What They Did Not Tell You About Algebraic (Non-)Existence, Mathematical (IR-)Regularity and (Non-)Asymptotic Properties of the Dynamic Conditional Correlation (DCC) Model*

Michael McAleer**

Department of Finance
Asia University
Taiwan
and
Discipline of Business Analytics
University of Sydney Business School
Australia
and
Econometric Institute
Erasmus School of Economics
Erasmus University Rotterdam
The Netherlands
and
Department of Economic Analysis and ICAE
Complutense University of Madrid
Spain
and
Institute of Advanced Sciences
Yokohama National University
Japan

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** Corresponding author: michael.mcaleer@gmail.com

Abstract

In order to hedge efficiently, persistently high negative covariances or, equivalently, correlations, between risky assets and the hedging instruments are intended to mitigate against financial risk and subsequent losses. If there is more than one hedging instrument, multivariate covariances and correlations will have to be calculated. As optimal hedge ratios are unlikely to remain constant using high frequency data, it is essential to specify dynamic time-varying models of covariances and correlations. These values can either be determined analytically or numerically on the basis of highly advanced computer simulations. Analytical developments are occasionally promulgated for multivariate conditional volatility models. The primary purpose of the paper is to analyse purported analytical developments for the only multivariate dynamic conditional correlation model to have been developed to date, namely Engle's (2002) widely-used Dynamic Conditional Correlation (DCC) model. Dynamic models are not straightforward (or even possible) to translate in terms of the algebraic existence, underlying stochastic processes, specification, mathematical regularity conditions, and asymptotic properties of consistency and asymptotic normality, or the lack thereof. The paper presents a critical analysis, discussion, evaluation and presentation of caveats relating to the DCC model, and an emphasis on the numerous dos and don'ts in implementing the DCC and related model in practice.

Keywords: Hedging, covariances, correlations, existence, mathematical regularity, invertibility, likelihood function, statistical asymptotic properties, caveats, practical implementation.

JEL: C22, C32, C51, C52, C58, C62, G32.

1. Introduction

Hedging financial investments is tantamount to insuring against possible losses arising from risky portfolio allocation. In order to hedge efficiently, persistently high negative covariances or, equivalently, correlations, between risky assets and the hedging instruments are intended to mitigate against financial risk and subsequent losses.

It is possible to hedge against risky assets using one or more hedging instruments as the benchmark, which requires the calculation of multivariate covariances and correlations. As optimal hedge ratios are unlikely to remain constant using high frequency data, it is essential to specify dynamic time-varying models of covariances and correlations.

Modelling, forecasting and evaluating dynamic covariances between hedging instrument and risky financial assets requires the specification and estimation of multivariate models of covariances and correlations. These values can either be determined analytically or numerically on the basis of highly advanced computer simulations. High frequency time periods such daily data can lead to either conditional or stochastic volatility, where analytical developments are occasionally promulgated for the former, but always numerically for the latter.

The purpose of the paper is to analyse purported analytical developments for the only multivariate dynamic conditional correlation model to have been developed to date, namely Engle's (2002) widely-used Dynamic Conditional Correlation (DCC) model. As dynamic models are not straightforward (or even possible) to translate in terms of the algebraic existence, underlying stochastic processes, specification, mathematical regularity conditions, and asymptotic properties of consistency and asymptotic normality, or the lack thereof, they will be evaluated separately.

For the variety of detailed possible outcomes mentioned above, where problematic issues arise constantly, and sometimes unexpectedly, a companion paper by the author evaluates the recent developments in modelling dynamic conditional covariances on the basis of the Full BEKK model (see McAleer (2019)).

The remainder of the paper is as follows. The DCC model is presented in Section 2, which will enable a subsequent critical analysis and emphasis on a discussion, evaluation and presentation of caveats in Section 3 of the numerous dos and don’ts in implementing the DCC model in practice.

2. Model Specification

Some, though not all, of the results in this section are available in the extant literature, but the interpretation of the models and their non-existent underlying stochastic processes, as well as the discussions and caveats in the following section, are not available. Much of the basic material relating to the univariate and multivariate specifications in Sections 2.1 - 2.3 overlap with the presentation in McAleer (2019).

The first step in estimating DCC is to estimate the standardized shocks from the univariate conditional mean returns shocks. The most widely used univariate conditional volatility model, namely GARCH, will be presented briefly, followed by DCC. Consider the conditional mean of financial returns, as follows:

$$y_t = E(y_t|I_{t-1}) + \varepsilon_t \tag{1}$$

where the returns, $y_t = \Delta \log P_t$, represents the log-difference in financial asset prices (P_t), I_{t-1} is the information set at time *t-1*, and ε_t is a conditionally heteroskedastic returns shock that has the same unit of measurement as the returns. In order to derive conditional volatility specifications, it is necessary to specify, wherever possible, the stochastic processes underlying the returns shocks, ε_t.

2.1 Univariate Conditional Volatility Models

Univariate conditional volatilities can be used to standardize the conditional covariances in alternative multivariate conditional volatility models to estimate conditional correlations, which are particularly useful in developing dynamic hedging strategies. The most widely-used univariate model, GARCH, is presented below as an illustration because the focus of the paper is on estimating and testing DCC.

2.2 Random Coefficient Autoregressive Process and GARCH

Consider the random coefficient autoregressive process of order one:

$$\varepsilon_t = \phi_t \varepsilon_{t-1} + \eta_t \tag{2}$$

where

$\phi_t \sim iid(0, \alpha)$,
$\eta_t \sim iid(0, \omega)$,
and $\eta_t = \varepsilon_t / \sqrt{h_t}$ is the standardized residual.

The standardized residual is unit-free of measurement, and is a financial fundamental as it represents a riskless asset.

Tsay (1987) derived the ARCH(1) model of Engle (1982) from equation (1) as:

$$h_t = E(\varepsilon_t^2 | I_{t-1}) = \omega + \alpha \varepsilon_{t-1}^2 \tag{3}$$

where h_t is conditional volatility, and I_{t-1} is the information set available at time *t*-1. The mathematical regularity condition of invertibility is used to relate the conditional variance, h_t , in equation (3) to the returns shocks, ε_t , which has the same measurement as y_t in equation (1), thereby yielding a valid likelihood function of the parameters given the data.

The use of an infinite lag length for the random coefficient autoregressive process in equation (2), with appropriate geometric restrictions (or stability conditions) on the random coefficients, leads to the GARCH model of Bollerslev (1986). From the specification of equation (2), it is clear that both ω and α should be positive as they are the unconditional variances of two independent stochastic processes. The GARCH model is given as:

$$h_t = E(\varepsilon_t^2 | I_{t-1}) = \omega + \alpha\varepsilon_{t-1}^2 + \beta h_{t-1}$$

where α is the short run ARCH effect, and β, which lies in the range (-1,1), is the GARCH contribution to the long run persistence of returns shocks.

2.3 Multivariate Conditional Volatility Models

Multivariate conditional volatility GARCH models are often used to analyze the interaction between the second moments of returns shocks to a portfolio of assets, and can model and the possible risk transmission or spillovers among different assets.

In order to establish volatility spillovers in a multivariate framework, it is useful to define the multivariate extension of the relationship between the returns shocks and the standardized residuals, that is, $\eta_t = \varepsilon_t / \sqrt{h_t}$. The multivariate extension of equation (1), namely:

$$y_t = E(y_t | I_{t-1}) + \varepsilon_t$$

can remain unchanged by assuming that the three components in the above equation are now $m \times 1$ vectors, where m is the number of financial assets.

The multivariate definition of the relationship between ε_t and η_t is given as:

$$\varepsilon_t = D_t^{1/2} \eta_t \tag{4}$$

where $D_t = diag(h_{1t}, h_{2t}, \dots, h_{mt})$ is a diagonal matrix comprising the univariate conditional volatilities. Define the conditional covariance matrix of ε_t as Q_t. As the $m \times 1$ vector, η_t, is assumed to be *iid* for all m elements, the conditional correlation matrix of ε_t, which is equivalent to the conditional correlation matrix of η_t, is given by Γ_t.

Therefore, the conditional expectation of the process in equation (4) is defined as:

$$Q_t = D_t^{1/2} \Gamma_t D_t^{1/2} \tag{5}$$

Equivalently, the conditional correlation matrix, Γ_t , can be defined as:

$$\Gamma_t = D_t^{-1/2} Q_t D_t^{-1/2} \tag{6}$$

Equation (5) is useful if a model of Γ_t is available for purposes of estimating the conditional covariance matrix, Q_t, whereas equation (6) is useful if a model of Q_t is available for purposes of estimating the conditional correlation matrix, Γ_t.

Both equations (5) and (6) are instructive for a discussion of asymptotic properties. As the elements of D_t are consistent and asymptotically normal, the consistency of Q_t in equation (5) depends on consistent estimation of Γ_t , whereas the consistency of Γ_t in equation (6) depends on consistent estimation of Q_t. As both Q_t and Γ_t are products of matrices, and the inverse of the matrix *D* is not asymptotically normal, even when *D* is asymptotically normal, neither the QMLE of Q_t nor Γ_t will be asymptotically normal, especially based on the definitions that relate the conditional covariances and conditional correlations given in equations (5) and (6).

The vector random coefficient autoregressive process of order one is the multivariate extension of equation (2), and is given as:

$$\varepsilon_t = \Phi_t \varepsilon_{t-1} + \eta_t \tag{7}$$

where

ε_t and η_t are $m \times 1$ vectors,
Φ_t is an $m \times m$ matrix of random coefficients,
$\Phi_t \sim iid(0, A)$,
$\eta_t \sim iid(0, QQ')$.

Technically, a vectorization of a full (that is, non-diagonal) matrix *A* to *vec A* can have dimension as high as $m^2 \times m^2$, whereas a half-vectorization of a symmetric matrix *A* to *vech A* can have dimension as low as $m(m-1)/2 \times m(m-1)/2$. The matrix *A* is crucial in the interpretation of symmetric and asymmetric weights attached to the returns shocks.

2.4 DCC Model

This section presents the DCC model, as given in Engle (2002), which does not have an underlying stochastic specification that leads to its derivation. Without distinguishing between dynamic conditional covariances and dynamic conditional correlations, Engle (2002) presented the DCC specification as:

$$Q_t = (1-\alpha-\beta)\overline{Q} + \alpha\eta_{t-1}\eta_{t-1}^{'} + \beta Q_{t-1} \tag{8}$$

where $\overline{Q}$ is assumed to be positive definite with unit elements along the main diagonal, the scalar parameters are assumed to satisfy the stability condition, $\alpha + \beta < 1$, where the two parameters do not have the same interpretation as in the univariate GARCH model, the standardized shocks, $\eta_t = (\eta_{1t,\ldots,}\eta_{mt})'$, which are not necessarily *iid*, are given as $\eta_{it} = \varepsilon_{it} / \sqrt{h_{it}}$, and D_t is a diagonal matrix with typical element $\sqrt{h_{it}}$, $i = 1,\ldots,m$. As m is the number of financial assets, the multivariate definition of the relationship between ε_t and η_t is given as $\varepsilon_t = D_t\eta_t$.

In view of equations (5) and (6), as the matrix in equation (8) does not satisfy the definition of a correlation matrix, Engle (2002) uses the following standardization:

$$R_t = (diag(Q_t))^{-1/2} Q_t (diag(Q_t))^{-1/2} \tag{9}$$

There is no clear explanation given for the standardization in equation (9) or, more recently, in Aielli (2013), especially as it does not satisfy the definition of a correlation matrix, as given in equation (6). The standardization in equation (9) might make sense if the matrix Q_t were the conditional covariance matrix of ε_t or η_t , although this is not made clear. Indeed, in the literature relating to DCC, it is not clear whether equation (8) refers to a conditional covariance or a conditional correlation matrix, although the latter is simply assumed without any clear explanation.

Despite the title of the paper, Aielli (2013) also does not provide any stationarity conditions for the DCC model, and does not mention the mathematical regularity condition of invertibility at the univariate of multivariate level. On the basis of equation (8), which does not relate the conditional covariance matrix, Q_t , in equation (8) to the returns shocks, ε_t , which has the same measurement as y_t in equation (1), invertibility does not hold. As there is no connection between Q_t and y_t, there is not a valid likelihood function of the parameters given the data.

It follows that there can be no first or second derivatives of the non-existent likelihood function, so the Jacobian and Hessian matrices do not exist. Therefore, there cannot be an analytical derivation of any asymptotic effects relating to consistency and asymptotic normality.

3. Vector Random Coefficient Moving Average Process

The random coefficient moving average process will be presented in its original univariate form in section 3.1, as in Marek (2005), with an extension to its multivariate counterpart in section 3.2, in order to derive the univariate and multivariate conditional volatility models, respectively, as well as the associated invertibility conditions at the univariate and multivariate levels.

3.1 Univariate process

Marek (2005) proposed a linear moving average model with random coefficients (RCMA).. In this section, we extend the univariate results of Marek (2005) using an *m*-dimensional vector random coefficient moving average process of order *p*, which is used as an underlying stochastic process to derive the DCC model. A novel conditional volatility model is derived in which is given as a function of the standardized shocks rather than of the returns shocks, as in the univariate ARCH model in equation (3).

Consider a univariate random coefficient moving average process given by:

$$\varepsilon_t = \theta_t \eta_{t-1} + \eta_t \tag{10}$$

where $\eta_t \sim iid\ (0,\omega)$. The sequence $\{\theta_t\}$ is assumed to be independent of $\eta_{t-1}, \eta_t, \eta_{t+1},...$, which is called the Future Independence Condition, with mean zero and variance α. It is also assumed to be measurable with respect to I_t, the information set generated by the random variable, $\{\varepsilon_{t,}\varepsilon_{t-1,...}\}$. Furthermore, it is assumed that the process $\{\varepsilon_t\}$ is stationary and invertible, such that $\eta_t \in I_t$.

Without the measurability assumption on $\{\theta_t\}$ it would be difficult to obtain results on invertibility. As demonstrated in McAleer (2018), an important special case of the model arises when $\{\theta_t\}$ is *iid*, that is, not measurable with respect to I_t, in which case the conditional and unconditional expectations of ε_t are zero, and the conditional variance of ε_t is given by:

$$h_t = E(\varepsilon_t^2 \mid I_{t-1}) = \omega + \alpha\eta_{t-1}^2 \qquad (11)$$

which differs from the ARCH model in equation (3) in that the returns shock is replaced by the standardized shock. This is a new univariate conditional volatility model, especially as conditional volatility is expressed in terms of a riskless random variable rather than the returns shock, which has the same measurement, and hence risk, as the returns, y_t

McAleer (2018) shows that, as $\eta_t \sim iid\ (0,\omega)$, the unconditional variance of ε_t is given as:

$$E(h_t) = (1+\alpha)\omega.$$

The use of an infinite lag length for the random coefficient moving average process in equation (10), with appropriate restrictions on θ_t, would lead to a generalized ARCH model that differs from the GARCH model of Bollerslev (1986) as the returns shock would be replaced with a standardized shock.

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A sufficient condition for stationarity is that the vector sequence $\upsilon_t = (\eta_t, \theta_t\eta_{t-1})'$ is stationary. Moreover, by Lemma 2.1 of Marek (2005), a new sufficient condition for invertibility is that:

$$E[\log|\theta_t|] < 0\,. \tag{12}$$

The stationarity of $\upsilon_t = (\eta_t, \theta_t \eta_{t-1})'$ and the invertibility condition in equation (12) are new results for the univariate ARCH(1) model given in equation (11), as well as its direct extension to GARCH models.

3.2 Multivariate process

Extending the analysis to a vector random coefficient moving average (RCMA) model of order p, McAleer (2018) derives a special case of DCC(p,q), namely DCC($p,0$), as follows:

$$\varepsilon_t = \sum_{j=1}^{p} \theta_{jt} \eta_{t-j} + \eta_t \tag{13}$$

where ε_t and η_t are both $m \times 1$ vectors, and θ_{jt} , $j = 1,...,p$ are random $m \times m$ matrices, independent of $\eta_{t-1}, \eta_t, \eta_{t+1},...$.

As the dimension of the unconditional variance of θ_{jt} is m, if the variance matrix is not restricted parametrically, the dynamic conditional covariance matrix of (13) would depend on the product of the variance of θ_{jt} , with dimension between $m(m-1)/2$ and m^2, neither of which would be conformable with the dimension of η_{t-j} . Under specific assumptions, it is possible to derive the conditional covariance matrix of ε_t in equation (11).

If θ_{jt} in equation (13) is given as:

$$\theta_{jt} = \lambda_{jt} I_m \text{, with } \lambda_{jt} \sim iid(0, \alpha_j)\,, \quad j = 1,\ ...,\ p,$$

where λ_{jt} is a scalar random variable, the dynamic conditional covariance matrix is given as:

$$Q_t = E(\varepsilon_t \varepsilon_t^{'} \mid I_{t-1}) = \Omega + \sum_{j=1}^{p} \alpha_j \eta_{t-j} \eta_{t-j}^{'} \; . \tag{14}$$

The DCC model in equation (8) is obtained by letting $p \to \infty$ in equations (13) and (14), setting $\alpha_j = \alpha \beta^{j-1}$, and standardizing Q_t in equation (14) to obtain a conditional correlation matrix. For the case $p = 1$ in equation (14), the appropriate univariate conditional volatility model is given in the new model in equation (11), which uses the standardized shocks, rather than the standard ARCH model in equation (3), which uses the returns shocks. Moreover, the DCC model adds only one parameter, α_1, when $p = 1$, so it is parsimonious in terms of its parametric representation.

The derivation in McAleer (2018) of DCC in equation (14) from a vector random coefficient moving average process is novel and important as it: (i) demonstrates that DCC is, in fact, a dynamic conditional **covariance** model of the returns shocks rather than a purported dynamic conditional **correlation** model; (ii) provides the motivation, which is presently missing, for standardization of the conditional covariance model in equation (9) to obtain the conditional correlation model; and (iii) shows that the appropriate ARCH or GARCH model as a first step in calculating DCC is based on the standardized shocks rather than on the returns shocks.

4. Discussion and Caveats of Dos and Don'ts Regarding DCC

The results in the previous section allow a clear discussion of the caveats associated with ***DCC***. The deficiencies and limitations in virtually all published papers that use the deeply flawed ***DCC*** model are given below. The discussion and caveats are presented in a clear and entirely straightforward manner that need no further elaboration.

(1) Engle (1982) developed an autoregressive model of conditional correlations, ***ARCH***, based on the conditional returns shocks.

(2) Bollerslev (1986) extended ***ARCH*** by adding a lagged dependent variable to obtain ***Generalized ARCH***, ***GARCH***.

(3) The ***GARCH(1,1)*** parameters must satisfy the regularity conditions of positivity as they are the unconditional variances from a univariate random coefficient autoregressive process (see Tsay, 1987; McAleer, 2014).

(4) However, the coefficient of the arbitrary lagged conditional variance is a positive or negative fraction (see Bollerslev (1986)).

(5) The results in Tsay (1987) were extended to a vector random coefficient stochastic process to derive ***Diagonal BEKK*** in McAleer et al. (2008):

(6) The ***DCC*** model does not satisfy the definition of a conditional correlation matrix, as the purported conditional correlations do not satisfy the definition of a correlation, except by an untenable assumption.

(7) There is no known underlying stochastic process that leads to the ***DCC*** model as a dynamic **correlation** matrix, so that there are no regularity conditions relating to its specification.

(8) The regularity conditions include invertibility, which is essential in relating the ***iid*** standardized residuals to the returns data.

(9) As invertibility does not hold, it follows that there is no likelihood function, and hence no derivatives that would enable the derivation of asymptotic properties for the ***Quasi-Maximum Likelihood Estimates (QMLE)*** of the estimated parameters.

(10) Therefore, any statements regarding the purported "statistical significance" are meaningless and lack statistical validity.

(11) It follows that any empirical results based on the ***DCC*** estimates are fatally flawed and lack statistical validity (see McAleer (2018) for a critical analysis).

(12) Marek (2005) proposed a univariate Random Coefficient Moving Average (RCMA) process that leads to a conditional heterosckedastic process based on the standardized residuals rather than the conditional returns shocks.

(13) McAleer (2018) extended the univariate RCMA stochastic process to a multivariate random coefficient Moving Average process, and demonstrated that ***DCC*** could be derived from a vector RCMA process as a dynamic **covariance** matrix.

(14) It follows that ***DCC*** is not a Dynamic Conditional **Correlation** model, but rather a Dynamic Conditional **Covariance** model, although both can retain the acronym ***DCC***.

(15) Several years earlier than Engle (2002), Tse and Tsui (2002) proposed a ***Varying Conditional Correlation (VCC)*** model that does satisfy the definition of a correlation recursively for each observation.

(16) As stated in the published versions of both papers, the paper by Tse and Tsui (2002) was submitted to the journal in December 1998, 25 months earlier than the submission of the paper by Engle (2002), and was accepted in May 2001, 7 months earlier than the submission of Engle (2002).

(17) However, as in the case of ***DCC***, the ***VCC*** model has no underlying stochastic process that leads to its specification as a dynamic **correlation** matrix.

(18) Therefore, there are no mathematical regularity conditions relating to ***VCC***, including invertibility, which is essential in relating the ***iid*** standardized residuals to the returns data.

(19) Consequently, the ***QMLE*** of the estimated parameters of ***VCC*** do not possess any asymptotic properties as the likelihood function does not exist.

(20) It is somewhat surprising that the DCC model has been widely estimated using real data, without any apparent computational difficulties, despite the fact that the model does not actually exist!

(21) Such computational outcomes would almost certainly arise from the parsimonious addition of only one parameter when $p = 1$, even when the value of m is high for the large financial portfolios that are observed in practice.

(22) In short, ***VCC*** shares all the existence, specification, mathematical and statistical deficiencies of ***DCC***.

(23) If these two models are to be considered at all, except in connection with the algebraic non-existence, absence of an underlying stochastic process, mathematical irregularity, and unknown asymptotic statistical properties, or alternatively, in the presence of problems that should be avoided at all costs, it is advisable that DCC and VCC be used with extreme and utter caution in empirical practice.

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