) the core is empty more often. For five players the fraction of instances with an empty core is never below 0.40. Additional results, not reported here, show that this fraction is decreasing in the number of players and that it drops to around 0.10 for $d = 0.5$ in case of ten players and individual type (ii) return functions. When there are very few or many resources, the core is (almost) never empty. This can be explained as follows. When there are very few resources, the return of individual players and small coalitions is almost zero (cf., Example 1). Only when all players cooperate and all resources can be given to one player, increased benefits can be expected. This implies that any coalition of players is already satisfied with a small share of the total return, which makes it easy to find a core allocation. In case of many resources, there are (almost) always some players that initially have many resources of which some are worth almost nothing to them. The players with fewer resources can then buy resources from these players for relatively low prices. This likely results in a non-empty core.

We can thus conclude that cooperation is plausible when there are almost no or many resources, but that one must be careful when the total amount of resources is moderate. In practice however, the case of relative scarcity might be the most interesting case to cooperate. When resources are relatively scarce, players are dependent on each other and they can expect to increase the total return through cooperation. Yet the results in this section show that, particularly in case of relative scarcity, finding a fair way to divide the total return among the players can be difficult and might not even be possible.

6.2 Market price allocations

When the core is non-empty, we would like to find a fair division of the total return. We have seen that market allocations are good candidates for fair division for some special cases considered in Section 5.2. In this section, we propose market allocations that might be useful in general.

A non-empty core does not have to imply that there are also market allocations in the core as we have seen in Example 3. Therefore, we first analyze how often there are market allocations in the core. We determine this by checking whether there are core allocations that satisfy equation (3) with $p_i = p_j$ for all $i, j \in N$ (see Appendix B.2 for a full description). We disregard the instances with an empty core. The results can be found in Figure 4.

The pattern in Figure 4 is comparable to the pattern in Figure 3, with always a market allocation in the core for very few resources or many resources. For limited amounts of resources with $d \in [0.5, 3]$, the fraction of instances with a market allocation in the core is lower. For those moderate amounts of resources, the total return that coalitions can obtain can vary substantially. For some players it is beneficial to cooperate,

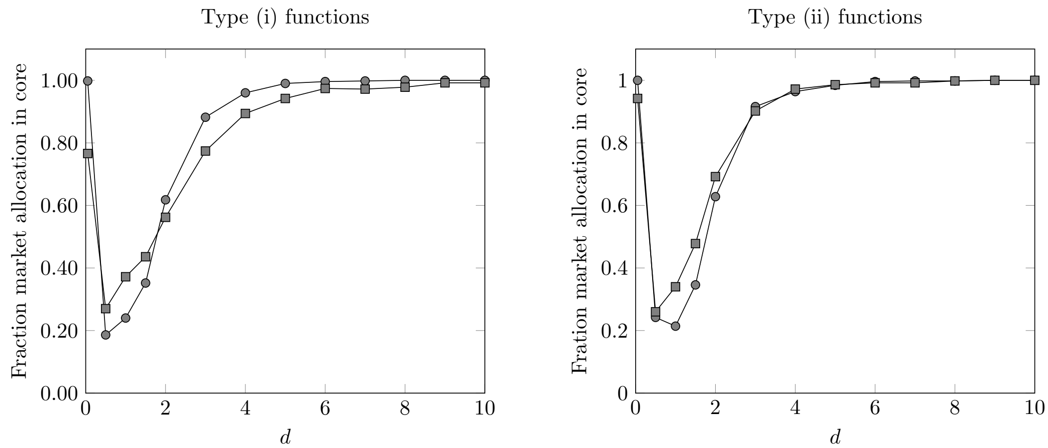

Figure 4: This figure presents the fraction of instances with five players for which there exists a market price allocation in the core. We take the average over 500 generated instances and we analyze both identical return functions (—●—) and individual return functions (—■—).

whereas others can hardly improve their return. This implies that in the grand coalition, some players are much better off, but others could have achieved also a high return in a smaller coalition. Hence, the players assign a different value to the resources that they receive in the grand coalition and a market price does not always exists.

Additional experiments, not reported here, show that the pattern in the curves in Figure 4 is similar for instances with more players. For $d \leq 0.05$ and $d > 5$ this fraction is not affected by the number of players and remains approximately equal to 1. However, for $d \in [0.5, 3]$ the fraction for which there is a market allocation in the core is decreasing in the number of players. This fraction even drops below 0.05 for $d = 0.5$ and instances with ten players and individual type (ii) return functions. To conclude, when the core is often empty, cooperation is complicated and a market allocation does not always exist in the core. But for settings in which the core is often non-empty, there is also likely to be a market allocation in the core.

It is therefore worthwhile to derive analytical expressions of market prices of which the corresponding market allocation is likely to be in the core. In order to do so, we make use of our theoretical results in Section 5.2. The market prices that result in core allocations in Theorems 3 and 4 are both equal to $A(N)$. Let us analyze the expression for $A(N)$ in more detail for the case of individual return functions. By equation (5):

$$
\begin{aligned}
A(N) = \frac{v(N) - \sum_{i\in N} M_i F_i(0)}{R(N)} &= \frac{1}{R(N)}\left[\sum_{i\in N} M_i \left(F_i(f_i^*) - F_i(0)\right)\right] \\
&= \frac{1}{R(N)}\left[\sum_{i\in N} M_i f_i^* D_{F_i}(f_i^*)\right]
\end{aligned}
\tag{6}
$$

Above analysis shows that $A(N)$ is a weighted average of the functions $D_{F_i}(\cdot)$ evaluated at the optimal solution $\mathbf{f}^*$. Under the assumption of identical return functions and under the conditions on $R(N)$ of Theorems 3 and 4 the expression for $A(N)$ can be simplified substantially and rewritten as follows: $A(N) = D_F(f_j^*)$ where $j \in N$ is such that $f_j^* > 0$. This result follows directly from the optimal solution $\mathbf{f}^*$ given in Theorem 2. The simplification indicates that it is worthwhile to consider other candidate market prices next to $A(N)$ that are also functions of $D_{F_i}(f_i^*)$ for all $i \in N$ for situations where the return functions are not identical.

Since we use an approximation of the value function, we do not know $\mathbf{f}^*$. We therefore use our heuristic solution to construct the market allocations. We propose four market prices, that bring about market allocations. A market allocation with price p has the following structure, where $\mathbf{g}^* \in \mathbb{R}^N$ denotes the heuristic solution for the grand coalition:

$$x_i = M_i F_i(g_i^*) + p(r_i - g_i^* M_i) \quad \forall i \in N.$$

We use the following indices to describe our market prices. Let $k \in \arg\min\{D_{F_j}(g_j^*) | g_j^* > 0, j \in N\}$, which always exists for $R(N) > 0$. Let $l \in \arg\min\{D_{F_j}(g_j^*) | g_j^* \geq \tilde{f}_{F_j}, j \in N\}$. The index l only exists if there are enough resources such that there is a player $j \in N$ for which $g_j^* \geq \tilde{f}_{F_j}$. We propose the following four market prices:

1. $p_1 = \frac{\sum_{i\in N} M_i F_i(g_i^*) - \sum_{i\in N} M_i F_i(0)}{R(N)}$,
2. $p_2 = D_{F_k}(g_k^*)$,
3. $p_3 = \begin{cases} D_{F_l}(g_l^*) & \text{if a player } l \text{ exists} \\ D_{F_k}(g_k^*) & \text{otherwise} \end{cases}$,
4. $p_4 = \begin{cases} F_l'(g_l^*) & \text{if a player } l \text{ exists} \\ F_k'(g_k^*) & \text{otherwise} \end{cases}$.

The first market price p_1 is equal to the weighted average of the functions $D_{F_i}(g_i^*)$ for all $i \in N$. This price follows the same structure as $A(N)$. For the second and third market price, we use one of the smaller values of $D_{F_i}(g_i^*)$ that contribute to the weighted average. Intuitively, these smaller values are good candidates for core allocations, because of the following. Players for which $g_i^* > 0$ have probably bought resources and they have often obtained a higher average revenue per resource than p_2 or p_3. Similarly, players for which $g_i^* = 0$ have sold their resources. For them a price p_2 or p_3 is such that the money they get from selling is higher than the return they would have obtained by keeping their resources.

Market price p_1 corresponds to the theoretical market price of Theorems 3 and 4 and is related to $A(N)$. In Theorem 5 we studied the case of many resources and derived that the market allocation in the core has a market price which is lower than $A(N)$. We therefore also propose a market price p_4 based on this result. The market allocation corresponding to p_4 is expected to perform well for instances with many resources.

6.3 Numerical results

To evaluate our proposed market allocations, we compare their performance with three proportional allocations as a benchmark. For both the market allocations and the proportional allocations we use the heuristic to determine the return that a coalition of players can achieve. In a proportional allocation, the total return $\bar{v}(N)$, determined with the heuristic, is divided proportionally among the players with respect to a certain player specific characteristic z_i:

$$x_i = \bar{v}(N)\left(\frac{z_i}{\sum_{j\in N} z_j}\right) \quad \forall i \in N.$$

We use the following benchmark allocations:

1. the proportional allocation based on size (i.e., pro rata): $z_i = M_i$ for all $i \in N$,
2. the proportional allocation based on initial resources: $z_i = r_i$ for all $i \in N$,
3. the proportional allocation based on initial return: $z_i = M_i F_i\left(\frac{r_i}{M_i}\right)$ for all $i \in N$.

We analyze how often an allocation lies in the core using the same settings as in Section 6.1 to generate instances. Figure 5 presents the results for $d \in \{0.05, 0.5, 1, ..., 4\}$. We cannot include $d = 0$, because several of our proposed market prices do not exists for $d = 0$.

We observe in Figure 5 that the total amount of resources, implicitly measured by d, has a big influence on the performance of the allocations. For very small amounts of resources ($d = 0.05$) the market allocations are in the core for almost all instances and cooperation can easily be achieved.

When resources are limited, but not very scarce ($d \in [0.5, 1.5]$), it heavily depends on the problem instance whether the proposed allocations are in the core. Although the market allocations are slightly more often in the core than the proportional allocations, most of the time none of the compared allocations is in the core.

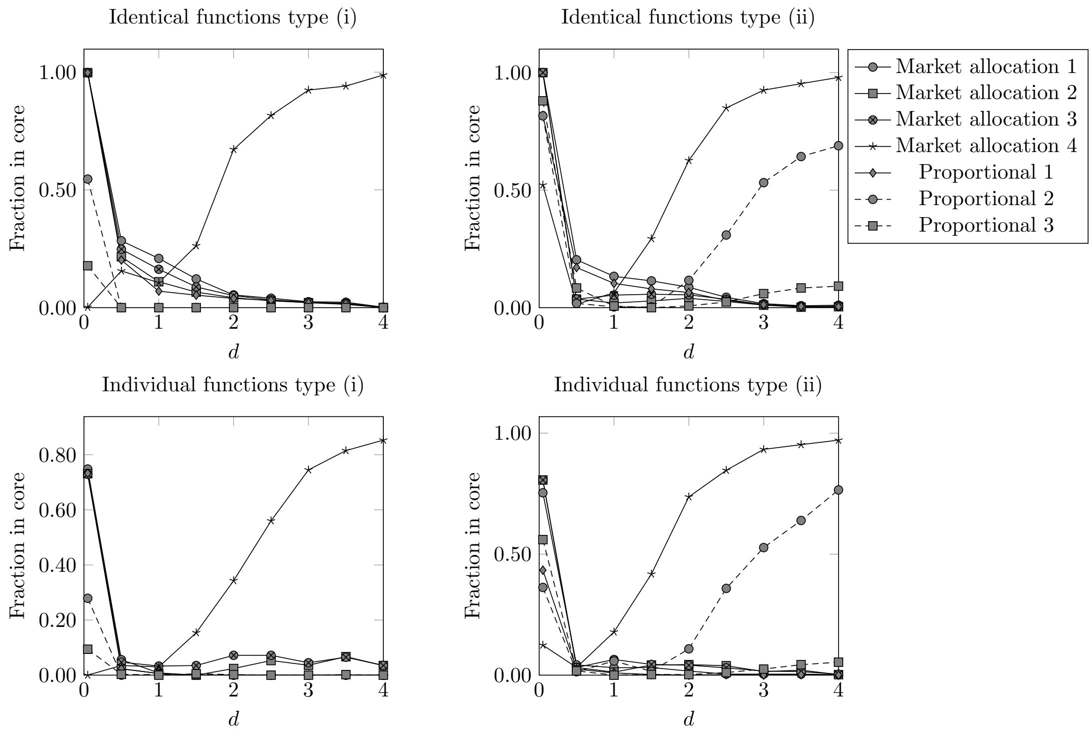

Figure 5: This figure presents the average proportion of instances for which the allocations belong to the core for various values of d. The average is taken over the number of instances for which the core was non-empty out of 500 generated instances.

This happens for example in more than 80% of the instances with a non-empty core for $d = 1$ and individual return functions of type (ii).

For large amounts of resources ($d > 2$) market allocation 4 performs the best. This market allocation corresponds to the theoretical results on large amounts of resources in Theorem 5. The good performance of market allocation 4 was therefore to be expected for identical return functions, but we also see this behavior for individual return functions of both type (i) and (ii).

For return functions of type (ii) also proportional allocation 2 performs well when there are many resources available. This proportional allocation divides the total return proportional to the initial amount of resources. Indeed, for return functions of type (ii) the initial amount of resources of a player provides a good proxy for the players contribution to a coalition. The parameters of the return function have a smaller influence on the contribution of a player to a coalition, because return functions of type (ii) level off for high amounts of resources.

Thus, we can conclude that when cooperation is plausible, market allocations often result in a fair allocation of the total return. When resources are scarce, market allocation 1 (corresponding to market price $A(N)$) is often in the core. When there are many resources, market allocation 4 is the best candidate. Cooperation is more complicated for moderate amounts of resources, even though market allocations still perform better than proportional allocations.

7 Case Study

In this section, we apply our results to a case study. We return to the application of vaccination that we discussed in the first sections of this paper. We describe the case in Section 7.1 and present our results in Section 7.2.

7.1 Case description

In the United States, the Centers for Disease Control and Prevention (CDC) is responsible for allocating influenza vaccines during an influenza epidemic. The policy is to allocate vaccines to geographical regions in proportion to their population size (Centers for Disease Control and Prevention, 2009a). In this section, we will investigate if, under such an initial distribution of vaccines, there is potential for collaboration during a influenza epidemic (e.g., by redistribution some of the vaccines). For that, we use the cooperative game as introduced in Section 3.

To model the influenza epidemic, we use the SIR model, which is a seminal model in epidemiology (Kermack & McKendrick, 1927). The US population is divided into the ten regions defined by Teytelman and Larson (2013), who study vaccine allocation during the 2009 H1N1 epidemic in the United States. We use the disease parameters of this epidemic as provided by Teytelman and Larson (2013). The regions and the corresponding number of inhabitants are presented in Table 1. The total population size equals 298,106,893 individuals. For a complete overview of which states belong to which region, we refer to Teytelman and Larson (2013). In contrast to Teytelman and Larson (2013), we assume a single vaccination moment.

To obtain good estimates of the value function, we use a realistic return function $F_i(\cdot)$ for every region i based on epidemiological models and disease parameters from the literature. The return function $F_i(\cdot)$ measures the fraction of people that escapes infection in region i. In line with Teytelman and Larson (2013), we assume that the epidemic in every region is independent of the epidemic in other regions. This is a reasonable assumption, because the interaction of individuals within a region is much higher than between

Region	**Size** (M_i)	**Region**	**Size** (M_i)
1. New England	14,429,720	6. Southwest	37,860,549
2. New York Area	19,949,192	7. The Plains	13,610,802
3. Mid-Atlantic	28,891,734	8. Rocky Mountains	10,787,806
4. Southeast	60,580,377	9. West	47,495,705
5. Great Lakes	51,766,882	10. Pacific Northwest	12,734,126

Table 1: Number of inhabitants of the ten regions in the United States (Teytelman & Larson, 2013).

regions. This implies that $F_i(\cdot)$ only depends on input variables related to region i. It has been shown that the function $F_i(\cdot)$ satisfies Assumption 1 if the epidemic is modeled with the SIR model (L. E. Duijzer, van Jaarsveld, Wallinga, & Dekker, 2018). We refer to Appendix C for a full characterization of this function and for an overview of the input parameters. This return function is more complex than the return functions studied before and has characteristics of both type (i) and type (ii) functions, which can be seen in Figure 6. The moderate increase resembles type (i) return functions, whereas the horizontal part matches the shape of type (ii) functions.

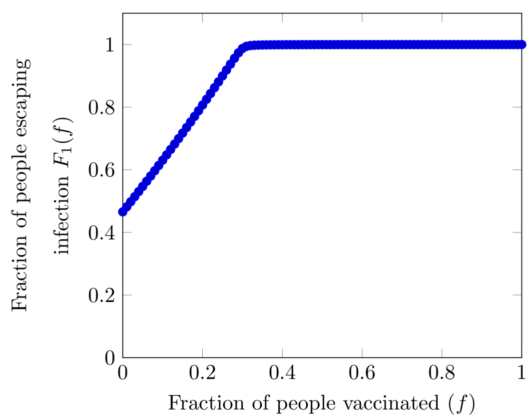

Figure 6: The return function $F_1(\cdot)$ for region 1 when vaccination takes place directly at the start of the epidemic. The full characterization of the function and the input parameters can be found in Appendix C.

The fraction of people that escapes infection can be translated to health benefits by multiplying with a factor that accounts for the average health benefits per individual spared from infection. Here, we assume that this factor is the same for every region, which is reasonable because all regions correspond to the same country. This implies that we can maximize the total number of people that escapes infection instead of the monetary health benefits.

7.2 Case results

In this section, we analyze cooperation between the regions. We vary the total amount of available vaccines and the moment of vaccination to analyze in which cases the core is non-empty. We also study whether there is a market allocation in the core.

Let V denote the total amount of vaccines of all regions together and let τ denote the moment of vaccination in days after the epidemic has started at time 0. Note that $V = R(N)$. In line with the policy of the CDC, we assume that the V available vaccines are initially distributed pro rata over the regions. We assume that the moment of vaccination is the same for every region. We let $\tau \in \{0, 5, 10, 15, 20, ..., 50\}$ and $V \in \{0, 0.5 \cdot 10^7, 1 \cdot 10^6, ..., 3 \cdot 10^8\}$. Figure 7 shows for every combination of τ and V whether the core is empty or not.

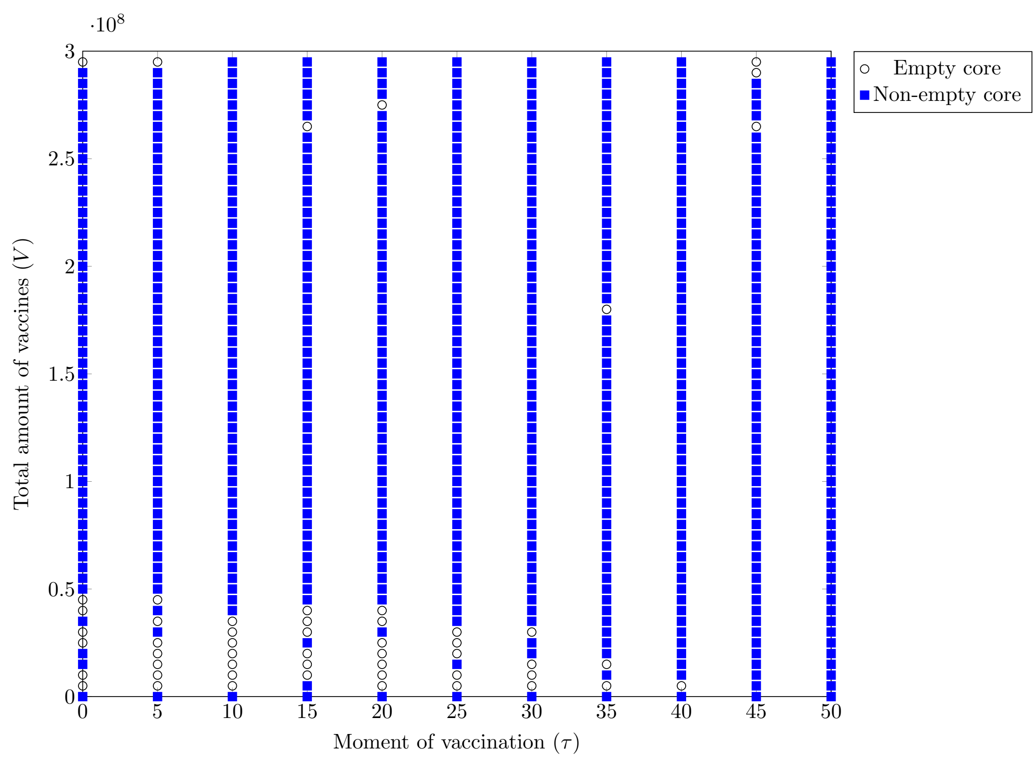

Figure 7: This figure illustrates the non-emptiness of the core for various combinations of the total vaccine stockpile (V) and the moment of vaccination (τ) using the disease parameters of Teytelman and Larson (2013).

Additional results, not reported here, show that the core is non-empty for $V \leq 2.5 \cdot 10^6$ for almost all $\tau \geq 0$. We therefore find that the core is non-empty when there are sufficient resources available or almost no resources. This is in line with the findings of Figure 3. Moreover, if we analyze Figure 7 in more detail, we can derive that the dark area with sufficient resources and a non-empty core has interesting characteristics. We note that for the return function considered here, $\tilde{f_{F_i}}$ depends on τ for all $i \in N$. Particularly, $\sum_{i \in N} \tilde{f_{F_i}} M_i$ decreases from 3.6×10^7 when $\tau = 0$ to 0 for $\tau \geq 40$. Based on this, we can approximately say that the core is non-empty when $V > \sum_{i \in N} \tilde{f_{F_i}} M_i$. This is in line with our theoretical results in Section 5.2. The return functions on which Figure 7 is based are different for every region. In Appendix C, we report similar results for the case of identical return functions.

The intuition behind the fact that $\tilde{f_{F_i}}$ is decreasing in τ is that the later you vaccinate, the more people are already infected and you are almost only vaccinating people that would not have become infected in the first place. If τ is very large, vaccination is too late to be very effective and the return functions do no longer have a convex part, which implies that $\tilde{f_{F_i}} = 0$. In that case, our cooperative game has concave return functions and it is equivalent to a market game for which the core is always non-empty.

Figure 7 also shows that outside the connected area with non-empty core and sufficient vaccines, there are only a few points for which the core is non-empty. There is no clear pattern for which combinations of V and τ this is the case. Sometimes these points are related to combinations of V and τ for which there are approximately enough vaccines to vaccinate a subset of the regions with their fraction $\tilde{f}_{F_i}$, but not always. This implies that for moderate amounts of resources it is very difficult to say beforehand whether the core is non-empty and a fair allocation of the total return exists.

In addition to analyzing the non-emptiness of the core, we also study whether there are market allocations in the core. These numerical results, not reported here, show that when the core is non-empty, there almost always exists a market allocation in the core. For large amounts of vaccines ($V > 5 \times 10^7$), the market allocation with market price p_4 is often in the core. This is in line with our findings in Figure 5. For moderate amounts of resources and a non-empty core, most of the time none of the market allocations that we proposed in Section 6.2 is in the core. Thus, even if cooperation can be achieved through trading all resources for a market price, there does not need to be an analytical expression for the market price that results in a fair allocation.

To conclude, the results for this case study confirm that cooperation is a delicate matter. Only when the amount of vaccines is very small or above a certain threshold, approximately characterized by $\sum_{i \in N} \tilde{f}_{F_i} M_i$, all players are willing to cooperate with each other and a market price is possible. In case of moderate shortages, cooperation is likely not possible. This is reasonable, because vaccines have the most value when they are reasonably scarce. Only in that case the health benefits in your region will be heavily affected when part of the available vaccines is given away. This reduces the willingness of regions to cooperate and share their vaccines. The CDC only allows health agencies to redistribute and trade their vaccines in case of emergencies, such a late deliveries and unexpected shortages (Centers for Disease Control and Prevention, 2009b). Our results thus show that when the CDC allows for cooperation, it is complicated to establish it and it might not even be possible. Only when there is a severe shortage of vaccines, cooperation might be an option and the vaccines can be traded for a market price. In all other realistic cases, cooperation is more complicated and perhaps not even possible.

8 Discussion and conclusion

This paper analyzes the redistribution of resources and cooperation between players. The return that players obtain from the resources is modeled via an S-shaped return function. Such a function captures convex returns for limited amounts of resources and concave returns in case of many resources. We use the concept of the core from cooperative game theory to analyze whether cooperation between the players is plausible. We derive theoretical conditions under which cooperation is plausible and we show that the resources can be traded for a market price in those cases. We perform numerical analysis to generalize these findings. Our numerical results show that parties are most likely willing to cooperate when the total amount of resources is limited or very large. A case study on the redistribution of vaccines confirms these findings.

With our analysis, we provide insights into general cooperation problems. Typically, cooperation leads to increased total benefits. This is particularly true for return functions with a convex part, such as the S-shaped return functions that are considered here. This seems to indicate that cooperation is worth pursuing. However, cooperation is only plausible when the resources can be traded among the players for prices that satisfy all participating players. Such prices may not always exist. Moreover, the S-shape of the return function actually makes it less likely that such prices exist. Hence, one must be careful when investigating potential cooperation between players. In addition, we find that, when cooperation is plausible, there is often just a single price for which resources are traded. Such a market price is likely to enhance cooperation in practice, because it prevents having dissatisfied players who found out that other players have bought resources for a lower price.

The results in this paper are established under some assumptions. First of all, we assume that the resources can be exchanged among players and that a player can assign a monetary value to the return he obtains from a certain number of resources. This allows a general formulation of the problem that can be applied to various types of resources. In case of vaccines, this implies that a health agency might sell some vaccines and receive money as compensation for an increased number of infections. Although it seems unethical to compare health with money, this approach is common in health economics literature (e.g., Neumann, Sanders, Russell, Siegel, & Ganiats, 2016).

Our theoretical results are only valid when players have identical return functions, but our numerical results in Section 6 extend these findings to individual return functions for all players. In addition, we assume that players are independent of each other. The return of one player does not depend on the amount of resources given to other players. It heavily depends on the context whether this assumption is reasonable. In case of vaccination, the assumption of no interaction is legitimate when the players correspond to geographically distant regions and when the interaction within a region is much larger than between regions (Mamani et al., 2013; Sun et al., 2009; Wu et al., 2007). It would be interesting to include interaction between the players, although this would complicate the analysis of the cooperative game in two dimensions. Firstly, the return functions are more complex if they dependent on each other. Secondly, determining the value function is more difficult, because it requires additional assumptions on the actions taken by players outside the coalition.

We use the concept of the core to determine whether cooperation is plausible and whether the total return can be divided in a fair way among the players. The advantage of core allocations is that they guarantee that any player will never be worse off by cooperating with all other players, neither by working on his own nor by cooperating with a subset of the players. A drawback is that core allocations do not always exist. As an alternative, one could exclude the coalitional stability conditions for some coalitions for which it is unlikely (e.g., due to geographical or political reasons) that they would cooperate. This results in a so-called restricted game (see, e.g., Faigle, 1989). In a restricted game the collection of coalitions that need to satisfy the stability conditions need not contain all subsets of players. As a consequence, the core of a restricted game is 'larger' than the original core and thus might be non-empty (while the core of the original game is empty). Another possibility is to keep all coalitions, but to relax some of the coalitional stability conditions with a fixed constant or factor. These generalization of the core, that do include some overhead for deviation of coalitions, are better known as the least-core (Maschler, Peleg, & Shapley, 1979) and the multiplicative ϵ-core (Faigle & Kern, 1993), respectively. Moreover, one could also focus on other well-known game theoretical solution concepts, like the Shapley value (Shapley, 1953), the nucleolus (Schmeidler, 1969), or the τ-value (Driessen & Tijs, 1985). These solutions concept have proven to be applicable in various settings and rely on other interesting fairness properties than coalitional stability. Further research could study such alternative fair allocations for our game.

Our results demonstrate that cooperation is a delicate matter. Often it is not possible to divide the total benefits in such a way that all parties are willing to cooperate. But if cooperation is an option, we provide an intuitive and fair way to divide the total benefits by trading all resources for a market price.

Appendices

Appendix A Proofs

Theorem 1. *In the case of two players (i.e., $|N| = 2$) the core is equal to the set of all market allocations with a market price*

$$p \in \left[\frac{M_s \left[F_s\left(\frac{r_s}{M_s}\right) - F_s(f_s^*) \right]}{r_s - M_s f_s^*}, \frac{M_b \left[F_b(f_b^*) - F_b\left(\frac{r_b}{M_b}\right) \right]}{M_b f_b^* - r_b} \right].$$

Proof. We use the subscripts 1 and 2 to refer to the two players without specifying which player is the seller and which player is the buyer. Furthermore, we assume by Lemma 1 that $f_1^* M_1 + f_2^* M_2 = r_1 + r_2$. We prove this theorem along the following lines:

(i) For two player games, all core allocations are market price allocations.

(ii) A market price allocation can only be in the core if p is in the given interval.

(iii) This interval is non-empty.

First, we show that all core-allocations are market price allocations in two player games. Let (z_1, z_2) denote a core allocation. By efficiency of the core, the following holds:

$$z_1 + z_2 = M_1 F_1(f_1^*) + M_2 F_2(f_2^*) = v(N) \tag{7}$$

Introduce a price p_j for player j, with:

$$p_j = \frac{z_j - M_j F_j(f_j^*)}{r_j - f_j^* M_j}$$

Because the core allocation is individually rational, we know that $z_j \geq M_j F_j(r_j/M_j)$. By the fact that the functions $F_j(f)$ are non-decreasing, this implies that $p_j \geq 0$ for $j = 1, 2$. The following analysis shows that $p_1 = p_2$:

$$\begin{aligned}
p_1 &= \frac{z_1 - M_1 F_1(f_1^*)}{r_1 - f_1^* M_1} && \text{by definition} \\
&= \frac{z_2 - M_2 F_2(f_2^*)}{f_1^* M_1 - r_1} && \text{by equation (7)} \\
&= \frac{z_2 - M_2 F_2(f_2^*)}{r_2 - f_2^* M_2} = p_2 && \text{by Lemma 1}
\end{aligned}$$

Since $p_1 = p_2 = p$, we conclude that p is a market price. Thus, any core allocation is a market price allocation.

Secondly, we show that a market price allocation can only be in the core if p is in the given interval. W.l.o.g. let player 1 be the seller and player 2 the buyer: i.e., $f_1^* M_1 < r_1$ and $f_2^* M_2 > r_2$. Consider the following allocation with market price p:

$$x_1 = M_1 F_1(f_1^*) + p(r_1 - f_1^* M_1)$$
$$x_2 = M_2 F_2(f_2^*) + p(r_2 - f_2^* M_2)$$

We will show that allocation (x_1, x_2) is in the core if and only if p is in the interval provided in the theorem.

Efficiency: By Lemma 1 we know that

$$M_1 f_1^* + M_2 f_2^* = r_1 + r_2. \tag{8}$$

This implies the following:

$$x_1 + x_2 = M_1 F_1(f_1^*) + p(r_1 - f_1^* M_1) + M_2 F_2(f_2^*) + p(r_2 - f_2^* M_2) = M_1 F_1(f_1^*) + M_2 F_2(f_2^*) = v(N).$$

The last equality follows from the optimality of (f_1^*, f_2^*).

Stability: Note that for $|N| = 2$, we only have to verify that $x_i \geq v(\{i\})$ for $i = 1, 2$. We start with $i = 1$:

$$\begin{aligned} & x_1 \geq v(\{1\}) \\ \Leftrightarrow \quad & M_1 F_1(f_1^*) + p(r_1 - f_1^* M_1) \geq M_1 F_1\left(\frac{r_1}{M_1}\right) \\ \Leftrightarrow \quad & p \geq \frac{M_1\left[F_1\left(\frac{r_1}{M_1}\right) - F_1(f_1^*)\right]}{r_1 - M_1 f_1^*} \end{aligned} \tag{9}$$

Recall that player 1 sold part of his recourses. Above inequality shows that he requires a minimum price per resource to be willing to sell. Analogously, we can derive that player 2 is willing to pay at most the following maximum price:

$$p \leq \frac{M_2\left[F_2(f_2^*) - F_2\left(\frac{r_2}{M_2}\right)\right]}{M_2 f_2^* - r_2}. \tag{10}$$

Hence, (x_1, x_2) is in the core if and only if p satisfies conditions (9) and (10), which corresponds to the interval for p given in the theorem.

We finish the proof by showing that the interval for p is non-empty. By optimality of (f_1^*, f_2^*) the following inequality holds:

$$
\begin{aligned}
& M_1F_1\left(\frac{r_1}{M_1}\right)+M_2F_2\left(\frac{r_2}{M_2}\right)\le M_1F_1(f_1^*)+M_2F_2(f_2^*)\\
\Leftrightarrow\quad & M_1\left[F_1\left(\frac{r_1}{M_1}\right)-F_1(f_1^*)\right]\le M_2\left[F_2(f_2^*)-F_2\left(\frac{r_2}{M_2}\right)\right]\\
\Leftrightarrow\quad & \frac{M_1\left[F_1\left(\frac{r_1}{M_1}\right)-F_1(f_1^*)\right]}{r_1-M_1f_1^*}\le\frac{M_2\left[F_2(f_2^*)-F_2\left(\frac{r_2}{M_2}\right)\right]}{M_2f_2^*-r_2}
\end{aligned}
$$

The first equivalence follows from rewriting and the second holds by (8) and by the assumption that $r_1 - M_1f_1^* > 0$. The final inequality implies that the interval for p is non-empty.

We have thus shown that the complete core of a two player game can be described by the market allocations with a market price in the given non-empty interval. This completes the proof of this theorem. □

Theorem 3. *Consider identical return functions ($F_i(\cdot) = F(\cdot)$ for all $i \in N$) and let $k \in \arg\min_{i\in N}\{M_i\}$. If $R(N) < \tilde{f}_F M_k$, then the market allocation with price $D_F\left(\frac{R(N)}{M_k}\right)$ is in the core. This implies that the core is non-empty.*

Proof. We will show that the allocation with market price $p = D_F\left(\frac{R(N)}{M_k}\right)$ is in the core, which directly implies that the core is non-empty.

By Theorem 2 we know that $f_k^* = \frac{R(N)}{M_k}$ and $f_j^* = 0$ for all $j \in N\backslash\{k\}$. The market price allocation is defined as follows:

$$
\begin{aligned}
x_k &= M_kF\left(\frac{R(N)}{M_k}\right)+p(r_k-R(N))\\
x_j &= M_jF(0)+pr_j \qquad\qquad \text{for all } j\in N\backslash\{k\}
\end{aligned}
$$

Efficiency: The market allocation satisfies efficiency by construction and by Lemma 1.

Stability: Recall that $R(S) = \sum_{i\in S} r_i$ for any $S \subseteq N$. To study the stability of our proposed allocation, we first note the following:

$$\frac{R(S)}{M_j}\le\frac{R(N)}{M_j}\le\frac{R(N)}{M_k}\le\tilde{f}_F\quad\forall S\subset N,\forall j\in N,$$

where the last inequality follows from the fact that $R(N) \le \tilde{f}_F M_k$. By Corollary 1 above relation implies that

$$D_F\left(\frac{R(S)}{M_j}\right)\le p\quad\forall S\subset N,\forall j\in N. \tag{11}$$

We will now analyse stability for all $S \subset N$. We distinguish between the following two cases: (i) $k \in S$ and (ii) $k \notin S$. We start with case (i):

$$
\begin{aligned}
v(S) &= M_k F\left(\frac{R(S)}{M_k}\right) + \sum_{i\in S\setminus\{k\}} M_i F(0) && \text{by Theorem 2}\\
&= x_k + p(R(N) - r_k) - M_k \left[F\left(\frac{R(N)}{M_k}\right) - F\left(\frac{R(S)}{M_k}\right)\right]\\
&\quad + \sum_{i\in S\setminus\{k\}} (x_i - pr_i) && \text{by definition of } [x_i]_{i\in N}\\
&= \sum_{i\in S} x_i - pR(S) + pR(N) - M_k \left[F\left(\frac{R(N)}{M_k}\right) - F\left(\frac{R(S)}{M_k}\right)\right]\\
&= \sum_{i\in S} x_i - pR(S) + M_k \left[F\left(\frac{R(N)}{M_k}\right) - F(0)\right]\\
&\quad - M_k \left[F\left(\frac{R(N)}{M_k}\right) - F\left(\frac{R(S)}{M_k}\right)\right] && \text{by definition of } p\\
&= \sum_{i\in S} x_i - pR(S) + D_F\left(\frac{R(S)}{M_k}\right) R(S) && \text{by definition of } D_F(\cdot)\\
&\leq \sum_{i\in S} x_i && \text{by (11)}
\end{aligned}
$$

Let us now consider case (ii), i.e., $k \notin S$. W.l.o.g. let $j \in \arg\min\{M_i : i \in S\}$. Because $R(S) \leq \tilde{f}_F M_k \leq \tilde{f}_F M_j$, it is optimal to give all resources to player j. This implies that:

$$
\begin{aligned}
v(S) &= M_j F\left(\frac{R(S)}{M_j}\right) + \sum_{i\in S\setminus\{j\}} M_i F(0) && \text{by Theorem 2}\\
&= \sum_{i\in S} x_i - pR(S) + M_j \left[F\left(\frac{R(S)}{M_j}\right) - F(0)\right] && \text{by definition of } [x_i]_{i\in N}\\
&= \sum_{i\in S} x_i - pR(S) + D_F\left(\frac{R(S)}{M_j}\right) R(S) && \text{by definition of } D_F(\cdot)\\
&\leq \sum_{i\in S} x_i && \text{by (11)}
\end{aligned}
$$

The market price allocation with $p = D_F\left(\frac{R(N)}{M_k}\right)$ satisfies both the efficiency condition and stability conditions. Hence, it is a core allocation which implies that the core is non-empty. This completes the proof.

□

Theorem 4. *Consider identical return functions ($F_i(\cdot) = F(\cdot)$ for all $i \in N$). If $R(N) = \tilde{f}_F \sum_{i\in K} M_i$ for some $K \subseteq N$, then the market allocation with price $D_F\left(\tilde{f}_F\right)$ is in the core. This implies that the core is non-empty.*

Proof. Let $(f_i^*)_{i\in N}$ denote the optimal solution for the problem underlying $v(N)$. By Theorem 2 this solution is of the following form: $f_i^* = \tilde{f}_F$ for all $i \in K$ and $f_i^* = 0$ for all $i \in N\setminus\{K\}$.

Let us now define the market price allocation as follows:

$$x_i = M_i F(f_i^*) + p\,(r_i - f_i^* M_i) \quad \forall i \in N$$

Efficiency: Using Lemma 1 we derive the efficiency of the market price allocation:

$$\begin{aligned}
\sum_{i\in N} x_i &= \sum_{i\in N} \left[M_i F(f_i^*) + p\,(r_i - f_i^* M_i)\right] \\
&= \sum_{i\in N} M_i F(f^*) + p\left(\sum_{i\in N} r_i - \sum_{i\in N} f^* M_i\right) \\
&= \sum_{i\in N} M_i F(f^*) = v(N).
\end{aligned}$$

Hence, the proposed allocation satisfies efficiency.

Stability: To study the stability of our proposed market price allocation, we will use duality theory for convex optimization problems. Since Problem (1) is not a convex optimization problem, we will introduce a new convex optimization problem using the upper convex envelope of the function $F(\cdot)$.

We introduce the function $G(\cdot)$, the upper convex envelope of $F(\cdot)$:

$$G(f) = \begin{cases} F(0) + f D_F(\tilde{f}_F) & \text{for } f < \tilde{f}_F \\ F(f) & \text{for } f \geq \tilde{f}_F \end{cases} \tag{12}$$

One can verify that $G(\cdot)$ is a concave function, because it is linear on $[0, \tilde{f}_F)$ and concave on $[\tilde{f}_F, +\infty)$ (L. E. Duijzer, van Jaarsveld, Wallinga, & Dekker, 2018). Furthermore, $G(f) \geq F(f)$ for all $f \geq 0$. The function $G(\cdot)$ is also continuous and differentiable, because $F'(\tilde{f}_F) = D_F(\tilde{f}_F)$ by (4). Let us introduce the following optimization problem:

$$\begin{aligned}
v_2(S) = \max \quad & \sum_{i\in S} M_i G(f_i) \\
\text{s.t.} \quad & \sum_{i\in S} f_i M_i \leq \sum_{i\in S} r_i \\
& f_i \geq 0 \qquad \forall i \in S
\end{aligned} \tag{13}$$

Problem (13) is a convex optimization problem, because it is a maximization problem with linear constraints involving a concave objective function. For this convex optimization problem, we can formulate the following

Wolfe Dual (Wolfe, 1961):

$$\begin{aligned} D(S) = \min \quad & \sum_{i \in S} M_i G(f_i) + \lambda \left[\sum_{i \in S} r_i - \sum_{i \in S} f_i M_i \right] + \sum_{i \in S} \mu_i f_i \\ \text{s.t.} \quad & M_i G'(f_i) - \lambda M_i + \mu_i = 0 \qquad \forall i \in S \\ & \mu_i \geq 0 \qquad \forall i \in S \\ & \lambda \geq 0 \end{aligned} \tag{14}$$

In above problem $G'(f)$ denotes the derivative of the function $G(f)$ with respect to f. Observe that the objective function of the Wolfe Dual problem is equal to the Lagrangian function of Problem (13) after relaxing all constraints. The KKT conditions of Problem (13) function as constraints in the dual problem. We will now construct a solution to the dual problem (14) using the optimal solution $(f_i^*)_{i \in N}$:

$$\begin{aligned} f_i &= f_i^* = \tilde{f}_F && \forall i \in K \\ f_i &= f_i^* = 0 && \forall i \in N \backslash \{K\} \\ \lambda &= G'(\tilde{f}_F) \\ \mu_i &= 0 && \forall i \in N \end{aligned} \tag{15}$$

By construction of the function $G(\cdot)$ we have that $G'(\tilde{f}_F) = G'(0)$. It can be verified that above solution satisfies the constraints of Problem (14). It is therefore a feasible solution for the dual problem. We analyze the objective function corresponding to this feasible solution:

$$\begin{aligned} \text{Objective} &= \sum_{i \in S} M_i G(f_i^*) + G'(\tilde{f}_F) \left[\sum_{i \in S} r_i - \sum_{i \in S} f_i^* M_i \right] + \sum_{i \in S} \mu_i f_i^* \\ &= \sum_{i \in S} \left[M_i G(f_i^*) + F'(\tilde{f}_F)(r_i - f_i^* M_i) \right] \\ &= \sum_{i \in S} \left[M_i F(f_i^*) + p(r_i - f_i^* M_i) \right] \\ &= \sum_{i \in S} x_i \end{aligned}$$

In above derivation, we use that $G'(\tilde{f}_F) = F'(\tilde{f}_F) = D(\tilde{f}_F) = p$. We also use that $G(f) = F(f)$ for $f = 0$ and for $f \geq \tilde{f}$. Above derivation shows that objective function of the Wolfe Dual problem for the solution in (15) is equal to the sum of the market allocations for all the players in S. We can now prove the stability of our allocation for any $S \subset N$:

$$\begin{aligned} \sum_{i \in S} x_i &\geq D(S) && \text{because of the feasibility of (15) in (14)} \\ &\geq v_2(S) && \text{because of weak duality} \\ &\geq v(S) && \text{because } G(f) \geq F(f) \end{aligned}$$

We have thus shown that the market price allocation with price $p = D_F\left(\tilde{f}_F\right)$, is both efficient and stable. Therefore, this allocation is in the core, which completes the proof.

$\square$

Theorem 5. *Consider identical return functions ($F_i(\cdot) = F(\cdot)$ for all $i \in N$). If $R(N) \geq \tilde{f}_F \sum_{i\in N} M_i$, then the allocation with the market price $F'\left(\frac{R(N)}{\sum_{i\in N} M_i}\right)$ is in the core. This implies that the core is non-empty.*

Proof. The proof of this theorem is very similar to the proof of Theorem 4. We therefore only present the points where the proof is different. First, we note that in our current theorem the optimal solution corresponding to $v(N)$ is of the following form by Theorem 2: $f_i^* = \frac{R(N)}{\sum_{i\in N} M_i}$ for all $i \in N$. Since f_i^* is the same for all $i \in N$, we denote this value with f^*.

Analogous to the proof of Theorem 4 we construct a solution to the dual problem (14) using the optimal solution $(f_i^*)_{i\in N}$:

$$\begin{aligned} f_i &= f_i^* = f^* && \forall i \in N \\ \lambda &= G'(f^*) && \\ \mu_i &= 0 && \forall i \in N \end{aligned} \tag{16}$$

One can verify that the solution proposed in (16) is feasible for the dual problem (14). We analyze the objective function corresponding to this feasible solution:

$$\begin{aligned} \text{Objective} &= \sum_{i\in S} M_i G\left(f^*\right) + G'(f^*)\left[\sum_{i\in S} r_i - \sum_{i\in S} f^* M_i\right] + \sum_{i\in S} \mu_i f^* \\ &= \sum_{i\in S} \left[M_i G\left(f^*\right) + F'(f^*)\left(r_i - f^* M_i\right)\right] \\ &= \sum_{i\in S} \left[M_i F\left(f^*\right) + p\left(r_i - f^* M_i\right)\right] \\ &= \sum_{i\in S} x_i \end{aligned}$$

In above derivation, we use that $f^* \geq \tilde{f}_F$, such that $G(f^*) = F(f^*)$ and $G'(f^*) = F'(f^*)$. Above derivation shows that objective function of the Wolfe Dual problem for the solution in (16) is equal to the sum of the market allocations for all the players in S. Analogous to the proof of Theorem 4 this implies that the market price allocation with $p = F'(f^*) = F'\left(\frac{R(N)}{\sum_{i\in N} M_i}\right)$ is in the core. This completes the proof.

$\square$

Appendix B Numerical results

In Section 6, we perform numerical experiments and we analyze cooperation for randomly generated instances. We use the core of a game to investigate whether cooperation is possible. In Sections B.1 and B.2, we respectively explain how we construct the core and how we determine whether it is empty or not.

B.1 Approximation of the value function

The conditions that describe the core depend on the value function $v(\cdot)$. Since the value function is difficult to determine in our case, due to the complexity of the underlying decision problem, we approximate it using the heuristic proposed by L. E. Duijzer, van Jaarsveld, Wallinga, and Dekker (2018), which is shown to perform close to optimal for a similar optimization problem. Let $[g_i^*]_{i \in S}$ denote the solution obtained by the heuristic for coalition $S \subseteq N$. This solution can be obtained as follows:

Heuristic Sort the players in S decreasingly based on $D_{F_i}(\tilde{f}_{F_i})$.

Case (a) If $R(S) < \sum_{i \in S} M_i \tilde{f}_{F_i}$, let $k = \min\{i \in S | \sum_{j=1}^{i} M_i \tilde{f}_{F_i} \geq R(S)\}$. Then,

$$g_i^* = \begin{cases} \tilde{f}_i & \text{for } i = 1, ..., k-1 \\ \frac{R(S) - \sum_{i=1}^{k-1} M_i \tilde{f}_{F_i}}{M_k} & \text{for } i = k \\ 0 & \text{otherwise.} \end{cases}$$

Case (b) If $R(S) \geq \sum_{i \in S} M_i \tilde{f}_{F_i}$, then

$$g_i^* = \tilde{f}_{F_i} + \frac{R(S) - \sum_{i \in S} M_i \tilde{f}_{F_i}}{\sum_{i \in S} M_i} \quad \forall i \in S.$$

The goal of the heuristic is to distribute the resources in such a way that as many players as possible receive their fraction $\tilde{f}_{F_i}$ (case (a)). To obtain the highest possible return, the 'profitable' players (i.e., players with a high additional return per resource at $\tilde{f}_{F_i}$) are prioritized over less profitable players by means of sorting. If there are many resources available (case (b)), then all players receive $\tilde{f}_{F_i}$ and the remaining resources are divided pro rata.

Let $v^H(S)$ denote the heuristic approximation of the value function for coalition $S \subseteq N$. Since the heuristic does not solve Problem 1 to optimality, $v^H(\cdot)$ does not always satisfy superadditivity. i.e., it is possible that $v^H(S \cup T) < v^H(S) + v^H(T)$ for some $S, T \subset N$ with $S \cap T = \emptyset$. For the original value function $v(\cdot)$ this is not possible: the players in S and T will never be worse off if they cooperate, because they can always decide to keep their own resources. If $v^H(\cdot)$ is not superadditive, then this will affect our analysis of the core. To prevent this from happening, we first determine the approximated value function for all $S \subseteq N$ and then check for all the subsets in lexicographic order whether $v^H(S)$ satisfies superadditivity. If not, we adjust the solution of the heuristic by setting $v^H(S) := \max_{K \subset S}\{v^H(K) + v^H(S \backslash K)\}$ to guarantee superadditivity.

We compare this final heuristic solution to the outcome of a built-in function from Matlab that minimizes non-linear programming problems. The solution approach of this built-in function is based on interior point methods and barrier functions (c.f., Waltz, Morales, Nocedal, & Orban, 2006). This approach cannot guarantee global optimality, but might outperform the heuristic on some instances. We use $\bar{v}(S)$ to denote the best approximation of the value function for all coalitions $S \subseteq N$, which is the maximum of $v^H(S)$ and the value of the solution found by the built-in function.

B.2 Analysis of the core

We explain in this section how we can determine whether the core is empty and whether a market price allocation exists in the core.

We determine whether the core is empty with the equations that characterize the core. Recall that when we talk about the core, we technically mean the ϵ-core of the approximated game. We use $\epsilon = 10^{-5}$. We conclude that the core is non-empty if there is a solution $[x_i]_{i \in N}$ to the following system of linear equations:

$$\begin{aligned} &\sum_{i \in N} x_i = \bar{v}(N) \\ &\sum_{i \in S} x_i \geq \bar{v}(S) - \epsilon \quad \forall S \subset N \end{aligned} \tag{17}$$

Similarly, we can use linear inequalities to determine whether there exists a market allocation in the core. We use the following linear programming problem:

$$\min \quad p_{max} - p_{min} \tag{18}$$

$$\sum_{i \in N} x_i = \bar{v}(N) \tag{19}$$

$$\sum_{i \in S} x_i \geq \bar{v}(S) - \epsilon \qquad \forall S \subset N \tag{20}$$

$$x_i - p_i\left(r_i - f_i^* M_i\right) = M_i F_i\left(f_i^*\right) \qquad \forall i \in N \tag{21}$$

$$p_{max} \geq p_i \qquad \forall i \in N \tag{22}$$

$$p_{min} \leq p_i \qquad \forall i \in N \tag{23}$$

Constraints (19) and (20) guarantee that allocation x is in the core. The individual prices for each player that correspond to allocation x are determined in constraints (21). The final constraints determine the maximum and the minimum price. The objective is to minimize the difference between the maximum and the minimum price, to find a solution for which all prices are (almost) equal. A market price is a price for which $p_i = p_j$ for all $i, j \in N$ and such a price would result in an objective value equal to zero. Here, we conclude that there exists a market allocation in the core if the objective value of Problem (18) - (23) is smaller than $\epsilon = 10^{-5}$.

Appendix C Case Study

In this appendix, we first describe the return functions that we use in the case study in Section 7. After that, we present some additional numerical results in Section C.2.

C.1 Return function

The return function, denoted by $F_i(\cdot)$ for region $i \in N$, measures the proportion of people that escape infection. We model the epidemic with the seminal SIR model (Kermack & McKendrick, 1927). This model divides the population into three compartments: people are either susceptible (S), infected (I) or removed (R). Let $S_i(t), I_i(t)$ and $R_i(t)$ denote the number of individuals in each of the three compartments at time t.

The return function requires a number of input parameters. One important parameter is the *basic reproduction number*, which is a measure for the speed of transmission: the higher the basic reproduction number, the more secondary infections will be caused by a single infected person. Let σ_i denote the basic reproduction number in region i, which is a measure for the severity of the disease, M_i denotes the number of inhabitants in region i and τ is the moment of vaccination.

The return function can be characterized as follows, where f_i is the fraction of the population in region i that is vaccinated:

$$F_i(f_i) = \frac{-1}{\sigma_i} W \left[-\sigma_i \left(1 - \frac{I_i(0)}{M_i} \right) \left(1 - \frac{f_i M_i}{S_i(\tau)} \right) \exp\{-\sigma_i(1 - f_i)\} \right] + f_i \frac{S_i(\tau)}{M_i} \tag{24}$$

In equation (24), $W(\cdot)$ denotes the Lambert W function (Corless, Gonnet, Hare, Jeffrey, & Knuth, 1996).

The return function $F_i(\cdot)$ represents the proportion of people in region i escaping infection. In the case study, we maximize the total number of people escaping infection: $\sum_{i \in N} M_i F_i(f_i)$. We could translate this objective function into the monetary health benefits of vaccination by introducing a parameter that represents the benefits of saving an individual from infection. We refer to L. E. Duijzer, van Jaarsveld, Wallinga, and Dekker (2018) for more details on this return function.

We conduct two experiments: one with individual return functions for all regions and one with identical return functions. Thereto, we respectively use the disease parameters from Teytelman and Larson (2013) and Nguyen and Carlson (2016). These parameters can be found in Table 2. For both experiments we use the following initial state: $I_i(0) = 10^{-4} M_i$ for all $i \in N$.

Parameter	**Teytelman and Larson (2013)**		**Nguyen and Carlson (2016)**
Reproduction	$\sigma_1 = 1.43$	$\sigma_6 = 1.18$	$\sigma_i = 2$ for all $i \in N$
number	$\sigma_2 = 1.26$	$\sigma_7 = 1.24$	
	$\sigma_3 = 1.32$	$\sigma_8 = 1.18$	
	$\sigma_4 = 1.18$	$\sigma_9 = 1.14$	
	$\sigma_5 = 1.21$	$\sigma_{10} = 1.16$	
Duration			
infectious period	2.3 days		$6\frac{2}{3}$ days

Table 2: Overview of the disease parameters.

C.2 Results

We analyze the non-emptiness of the core for various combinations of the moment of vaccination τ and the total amount of available vaccines V. In Section 7, we reported the results for the case of individual return functions. The results for the identical return functions are comparable, which can be seen in Figure 8.

We note that for the return function considered in Figure 8 $\tilde{f}_F$ depends on τ and decreases from around 0.50 when $\tau = 0$ to 0.20 for $\tau = 45$ and $\tilde{f}_F$ equals 0 for $\tau \geq 50$. Based on this, we can derive that the small dark area at the bottom of the graph with few resources and a non-empty core is approximately characterized by $V < \tilde{f}_F \min_{i \in N}\{M_i\}$ and the area with many resources satisfies $V > \tilde{f}_F \sum_{i \in N} M_i$. This is in line with our theoretical results in Section 5.2.

Regarding which allocations are in the core, we find that for very few or many vaccines there almost always exists a market allocation in the core. This is more often the case than in the case of individual return functions. We also note that for large amounts of resources, the initial pro rata distribution of vaccines is also optimal in the grand coalition. Hence, there is no need to redistribute resources when all

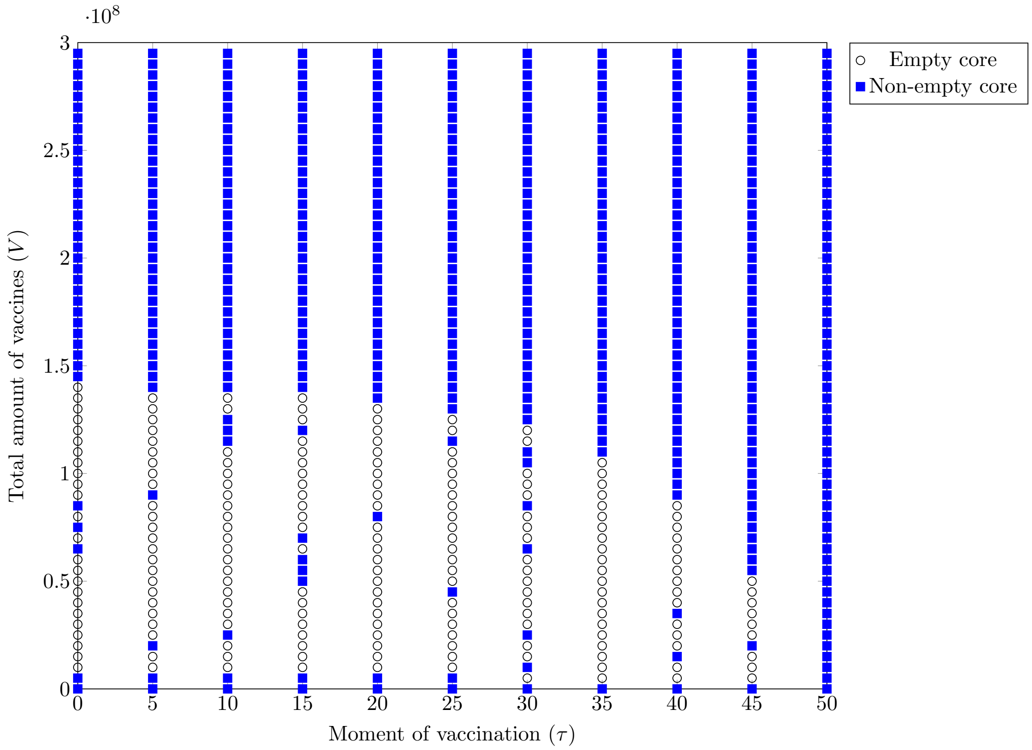

Figure 8: This figure illustrates the non-emptiness of the core for various combinations of the total amount of vaccines (V) and the moment of vaccination (τ) using the disease parameters of Nguyen and Carlson (2016).

regions cooperate and any market price would result in a core allocation. In line with our results in Section 7 we find that for moderate amounts of resources and a non-empty core, often none of our proposed market allocations is in the core.

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