Physical Depreciation and Discounting in Dynamic Markets with Adverse Selection

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Abstract: We investigate the nature of the adverse selection problem in a market for a durable good where trading and entry of new buyers and sellers takes place in continuous time. We focus on the role of the interest rate and physical depreciation. We show that when the physical depreciation rate is relatively small, infinitely many equilibria exist where all goods are traded within finite time after their appearance in the market. In contrast, when the physical depreciation rate is relatively large the trade of new goods will stop in any equilibrium after a finite moment in time. At intermediate values, stationary equilibria, different from the static equilibria, may emerge. For any given level of the relative depreciation rate the interest rate only determines the speed of evolution along the equilibrium path.

Key Words: Dynamic Trading, Asymmetric Information, Entry, Durable Goods.

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1. Introduction

Since the pioneering work of Akerlof (1970) and Wilson (1979, 1980), adverse selection is regarded as one of the main sources of market failure. In a few recent articles Janssen and Roy (1999a, 1999b) and Janssen and Karamychev (2000) have considered the adverse selection problem in a dynamic perspective. They ask the question whether the market mechanism, by changing prices over time, provide adequate incentives for sellers of different qualities to sort themselves over time. This question is relevant in markets where goods have a use value that extends over some time periods and where high quality goods have a higher use value than low quality goods. They show in a variety of settings that there exist dynamic competitive equilibria where all goods are traded after a finite number of periods. The main idea behind this result is that given a sequence of prices high quality sellers have more incentives to wait (and enjoy a higher use value before selling) than low quality sellers do. Once certain (low) qualities are sold, only relatively high qualities remain in the market. Risk-neutral consumers can predict that sellers of different qualities will sort themselves into different time periods and, hence, they are willing to pay higher prices in later periods. The equilibria are thus such that higher qualities are sold in later periods at higher prices.

In these dynamic models, two exogenous factors account for the decision of an individual seller with a given quality to sell in a certain time period: the rate of time preferences (discount factor) and the rate at which goods depreciate. Given the complexity of the models, it is, however, very difficult (if not impossible) to evaluate the role of each of these factors in determining the dynamic equilibrium path. The discount factor plays a role, but because of the discreteness of the time intervals it is difficult to analyze whether the qualitative properties of equilibria are affected by changes in the discount factor. Physical depreciation of the goods, on the other hand, is assumed to be absent in the models considered by Janssen and Roy (1999a, b) and Janssen and Karamychev (2000). In this paper we analyze the role of both factors in a continuous time version of the model. As the continuous time version is easier to analyze we are, at the same time, able to generalize the demand side of the model considerably by allowing consumers not to be risk-neutral.

We consider a competitive market for an imperfectly durable good where potential sellers are privately informed about the quality of the goods they own. Each moment in time a constant flow of sellers with an identical but arbitrary distribution of quality enters the market. All buyers are identical, have unit demand and for any given quality, a buyer's willingness to pay exceeds the reservation price of a seller for that quality. As buyers do not know the quality, their willingness to pay equals the expected valuation of goods traded at a certain time. The flow of such buyers into the market is larger than the flow of sellers so that, in equilibrium, prices are equal to the expected buyers' valuation. Once traded, goods are not re-sold in the same market.[1]

A first result says that only the relative depreciation rate, defined as the ratio of the physical depreciation rate and the interest rate, determines the qualitative properties of the equilibrium path. In particular, changes in the interest rate and the absolute physical depreciation rate that do not affect the relative depreciation rate do not affect the nature of equilibria in any way. The interest rate itself only determines the speed of evolution along an equilibrium path and in particular, higher interest rates implies a higher volume of trade at each moment as it is easier to separate goods of different quality.

Next, we analyze the role of the relative depreciation rate. We first consider the limit case where goods are perfectly durable so that the relative depreciation rate equals to zero for any positive value of the interest rate. The main result here says that there exist an infinite number of equilibria where *every* potential seller entering the market trades within a finite time after entering the market. When the quality distribution is such that there are relatively few sellers around the static equilibrium quality such equilibria only exist when we allow price to be a discontinuous function of time before all goods are sold. We then consider the case where goods are imperfectly durable and depreciate over time. A first observation is that with depreciation the infinitely repeated version of a static equilibrium outcome is not an equilibrium anymore in our dynamic model. Apart from this observation we have three types of results. A first result is that if the depreciation rate is small enough, i.e., if goods are "almost perfectly durable", the qualitative properties of the dynamic equilibria without physical depreciation hold true. A second result is that if the depreciation rate is relatively large, all equilibria result in no trade of new goods after a finite point in time. Finally, at intermediate levels of the depreciation rate, stationary equilibria, different from the static equilibria, may then emerge where low quality "new" goods and depreciated goods that originally were of high quality are traded at the same

[1] For example, in car markets, it is publicly observable how many owners a car has had up to particular point in time. Hence, second hand markets may be distinguished from third-hand markets, and so on.

time. A full characterization of the uniform case is given for different values of the depreciation rate.

The paper is organized as follows. Section 2 sets out the model and the equilibrium concept. Section 3 shows that only the relative depreciation rate has an effect on the qualitative properties of equilibria. Sections 4 and 5 discuss the role of the physical depreciation rate by analyzing the case of perfectly durable goods and imperfectly durable goods, respectively. Section 6 concludes. Proofs are contained in the Appendix.

2. The Model

Consider a Walrasian market for a good whose quality, denoted by θ, depreciates over time with a rate $\delta \in [0, \infty)$. Time, denoted by t, is continuous and $t \in [0, \infty)$. For every time moment t a constant flow of sellers I enters the market. Let t_i be the entry time of seller i and let θ_i be the quality he is endowed with at the time of entry. This implies that the quality owned by a seller i at time t is a function $\theta_i(t) = \theta_i \mathrm{e}^{-\delta(t-t_i)}$. We assume that θ_i varies between $\underline{\theta}$ and $\overline{\theta}$, where $0 < \underline{\theta} < \overline{\theta} < \infty$. The set of all sellers, therefore, is $\mathbf{I} = \{i\} = \{(\theta_i, t_i)\}$. We denote by $\mu(\theta)$ the Lebesgue measure of sellers in the flow I who own a good of quality less than or equal to θ. We assume that $\mu(\theta)$ is strictly increasing, absolutely continuous with respect to the Lebesgue measure and constant over time.

Each seller i knows the quality θ_i of the good he is endowed with and derives flow utility from ownership of the good until he sells it. Therefore, the seller's reservation price is the present discounted value of the flow of gross utility and we normalize this to be equal to θ_i. This implies that the gross utility flow is $(r+\delta)\theta_i(t)$, where r is the interest rate:

$$\theta_i = \int_{t_i}^{\infty} \mathrm{e}^{-r(t-t_i)} (r+\delta)\theta_i(t)\, dt .$$

On the demand side there is an inflow of new buyers at every time moment, which is larger in size than $\mu(\overline{\theta})$. All buyers are identical and have unit demand and quasi-linear preferences. A buyer's valuation of quality θ is $v(\theta)$, where $v(\theta) > \theta$ and $v' > 0$. Thus, under full information, first, there are always gains from trade and, second, higher quality goods are valued more than lower qualities. Having bought a good of quality θ at time

$t \geq t_i$ against price $p(t)$, a buyer derives utility $u(v(\theta)-p)$, where $u' > 0$, and we normalize $u(0)=0$.

All buyers know the *ex ante* quality distribution $\mu(\theta)$ but do not know the quality of the good offered by a particular seller. Goods that are once bought are not re-sold in the same market. Buyers and sellers discount the future at the common interest rate *r* and maximize their expected utility.

Before proceeding, we introduce some additional notation. Let $\pi(I',t)$ be the price that makes a buyer indifferent between buying a good from a seller *i* that belongs to a certain subset $I' \subset \mathbf{I}$ at time *t* and not buying at all. It follows that $\int_{i \in I', t_i \leq t} u\left(v\left(e^{-\delta(t-t_i)}\theta_i\right)-\pi(I',t)\right)d\mu = 0$. Adverse selection implies that $\pi(\mathbf{I},t) < \bar{\theta}$, i.e., the willingness to pay for the average quality in the population is lower than the reservation price of the seller of the best quality. Thus, the static Akerlof-Wilson version of the model has a largest equilibrium quality, which we denote by θ_S, satisfying $\pi(\{i|\theta_i \leq \theta_S\},0) = \theta_S$. The measure of sellers in the set I' is denoted as $\mu(I') = \mu(\{i|i \in I'\})$ such that $\mu(\theta) = \mu(\{i|i \in [\underline{\theta},\theta], t_i = t\})$ for all *t*.

To simplify our analysis we introduce the following regularity assumptions. Throughout this paper, we assume that these assumptions hold.

Assumption 1. The measure function $\mu(\theta)$ is differentiable on [$\underline{\theta},\bar{\theta}$] with strictly positive and Lipschitz-continuous density function $f(\theta)=\mu'(\theta)$, i.e., $f(\theta) \geq \varepsilon_f > 0$ and $|f(\theta')-f(\theta)| < M_f|\theta'-\theta|$ for some ε_f, M_f and for all $\theta',\theta \in$ [$\underline{\theta},\bar{\theta}$].

Assumption 2. The buyers' valuation function $v(\theta)$ is continuously differentiable on [$\underline{\theta},\bar{\theta}$] and there exists $\varepsilon_v > 0$ such that for all θ: $v(\theta)-\theta > \varepsilon_v$ and $v'(\theta) > \varepsilon_v$.

Assumption 3. The buyers' utility function $u(m)$ is continuously differentiable on [$v(\underline{\theta})-v(\bar{\theta}), v(\bar{\theta})-\underline{\theta}$] and there exist $M_u > \varepsilon_u > 0$ such that $M_u > u'(m) > \varepsilon_u$ and $u(0)=0$.

Given an evolution of market prices $p(t)$, $t \in [0,\infty)$, each seller *i* chooses whether or not to sell and if he decides to sell, the selling time. If he chooses not to sell his gross surplus is equal to θ_i and, therefore, his net surplus equals to zero. If, on the other hand, he decides to sell at time $t \geq t_i$ his gross surplus becomes

$$\int_{t_i}^{t} (r+\delta)\theta_i(t)\mathrm{e}^{-r(t-t_i)}\,dt + \mathrm{e}^{-r(t-t_i)}\,p(t) = \theta_i + \mathrm{e}^{-r(t-t_i)}\left(p(t) - \mathrm{e}^{-\delta(t-t_i)}\,\theta_i\right),$$

and, therefore, his net discounted surplus is equal to

$$s_i(t) = \mathrm{e}^{-rt}\left(p(t) - \mathrm{e}^{-\delta(t-t_i)}\,\theta_i\right).$$

The set of time moments in which it is optimal to sell for a seller *i* is given by

$$T_i(p(t)) \equiv \arg\max_{t \geq t_i}\{s_i(t) | s_i(t) \geq 0\}.$$

If $p(t) - \mathrm{e}^{-\delta(t-t_i)}\,\theta_i < 0$ for all $t \geq t_i$ then $T_i(p(t)) = \varnothing$. Each potential seller *i* chooses a time $\tau_i \in T_i$ when to sell. Let $\boldsymbol{\tau} = \{\tau_i\}_{i \in \mathbf{I}}$ be a set of all selling decisions. This implies that there is a flow of goods being offered for sale. We will denote this flow by J_t and it follows that $J_t \equiv \{i | \tau_i = t\}$.

At any time moment *t* buyers expect a certain quality distribution to be offered for sale. This determines their subjective willingness to pay for the average good at time *t*, which will be denoted by $\pi(t)$.

A dynamic equilibrium is an equilibrium where all players rationally maximize their objectives, expectations are fulfilled and the market always clears.

Definition 1. A dynamic equilibrium is described in terms of a path of prices $p(t)$, buyers' willingness to pay $\pi(t)$ and a set of selling decision $\boldsymbol{\tau} = \{\tau_i\}_{i \in \mathbf{I}}$ such that:

a) **Sellers maximize:** $\tau_i \in T_i(p(t))$ for all $i \in I$, i.e., every seller *i* chooses time τ_i to trade optimally.

b) **Buyers maximize and market clear:** If $\mu(J_t) > 0$ then $p(t) = \pi(t)$, i.e., if there is a strictly positive flow of goods offered for sale at time *t*, then each buyer gets zero net expected utility so that he is indifferent between buying and not buying and market clears. If $\mu(J_t) = 0$ then $p(t) \geq \pi(t)$, i.e., if there are (almost) no goods for sale at time *t* then each buyer can get at most zero net expected utility. Hence, it is optimal for him not to buy at that time as well.

c) **Expectations are fulfilled when trade occurs:** If $\mu(J_t) > 0$ then $\pi(t) \equiv \pi(J_t, t)$.

d) **Expectations are reasonable even if no trade occurs:** If trade stops at time $\hat{t}$ and the lowest quality left in the market is $\hat{\theta}$ then for all $t > \hat{t}$ until the trade starts again: $\pi(t) \geq v\left(\mathrm{e}^{-(t-\hat{t})\delta}\,\theta_{\min}\right)$, where $\theta_{\min} = \min\{\underline{\theta}, \hat{\theta}\}$.

Given the set-up described above, conditions (a)-(c) are quite standard. Condition (d) is introduced for the formal reason that the buyers' willingness to pay is not defined when no trade occurs. The condition says that even if the flow of goods offered for sale is zero, buyers should believe that the lowest quality offered is larger than the *apriori* lowest possible quality at that time. This condition assures that autarky, i.e., no trade at any time, cannot be sustained in an equilibrium of the dynamic model.

3. Equilibrium Dynamics and The Role of The Interest Rate

The model to be solved includes three exogenous functions $\mu(\theta)$, $v(\theta)$ and $u(m)$, and, in addition, two exogenous scalar parameters, r and δ. Obviously, all of them could affect the model's outcome and our aim is to understand the way they do. In this section we first describe the system of differential equations determining the dynamic equilibrium path. We then show that the interest rate r and the physical depreciation rate δ only effect the set of dynamic equilibria of the model through their effect on the relative depreciation rate $\frac{\delta}{r}$.

We start our analysis by arguing that if a good of certain quality sells at time t, then all goods with lower qualities at that time that have entered the market before (and are still in the market) will also sell at that time. Given any $p(t)$ a seller i by selling at time t earns a net discounted surplus $\mathrm{e}^{-rt}\left(p(t) - \mathrm{e}^{-\delta(t-t_i)}\,\theta_i\right)$. Maximizing this expression yields the first order conditions:

a) $\tau(\theta_i) = t_i$, if $\dot{p}(t_i) + (r+\delta)\theta_i(t) \leq rp(t_i)$, or

b) $\dot{p}(\tau(\theta_i)) + (r+\delta)\theta_i(t) = rp(\tau(\theta_i))$, if $\dot{p}(t_i) + (r+\delta)\theta_i(t) > rp(t_i)$.

The second order condition is simply $\ddot{p}(\tau) < r\dot{p}(\tau) + \delta(r+\delta)\theta_i(t)$ if $\tau(\theta_i) > t_i$. Implicitly, we have assumed that $p(t)$ is twice differentiable. As we will see, the solution we obtain is such that this assumption is satisfied.

At first we will look for equilibria that satisfy the second order condition for all θ_i. This implies that for any given θ_i the optimal selling time $\tau(\theta_i)$, if it exists, is unique. We will see that there are equilibria such that high quality sellers will never sell. If this is the case then the first order conditions are never satisfied for them and the optimal selling time does not exist. Then, the first order condition (a) says that a seller should sell immediately upon entering the market, i.e., at time t_i, if the benefit of using a good rather than selling at time t_i, i.e., the use value of the good $(r+\delta)\theta_i(t)$ plus capital gain $\dot{p}(t_i)$, is smaller than the opportunity cost of owing the good at the entry time, which is $rp(t_i)$. If, on the other hand, the benefit is larger than the cost, then the seller should wait and use the good himself until the moment they are equal to each other and sell at that time, condition (b).

It is clear now that if a seller of quality θ_i sells at time *t* then all sellers with qualities from the range $[\underline{\theta},\theta_i]$, who are in the market at time *t*, also prefer to sell at that time *t*. This allows us to define for any *t* a marginal seller $\theta(t)$ as the seller of the highest quality at that time:

$$\theta(t)=\sup_i\{\theta_i | i\in J_t\}=\tfrac{1}{r+\delta}\left(rp(t)-\dot{p}(t)\right),$$

or

$$\dot{p}(t)=rp(t)-(r+\delta)\theta(t). \tag{1}$$

Differentiating (1) yields:

$$\ddot{p}-r\dot{p}-\delta(r+\delta)\theta=r\dot{p}-(r+\delta)\dot{\theta}-r\dot{p}-\delta(r+\delta)\theta=-(r+\delta)\left(\dot{\theta}+\delta\theta\right),$$

which implies that the second order condition requires $\dot{\theta}+\delta\theta>0$ and, therefore, $e^{\delta t}\theta(t)>0$ to be an increasing function.

Now we are able to derive the main equation that must be satisfied along the equilibrium path. Let us consider an infinitely short time interval $(t,t+dt)$. All qualities that entered before and at time *t* from the interval $[\underline{\theta},\theta(t)]$ have already been traded and all qualities from $(\theta(t),\overline{\theta}]$ are still in the market. Expected utility at time $t+dt$ from buying goods of quality from the range $[\underline{\theta},\theta(t)]$ is

$$dt\cdot\int_{\underline{\theta}}^{\theta(t)}u\left(v(x)-p(t)\right)f(x)dx.$$

In addition to these, "new" goods, some "old" goods will be also traded. Those are goods that were in the market, which quality was higher than $\theta(t)$ at time *t*, and which quality becomes smaller than $\theta(t)+\dot{\theta}(t)dt$ at time $t+dt$. We have to calculate the expected utility of buying these goods.

Let us consider an infinitely short quality range $(\theta(t),\theta(t)+d\theta)$ at time *t*. All the goods of quality from the interval $\left(e^{\delta\tau}\,\theta(t),e^{\delta\tau}(\theta(t)+d\theta)\right)$ that entered at time $t-\tau$ will fall in that range after depreciation. The measure of such goods is $\mu\left(e^{\delta\tau}(\theta+d\theta)\right)-\mu\left(e^{\delta\tau}\,\theta\right)$ or, in first-order term, $d\theta\cdot e^{\delta\tau} f\left(e^{\delta\tau}\,\theta\right)$. Integrating out the corresponding density $e^{\delta\tau} f\left(e^{\delta\tau}\,\theta\right)$ over $\tau\in(0,t)$ yields the following density of goods of quality just above $\theta(t)$ at time *t*:

$$\int_0^t e^{\delta\tau} f\left(e^{\delta\tau}\,\theta\right)d\tau=\frac{1}{\delta\theta}\int_0^t f\left(e^{\delta\tau}\,\theta\right)d\left(e^{\delta\tau}\,\theta\right)=\frac{\mu\left(e^{\delta t}\,\theta\right)-\mu(\theta)}{\delta\theta}.$$

The expected utility from buying these goods is

$$\int_{\theta(t)-\delta\theta(t)dt}^{\theta(t)+\dot{\theta}(t)dt} u(v(x)-p(t))\frac{\mu\left(e^{\delta t}\,\theta\right)-\mu(\theta)}{\delta\theta}dx.$$

Taking the first order term yields

$$dt\cdot u(v(\theta(t))-p(t))\frac{\mu\left(e^{\delta t}\,\theta(t)\right)-\mu(\theta(t))}{\delta\theta(t)}\left(\dot{\theta}(t)+\delta\theta(t)\right).$$

Hence, the unconditional expected utility from buying a good at time $t+dt$ is

$$dt\cdot\left\{\int_{\underline{\theta}}^{\theta(t)} u(v(x)-p(t))f(x)dx+u(v(\theta(t))-p(t))\frac{\mu\left(e^{\delta t}\,\theta(t)\right)-\mu(\theta(t))}{\delta\theta(t)}\left(\dot{\theta}(t)+\delta\theta(t)\right)\right\},$$

which must be zero for all *dt*. Hence,

$$\int_{\underline{\theta}}^{\theta(t)} u(v(x)-p(t))f(x)dx+u(v(\theta(t))-p(t))\frac{\mu\left(e^{\delta t}\,\theta(t)\right)-\mu(\theta(t))}{\delta\theta(t)}\left(\dot{\theta}(t)+\delta\theta(t)\right)=0,$$

or

$$\dot{\theta}=\delta\theta\left(\frac{F(\theta,p)}{u(v(\theta)-p)\left(\mu\left(e^{\delta t}\,\theta\right)-\mu(\theta)\right)}-1\right),$$

where $F(\theta,p)\equiv-\int_{\underline{\theta}}^{\theta}u(v(x)-p)f(x)dx$ is the expected disutility of buying goods of quality from the range $[\underline{\theta},\theta]$ against the price *p* in the corresponding static Akerlof model. *F* is differentiable in both arguments and strictly increasing in *p* function. By definition, $F(\theta_S,\theta_S)=0$. Together with (1) we have finally obtained the following system

$$\begin{cases}\dot{p}=rp-(r+\delta)\theta\\ \dot{\theta}=\delta\theta\left(\dfrac{F(\theta,p)}{u(v(\theta)-p)\left(\mu\left(e^{\delta t}\,\theta\right)-\mu(\theta)\right)}-1\right)\end{cases},\qquad (2)$$

which describes the evolution of price and marginal quality along an equilibrium path.

If we rescale time by the parameter *r* as $\psi=rt$, then the system becomes

$$\begin{cases} \dfrac{dp}{d\psi} = p - \left(1 + \dfrac{\delta}{r}\right)\theta \\ \dfrac{d\theta}{d\psi} = \dfrac{\delta}{r}\theta\left(\dfrac{F(\theta, p)}{u(v(\theta) - p)\left(\mu\left(e^{\frac{\delta}{r}\psi}\theta\right) - \mu(\theta)\right)} - 1\right) \end{cases}.$$

One may easily note that the interest rate r and the physical depreciation rate δ can influence the solution only through the ratio $\frac{\delta}{r}$, which we will call the "relative depreciation rate". This is our first result.

Proposition 1. For any given level of $\frac{\delta}{r}$, the interest rate r and the physical depreciation rate δ only determine the speed of the evolution along an equilibrium path. The set of dynamic equilibria only depends on $\frac{\delta}{r}$.

Having established the way interest rate influences dynamic equilibria in what follows we take $r = 1$ without loss of generality assuming that now δ is a relative rate of depreciation.

4. Dynamic Equilibria for Perfectly Durable Goods

We start investigating equilibrium properties for the case of perfectly durable goods. In view of the fact that $\delta = 0$ is a singular point of (2) we have to take a limit of its second equation when $\delta \to 0$ to do this. As

$$\lim_{\delta \to 0} \frac{\mu\left(e^{\delta t}\theta\right) - \mu(\theta)}{\delta\theta} = \lim_{\delta \to 0} \frac{\theta\, e^{\delta t}\, tf\left(e^{\delta t}\theta\right)}{\theta} = tf(\theta)$$

uniformly for all finite t, system (2) could be written as

$$\begin{cases} \dot{p} = p - \theta \\ \dot{\theta} = \dfrac{F(\theta, p)}{tf(\theta)u(v(\theta) - p)} \end{cases}. \tag{3}$$

As $\dfrac{\partial F}{\partial p} > 0$ we define a function $\pi_S(\theta)$ as $F(\pi_S(\theta), \theta) = 0$, which is just a buyers' willingness to pay for a good of quality from the range $[\underline{\theta}, \theta]$ in the static model. Then, we define $a(\theta)$ as

$$a(\theta) \equiv \frac{d\pi_S(\theta)}{d\theta} = -\left(\frac{\frac{\partial F}{\partial \theta}}{\frac{\partial F}{\partial p}}\right)_{F(\theta,\pi_S(\theta))=0} = \frac{u(v(\theta)-\pi_S(\theta))f(\theta)}{\int_{\underline{\theta}}^{\theta} u'(v(x)-\pi_S(\theta))f(x)dx} > 0\,. \tag{4}$$

Hence, $\pi_S(\theta)$ is strictly increasing function.

Figure 1 shows the vector field of the system for some fixed $t > 0$, which is given by $\frac{dp}{d\theta} = \frac{\dot{p}}{\dot{\theta}} = t\frac{f(\theta)u(v(\theta)-p)(p-\theta)}{F(\theta,p)}$.

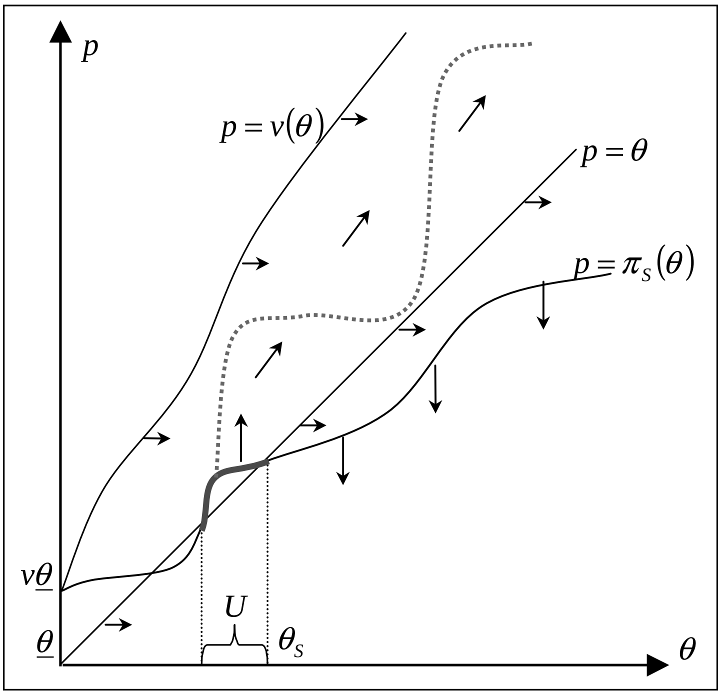

Figure 1.

As $p < v(\theta)$ no dynamic path that is a solution to system (3) can be above the line $p = v(\theta)$. On the other hand, for any solution to be a dynamic equilibrium it must satisfy $p \geq \theta$, i.e., the surplus of the marginal seller may not be negative. For all intermediate values of prices where $\theta < p < v(\theta)$, $\frac{dp}{d\theta} > 0$. Finally, if $F(p,\theta) = 0$ then $\dot{\theta} = 0$, $\dot{p} > 0$ and tangents at such points are vertical for any $t > 0$.

Multiplicity

We will show that there exists a neighborhood U such that for any $\theta_0 \in U$ system (3) with initials $\theta(0) = \theta_0$ and $p(0) = p_0 = \pi_S(\theta_0)$ has a solution $(\theta(t,\theta_0), p(t,\theta_0))$, like the solution denoted by the dotted line in Figure 1, with $p \geq \theta$. What we will prove then in Proposition

2 is that for all $\theta_0 \in U$ there exists a $T(\theta_0) > 0$ such that all equilibrium conditions are fulfilled, price and marginal quality increase over the time interval $(0, T(\theta_0))$ and either $\theta(T, \theta_0) = \overline{\theta}$ or $\theta(T, \theta_0) = p(T, \theta_0) > \theta_S$. In both cases $\theta(T, \theta_0)$ is the largest quality that can be traded in that equilibrium and we can extend $(\theta(t, \theta_0), p(t, \theta_0))$ in a periodic way, namely by defining $p(t+T) = p(t)$ and $\theta(t+T) = \theta(t)$, in this way we obtain a dynamic equilibrium where all goods from the range $[\underline{\theta}, \hat{\theta}]$ are traded, where $\hat{\theta} = \theta(T)$.

Proposition 2. For $\delta = 0$ there exists an infinite number of dynamic equilibria trading all goods from a certain range $[\underline{\theta}, \hat{\theta}$**]**, where $\hat{\theta} \in (\theta_S, \overline{\theta}$].

Proposition 2 implies that the repetition of the static equilibrium is the only stationary equilibrium. If we choose any arbitrary $\theta_0 < \theta_S$, the dynamic path will be such that eventually more than the static equilibrium amount of goods will be sold.

Equilibria Trading All Goods

So far, we have shown that for all distributions we can trade more than the static equilibrium quality if we allow for trade to take place over time. In this section we extend this result by showing that all goods can be traded if we relax the assumption about continuity of $p(t)$.

In the following Proposition 3 we show that there exists an infinite number of cyclical dynamic equilibria where all goods are traded at time $T, 2T, 3T, \ldots$

Proposition 3. For $\delta = 0$ there exists an infinite number of dynamic equilibria $(\theta(t), p(t))$ such that for some *T*:

a) $p(t+T) = p(t)$ and $\theta(t+T) = \theta(t)$;

b) $\theta(T) = \overline{\theta}$;

c) $\theta(t)$ and $p(t)$ are strictly increasing functions for all $t \in (0, T)$ except (at most) at a finite number of points $\{t^{(k)}\}_{k=1}^{K}$ where both functions are discontinuous.

Figure 2 represents a typical equilibrium path $\theta(t)$. Within each cycle *n*, where $t \in (nT, (n+1)T$], the path is piecewise continuous, i.e., $\theta(t)$ is a solution of (3) for every

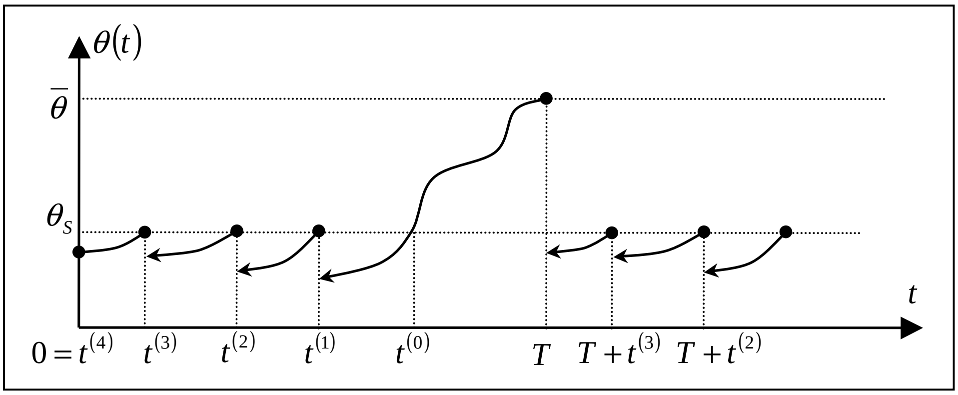

Figure 2.

subcycle $t \in (t^{(k)}, t^{(k-1)}]$, $k = 1,2,\ldots,K$, where K is a finite number defined in the proof of Proposition 3. The equilibrium construction is such that all sellers of quality θ_S earn the same discounted surplus by selling at $t = t^{(k)}$, $k = 0,1,\ldots,K-1$. Hence, they are indifferent between selling at each of these moments.

The discontinuities described in Figure 2 are used to build up enough time and high quality goods to allow the expected quality to improve enough to trade all goods. One may wonder whether these discontinuities are required for all distributions of quality. Next, we will show that for certain distributions we can construct infinitely many equilibrium paths with $\theta(t)$ and $p(t)$ being continuous and strictly increasing over the whole cycle $(0,T)$. In the proof of Proposition 2 we have defined $a = a(\theta_S)$. We will show that this parameter *a* plays a crucial role in analyzing when continuous price equilibria exist. First, it must be that $0 < a < 1$. To this end, consider the surplus of the marginal seller in the *static model*, denoted by $s^{(S)}$, as a function of *θ*: $s^{(S)}(\theta) \equiv p(\theta) - \theta = \pi_S(\theta) - \theta$, and, then

$$\frac{ds^{(S)}(\theta_S)}{d\theta} = a - 1.$$

Suppose then that $a > 1$. This would imply that $s^{(S)}(\theta) > 0$ in some right neighborhood of θ_S. But this contradicts the assumption that θ_S is the highest static equilibrium quality. Hence, generically, $a < 1$. Lastly, $a(\theta) > 0$ under assumptions 1, 2 and 3. The case where $a = 1$ is a non-generic case.

It turns out that the value $\frac{1-a}{a}$ determines the qualitative behavior of (θ, p) in the neighborhood of (θ_S, θ_S). Functions $\hat{x}(t)$ and $\hat{y}(t)$ that are the solutions of the corresponding linearized system behave quite differently depending on whether $\frac{1-a}{a}$ is smaller or larger than 1, i.e., whether *a* is larger or smaller than $\frac{1}{2}$. Figure 3 shows the

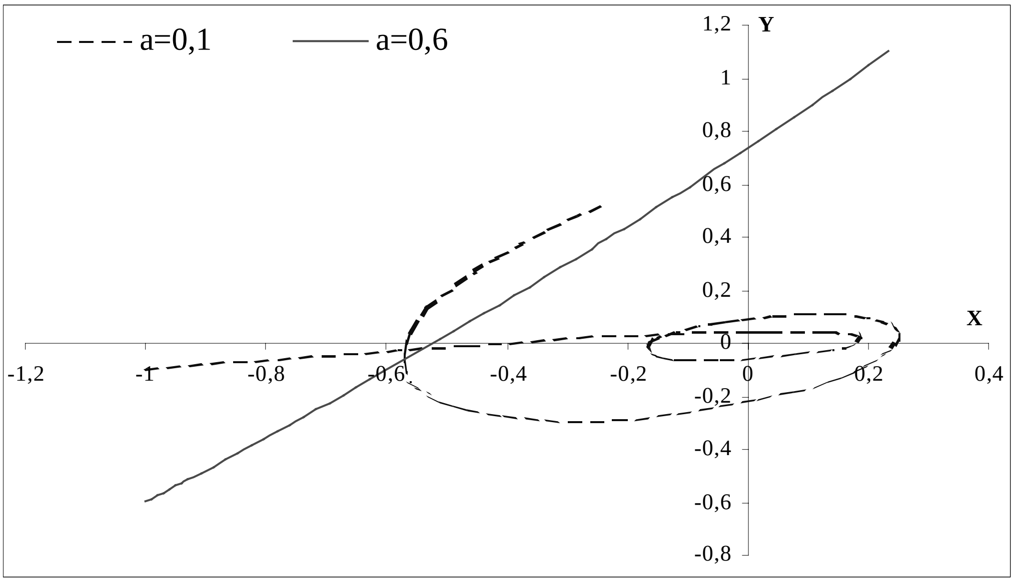

Figure 3.

solution $(\hat{x}, \hat{y})$ as a parametric function $\hat{y}(\hat{x})$ with the parameter *t*, for two different values of *a*, $a = 0.1$ and $a = 0.6$. One can see that in the former case $\hat{y}(\hat{x})$ oscillates around the origin so that the second order condition ($\dot{\theta} > 0$) is not satisfied. In case $a = 0.6$ $\hat{x}(t)$ and $\hat{y}(t)$ are increasing functions so that in the neighborhood of the static equilibrium quality, prices and marginal qualities are increasing functions as well.

Proposition 4 constructs for quality distributions with $a > \frac{1}{2}$ equilibria trading all goods where price and marginal quality are continuous in every cycle.

Proposition 4. For $\delta = 0$ and $a > \frac{1}{2}$ there exists an infinite number of cyclical dynamic equilibria $(\theta(t), p(t))$ such that:

a) $p(t+T) = p(t)$ and $\theta(t+T) = \theta(t)$;

b) $\theta(T) = \overline{\theta}$;

c) $\theta(t)$ and $p(t)$ are strictly increasing and continuous functions on $(0, T)$.

The result obtained in Proposition 4 says that in case $a > \frac{1}{2}$ we can choose θ_0 sufficiently close to θ_S such that we do not need to build more than one subcycle in order to build up enough time and high quality goods to allow the expected quality to improve enough to trade all goods. Basically, the condition $a > \frac{1}{2}$ says that in a neighborhood of

θ_S there is a sufficient mass of goods so that at the moment when the marginal quality becomes larger then θ_S, the marginal seller is able to make a positive surplus.

5. Dynamic Equilibria for Imperfectly Durable Goods

Now we turn back to analyze system (2) for the case where the good under consideration is not durable, i.e., $\delta > 0$. We start our analysis by considering two extreme cases, when δ is very large and when δ is close to zero. Then we will consider intermediate values of δ. As the expressions to be discussed are relatively difficult to interpret we use the linear model, where $v(\theta) = v\theta$, $u(m) = m$ and $f(\theta) = 1$, for illustration purposes. In the linear model $1 < v < 2$ and $\mu(\theta) = \theta - \underline{\theta}$.

High Relative Depreciation Rate

Suppose first that $\delta > \max_\theta \left(\frac{v(\theta)-\theta}{\theta}\right) \equiv \overline{\delta}$. We will show that in this case, we get back a type of traditional "Akerlof-result": after some moment in time no new goods that come to the market will be traded. The argument can be sketched as follows. In any continuous equilibrium path $(\theta(t), p(t))$ price decreases over time as $\dot{p} = p - (1+\delta)\theta \le v(\theta) - (1+\delta)\theta < 0$. This price dynamics leads to the following: either at a certain moment t' price $p(t')$ becomes smaller than $\theta(t')$, i.e., the marginal seller earns a negative surplus $p(t') - \theta(t')$ and the path $(\theta(t), p(t))$ cannot be a dynamic equilibrium, or price $p(t')$ becomes equal to $v(\underline{\theta})$. In the latter case the marginal quality at that moment $\theta(t')$ must be equal to $\underline{\theta}$, in other words, only the lowest quality $\underline{\theta}$ is traded at that time and there is no uncertainty about quality. Indeed, if it had been $\theta(t') > \underline{\theta}$, then some higher quality goods would have been traded and the price must have been higher than $v(\underline{\theta})$.

From that moment on the price $p(t)$ will always be strictly smaller than $v(\underline{\theta})$, hence, the marginal quality $\theta(t)$ will always be strictly smaller than $\underline{\theta}$. Therefore, the second dynamic equation of (2) is not valid any more. This equation should be replaced by a simple dependence $p = v(\theta)$ as $\theta \le \underline{\theta}$ for all $t > t'$ and the only quality that will be traded at time t is $\theta(t)$, hence the price must be equal to $v(\theta)$. The first equation of (2) is still valid and we get the following system

$$\begin{cases} \dot{p} = p - (1+\delta)\theta \\ p = v(\theta) \end{cases},$$

which can be rewritten as

$$\dot{\theta} = \frac{v(\theta) - (1+\delta)\theta}{v'(\theta)}. \tag{5}$$

Whatever solution with initials $\theta(t') = \underline{\theta}$ the above equation has it is clear that $\theta < \underline{\theta}$ for all $t > t'$, i.e., no new goods that just entered the market will be traded.

One may wonder what happens if we allow $\theta(t)$ to be discontinuous. It turns out that the results remain the same if we require that trade takes place at every moment in time. Indeed, suppose that at a certain time moment t' the marginal quality is discontinuous. Then, the price at that time must be either continuous, hence the first equation of (2) is valid, or the price has a strictly negative increment. In the latter case the first equation of (2) will be working just after time t' and, again, $\dot{p} = p - (1+\delta)\theta \le v(\theta) - (1+\delta)\theta < 0$. Therefore, in both cases the above analysis leads to no new goods trading after a certain time. The reason why price may not have a positive increment at t' is that in this case all the sellers who sell just before t' would wait the instant price increase and there would be no trade just before t'. The above argument is summarized in the following proposition.

Proposition 5. If $\delta > \overline{\delta}$ then there is no trade of new goods after a certain time moment along any dynamic equilibrium path where trade takes place at every moment.

For the linear model the expression for $\overline{\delta}$ simplifies to $\overline{\delta} = v - 1$ and equation (5) takes the form $\dot{\theta} = -\frac{\delta - (v-1)}{v}\theta$. The solution in this case exponentially converges to zero, which implies that the marginal quality traded becomes almost worthless. Hence, when time passes, almost no gains from trade are realized as in the pure adverse selection result in the static model analyzed by Akerlof (1970).

Low Relative Depreciation Rate

For the case of a relatively small value of δ, we rewrite system (2) as

$$\begin{cases} \dot{p} = p - \theta - \delta\theta \\ \dot{\theta} = \dfrac{F(\theta, p)}{u(v(\theta) - p)\dfrac{\mu(e^{\delta t}\theta) - \mu(\theta)}{\delta\theta}} - \delta\theta = \dfrac{F(\theta, p)}{u(v(\theta) - p)tf(e^{\xi_1 \delta t}\theta)} - \delta\theta \end{cases},$$

where $\xi_1 \in (0,1)$. Then,

$$\frac{F(\theta,p)}{u(v(\theta)-p)tf\left(e^{\xi_1\delta t}\theta\right)} = \frac{F(\theta,p)}{u(v(\theta)-p)tf(\theta)}\left(1-\delta\frac{\theta g\xi t\, e^{\xi_1\xi_2\delta t}}{f\left(e^{\xi\delta t}\theta\right)}\right),$$

where

$$g = \frac{f\left(e^{\xi_1\delta t}\theta\right)-f(\theta)}{\theta\left(e^{\xi_1\delta t}-1\right)} \in \left(-M_f, M_f\right)$$

for all $\theta < \overline{\theta} - \varepsilon$, and $\xi_2 \in (0,1)$. Hence, system (2) can be written as

$$\begin{cases} \dot{p} = p - \theta - \delta\theta \\ \dot{\theta} = \dfrac{F(\theta,p)}{u(v(\theta)-p)tf(\theta)} - \delta\left(\dfrac{AF(\theta,p)}{u(v(\theta)-p)tf(\theta)} - \theta\right), \end{cases}$$

where A is bounded for all $\theta < \overline{\theta} - \varepsilon$. Hence, any solution of (2), being written as $(\theta(t,\theta_0,\delta), p(t,\theta_0,\delta))$, uniformly converges to the solution $(\theta(t,\theta_0), p(t,\theta_0))$ of (3) for all θ_0 and all t such that $\theta(t,\theta_0) < \overline{\theta} - \varepsilon$ when δ converges to zero. Now, let us define $\tilde{t}(\varepsilon)$ as $\theta(\tilde{t}(\varepsilon),\theta_0,\delta) = \overline{\theta} - \varepsilon$. It follows that $\lim\limits_{\varepsilon\to 0}\dot{\theta}(t,\theta_0,\delta) = +\infty$ for $t \geq \tilde{t}(\varepsilon)$. Hence, $\exists T^{\delta} > \tilde{t}$ such that $\theta(T^{\delta},\theta_0,\delta) = \overline{\theta}$. We can state this result as the following proposition.

Proposition 6. If $a > \frac{1}{2}$, then there exists a $\underline{\delta} > 0$ such that for all $\delta \in [0, \underline{\delta})$ there exist an infinite number of cyclical dynamic equilibria $(\theta(t,\delta), p(t,\delta))$ such that for some $T^{\delta} > 0$:

a) $p(t+T^{\delta}) = p(t)$ and $\theta(t+T^{\delta}) = \theta(t)$;

b) $\theta(T^{\delta}) = \overline{\theta}$;

c) $\theta(t,\delta)$ and $p(t,\delta)$ are continuous functions on $t \in (0, T^{\delta})$.

Let us consider an example, where $\underline{\theta} = 10$, $\overline{\theta} = 20$, $f(\theta) = e^{0.1(\theta-\underline{\theta})}$, $v(\theta) = 1.2\theta$, $u(m) = m$ and $\delta = 0.01$. Figure 4 shows that all goods are sold in finite time. Proposition 6 generalizes this example and argues that we can extend the conclusion of Proposition 4 to the case where δ is small enough.

Although the result obtained in Proposition 6 is proved for the case $a > \frac{1}{2}$ only, we think that it is valid for any value of *a*. As for $0 < a \leq \frac{1}{2}$ without depreciation the equilibrium path is piece-wise continuous the proof would be much more technical without adding new insights.

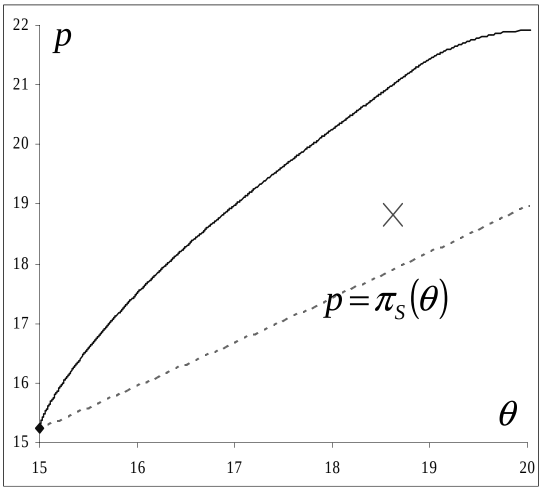

Figure 4: An example with a unique unstable steady-state ($\underline{\theta} = 10$, $\overline{\theta} = 20$, $f(\theta) = \mathrm{e}^{0.1(\theta - \underline{\theta})}$, $v(\theta) = 1.2\theta$, $u(m) = m$, $\delta = 0.01$).

Intermediate Range of Relative Depreciation Rate

Having established dynamic equilibrium properties for two extreme cases, when the good is almost perfectly durable or when it depreciates quickly, we will analyze the properties of the equilibrium path at intermediate values of the relative depreciation rate. Having initially been time-dependent the system becomes autonomous for large *t* when $\theta\,\mathrm{e}^{\delta t} \geq \overline{\theta}$, which is guaranteed by the second order condition $\dot{\theta} + \delta\theta > 0$, and, therefore, $\mu\left(\theta\,\mathrm{e}^{\delta t}\right) = \mu\left(\overline{\theta}\right)$. We consider the corresponding autonomous system:

$$\begin{cases} \dot{p} = rp - (r + \delta)\theta \\ \dot{\theta} = \delta\theta\left(\dfrac{F(\theta, p)}{u(v(\theta) - p)\left(\mu\left(\overline{\theta}\right) - \mu(\theta)\right)} - 1\right) \end{cases} \tag{6}$$

We will show that there exists a steady state $\left(\theta^*, p^*\right)$, not necessarily unique, such that $(\theta(t), p(t)) = \left(\theta^*, p^*\right)$ is a solution of system (6) for all $t \geq \frac{1}{\delta}\ln\frac{\overline{\theta}}{\theta^*}$ when $\theta(t)\mathrm{e}^{\delta t} \geq \overline{\theta}$. Indeed, solving for $\dot{\theta}(t) = \dot{p}(t) = 0$ yields:

$$\begin{cases} p = (1 + \delta)\theta \\ F(\theta, (1 + \delta)\theta) = u(v(\theta) - (1 + \delta)\theta)\left(\mu\left(\overline{\theta}\right) - \mu(\theta)\right) \end{cases} \tag{7}$$

Suppose that $\delta < \frac{v(\underline{\theta}) - \underline{\theta}}{\underline{\theta}} \equiv \overline{\overline{\delta}} \leq \overline{\delta}$. As $F(\underline{\theta}, (1 + \delta)\underline{\theta}) = 0 < u(v(\underline{\theta}) - (1 + \delta)\underline{\theta})\left(\mu\left(\overline{\theta}\right) - \mu(\underline{\theta})\right)$ and, on the other hand, $F\left(\overline{\theta}, (1 + \delta)\overline{\theta}\right) > 0 = u\left(v\left(\overline{\theta}\right) - (1 + \delta)\overline{\theta}\right)\left(\mu\left(\overline{\theta}\right) - \mu\left(\overline{\theta}\right)\right)$, there exists at least one point $\theta^* \in \left(\underline{\theta}, \overline{\theta}\right)$ that solves (7). Hence, there is at least one steady state.

For the linear model $F(\theta, p) = (\theta - \underline{\theta})\left(p - \frac{1}{2}v(\theta + \underline{\theta})\right)$ and system (7) simplifies to

$$\begin{cases} p = (1+\delta)\theta \\ (\theta - \underline{\theta})\left((1+\delta)\theta - \frac{1}{2}v(\theta + \underline{\theta})\right) = \left(v\theta - (1+\delta)\theta\right)\left(\overline{\theta} - \theta\right) \end{cases}.$$

Having been considered as a function of θ the left-hand side of the second equation is a quadratic function decreasing at $\underline{\theta}$ where it equals to zero and increases at $\overline{\theta}$ where it is positive. Then, the right-hand side is a quadratic function as well, which has zeros at 0 and $\overline{\theta}$ and is positive between them. Hence, the system has two solutions, the first one between 0 and $\underline{\theta}$, which is not feasible, and the second, which is being searched for, between $\underline{\theta}$ and $\overline{\theta}$. So, for the linear model the steady state is unique.

The natural question about the local stability of the singular point (θ^*, p^*) for the general model can be resolved by taking a linear analysis of (6) in the neighborhood of (θ^*, p^*). The corresponding linear system is:

$$\begin{cases} \dot{y} = y - (1+\delta)x \\ \dot{x} = \dfrac{\delta\theta^*}{u\left(v(\theta^*) - p^*\right)\left(\mu(\overline{\theta}) - \mu(\theta^*)\right)} \times \Big\{ - xu'\left(v(\theta^*) - p^*\right)\left(\mu(\overline{\theta}) - \mu(\theta^*)\right)v'(\theta^*) + \\ \quad + y\left(\int_{\underline{\theta}}^{\theta^*} u'\left(v(x) - p^*\right) f(x)dx + u'\left(v(\theta^*) - p^*\right)\left(\mu(\overline{\theta}) - \mu(\theta^*)\right)\right)\Big\} \end{cases},$$

where $p = p^* + y$ and $\theta = \theta^* + x$. (θ^*, p^*) is stable if

$$1 + \frac{-\delta\theta^*}{u\left(v(\theta^*) - p^*\right)\left(\mu(\overline{\theta}) - \mu(\theta^*)\right)} u'\left(v(\theta^*) - p^*\right)\left(\mu(\overline{\theta}) - \mu(\theta^*)\right)v'(\theta^*) < 0,$$

which can be stated as $\delta > \dfrac{u\left(v(\theta^*) - (1+\delta)\theta^*\right)}{u'\left(v(\theta^*) - (1+\delta)\theta^*\right)v'(\theta^*)\theta^*}$. The latter inequality cannot be solved at the current level of generality. For the linear model, the inequality simplifies to

$$\delta > \frac{v-1}{v+1} \equiv \delta^*.$$

For the general model we can show by means of examples the richness of qualitatively new phenomena that may emerge along possible equilibrium paths. The examples are presented in Figure 4 and Figure 5, where a cross denotes a steady state. Figure 4 shows a case where a steady state exists, but is unstable. An equilibrium path is depicted where all goods are eventually traded, even new goods of relatively high quality. Figure 5 sows that a stable steady state can be either below the static equilibrium as in the left graph, or above it as in the right graph. Unlike the static equilibrium in these stationary equilibria all

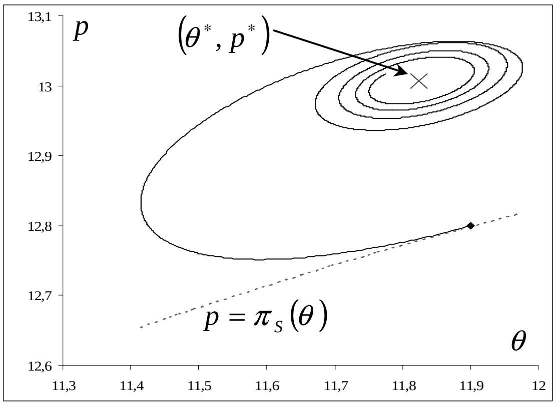

Figure 5a.

$\underline{\theta}=10$, $\overline{\theta}=20$, $f(\theta)=\mathrm{e}^{-(\theta-\underline{\theta})}$, $v(\theta)=1.2\theta$, $u(m)=m$, $\delta=0.1$, $\theta^*\approx 11.82$ and $\theta_S\approx 13.01$

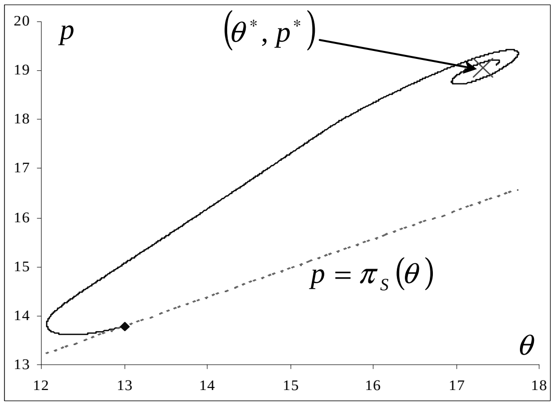

Figure 5b.

$\underline{\theta}=10$, $\overline{\theta}=30$, $f(\theta)=\mathrm{e}^{-0.01(\theta-\underline{\theta})}$, $v(\theta)=1.2\theta$, $u(m)=m$, $\delta=0.1$, $\theta^*\approx 17.32$ and $\theta_S\approx 14.94$

qualities are eventually traded in the market. However, owners of qualities $\theta_i>\theta^*$ first wait until their good has depreciated to θ^* before selling.

Figure 6 finally shows an example of unstable steady state with a periodic cycle. Here δ is "slightly" below δ^*. Then (θ^*, p^*) is not stable but there exists a cycle, i.e., a “closed loop” or a periodical solution of the corresponding autonomous system. In the long run price as well as marginal quality fluctuate with an asymptotically constant period.

The three cases discussed above are summarized for the simple case of the linear model in Figure 7, where the system (2) takes the form

$$\begin{cases} \dot{p}=p-(r+\delta)\theta \\ \dot{\theta}=\delta\theta\left(\dfrac{(\theta-\underline{\theta})\left(p-\frac{v}{2}(\theta+\underline{\theta})\right)}{(v\theta-p)\left(\mathrm{e}^{\delta t}-1\right)\theta}-1\right). \end{cases}$$

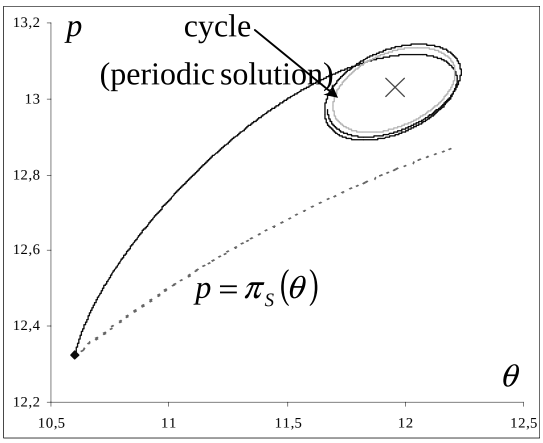

Figure 6.

$\underline{\theta}=10$, $\overline{\theta}=20$, $f(\theta)=\mathrm{e}^{-(\theta-\underline{\theta})}$, $v(\theta)=1.2\theta$, $u(m)=m$ and $\delta=0.09$

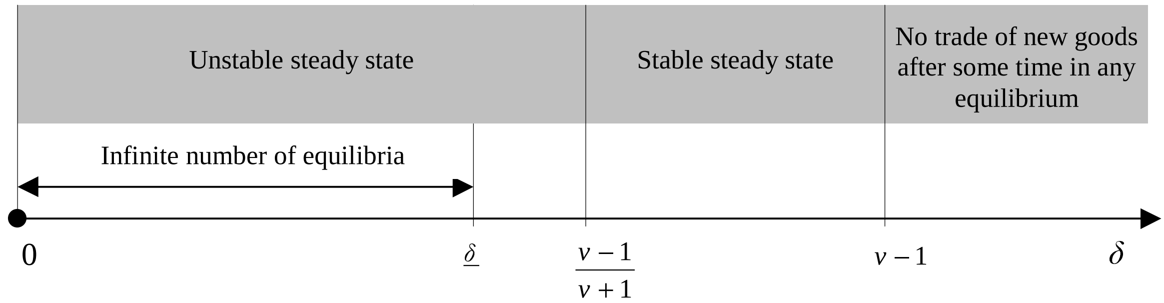

Figure 7.

The figure shows the different possible equilibrium paths for different values of the relative depreciation rate. There are four relevant regions. First, for all δ larger than $v-1$ trade of new goods stops after some time. Second, if δ smaller than $v-1$ bus still larger than $\frac{v-1}{v+1}$ there exists a steady state, which attracts the evolution path, and in the long run price as well as marginal quality are constant. Third, for all δ smaller than $\frac{v-1}{v+1}$ the steady state is not stable but, and this is the fourth region, if δ is small enough than there are infinite number of dynamic equilibria where marginal quality increases up to the highest quality $\overline{\theta}$.

6. Conclusions

In this paper, we have studied the role the interest rate and the physical depreciation rate play in dynamic competitive equilibria under adverse selection. Without physical depreciation, the infinite repetition of the static (Akerlof) equilibrium is one of the equilibria in the dynamic model. There are, however, infinitely many other equilibria where all goods are sold within finite time after entering the market. These results change when physical depreciation is taken into account. When the depreciation rate is small, repetition of the static equilibrium stops being a dynamic equilibrium, but there are still infinitely many other equilibria where all goods are sold. When the depreciation rate is relatively high, all equilibria exhibit no trade of new goods after some finite moment in time. At intermediate values of the depreciation rate, new stationary equilibria may emerge where all goods are eventually traded. In this type of equilibrium, owners of high quality goods sell only after the good has depreciated enough. All these results are independent of the value of the interest rate.

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Appendix.

Proof of Proposition 2

Under assumptions 1, 2 and 3 for any $t_0 > 0$ and (θ_0, p_0) such that $v(\theta_0) \neq p_0$ system (3) with initial conditions $\theta(t_0) = \theta_0$ and $p(t_0) = p_0$ has a unique solution $(\theta(t,\theta_0, p_0, t_0), p(t,\theta_0, p_0, t_0))$, which is continuous w.r.t. θ_0 and p_0. Considering $\dot{\theta}$ as a function of *θ*, *p* and *t*, i.e., $\dot{\theta} = \Theta(\theta, p, t) \equiv \frac{F(\theta, p)}{tf(\theta)u(v(\theta) - p)}$, allows us to write:

$$\lim_{t\to 0}(t\Theta(\theta, p, t)) = \lim_{t\to 0}\frac{F(\theta, p)}{f(\theta)u(v(\theta) - p)} = \frac{F(\theta(0), p(0))}{f(\theta(0))u(v(\theta(0)) - p(0))} < \infty .$$

Hence, system (3) has a solution even for $t_0 = 0$, but not necessarily unique. The uniqueness is guaranteed by the fact that $t\Theta(\theta, p, t) \not\equiv \theta + const$. Finally, that solution is differentiable at $t = 0$ as long as $\lim_{t\to 0}(t\Theta(\theta, p, t)) = 0$, i.e., $p_0 = \pi_S(\theta_0)$, and we will denote it as $(\theta(t,\theta_0), p(t,\theta_0)) \equiv (\theta(t,\theta_0,\pi_S(\theta_0),0), p(t,\theta_0,\pi_S(\theta_0),0))$. Indefiniteness of $\dot{\theta}(0,\theta_0)$ is resolved by continuity:

$$\dot{\theta}(0,\theta_0) = \lim_{t\to 0}\dot{\theta}(t,\theta_0) = \lim_{t\to 0}\frac{F(\theta, p)}{tf(\theta)u(v(\theta) - p)} = \frac{1}{f(\theta_0)u(v(\theta_0) - p_0)}\lim_{t\to 0}\frac{F(\theta, p)}{t}.$$

Taking the latter limit explicitly yields:

$$\lim_{t\to 0}\frac{F(\theta, p)}{t} = \lim_{t\to 0}\frac{d}{dt}F(\theta, p) = \frac{\partial F}{\partial \theta}(\theta_0, p_0)\dot{\theta}(0,\theta_0) + \frac{\partial F}{\partial p}(\theta_0, p_0)\dot{p}(0,\theta_0) =$$

$$= -f(\theta_0)u(v(\theta_0) - p_0)\dot{\theta}(0,\theta_0) + \dot{p}(0,\theta_0)\int_{\underline{\theta}}^{\theta_0} u'(v(x) - p_0)f(x)dx.$$

Hence, $\dot{\theta}(0,\theta_0) = \frac{1}{2a(\theta_0)}\dot{p}(0,\theta_0)$, where $a(\theta_0)$ is defined in (4) and, therefore,

$$\left(\frac{dp}{d\theta}\right)_{t=0} = \frac{\dot{p}_0}{\dot{\theta}_0} = 2a(\theta_0) > a(\theta_0) = \left(\frac{d\pi_S}{d\theta}\right)_{t=0}.$$

This implies that for small $t > 0$ $p(t,\theta_0) > \pi_S(\theta(t,\theta_0))$ as long as $\dot{p}_0 = \pi_S(\theta_0) - \theta_0 > 0$.

Now we define $\underline{\theta_0}$ as $\underline{\theta_0} = \inf\{\theta' | \forall\theta \in (\theta', \theta_S) : \pi_S(\theta) > \theta\}$ and $\hat{U}$ as $\hat{U} = (\underline{\theta_0}, \theta_S)$ such that for all $\theta_0 \in \hat{U}$ $\pi_S(\theta_0) > \theta_0$.

Finally, we will show that there exists a neighborhood $U \subset \hat{U}$ such that for any solution $(\theta(t,\theta_0), p(t,\theta_0))$ of (3), where $\theta_0 \in U$, there exists a time $T(\theta_0) > 0$ such that for all $t \in (0,T)$ $p > \theta$ and $(p(T) - \theta(T))(\overline{\theta} - \theta(T)) = 0$, i.e., either all goods are sold or the marginal surplus is zero at time T. If this were not the case then there would have been $\lim_{t\to\infty} p = \lim_{t\to\infty}\theta = \theta'$ as p and θ are increasing and bounded: $\theta < \bar{\theta}$, $p < v(\theta) \le v(\bar{\theta})$. But then the equation $\dot{\theta} = \dfrac{F(\theta, p)}{tf(\theta)u(v(\theta)-p)}$ for large t becomes

$$\dot{\theta} = \frac{1}{t}\left(\frac{F(\theta',\theta')}{f(\theta')u(v(\theta')-\theta')} + \varepsilon(t)\right),$$

where $\lim_{t\to\infty}\varepsilon(t) = 0$, and, therefore,

$$\theta(t) > \frac{F(\theta',\theta')}{2f(\theta')u(v(\theta')-\theta')}\ln t + const$$

for sufficiently large t. Hence, $\lim_{t\to\infty}\theta(t) = \infty$ unless $F(\theta',\theta') = 0$, i.e., $\theta' = \theta_S$.

In order to rule out the possibility that $\theta' = \theta_S$ (and, hence, that $\theta(t,\theta_0)$ and $p(t,\theta_0)$ converge to $\theta' < \bar{\theta}$) we rewrite system (3) as follows

$$\begin{cases} \dot{p} = (p-\theta_S) - (\theta - \theta_S) \\ \dot{\theta} = \dfrac{1}{at}\left((p-\theta_S) - a(\theta-\theta_S) + \begin{pmatrix}(\theta-\theta_S)\\(p-\theta_S)\end{pmatrix}' \mathbf{B}(\theta, p)\begin{pmatrix}(\theta-\theta_S)\\(p-\theta_S)\end{pmatrix}\right) \end{cases},$$

where $a = a(\theta_S)$ and $\|\mathbf{B}(\theta,p)\| < \infty$ uniformly in a certain neighborhood $U \subset \hat{U}$ of θ_S. Thus, for all $\theta_0 \in U$ the solution $(\theta(t,\theta_0), p(t,\theta_0))$ can be written as

$$\begin{cases} \theta(t,\theta_0) = \theta_S + (\theta_S - \theta_0)\hat{x}(t) + o(\theta_S - \theta_0) \\ p(t,\theta_0) = \theta_S + (\theta_S - \theta_0)\hat{y}(t) + o(\theta_S - \theta_0) \end{cases},$$

where $(\hat{x}(t), \hat{y}(t))$ solves the corresponding linearized system

$$\begin{cases} \dot{y} = y - x \\ \dot{x} = \frac{1}{at}(y - ax) \end{cases} \tag{A.1}$$

with initials $x(0) = -1$, $y(0) = -a$. Defining $k(t) = \frac{y(t)}{x(t)}$ allows us to rewrite (A.1) as $\dot{k} = -\left(1 - k - k\frac{a-k}{at}\right)$. As $\theta < \theta_S$, $p < \theta_S$ and $p > \pi_S(\theta)$ for all $t \in (0,\infty)$ then $x < 0$, $y < 0$ and $y > ax$. Hence, $k \in (0,a)$ and $\dot{k} < -\frac{1-a}{2} < 0$ for sufficiently large t. Therefore, $\lim_{t\to\infty} k(t) = -\infty$, which contradicts $k \in (0,a)$.

Hence, for any $\theta_0 \in U$ $\exists T(\theta_0) > 0$ such that either $\theta(T,\theta_0) = \bar{\theta}$ or $\theta(T,\theta_0) = p(T,\theta_0)$. In both cases we extend (θ, p) in a periodic way, namely $p(t+T) = p(t)$ and

$\theta(t+T)=\theta(t)$. In order to show that $\theta(T)>\theta_S$ when $\theta(T)=p(T)<\overline{\theta}$ let us consider two cases.

a) $\theta(T)=\theta_S$. This contradicts with the uniqueness of the solution with initials $\theta(T)=\theta_S$. Indeed, we always have a static solution $\theta(t,\theta_S)=p(t,\theta_S)=\theta_S$ and we have found another, namely $(\theta(t,\theta_0),p(t,\theta_0))$, such that $\theta(T,\theta_0)=p(T,\theta_0)=\theta_S$.

b) $\theta(T)<\theta_S$. This implies that $\pi_S(\theta(T))>\theta(T)$, which can never happen as for small t $\pi_S(\theta(t))<\theta(t)$ and no solution may cross the curve $p=\pi_S(\theta)$ from above at $\theta<\theta_S$.

Hence, $\theta(T)>\theta_S$.

As we have obtained a discontinuous function $p(t)$, we lost the sufficiency of the first and second order conditions. So, we must check the optimality of the stipulated sellers' behavior directly.

Let us take a seller i with quality θ_i and entry time $t_i\in[nT,(n+1)T)$, where n is the entry cycle's number. If $\theta_i>\theta(T)$ then he will never sell, which is clearly optimal for him. If, on the other hand, $\theta_i<\theta(T)$ then there are two possibilities.

a) $\theta_i>\theta(t_i)$. In this case he maximizes his surplus by selling in the current cycle n at time $\tau_i(\theta_i)$, where $\theta(\tau_i)=\theta_i$, and getting $s_i(\tau_i)=s(\tau_i)$, see Figure A.1 (a), and it follows that $\tau_i>t_i$. Indeed, within a cycle the first and second order conditions still work so there is a unique optimal selling time τ_i. If he, however, had been waiting for the next cycle $(n+1)$ he would have chosen time $\tau_i'(\theta_i)=\tau_i+T$ to sell, where $\theta(\tau_i')=\theta_i$, and got $s_i(\tau_i')=s(\tau_i')$. But $p(\tau_i)=p(\tau_i+T)=p(\tau_i')$ and the seller i will certainly choose the earlier time τ_i.

b) $\theta_i<\theta(t_i)$. In this case let us first investigate the marginal surplus function $s(t)\equiv e^{-t}(p-\theta)$. Differentiating yields $\dot{s}=-e^{-t}\dot{\theta}$ and, finally,

$$\frac{ds}{d\theta}=\frac{\dot{s}}{\dot{\theta}}=-e^{-t}.$$

Although the above expression has been obtained only for $t\in[0,T]$ it holds for any $t>0$. To see this, suppose $t\in(nT,(n+1)T]$. It then follows that

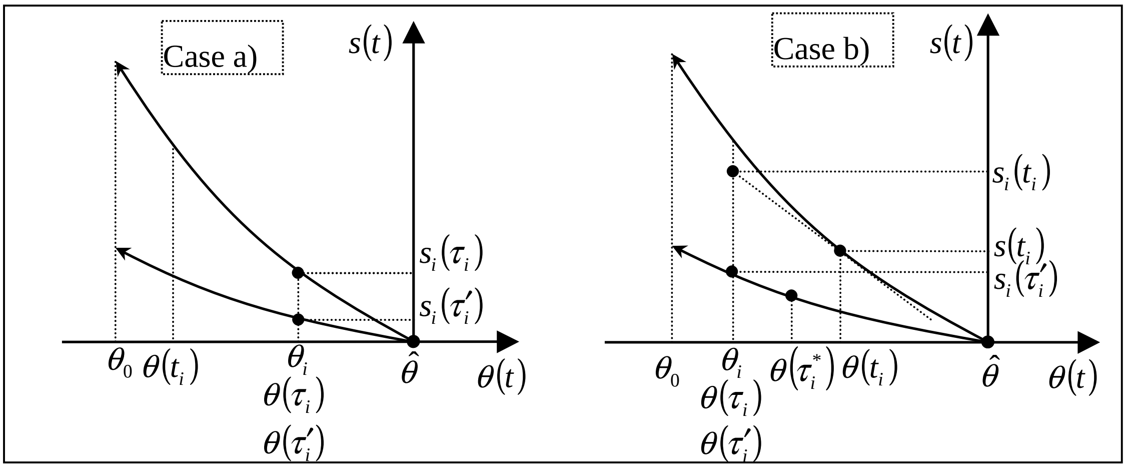

Figure A.1.

$$\frac{ds}{d\theta}(t) = -\mathrm{e}^{-nT}\frac{ds}{d\theta}(t-nT) = -\mathrm{e}^{-nT}\mathrm{e}^{-(t-nT)} = -\mathrm{e}^{-t} \text{ for } t > 0. \quad \text{(A.2)}$$

Hence, $s(t)$ is a positive, decreasing and convex function on $(nT,(n+1)T)$. These properties allow us to validate the maximum principle across different continuous segments of an equilibrium path.

Now let us define $\tau_i(\theta_i)$ such that $\tau_i \in (nT,(n+1)T]$ and $\theta(\tau_i) = \theta_i$, in this case $\tau_i < t_i$, see Figure A.1(b). If seller *i* sells immediately after the entry, i.e., at time t_i, he gets

$$s_i(t_i) = \mathrm{e}^{-t_i}(p(t_i) - \theta_i) = \mathrm{e}^{-t_i}(p(t_i) - \theta(t_i)) + \mathrm{e}^{-t_i}(\theta(t_i) - \theta_i) = s(t_i) + \mathrm{e}^{-t_i}(\theta(t_i) - \theta_i),$$

while if he waits until the next cycle $(n+1)$ he will choose time $\tau_i'(\theta_i) = \tau_i + T$ to sell, where $\theta(\tau_i') = \theta_i$, and, therefore, his surplus becomes

$$s_i(\tau_i') = \mathrm{e}^{-\tau_i'}(p(\tau_i') - \theta_i) = \mathrm{e}^{-\tau_i}(p(\tau_i') - \theta(\tau_i')) = s(\tau_i').$$

In order to show that $s_i(t_i) > s_i(\tau_i)'$ for all $\theta_i < \theta(t_i)$ let us consider $\Delta(\theta_i) \equiv s_i(t_i) - s_i(\tau_i')$ as a function of θ_i and apply the mean-value theorem:

$$s(\tau_i') = s(t_i + T) + (\theta(\tau_i') - \theta(t_i + T))\tfrac{ds}{d\theta}(\tau_i^*) = \mathrm{e}^{-T}s(t_i) + (\theta(t_i) - \theta_i)\mathrm{e}^{-\tau_i^*}$$

for some $\tau_i^* \in (\tau_i', t_i + T)$. Then

$$\begin{aligned}\Delta(\theta_i) &= s(t_i) + \mathrm{e}^{-t_i}(\theta(t_i) - \theta_i) - \mathrm{e}^{-T}s(t_i) - (\theta(t_i) - \theta_i)\mathrm{e}^{-\tau_i^*} = \\ &= s(t_i)(1 - \mathrm{e}^{-T}) + (\theta(t_i) - \theta_i)(\mathrm{e}^{-t_i} - \mathrm{e}^{-\tau_i^*}) > 0.\end{aligned}$$

Therefore, we have shown that for any $\theta_0 \in U$ $(\theta(t,\theta_0), p(t,\theta_0))$ constitutes a dynamic equilibrium trading all goods from the range $[\underline{\theta},\hat{\theta}]$, where $\hat{\theta} = \theta(T) > \theta_S$. ■

In order to prove the following propositions we need the following lemma.

Lemma 1. For any numbers $\theta_0 \in [\theta_S, \overline{\theta})$ and $p_0 \in (\theta_0, v(\theta_0))$ there exists $\hat{t} > 0$, depending on $p_0 - \theta_0$, such that for all $t_0 > \hat{t}$ system (3) has a unique solution with initials $\theta(t_0) = \theta_0$ and $p(t_0) = p_0$. Moreover, there exists a finite time $T > \hat{t}$ such that $\theta(T) = \overline{\theta}$.

Proof. Under assumptions 1,2 and 3 for any t_0 system (3) has a unique solution passing through (θ_0, p_0), where $\theta_0 \geq \theta_S$ and $p_0 \in (\theta_0, v(\theta_0))$. All we need to show then is the existence of *T* if t_0 is taken to be sufficiently large. We define $\alpha \equiv \min\{p_0 - \theta_0, \frac{1}{2}\varepsilon_v\}$ and

$$\hat{t} \equiv \frac{2F(\theta_S, v(\overline{\theta}))}{\varepsilon_f \varepsilon_u \varepsilon_v a}. \tag{A.3}$$

As α is a function of $p_0 - \theta_0$ so is $\hat{t}$. Now let us consider the solution mentioned above when $t_0 > \hat{t}$ and suppose that $p(t) - \theta(t) = \alpha$ for some $t \geq t_0$. Then

$$\frac{d}{dt}(p-\theta) = \dot{\theta}\frac{d}{d\theta}(p-\theta) = \dot{\theta}\left(\frac{dp}{d\theta} - 1\right) = \dot{\theta}\left(\frac{\dot{p}}{\dot{\theta}} - 1\right).$$

Using (3) yields

$$\frac{d(p-\theta)}{dt} = \dot{\theta}\left(t\frac{f(\theta)u(v(\theta)-p)(p-\theta)}{F(\theta,p)} - 1\right) > \dot{\theta}\left(t\frac{\varepsilon_f \varepsilon_u (v(\theta)-p)\alpha}{F(\theta_S, v(\overline{\theta}))} - 1\right) >$$
$$> \dot{\theta}\left(t\frac{\varepsilon_f \varepsilon_u (\theta + \varepsilon_v - p)\alpha}{F(\theta_S, v(\overline{\theta}))} - 1\right) = \dot{\theta}\left(t\frac{\varepsilon_f \varepsilon_u (\varepsilon_v - \alpha)\alpha}{F(\theta_S, v(\overline{\theta}))} - 1\right) >$$
$$> \dot{\theta}\left(t\frac{\varepsilon_f \varepsilon_u \varepsilon_v \alpha}{2F(\theta_S, v(\overline{\theta}))} - 1\right) > \dot{\theta}\left(\hat{t}\frac{\varepsilon_f \varepsilon_u \varepsilon_v \alpha}{2F(\theta_S, v(\overline{\theta}))} - 1\right) = 0.$$

Thus $p(t) - \theta(t) > \alpha > 0$ for all $t \geq t_0$. Now it becomes clear (see proof of Proposition 2) that for some *T* we must have $\theta(T) = \overline{\theta}$. ■

Proof of Proposition 3

We first define functions $(\theta(t), p(t))$ for all $t \in [0, t_0]$ and some t_0 such that the condition of Lemma 1 is satisfied, i.e., $t_0 > \hat{t}(a)$, where $\alpha = p(t_0) - \theta(t_0)$, and $\theta(t_0) = \theta_S$. Then we show that $(\theta(t), p(t))$ actually is an equilibrium path. Lastly, Lemma 1 says that all goods are traded by a certain time *T*.

We have already shown in proof of Proposition 2 that for any $\theta_0 \in U$ and $p_0 = \pi_S(\theta_0)$ system (3) has a solution $(\theta(t,\theta_0), p(t,\theta_0))$ and $\theta(T) > \theta_S$ for some *T*. As $\dot{\theta}(t,\theta_0) > 0$, it follows that for all $\beta \in [\theta_0, \theta(T)]$ there exists an inverse $t(\beta,\theta_0)$ such that $\theta(t(\beta,\theta_0),\theta_0) = \beta$. Function $t(\beta,\theta_0)$ is continuously differentiable on $\beta \in [\theta_0, \theta(T)]$ and continuous w.r.t. θ_0. Hence, $p(t(\beta,\theta_0),\theta_0)$ is continuous w.r.t. θ_0 as well. We define $\tau(\theta_0)$ by $\tau(\theta_0) = t(\theta_S,\theta_0)$, so that $\theta(\tau(\theta_0),\theta_0) = \theta_S$ for all $\theta_0 \in U$. Note that $p(\tau(\theta_0),\theta_0) > \theta(\tau(\theta_0),\theta_0) = \theta_S$.

Now we will solve the linearized system (A.1). First, it can be rewritten as *Kummer's equation* (see Abramowitz and Stegun, 1972, pp. 504-515):

$$t\ddot{x} + (2-t)\dot{x} - x\left(-\tfrac{1-a}{a}\right) = 0$$

with initials $x(0) = -1$, $\dot{x}(0) = \frac{1-a}{2a}$. The unique solution is the negative to the so-called *Kummer's function* $M(a_1, a_2, t)$, with $a_1 = -\frac{1-a}{a}$ and $a_2 = 2$. Turning back to (A.1) we get:

$$\begin{cases} \hat{x}(t) = -M\left(-\tfrac{1-a}{a}, 2, t\right) = -1 + \dfrac{1-a}{a} \displaystyle\sum_{n=1}^{\infty} \dfrac{\Gamma\left(n - \frac{1-a}{a}\right)}{n!(n+1)!\Gamma\left(1-\frac{1-a}{a}\right)} t^n \\ \hat{y}(t) = -a \tfrac{d}{dt}\left(tM\left(-\tfrac{1-a}{a}, 2, t\right)\right) = a\left(-1 + \dfrac{1-a}{a} \displaystyle\sum_{n=1}^{\infty} \dfrac{\Gamma\left(n-\frac{1-a}{a}\right)}{(n!)^2 \Gamma\left(1-\frac{1-a}{a}\right)} t^n \right) \end{cases}, \qquad \text{(A.4)}$$

where $\Gamma(x) = \int_0^\infty \mathrm{e}^{-t} t^{x-1} dt$ is the Gamma-function, and

$$\frac{\Gamma\left(n - \frac{1-a}{a}\right)}{\Gamma\left(1-\frac{1-a}{a}\right)} = \left(2-\tfrac{1}{a}\right)\left(3-\tfrac{1}{a}\right) \times \ldots \times \left(n - \tfrac{1}{a}\right).$$

Now we define *ω* as $\hat{x}(\omega) = 0$ and it follows that $\omega > 0$ and $\hat{y}(\omega) > 0$. As

$$\begin{cases} \theta(t,\theta_0) = \theta_S + (\theta_S - \theta_0)\hat{x}(t) + o(\theta_S - \theta_0) \\ p(t,\theta_0) = \theta_S + (\theta_S - \theta_0)\hat{y}(t) + o(\theta_S - \theta_0) \end{cases},$$

the functions $\hat{x}(t)$ and $\hat{y}(t)$ describe the behavior of the solution $(\theta(t,\theta_0), p(t,\theta_0))$ in the neighborhood of $p = \theta = \theta_S$ and it follows that $\lim_{\theta_0 \to \theta_S} \tau(\theta_0) = \omega$ and

$$\lim_{\theta_0 \to \theta_S} \frac{p(\tau(\theta_0),\theta_0) - \theta_S}{\theta_S - \theta_0} = \hat{y}(\omega).$$

Therefore, there exists a left neighborhood of the static equilibrium quality $U^\omega \equiv \left(\underline{\theta_0^\omega}, \theta_S\right) \in U$ such that for all $\theta_0 \in U^\omega$ $\tau(\theta_0) > \frac{1}{2}\omega$.

Now we are ready to construct the pair of functions $(\theta(t), p(t))$. Let us take any $\theta_0^{(1)} \in U^{\omega}$ and define $\tau^{(1)} \equiv \tau(\theta_0^{(1)})$, $\theta^{(1)}(t) \equiv \theta(t, \theta_0^{(1)})$ and $p^{(1)}(t) \equiv p(t, \theta_0^{(1)})$ on $t \in (0, \tau^{(1)}]$, $s^{(1)}(t) \equiv \mathrm{e}^{-t}\left(p^{(1)}(t) - \theta^{(1)}(t)\right)$. By construction we have $\tau^{(1)} > \frac{1}{2}\omega$ and $s^{(1)}(\tau^{(1)}) > 0$.

Let us now consider the function $\rho^{(1)}(\theta_0) \equiv p(\tau(\theta_0), \theta_0) - \theta_S - s^{(1)}(\tau^{(1)})$ as a function of θ_0. It is continuous on $[\theta_0^{(1)}, \theta_S]$. Moreover,

$$\rho^{(1)}(\theta_0^{(1)}) = p(\tau(\theta_0^{(1)}), \theta_0^{(1)}) - \theta_S - s^{(1)}(\tau^{(1)}) = \theta_S + \mathrm{e}^{\tau^{(1)}} s^{(1)}(\tau^{(1)}) - \theta_S - s^{(1)}(\tau^{(1)}) > 0,$$

and

$$\rho^{(1)}(\theta_S) = p(\tau(\theta_S), \theta_S) - \theta_S - s^{(1)}(\tau^{(1)}) = \theta_S - \theta_S - \mathrm{e}^{\tau^{(1)}} s^{(1)}(\tau^{(1)}) < 0.$$

Therefore, $\exists \theta_0^{(2)} \in (\theta_0^{(1)}, \theta_S)$ depending on $\tau(\theta_0)$ such that $\rho^{(1)}(\theta_0^{(2)}) = 0$, i.e., $p(\tau(\theta_0^{(2)}), \theta_0^{(2)}) = s^{(1)}(\tau^{(1)})$. Again, we define $\tau^{(2)} \equiv \tau(\theta_0^{(2)})$, $\theta^{(2)}(t) \equiv \theta(t, \theta_0^{(2)})$ and $p^{(2)}(t) \equiv p(t, \theta_0^{(2)})$ on $t \in (0, \tau^{(2)}]$, $s^{(2)}(t) \equiv s(t, \theta_0^{(2)})$ and, again, $\tau^{(2)} > \frac{1}{2}\omega$.

Repeating this process, we get a sequence $\{\tau^{(k)}\}$ such that $\lim_{k\to\infty} \sum_{j=1}^{k} \tau^{(j)} = \infty$. We define $K \geq 1$ to be the smallest number such that $\sum_{j=1}^{k} \tau^{(j)} > \hat{t}\left(p^{(1)}(\tau^{(1)}) - \theta^{(1)}(\tau^{(1)})\right)$. Then, we define $t^{(k)}$ as $t^{(k)} = \sum_{j=k+1}^{K} \tau^{(j)}$, so that $t^{(K)} = 0$, $t^{(K-1)} = \tau^{(K)}$, $t^{(K-2)} = \tau^{(K)} + \tau^{(K-1)}, \ldots$, and $t^{(0)} = \sum_{j=1}^{K} \tau^{(j)} > \hat{t}$. Finally, we define an equilibrium path for $t \in (0, t^{(0)}]$ as follows: $p(t, \theta_0) = p^{(k)}(t - t^{(k)})$, $\theta(t, \theta_0) = \theta^{(k)}(t - t^{(k)})$ and, therefore, $s(t, \theta_0) = \mathrm{e}^{-t^{(k)}} s^{(k)}(t - t^{(k)})$ for $t \in (t^{(k)}, t^{(k-1)}]$. For $t > t^{(0)}$ we take the solution $\left(\theta(t, \theta^{(1)}(t^{(0)}), p^{(1)}(t^{(0)}), t^{(0)}), p(t, \theta^{(1)}(t^{(0)}), p^{(1)}(t^{(0)}), t^{(0)})\right)$ of (3) as an equilibrium path.

It can be easily seen that within every interval $(t^{(k)}, t^{(k-1)}]$ a seller chooses the time to trade optimally. In order to check that he behaves optimally even across those intervals (*subcycles*) and across cycles, again, like in the proof of Proposition 2, we use (A.2) and considering a seller i with quality θ_i and entry time $t_i \in (nT + t^{(k)}, nT + t^{(k-1)}]$. As the arguments are quite similar to the ones given in the proof of Proposition 2 we skip the details here.

Like in Proposition 2 any seller i optimally waits until the first moment after entry when the marginal quality is larger than or equal to his own quality. Hence, the pair of functions $(\theta(t), p(t))$ constructed above satisfies all equilibrium requirements. Then, it follows from Lemma 1 that $\theta(T)=\overline{\theta}$ for some *T*. The constructed equilibrium path is entirely determined by choosing $\theta_0^{(1)}$ which is an arbitrary point from U^{ω}. Therefore, we have obtained infinitely many (continuum of) equilibria. ■

Proof of Proposition 4

If $a>\frac{1}{2}$, then each term (apart from the constant) in the Tailor expansion (A.4) is positive as $\Gamma\left(1-\frac{1-a}{a}\right)>0$ and its radius of convergence is infinity. Hence, $\hat{x}$ and $\hat{y}$ are defined by (A.4) for all $t\in[0,\infty)$, $\lim_{t\to+\infty}\hat{y}(t)=\lim_{t\to+\infty}\hat{x}(t)=\lim_{t\to+\infty}\dot{\hat{y}}(t)=\lim_{t\to+\infty}\dot{\hat{x}}(t)=+\infty$, and

$$\lim_{t\to+\infty}\frac{\hat{y}}{\hat{x}}=\lim_{t\to+\infty}\frac{\dot{\hat{y}}}{\dot{\hat{x}}}=\lim_{t\to+\infty}\frac{\hat{y}-\hat{x}}{\frac{\hat{y}-a\hat{x}}{at}}=\lim_{t\to+\infty}\frac{\dot{\hat{y}}-\dot{\hat{x}}}{\frac{d}{dt}\frac{\hat{y}-a\hat{x}}{at}}=\lim_{t\to+\infty}\frac{\dot{\hat{y}}-\dot{\hat{x}}}{\frac{\dot{\hat{y}}-a\dot{\hat{x}}}{at}-\dot{\hat{x}}\frac{1}{t}}=\lim_{t\to+\infty}at\left(1+\frac{(2a-1)\dot{\hat{x}}}{\dot{\hat{y}}-2a\dot{\hat{x}}}\right)=+\infty$$

as $\dot{\hat{y}}-a\dot{\hat{x}}>\dot{\hat{y}}-\dot{\hat{x}}>\dot{\hat{y}}-2a\dot{\hat{x}}=(1-a)\sum_{n=2}^{\infty}\frac{\Gamma\left(n-\frac{1-a}{a}\right)t^{n-1}}{(n-2)!\,(n+1)!\,\Gamma\left(1-\frac{1-a}{a}\right)}>0$ for all $t>0$.

This implies that for $a>\frac{1}{2}$

$$\lim_{t\to+\infty}\lim_{\theta_0\to\theta_S}\frac{p(t,\theta_0)-\theta_S}{\theta(t,\theta_0)-\theta_S}=\lim_{t\to+\infty}\lim_{\theta_0\to\theta_S}\frac{\hat{y}(t)+O(\theta_S-\theta_0)}{\hat{x}(t)+O(\theta_S-\theta_0)}=\lim_{t\to+\infty}\frac{\hat{y}(t)}{\hat{x}(t)}=+\infty.$$

In other words, for any $M>0$ $\exists t'(M)$ such that for all $t>t'$ $\exists U^a(t,M)=\left(\underline{\theta}_0^a(t,M),\theta_S\right)$ such that $\frac{p(t,\theta_0)-\theta_S}{\theta(t,\theta_0)-\theta_S}>M$ for all $\theta_0\in U^a$. We take $\alpha=\frac{1}{2}\varepsilon_v$, $M=1+\max\left\{\frac{\alpha}{\overline{\theta}-\theta_S},\max_{\theta\in[\theta_S,\overline{\theta}]}a(\theta)\right\}$ and $\tau=\max\left\{\hat{t}(\alpha),t'(M),\frac{2M^2M_u\mu(\overline{\theta})}{\varepsilon_f\varepsilon_u\varepsilon_v(M-1)}\right\}$, where $\hat{t}(a)$ is as defined in (A.3), Lemma 1. For this τ there exists a neighborhood $U^a(\tau,M)$ such that $\frac{p(\tau,\theta_0)-\theta_S}{\theta(\tau,\theta_0)-\theta_S}>M$ for all $\theta_0\in U^a$.

We will show that $\exists t''\geq\tau$ such that $p(t'',\theta_0)\geq\theta(t'',\theta_0)+\alpha$. Suppose to the contrary that $p(t,\theta_0)<\theta(t,\theta_0)+\alpha$ for all $t\geq\tau$. Then it must be the case that $\frac{p(t,\theta_0)-\theta_S}{\theta(t,\theta_0)-\theta_S}>M$ for all $t>\tau$, otherwise there would have been some $t'''>\tau$ such that $\frac{p(t''',\theta_0)-\theta_S}{\theta(t''',\theta_0)-\theta_S}=M$ and, therefore, for $t=t'''$:

$$\frac{d\left(\frac{p-\theta_S}{\theta-\theta_S}\right)}{dt}=\frac{\dot p}{\theta-\theta_S}-\frac{M\dot\theta}{\theta-\theta_S}=\frac{\dot\theta}{\theta-\theta_S}\left(\frac{\dot p}{\dot\theta}-M\right)=\frac{\dot\theta}{\theta-\theta_S}\left(t'''\frac{f(\theta)u(v(\theta)-p)(p-\theta)}{F(\theta,p)-F(\theta,\pi_S(\theta))}-M\right)>$$

$$>\frac{\dot\theta}{\theta-\theta_S}\left(\tau\frac{\varepsilon_f\varepsilon_u(v(\theta)-p)\left(\frac{p-\theta_S}{\theta-\theta_S}-\frac{\theta-\theta_S}{\theta-\theta_S}\right)}{\frac{(p-\pi_S(\theta))\frac{\partial F}{\partial p}(\theta,\pi_S(\theta)+\xi(p-\pi_S(\theta)))}{\theta-\theta_S}}-M\right),$$

where $\xi\in(0,1)$ and $F(\theta,\pi_S(\theta))=0$ by definition. Then, as we have supposed $\theta>p-\alpha$,

$$\frac{d\left(\frac{p-\theta_S}{\theta-\theta_S}\right)}{dt}>\frac{\dot\theta}{\theta-\theta_S}\left(\tau\frac{\varepsilon_f\varepsilon_u(\varepsilon_v+\theta-p)(M-1)}{\frac{p-\pi_S(\theta)}{\theta-\theta_S}\int_{\underline{\theta}}^{\theta}u'(v(x)-\pi_S(\theta)-\xi(p-\pi_S(\theta)))f(x)dx}-M\right),$$

$$\frac{d\left(\frac{p-\theta_S}{\theta-\theta_S}\right)}{dt}>\frac{\dot\theta}{\theta-\theta_S}\left(\tau\frac{\varepsilon_f\varepsilon_u(\varepsilon_v-\alpha)(M-1)}{\left(\frac{p-\theta_S}{\theta-\theta_S}-\frac{\pi_S(\theta)-\theta_S}{\theta-\theta_S}\right)\int_{\underline{\theta}}^{\theta}M_u f(x)dx}-M\right)>$$

$$>\frac{M\dot\theta}{\theta-\theta_S}\left(\tau\frac{\varepsilon_f\varepsilon_u\varepsilon_v(M-1)}{2M(M-a(\theta'))M_u\mu(\overline{\theta})}-1\right)>\frac{M\dot\theta}{\theta-\theta_S}\left(\tau\frac{\varepsilon_f\varepsilon_u\varepsilon_v(M-1)}{2M^2M_u\mu(\overline{\theta})}-1\right)\geq 0,$$

for some $\theta'\in(\theta_S,\overline{\theta})$. But then $p-\theta_S>M(\theta-\theta_S)$ for all $t\geq\tau$ that can be rewritten as $\alpha>p-\theta>(M-1)(\theta-\theta_S)$ and, further, as $\theta<\frac{\alpha}{M-1}+\theta_S$. Hence, on one hand, $\lim_{t\to\infty}\theta\leq\frac{\alpha}{M-1}+\theta_S<\overline{\theta}$, on the other hand, $\dot p=p-\theta>(M-1)(\theta-\theta_S)>0$ and $\lim_{t\to\infty}p=+\infty$, that is a contradiction with the assumption that that $p<\theta+\alpha$ for all $t\geq\tau$.

Hence, $p(t'',\theta_0)\geq\theta(t'',\theta_0)+\alpha$ for some $t''\geq\tau\geq\hat t(\alpha)$ and Lemma 1 applies. ■