Dynamic Time Window Adjustment

Kevin Dalmeijer* , Remy Spliet and Albert P.M. Wagelmans

Econometric Institute
Erasmus School of Economics
Erasmus University Rotterdam

May 22, 2019
EI2019-22

Abstract

To improve customer satisfaction in a delivery network with uncertain travel times, we propose to communicate time window adjustments to the customers throughout the day. We refer to these updates as dynamic time window adjustments. Dynamic time window adjustments are often used in practice, but have not yet been considered in the scientific literature. We provide a general model and we present the Dynamic Time Window Adjustment Problem (DTWAP). The DTWAP is the problem of optimizing the dynamic time window adjustments to maximize the expected customer satisfaction for a given route. Instead of solving the DTWAP in a specific setting, we derive general properties and we present three different solution methods. We also introduce the simple DTWAP, which is a special case that we analyze in more detail. The use of our results is demonstrated with an illustrative example concerning attended home delivery.

Keywords: *Dynamic Time Window Adjustment, Customer Satisfaction, Stochastic Programming, Dynamic Programming, Attended Home Delivery.*

*E-mail: `dalmeijer@ese.eur.nl`; Phone: `+3110 408 9059`; Address: `PO Box 1738, 3000DR Rotterdam, The Netherlands`; Corresponding author

1 Introduction

In this paper, we study the problem of communicating to customers in a delivery network with uncertain travel times. For logistics service providers, it is common to update the customers about the estimated time window of delivery throughout the day, e.g., through text messages, app notifications, or by providing tracking information online. For example, major logistics companies FedEx, UPS and DHL all provide such services.

We use the term *dynamic time window adjustment* for the act of changing the time window that has been communicated to the customer. Dynamic time window adjustments are often used in practice. However, to the best of our knowledge, optimizing the dynamic time window adjustments has not been considered in the current scientific literature.

We introduce a general model for dynamic time window adjustment, and we consider the problem of determining adjustments that maximize the expected customer satisfaction for a given route. This optimization problem is referred to as the *Dynamic Time Window Adjustment Problem (DTWAP)*.

Two types of adjustment are considered: time window extension (EXT) and time window postponement (POS). For EXT, the start of the time window is fixed, and only the deadline can be adjusted. For example, a customer may be told that the original 1pm-3pm time window is extended to 1pm-4pm. For POS, the width of the time window is fixed, and the time window can be moved. For example, the 1pm-3pm time window may be postponed to 2pm-4pm.

We do not restrict ourselves to specific customer satisfaction functions. Instead, we discuss multiple solution methods, and we present an overview of which methods can be applied, depending on the properties of the customer satisfaction functions. To demonstrate how our work can be applied, an illustrative example is presented for which we optimize the customer satisfaction in an attended home delivery setting.

In the literature, we see an increasing interest in improving customer satisfaction, with different lines of research directly or indirectly addressing this topic. Examples are studies considering *consistent delivery* (Groër et al., 2009), *self-imposed time windows* (Jabali et al., 2015), *attended home delivery* (Agatz et al., 2011), *arrival time prediction* (Ulmer and Thomas, 2019), and *stochastic vehicle routing* (Gendreau et al., 1996).

It should be noted that in the literature, improving customer satisfaction is often combined with vehicle routing. In this paper, we limit ourselves to studying the DTWAP, for which the route is assumed to be given. This choice is motivated by the fact that, for general customer satisfaction functions, the DTWAP is already a challenging problem in itself. To be more precise, it is strongly NP-hard to solve the DTWAP, even for deterministic travel times. This result is proven in Section 2.

1.1 Literature review

To improve customer satisfaction, Groër et al. (2009) propose consistent delivery over multiple days or scenarios, e.g., always serve a customer with the same driver, or at approximately the same time. It is assumed that when the day or the scenario is known, the remaining vehicle routing problem is deterministic. This differs from our setting, where we assume that information is revealed during the day, and we are able to react dynamically.

Jabali et al. (2015) introduce the concept of self-imposed time windows. Self-imposed time windows are time windows that are chosen and communicated by the distributor to improve customer satisfaction. The main difference between assigning self-imposed time windows and dynamic time window adjustment, is that the former is done once, while the latter can be repeated throughout the day.

Han et al. (2017) consider randomness in attended home delivery due to customer no-show and customer response time, e.g., the time to answer the door. After completing an appointment, the authors dynamically determine the starting time of the next appointment and the maximum waiting time. This appointment scheduling subproblem has similarities with the special cases that we consider in Section 5. In general, an important difference with our work is that we communicate to all customers, not just to the one that will be visited next.

Ulmer and Thomas (2019) study the case where customers arrive dynamically, and immediately have to be given a (static) estimate of the arrival time. This differs from our setting, as we assume that the customers are static, and we can dynamically adjust the time windows.

Authors who study the Stochastic Vehicle Routing Problem (SVRP, Gendreau et al. (1996)) propose various ways to improve customer satisfaction. Zhang et al. (2013) select routes that guarantee a minimum on-time arrival probability at each customer location, while Taş et al. (2014) use linear penalties for early and late deliveries. Zhang et al. (2016) consider the SVRP with stochastic demands and propose different models to benefit the customer. They propose, among other things, to maximize the sum of the on-time probabilities and to consider earliness and tardiness penalties.

Jaillet et al. (2016) introduce a riskiness measure for time window violations based on the work of Aumann and Serrano (2008). Zhang et al. (2018) propose the 'essential riskiness index' which has properties similar to the index used by Jaillet et al. (2016) but gives some computational advantages. In both papers, minimizing the total riskiness is part of the objective when constructing the routes.

A different approach is taken by Errico et al. (2016). The authors consider the SVRP with stochastic service times, but explicitly do not allow the time windows to be adjusted. To deal with uncertainty, they introduce two recourse actions: *skip the current customer* and *skip the next customer*. Customers that are skipped receive emergency service, or are postponed to the following day.

For the SVRP variants mentioned above, determining the optimal actions for a given route is relatively easy. The main difficulty is due to integrating these decisions into the routing phase. For the DTWAP, we do not consider routing decisions. However, determining the optimal actions for a given route is significantly more difficult in this case.

The SVRP and the DTWAP are complementary, as one may first solve the SVRP to obtain high quality routes, and then solve the DTWAP for each route to improve customer satisfaction further. Integrating the DTWAP and the SVRP may yield additional benefits, and is an interesting direction for future research.

1.2 Contribution and outline

In this paper, we introduce dynamic time window adjustment into the literature. Dynamic time window adjustment is common in practice, but has not yet attracted the attention of researchers. We provide a general model and we present the DTWAP, which is the problem of optimizing the dynamic time window adjustments to maximize the expected customer satisfaction.

Instead of solving the DTWAP in a specific setting, we present three different solution methods. We discuss when each method is applicable, and we consider its advantages and disadvantages. As a result, our work can be applied to different settings, including parcel delivery, attended home delivery, and retailer distribution. We illustrate this with an attended home delivery example, for which we use the results of this paper to maximize customer satisfaction.

In Section 2, we formally introduce the DTWAP, and we prove that the problem is strongly NP-hard. In Section 3, we formulate the DTWAP as a multi-stage stochastic programming problem, and we present three solution methods. Section 4 is used to discuss the properties of the DTWAP, and to state sufficient conditions such that the methods in Section 3 can be applied.

In Section 5, we analyze the *simple DTWAP*, which is a special case of the DTWAP with additional structure. By exploiting this structure, the solution methods can be made more efficient. In Section 6, we consider the effect of discretizing time. We show that discretization may lead to suboptimal solutions in general. However, under certain assumptions, we prove that discretizing time yields an optimal solution for the simple DTWAP.

Our illustrative example is presented in Section 7. This example demonstrates how the results of this paper can be used to maximize customer satisfaction in an attended home delivery setting. We present a model with customer dissatisfaction functions that are inspired by Section 5, and we make use of Section 3 and Section 4 to construct a solution method. We also present heuristics motivated by Sections 5 and 6. In the final section, we give a conclusion and we present some directions for further research.

2 Dynamic Time Window Adjustment Problem

In this section, we formally introduce the DTWAP. Assume that a fixed route is given, starting at the depot, visiting locations 1 up to n (in order), and returning to the depot. Locations $V' = \{1, \ldots, n\}$ correspond to the customers and locations 0 and $n+1$ correspond to the depot. Let $V = V' \cup \{0, n+1\}$.

The travel time between location i and location $i+1$ for $i \in V' \cup \{0\}$ is given by the non-negative random variable t_{ii+1}. We assume that all travel times are *stagewise independent*, i.e., t_{ii+1} is independent of $t_{01}, \ldots, t_{i-1,i}$, for all $i \in V' \cup \{0\}$ (Shapiro et al., 2009).

Let $[a_i, b_i]$ be the initial time window of customer $i \in V'$, with $a_i, b_i \in \mathbb{R}$ and $a_i \leq b_i$. One-sided time windows can be expressed by setting $a_i = -\infty$ or $b_i = \infty$. Time windows can be adjusted in two different ways: they can be extended (EXT) or postponed (POS). Let $\mathcal{P}_i$ be the set of allowed adjustments for customer i. For a given adjustment $p \in \mathcal{P}_i$, the EXT time window is given by $[a_i, b_i + p_i]$. We assume that $a_i \leq b_i + p$ for all $p \in \mathcal{P}_i$. The POS time window is given by $[a_i + p, b_i + p]$. We assume that the set $\mathcal{P}_i$ is closed and that $0 \in \mathcal{P}_i$ for all $i \in V'$. Note that $|\mathcal{P}_i|$ is allowed to be infinite.

On arrival at customer location $i \in V'$ (possibly before the time window opens), the following decisions are made. Let $p^i_j \in \mathcal{P}_j$ for $i < j$, $i \in V' \cup \{0\}$, $j \in V'$, be the adjustment of the time window of customer j, as decided when arriving at customer i. Note that the adjustment is defined with respect to the initial time window, i.e., the time window before any adjustments are made. If $i = 0$, then p^i_j is the adjustment for customer j as decided before leaving the depot. Let $p^i = (p^i_{i+1}, \ldots, p^i_n)$ be the vector of adjustments as determined at location $i \in \{0, 1, \ldots, n-1\}$. For convenience, we define $p^i_j = 0$ for $i = -1$.

The arrival time at customer $i \in V'$ is denoted by t_i and the departure time from customer i is denoted by x_i. Note that t_i and x_i both depend on earlier decisions and realizations of the random travel times. For now, we assume that the service times are equal to zero, such that customer i is served at time x_i. Let the parameter $x_0 \geq 0$ be the time at which the vehicle leaves the depot. For convenience, we define $t_0 = x_0$.

When the vehicle arrives at customer i, we consider three possible waiting behaviors.

1. Never Wait (NW): customer i is served immediately upon arrival, which may be outside of the time window.

2. Always Wait (AW): if necessary, the vehicle waits until the time window opens, and

then immediately serves customer i. That is, the vehicle waits until a_i for EXT and until $a_i + p_i^{i-1}$ for POS.

3. Voluntarily Wait (VW): we decide dynamically how long the vehicle waits before serving customer i. Note that it is allowed to serve customer i early, even after waiting.

In all cases, customer i may be served after the deadline if the vehicle arrives late or waits.

We assume that the time window adjustment type (EXT or POS) and the waiting behavior (NW, AW, or VW) are the same for all customers. This results in six cases, which we abbreviate by EXT-NW, EXT-AW, EXT-VW, POS-NW, POS-AW, and POS-VW, respectively.

After the adjustments for the customers $j > i$ are decided, i.e., the vector p^i is chosen, the adjustments are communicated to the customers. We assume this communication to be instantaneous. Next, in the case of waiting behavior VW, we decide on how long to wait before serving customer i. We call this the voluntary waiting time $w_i \in W_i$, with $W_i = \mathbb{R}_{\geq 0}$. For notational convenience, we define the same decision for NW and AW, but with $W_i = \{0\}$, such that the voluntary waiting time is zero. At the depot, only the adjustment vector p^0 is determined, and no voluntary waiting decision is made.

For a given customer $i \in V'$, let $d_i(t_i, w_i, p_i^{i-1})$ return the departure time x_i, given the arrival time t_i, the voluntary waiting time w_i, and the adjustment p_i^{i-1}. The function $d_i(t_i, w_i, p_i^{i-1})$ depends on the type of adjustment (EXT or POS) and on the waiting behavior (NW, AW, or VW). Explicitly, we have:

$$d_i^{EXT-NW}(t_i, w_i, p_i^{i-1}) = t_i, \tag{1}$$
$$d_i^{EXT-AW}(t_i, w_i, p_i^{i-1}) = \max\{t_i, a_i\}, \tag{2}$$
$$d_i^{EXT-VW}(t_i, w_i, p_i^{i-1}) = t_i + w_i, \tag{3}$$
$$d_i^{POS-NW}(t_i, w_i, p_i^{i-1}) = t_i, \tag{4}$$
$$d_i^{POS-AW}(t_i, w_i, p_i^{i-1}) = \max\{t_i, a_i + p_i^{i-1}\}, \tag{5}$$
$$d_i^{POS-VW}(t_i, w_i, p_i^{i-1}) = t_i + w_i, \tag{6}$$

with w_i on the relevant domain W_i.

Without loss of generality, we minimize *customer dissatisfaction*, instead of maximizing customer satisfaction. We assume that the dissatisfaction of customer j can be written as

$$\sum_{i=0}^{j-1} g_j(t_i, p_j^{i-1}, p_j^i) + h_j(x_j, p_j^{j-1}), \tag{7}$$

for some functions g_j and h_j for all $j \in V'$.

The function g_j models the dissatisfaction that results from changing the adjustment of customer j from p_j^{i-1} to p_j^i. Note that the adjustment is communicated immediately upon arrival at customer i. The function h_j represents dissatisfaction due to serving customer j outside of the communicated time window. As such, it is dependent on the time that customer j is served and on the final adjustment p_j^{j-1}. We assume that the total dissatisfaction of the customers is given by the sum of the individual dissatisfaction functions.

We now formally introduce the DTWAP. Let a *state* be a triple of location $i \in V' \cup \{0\}$, arrival time t_i, and adjustment vector p^{i-1}. An *action* consists of choosing a new adjustment vector p^i and a voluntary waiting time w_i. The DTWAP is the problem of finding an optimal action for every state that is encountered by the vehicle, such that the total expected customer dissatisfaction is minimized.

We point out that driver costs and stochastic service times can easily be incorporated into the model. To incur driver costs, we add a dummy customer n' after customer n and we choose an appropriate function $h_{n'}$. Service time at customer i can be included by inserting a dummy customer i' after customer i. The travel time between customers i and i' is taken to be the (stochastic) service time, and customer i' is given the non-adjustable time window $[0, \infty)$. Note that in this way, we can only model service times that are stagewise independent.

Finally, we prove that the DTWAP is strongly NP-hard, even for deterministic travel times, and for dissatisfaction functions that can be evaluated in polynomial time. Note that if we do not assume that the dissatisfaction functions can be evaluated in polynomial time, then the DTWAP is obviously hard to solve.

Proposition 1. *The DTWAP is strongly* NP*-hard, even for deterministic travel times, and for dissatisfaction functions that can be evaluated in polynomial time.*

Proof. See Appendix A. □

3 Formulations and solution methods

The DTWAP can be seen as a *multi-stage stochastic programming problem*. We formalize this by stating the DTWAP in recursive form, i.e., as a stochastic dynamic program. Next, we consider three solution methods for multi-stage stochastic programming problems. We do so to identify properties that need to be satisfied by the DTWAP such that existing methods can be applied. For more details, we make references to the books by Bertsekas (2005), Kall and Mayer (2011), and Shapiro et al. (2009).

We consider the following three solution methods. In Section 3.2, we present a deterministic equivalent mathematical program for the DTWAP (Shapiro et al., 2009). In certain cases, this mathematical program can be solved efficiently by standard solvers.

In Section 3.3, we consider dual decomposition methods (Kall and Mayer, 2011). In Section 3.4, we discuss discretizing time, such that the stochastic dynamic programming recursions can be solved directly (Bertsekas, 2005).

3.1 Stochastic dynamic programming formulation

Let $c_i(t_i, p^{i-1})$ be the *value function* of customer i. The value function is defined as the minimum expected total dissatisfaction of customers i through n, given that the vehicle arrives at customer i at time t_i, with adjustment vector p^{i-1}.

In the remainder, we assume that the DTWAP instances are *sufficiently expensive.* That is, for every attainable state, we assume that $c_i(t_i, p^{i-1}) > -\infty$. This condition is satisfied, for example, when the dissatisfaction functions are non-negative. Furthermore, we assume that the optimal action is attainable, i.e., there exist finite optimal adjustments and voluntary waiting times. We do not consider these assumptions to be restrictive in practice.

The DTWAP can then be stated as follows.

$$c_n(t_n, p^{n-1}) = \min_{w_n \in W_n} \left\{ h_n\left(x_n, p_n^{n-1}\right) \right\}, \qquad \forall t_n \geq 0,\ p^{n-1} \in \mathcal{P}_n, \tag{8}$$

$$c_i\left(t_i, p^{i-1}\right) = \min_{\substack{p^i \in \prod_{j=i+1}^n \mathcal{P}_i \\ w_i \in W_i}} \left\{ \sum_{j=i+1}^{n} g_j(t_i, p_j^{i-1}, p_j^i) + h_i\left(x_i, p_i^{i-1}\right) + \mathbb{E}\left[c_{i+1}\left(x_i + t_{ii+1}, p^i\right)\right] \right\},$$
$$\forall i \in V' \backslash \{n\},\ t_i \geq 0,\ p^{i-1} \in \prod_{j=i}^{n} \mathcal{P}_i, \tag{9}$$

$$c_0\left(t_0\right) = \min_{p^0 \in \prod_{j=1}^n \mathcal{P}_i} \left\{ \sum_{j=1}^{n} g_j(t_0, 0, p_j^0) + \mathbb{E}\left[c_1\left(x_0 + t_{01}, p^0\right)\right] \right\}, \tag{10}$$

with $x_i = d_i(t_i, w_i, p_i^{i-1})$ for all $i \in V'$. Recall that x_0 and t_0 are parameters with the same value.

Equations (9) define the value function at customer $i \in V' \backslash \{n\}$. At customer i, and based on t_i and p^{i-1}, we decide on the optimal amount of voluntary waiting, w_i, and on the new adjustment vector p^i. The first term within the minimum represents the cost of all changes to the adjustment vector. The h_i term gives the cost of early and late deliveries at customer i. The third term is the expected cost from arriving at customer $i+1$ onwards.

Equations (8) and (10) are the boundary cases for the last customer and the depot, respectively. For the last customer, we only incur the penalty for early and late delivery. For the depot, we only determine the initial adjustment vector.

The DTWAP can be seen as a multi-stage stochastic optimization problem with $n+1$ stages. The first stage problem is to find the optimal adjustment vector p^0 before the vehicle leaves the depot. This corresponds to Equation (10). Next, the travel time to the first customer is revealed, and the vehicle arrives at customer 1 at time t_1 with postponement vector p^0. The second stage problem is to determine the optimal adjustment vector p^1 and voluntary waiting time w_1, which corresponds to Equations (9). Stages three up to n also correspond to Equations (9). The $n+1$'th stage problem is to decide w_n, as in Equations (8).

3.2 Solving the deterministic equivalent mathematical program

If the travel time distributions are discrete with finite support, we can formulate a deterministic mathematical program that is equivalent to (8)-(10). This construction is based on the scenario tree, which is discussed in more detail by Shapiro et al. (2009).

A *scenario* is a vector of realizations of all random variables. We denote scenario $k \in K$ by $\xi^k = (\xi_j^k)_{j=2,\ldots,n+1}$, with ξ_j^k the information that is revealed before making the j'th stage decision. In our case, ξ_j^k is the travel time from customer $j-2$ to customer $j-1$ under scenario k. For example, at customer two we make the third stage decision, and the travel time from customer one to customer two is known at this point. The probability that a random scenario ξ is equal to scenario ξ^k is given by $\mathbb{P}(\xi = \xi^k) = \prod_{j=2}^{n+1} \mathbb{P}(t_{j-2,j-1} = \xi_j^k)$. This follows from the stagewise independence of the random travel times.

Next, we present the deterministic equivalent mathematical program. Each decision variable is duplicated $|K|$ times to represent the decisions in the different scenarios. That is, $p^i(\xi^k)$ is the adjustment vector that is determined at customer i under scenario ξ^k and $w_i(\xi^k)$ is the voluntary waiting time at customer i under scenario ξ^k. For clarity, we denote the travel time between customer i and customer $i+1$ by $t_{ii+1}(\xi^k)$. Note that $t_{ii+1}(\xi^k)$ is now a deterministic parameter. The variables $t_i(\xi^k)$ and $x_i(\xi^k)$ represent the arrival and departure time at customer i in scenario ξ^k.

The deterministic equivalent mathematical program can be stated as follows.

$$\min \sum_{k\in K}\left(\mathbb{P}(\xi=\xi^k)\sum_{j\in V'}\left(\sum_{i=0}^{j-1} g_j(t_i(\xi^k),p_j^{i-1}(\xi^k),p_j^i(\xi^k)) + h_j(x_j(\xi^k),p_j^{j-1}(\xi^k))\right)\right) \tag{11}$$

$$\begin{aligned}
\text{s.t.,}\quad & t_i(\xi^k) = x_{i-1}(\xi^k) + t_{i-1,i}(\xi^k), && \forall i\in V', k\in K, && (12)\\
& x_i(\xi^k) = d_i(t_i(\xi^k), w_i(\xi^k), p_i^{i-1}(\xi^k)), && \forall i\in V', k\in K, && (13)\\
& p^i(\xi^k) = p^i(\xi^l), && \forall i\in V'\backslash\{n\}, k,l\in K \text{ s.t. } \xi_j^k=\xi_j^l\ \forall j\le i+1, && (14)\\
& w_i(\xi^k) = w_i(\xi^l), && \forall i\in V', k,l\in K \text{ s.t. } \xi_j^k=\xi_j^l\ \forall j\le i+1, && (15)\\
& p^i(\xi^k) \in \prod_{j=i+1}^{n}\mathcal{P}_j, && \forall i\in\{0,\dots,n-1\}, k\in K, && (16)\\
& w_i(\xi^k)\in W_i, && \forall i\in V', k\in K, && (17)\\
& t_i(\xi^k)\in\mathbb{R}, && \forall i\in V', k\in K, && (18)\\
& x_i(\xi^k)\in\mathbb{R}, && \forall i\in V', k\in K. && (19)
\end{aligned}$$

The Objective (11) is to minimize the expected total customer dissatisfaction over all scenarios. Constraints (12) define the arrival time at customer i for all $i\in V'$. Constraints (13) define the departure time from customer i, using the appropriate d_i function from (1)-(6).

Constraints (14)-(15) are the *nonanticipativity constraints* (Shapiro et al., 2009), which prevent decisions to be based on future information. For example, the value of $p^i(\xi^k)$ is a stage $i+1$ decision. As such, Constraints (14) enforce that if two scenarios k and l are the same up to stage $i+1$, then the decisions $p^i(\xi^k)$ and $p^i(\xi^l)$ are the same. It can be seen that nonanticipativity for $t_i(\xi^k)$ and $x_i(\xi^k)$ is implied by nonanticipativity for $p^i(\xi^k)$ and $w_i(\xi^k)$, and Equalities (12)-(13).

Solving the deterministic equivalent mathematical program (11)-(19) may be difficult. First, the number of scenarios, $|K|$, can be extremely large. For example, for n customers and only two possible travel time realizations per arc, we already have 2^n scenarios. To limit the number of scenarios, Sample Average Approximation (SAA, Kleywegt et al. (2002)) may be applied. However, especially for multi-stage problems, a large number of samples may still be necessary to obtain a good solution (Shapiro and Nemirovski, 2005).

Second, as a consequence of Proposition 1, the mathematical program (11)-(19) may already be difficult to solve for a single scenario. In Section 4, we give conditions under which the deterministic equivalent mathematical program can be solved efficiently for a moderate number of scenarios.

3.3 Dual decomposition methods

An important difficulty in solving stochastic dynamic programming recursions like (8)-(10) directly, is that the expected future value function is defined implicitly (Shapiro and Nemirovski, 2005). That is, for $F_{i+1}(x_i, p^i) = \mathbb{E}\left[c_{i+1}(x_i + t_{ii+1}, p^i)\right]$, we do not have a closed form analytic expression for the function F_{i+1}. On the other hand, if F_{i+1} were known explicitly, the DTWAP could be solved independently per stage as a straightforward mathematical program.

Dual decomposition methods construct an outer approximation of F_{i+1} by using linear inequalities, which is possible if F_{i+1} is convex. This idea goes back to Benders' decomposition (Benders, 1962) and the L-shaped method (Van Slyke and Wets, 1969). Pereira and Pinto (1991) introduce the Stochastic Dual Dynamic Programming (SDDP) method which includes Monte Carlo simulation to approximate F_{i+1}.

For more information on dual decompostion methods, we refer to Kall and Mayer (2011) and Pereira and Pinto (1991). For a detailed analysis of the SDDP method and the amount of samples that is needed to obtain good quality solutions, we refer to Shapiro (2011). For our discussion, the most important property to note is that dual decomposition methods rely on all functions F_{i+1} to be convex. In Section 4 we give sufficient conditions for the DTWAP such that this is the case.

3.4 Solving the discretized stochastic dynamic program

For specific models, it may be possible to solve the stochastic dynamic program (8)-(10) directly. For the appropriate solution methods, we refer to Bertsekas (2005). In general, however, solving the stochastic dynamic program is complicated due to the potentially infinite state-space, the potentially infinite action-space, and the potentially continuous travel time distributions.

By discretizing time, we obtain a DTWAP instance for which (8)-(10) is straightfoward to solve. Specifically, we approximate the vehicle starting time and the initial time windows by integers. Furthermore, we approximate each set of possible adjustments and each set of possible voluntary waiting times by a finite set of integer values. The travel time distributions are approximated by discrete distributions with finite support.

For the discretized instance, the number of possible states, possible actions, and possible travel time realizations are all finite. This allows us to solve the (8)-(10) directly by *backward recursion*. That is, we first use (8) to determine c_n. For every possible t_n and p^{n-1}, we enumerate $w_n \in W_n$ to find the minimum costs. When c_n is determined, we use (9) to determine c_{n-1}. In this case, we enumerate $p^{n-1} \in \mathcal{P}_n$ and $w_{n-1} \in W_{n-1}$. Note that, by assumption, the expectation can be written as a finite sum. We repeat the process until all value functions are completely determined.

Alternatively, one can solve (8)-(10) by *forward recursion*. We start by calculating

$c_0(t_0)$, the optimal expected total dissatisfaction. To calculate this value, we must first calculate c_1 for all possible future states. For each such state, we calculate c_2 for all possible future states, etc. Once a value has been calculated, it is stored in memory to prevent that the same calculation is made twice. The advantage of forward recursion is that we only calculate the values of the states that are necessary to determine the optimal dynamic time window adjustments. This is different from backward recursion, where we calculate the values of all states.

The discretization approach is straightforward and does not require any assumptions on the convexity of the dissatisfaction functions. For this reason, the same approach can be used when the model is extended, for example, by including time-dependent travel times (Ichoua et al., 2003).

The clear downside of the discretization approach is that the number of states may be very large, especially because the discretization must be sufficiently fine to obtain a good quality solution. Furthermore, solving the minimization problems by enumerating $p^i \in \mathcal{P}_{i+1} \times \ldots \times \mathcal{P}_n$ and $w_i \in W_i$ may be computationally expensive.

In Section 6, we consider the effect of discretization in more detail. We show that the discretization approach can result in suboptimal solutions, even if the vehicle starting time is integer, the initial time windows are integer, and the travel times have finitely many realizations that are all integer. On the other hand, we prove in the same setting that the discretization approach is exact for the simple DTWAP introduced in Section 5.

4 Properties

In Section 3 we have considered different solution methods for multi-stage stochastic programming problems, and we have identified the relevant properties such that they can be applied to the DTWAP. In this section, we give sufficient conditions under which these properties hold.

First, we consider properties such that the deterministic equivalent mathematical program (Section 3.2) can be solved as a convex or linear program, respectively. As in Section 3.2, we assume that the number of scenarios is finite.

Proposition 2. *Consider the following six conditions:*

1. $\mathcal{P}_i$ *is a closed continuous interval,* $\forall i \in V'$,
2. $g_j(t_i, p_j^{i-1}, p^i)$ *is convex,* $\forall j \in V'$,
3. $g_j(t_i, p_j^{i-1}, p^i)$ *is non-decreasing in* t_i, $\forall j \in V'$,
4. $h_i(x_i, p_i^{i-1})$ *is convex,* $\forall i \in V'$,
5. $h_i(x_i, p_i^{i-1})$ *is non-decreasing in* x_i *for* $x_i \geq a_i$, $\forall i \in V'$,

6. $h_i(x_i, p_i^{i-1})$ *is non-decreasing in* x_i *for* $x_i \geq a_i + p_i^{i-1}$, $\forall i \in V'$.

The deterministic equivalent mathematical program (11)-(19) *can be solved as a convex program in the following cases:*

- *The waiting behavior is NW or VW, and Conditions 1, 2, and 4 hold.*
- *The time window adjustment type is EXT, the waiting behavior is AW, and Conditions 1-5 hold.*
- *The time window adjustment type is POS, the waiting behavior is AW, and Conditions 1-4, and 6 hold.*

Proof. Under Conditions 2 and 4, we have that the Objective (11) is a convex function. By definition, Equations (12), (14)-(15), and (17) are linear constraints. Condition 1 ensures that (16) can be replaced by box constraints.

It remains to consider Constraints (13), i.e., $x_i(\xi^k) = d_i(t_i(\xi^k), w_i(\xi^k), p_i^{i-1}(\xi^k))$. Under waiting behavior NW or VW, we have that d_i is a linear function, which implies that (13) are linear constraints. Hence, (11)-(19) can be solved as a convex program in the first case.

Next, consider the combination of EXT and waiting behavior AW. We replace Constraints (13) by the two linear constraints $x_i(\xi^k) \geq t_i(\xi^k)$ and $x_i(\xi^k) \geq a_i$, for every $i \in V'$ and $k \in K$. By Conditions 3 and 5, it is optimal to choose $x_i(\xi^k)$ as small as possible, i.e., $x_i(\xi^k) = \max\{t_i(\xi^k), a_i\} = d_i(t_i(\xi^k), w_i(\xi^k), p_i^{i-1}(\xi^k))$. Hence, (11)-(19) can be solved as a convex program in the second case.

The proof for the third case is similar to the second case, and follows from replacing the Constraints (13) by $x_i(\xi^k) \geq t_i(\xi^k)$ and $x_i(\xi^k) \geq a_i + p_i^{i-1}$, for every $i \in V'$ and $k \in K$. □

Corollary 3. *If, in addition to the conditions of Proposition 2, the dissatisfaction functions are polyhedral convex functions, then the deterministic equivalent mathematical program* (11)-(19) *can be solved as a linear program.*

Proof. The convex programs that are constructed in the proof of Proposition 2 all have a convex objective function and linear constraints.

In the objective function, we may replace each $g_j(t_i(\xi^k), p_j^{i-1}(\xi^k), p_j^i(\xi^k))$ by a new variable $y_{ij}(\xi^k) \in \mathbb{R}$, if we also add the constraints $y_{ij}(\xi^k) \geq g_j(t_i(\xi^k), p_j^{i-1}(\xi^k), p_j^i(\xi^k))$. By the definition of polyhedral convex functions, the new constraints can be represented by a finite number of linear inequalities.

For the h_j terms, we can use a similar transformation. It follows that (11)-(19) can be solved as a linear program. □

Proposition 2 and Corollary 3 give conditions for which the deterministic equivalent mathematical program can be solved efficiently if the number of scenarios is not too big. In the linear case, general purpose solvers like CPLEX and Gurobi can be used immediately.

We remark that if the dissatisfaction functions can be modeled with integer variables and linear constraints, then the deterministic equivalent mathematical program can be solved as a mixed integer linear programming problem by the same commercial solvers.

Next, we consider conditions such that dual decomposition methods can be applied. As discussed in Section 3.3, we require that $F_{i+1}(x_i, p^i) = \mathbb{E}\left[c_{i+1}(x_i + t_{ii+1}, p^i)\right]$ is convex for all $i \in \{0, \dots, n-1\}$. Note that we no longer assume that the travel times are discretely distributed.

Proposition 4. *Given the same conditions as in Proposition 2, the function $F_{i+1}(x_i, p^i)$ is convex for all $i \in \{0, \dots, n-1\}$.*

Proof. We prove this by induction. That is, we assume that $F_{i+1}(x_i, p^i)$ is convex and we use (9) to prove that $F_i(x_{i-1}, p^{i-1})$ is convex. The proofs for the boundary cases F_n and F_1 are analogous.

If $g_j(t_i, p_j^{i-1}, p_j^i)$, $h_i(x_i, p_i^{i-1})$, and $F_{i+1}(x_i, p^i)$ are convex, then the right-hand side of (9) can be stated as a convex minimization problem. To do so, we replace $x_i = d_i(t_i, w_i, p_i^{i-1})$ by the appropriate linear constraints, similar as in the proof of Proposition 2, and we minimize over x_i. For EXT-AW, for example, we replace $x_i = d_i(t_i, w_i, p_i^{i-1})$ by $x_i \geq t_i$ and $x_i \geq a_i$. Note that for this proof, we do not require an explicit description of the function $F_{i+1}(x_i, p^i)$.

The function $c_i(t_i, p^{i-1})$ is the result of minimizing a convex function over a convex set for a subset of the variables. It follows that c_i is convex (Boyd and Vandenberghe, 2004). By definition, $F_i(x_{i-1}, p^{i-1}) = \mathbb{E}[c_i(x_{i-1} + t_{i-1,i}, p^{i-1})]$, which is a (possibly infinite) non-negative sum of convex functions, which is itself convex. □

Corollary 5. *If, in addition to the conditions of Proposition 2, the dissatisfaction functions are polyhedral convex functions and the travel time distributions are discrete with finite support, then the function $F_{i+1}(x_i, p^i)$ is polyhedral convex for all $i \in \{0, \dots, n-1\}$.*

Proof. If $g_j(t_i, p_j^{i-1}, p_j^i)$, $h_i(x_i, p_i^{i-1})$, and $F_{i+1}(x_i, p^i)$ are polyhedral convex functions, then the minimization problems at the right-hand sides of (8)-(10) can be solved as linear programs. The required transformations are similar to those described in the proof of Corollary 5.

From linear programming duality, it follows that $c_i(t_i, p^{i-1})$ is a polyhedral convex function. By definition, $F_i(x_{i-1}, p^{i-1})$ is the sum of finitely many polyhedral convex functions, which is polyhedral convex. □

Proposition 4 and Corollary 5 give sufficient conditions such that dual decomposition methods, including SDDP, can be applied. These conditions are similar to the conditions that ensure that the deterministic equivalent mathematical program can be solved efficiently for a moderate number of scenarios. As discussed in Section 3.4, the discretization approach does not require any assumptions on the dissatisfaction functions.

5 Simple DTWAP

In this section, we introduce the *simple DTWAP*, a special case of the DTWAP, which is defined in Section 5.1. In Section 5.2, we show that the simple DTWAP can be solved relatively efficiently by the methods discussed in Section 3. Furthermore, we show that for EXT-NW, POS-NW, and EXT-AW, the simple DTWAP can be decomposed into n independent problems. This fact can be exploited by the solution methods. In Section 5.3, we demonstrate that ignoring one of the main assumptions of the simple DTWAP may result in a severely more complicated problem.

5.1 Assumptions

First, we assume that $\mathcal{P}_i$ is a continuous interval for all $i \in V'$. Next, we discuss our assumptions on the dissatisfaction functions.

We assume that the time at which an adjustment is communicated to the customer does not affect the amount of dissatisfaction received. That is, we can write $g_j(t_i, p_j^{i-1}, p_j^i)$ as $g_j(p_j^{i-1}, p_j^i)$. This assumption is restrictive, but may be appropriate if travel times are relatively long. In this case, updating the next customer before driving there may be sufficiently far in advance, such that the customer is indifferent about the timing of the information. In Section 5.3, we show that omitting the above assumption may result in a considerably more complicated problem.

For the simple DTWAP, we assume that increasing and decreasing the adjustment both yield a penalty that is linear in the amount of change. That is, we assume that

$$g_j(p_j^{i-1}, p_j^i) = \left(p_j^i - p_j^{i-1}\right)^+ \alpha_j + \left(p_j^{i-1} - p_j^i\right)^+ \beta_j, \tag{20}$$

for some parameters $\alpha_j, \beta_j \in \mathbb{R} \cup \{\infty\}$, and using $y^+ = \max\{0, y\}$.

If any of the parameters is chosen to be ∞, we replace the corresponding term by a domain restriction. For example, if decreasing the adjustment is not allowed, we set $\beta_j = \infty$ and (20) reduces to $g_j(p_j^{i-1}, p_j^i) = \left(p_j^i - p_j^{i-1}\right) \alpha_j$ on domain $p_j^{i-1} - p_j^i \leq 0$.

We assume that $\alpha_j + \beta_j \geq 0$, which implies that $g_j(p_j^{i-1}, p_j^i)$ is convex. This assumption ensures that the satisfaction of customer j cannot be improved by first increasing the adjustment and then decreasing it again.

In a similar way, we define linear penalties for early and late delivery, for both EXT and POS:

$$h_j^{EXT}(x_j, p_j^{j-1}) = \left(x_j - (b_j + p_j^{j-1})\right)^+ \gamma_j + (a_j - x_j)^+ \delta_j, \tag{21}$$
$$h_j^{POS}(x_j, p_j^{j-1}) = \left(x_j - (b_j + p_j^{j-1})\right)^+ \gamma_j + \left((a_j + p_j^{j-1}) - x_j\right)^+ \delta_j, \tag{22}$$

for some parameters $\gamma_j, \delta_j \in \mathbb{R} \cup \{\infty\}$. We assume that $\gamma_j, \delta_j \geq 0$, such that h_j is convex. That is, early and late delivery are both disliked by the customer.

We assume that the parameters α_j, β_j, γ_j, and δ_j are chosen such that the DTWAP is sufficiently expensive and that the optimal adjustments and voluntary waiting times can be attained. It is sufficient to make the additional assumption that $\alpha_j \geq 0$ and $\beta_j \geq 0$, or that $\mathcal{P}_j$ is bounded. For specific cases, weaker conditions may apply. In the EXT case, for example, it can be seen that $\beta_j < 0$ is allowed, even if $\mathcal{P}_j$ is unbounded.

It is straightforward to verify that the conditions of Proposition 2 are satisfied for every combination of EXT and POS with NW, AW and VW. By Proposition 2, it follows that the deterministic equivalent mathematical program can be solved as a convex program, given that a finite sample of scenarios is used. By Proposition 4, it follows that $F_{i+1}(x_i, p^i) = \mathbb{E}\left[c_{i+1}(x_i + t_{ii+1}, p^i)\right]$ is convex for all $i \in \{0, \ldots, n-1\}$.

Hence, the solution methods discussed in Section 3 can all be used to solve the simple DTWAP. Additionally, we note that g_j and h_j are polyhedral convex functions. It follows from Corollaries 3 and 5 that if the travel time distributions are discrete with finite support, then the deterministic equivalent mathematical program reduces to a linear program, and the functions $F_{i+1}(x_i, p^i)$ are polyhedral convex functions.

5.2 Properties of the simple DTWAP

In this section, we analyze the simple DTWAP. We first show that all solution methods discussed in Section 3 are more efficient for the simple DTWAP than in general. Afterwards, we consider every combination of EXT and POS with NW, AW and VW in more detail. We show which cases are decomposable, and thus allow for even more efficient solution methods. Our analysis is also useful for Section 6, where we prove that the discretization approach (see Section 3.4) is exact for the simple DTWAP, under certain assumptions.

We make the following observation:

Observation 6. *The cost of informing customer j is not dependent on customer i or on the current time. Nor can we benefit from increasing and then decreasing the adjustment (Section 5.1). The longer we wait with informing the customer, the more stochastic variables have been realized. As such, it is optimal to determine the adjustment for customer j on arrival at customer $j-1$. In other words, there exists an optimal solution with $p_j^0 = p_j^1 = \ldots = p_j^{j-2} = 0$.*

Observation 6 implies that all value functions $c_i(t_i, p^{i-1})$ can be replaced by two-dimensional value functions $c_i(t_i, p_i^{i-1})$. Furthermore, for the simple DTWAP, we have that $g_j(0,0) = 0$ by definition. It follows that (8)-(10) simplify to

$$c_n(t_n, p_n^{n-1}) = \min_{w_n \in W_n} \left\{ h_n\left(x_n, p_n^{n-1}\right) \right\}, \qquad \forall t_n \geq 0,\ p_n^{n-1} \in \mathcal{P}_n, \tag{23}$$

$$c_i\left(t_i, p_i^{i-1}\right) = \min_{\substack{p_{i+1}^i \in \mathcal{P}_{i+1} \\ w_i \in W_i}} \left\{ g_{i+1}(0, p_{i+1}^i) + h_i\left(x_i, p_i^{i-1}\right) + \mathbb{E}\left[c_{i+1}\left(x_i + t_{ii+1}, p_{i+1}^i\right)\right] \right\},$$

$$\forall i \in V' \backslash \{n\},\ t_i \geq 0,\ p_i^{i-1} \in \mathcal{P}_i, \tag{24}$$

$$c_0\left(t_0\right) = \min_{p_1^0 \in \mathcal{P}_1} \left\{ g_1(0, p_1^0) + \mathbb{E}\left[c_1\left(x_0 + t_{ii+1}, p_1^0\right)\right] \right\}, \qquad \forall t_0 \geq 0. \tag{25}$$

with $x_i = d_i(t_i, w_i, p_i^{i-1})$.

Next, we define an equivalent stochastic dynamic program with one-dimensional value functions. Without loss of generality, we may reverse the order of determining the adjustment p_{i+1}^i and serving customer i. This follows from the fact that customer $i+1$ is indifferent about the timing of the information.

Let $\bar{c}_i(x_i)$ be the expected total dissatisfaction incurred after serving customer i at time x_i, but before determining the adjustment p_{i+1}^i. That is,

$$\bar{c}_n(x_n) = 0, \qquad \forall x_n \geq 0, \tag{26}$$

$$\bar{c}_i(x_i) = \min_{p_{i+1}^i \in \mathcal{P}_{i+1}} \left\{ g_{i+1}(0, p_{i+1}^i) + \mathbb{E}\left[c_{i+1}\left(x_i + t_{ii+1}, p_{i+1}^i\right)\right] \right\}, \quad \forall i \in \{0, \ldots, n-1\}, x_i \geq 0. \tag{27}$$

From the definition, it is straightforward to show that $\bar{c}_i$ is convex. Furthermore, if the travel time distributions are discrete with finite support, then $\bar{c}_i$ is a polyhedral convex function.

The stochastic dynamic program (23)-(25) can be rewritten as follows.

$$\bar{c}_n(x_n) = 0, \qquad \forall x_n \geq 0, \tag{28}$$

$$\bar{c}_i(x_i) = \min_{p_{i+1}^i \in \mathcal{P}_{i+1}} \left\{ g_{i+1}(0, p_{i+1}^i) + \mathbb{E}\left[\min_{w_{i+1} \in W_{i+1}} \left\{ \bar{c}_{i+1}\left(x_{i+1}\right) + h_{i+1}(x_{i+1}, p_{i+1}^i) \right\} \right] \right\},$$

$$\forall i \in \{0, \ldots, n-1\},\ x_i \geq 0,\ p_i^{i-1} \in \mathcal{P}_i, \tag{29}$$

with $x_{i+1} = d_{i+1}(t_{i+1}, w_{i+1}, p_{i+1}^i)$ and $t_{i+1} = x_i + t_{ii+1}$ in the right-hand side of (29).

The structural properties of the simple DTWAP yield computational advantages for all solution methods discussed in Section 3. First, if we construct a deterministic equivalent mathematical program for (28)-(29), we require significantly less variables than in Section 3.2. This is due to Observation 6.

For the dual decomposition methods (Section 3.3), the expected future value function

F_{i+1} is defined as $F_{i+1}(x_i, p^i) = \mathbb{E}\left[c_{i+1}(x_i + t_{ii+1}, p^i_{i+1})\right]$. Note that F_{i+1} only depends on x_i and on p^i_{i+1}, and can thus be seen as a two-dimensional function. Due to the lower dimensionality, one may expect it to be easier to approximate F_{i+1} for the simple DTWAP than in the general case.

Finally, for the discretization approach (Section 3.4), the number of possible states decreases significantly. Furthermore, the minimization problems that are solved by enumeration have a significantly smaller feasible set. As a result, the discretization approach is less computationally expensive for the simple DTWAP.

5.2.1 Never Wait

For waiting behavior NW, the set of possible voluntary waiting times is given by $W_i = \{0\}$, and the departure time is given by $d_{i+1}(t_{i+1}, w_{i+1}, p^i_{i+1}) = t_{i+1} = x_i + t_{ii+1}$. It follows that Equation (29) simplifies to

$$\begin{aligned} \bar{c}_i\left(x_i\right) &= \min_{p^i_{i+1} \in \mathcal{P}_{i+1}} \left\{ g_{i+1}(0, p^i_{i+1}) + \mathbb{E}\left[\bar{c}_{i+1}\left(x_i + t_{ii+1}\right) + h_{i+1}(x_i + t_{ii+1}, p^i_{i+1})\right] \right\} \\ &= \min_{p^i_{i+1} \in \mathcal{P}_{i+1}} \left\{ g_{i+1}(0, p^i_{i+1}) + \mathbb{E}\left[h_{i+1}(x_i + t_{ii+1}, p^i_{i+1})\right] \right\} + \mathbb{E}\left[\bar{c}_{i+1}\left(x_i + t_{ii+1}\right)\right]. \end{aligned} \tag{30}$$

Equation (30) reveals that the simple DTWAP is decomposable in the NW case. This follows from the fact that the minimization problem in Equation (30) does not depend on $\bar{c}_{i+1}$, the value function of the next customer. The optimal adjustments can then be determined independently per customer, and there is no need to calculate the value functions to determine the optimal actions. This is true for both EXT-NW and POS-NW.

Decomposing the problem makes it easier to solve. If the expectation of h_{i+1} can be evaluated efficiently, then the optimal action can be found efficiently for any given state. This follows from the convexity of g_{i+1} and h_{i+1}. In practice, this means that we do not require extensive calculations up front to determine the optimal actions. Instead, we can solve an easy problem for each state that we encounter.

The independence of the customers is a result of the following fact: if vehicles never wait, adjusting the time window of customer i changes the dissatisfaction of customer i, but has no affect on the arrival times at the other customers. A similar observation is made by Taş et al. (2014), who use this fact to specify the arrival time distributions at the customers.

5.2.2 Always Wait

For waiting behavior AW, we also have that the domain of the voluntary waiting time is given by $W_i = \{0\}$. We first consider EXT-AW specifically. In this case, we have

$$d_{i+1}(t_{i+1}, w_{i+1}, p^i_{i+1}) = \max\{t_{i+1}, a_{i+1}\} = \max\{x_i + t_{ii+1}, a_{i+1}\}, \tag{31}$$

such that (29) reduces to

$$\bar{c}_i(x_i) = \min_{p_{i+1}^i \in \mathcal{P}_{i+1}} \left\{ g_{i+1}(0, p_{i+1}^i) + \mathbb{E}\left[h_{i+1}(\max\{x_i + t_{ii+1}, a_{i+1}\}, p_{i+1}^i) \right] \right\} + \\ \mathbb{E}\left[\bar{c}_{i+1}(\max\{x_i + t_{ii+1}, a_{i+1}\}) \right]. \tag{32}$$

Equation (32) shows that the EXT-AW case decomposes into n independent problems, just like the EXT-NW case and the POS-NW case. This is a result of the fact that the vehicle always waits until the start of the time window. When the time window is extended, the start of the time window does not change. As a result, the arrival time distributions are not affected by the adjustments. It follows that the adjustments can be determined independently per customer.

Next we consider POS-AW. In this case, we have

$$d_{i+1}(t_{i+1}, w_{i+1}, p_{i+1}^i) = \max\{t_{i+1}, a_{i+1} + p_{i+1}^i\} = \max\{x_i + t_{ii+1}, a_{i+1} + p_{i+1}^i\}, \tag{33}$$

such that (29) simplifies to

$$\bar{c}_i(x_i) = \min_{p_{i+1}^i \in \mathcal{P}_{i+1}} \left\{ g_{i+1}(0, p_{i+1}^i) + \mathbb{E}\left[h_{i+1}(\max\{x_i + t_{ii+1}, a_{i+1} + p_{i+1}^i\}, p_{i+1}^i) \right] + \\ \mathbb{E}\left[\bar{c}_{i+1}\left(\max\{x_i + t_{ii+1}, a_{i+1} + p_{i+1}^i\}\right) \right] \right\}. \tag{34}$$

Note that we are not be able to take the $\mathbb{E}[\bar{c}_{i+1}]$ term out of the minimum, due to its dependency on p_{i+1}^i. Hence, the POS-AW case cannot be decomposed in the same way as EXT-NW, POS-NW, and EXT-AW.

5.2.3 Voluntarily Wait

In the VW case, the voluntary waiting time $w_i \in W_i = \mathbb{R}_{\geq 0}$ is a decision variable. The departure time $d_{i+1}(t_{i+1}, w_{i+1}, p_{i+1}^i) = t_{i+1} + w_{i+1}$ is the sum of the arrival time and the voluntary waiting time.

For both EXT-VW and POS-VW, we have that the simple DTWAP cannot be decomposed in the same way as EXT-NW, POS-NW, and EXT-AW. Voluntarily waiting at customer i affects the arrival time at customer $i+1$. As such, we are unable to determine the optimal waiting time independently per customer. Our observations on the VW cases are detailed in Appendix B.

5.3 Time-dependent linear penalties

For the simple DTWAP, we have made the important assumption that the time at which an adjustment is communicated to the customer does not affect the amount of dissatisfaction received. That is, we have replaced $g_j(t_i, p_j^{i-1}, p_j^i)$ by $g_j(p_j^{i-1}, p_j^i)$. In this section, we prove that dropping this assumption may lead to a severely more complicated problem. We show this by introducing time-dependent linear penalties into the simple DTWAP.

Recall that

$$g_j(p_j^{i-1}, p_j^i) = \left(p_j^i - p_j^{i-1}\right)^+ \alpha_j + \left(p_j^{i-1} - p_j^i\right)^+ \beta_j, \tag{20}$$

for some parameters $a_j, b_j \in \mathbb{R} \cup \{\infty\}$, $a_j + b_j \geq 0$. If any of the parameters is chosen to be ∞, we replace the corresponding term by a domain restriction. Note that both increasing and decreasing the adjustment is penalized linearly.

Next, we consider time-dependent linear penalties of the form

$$g_j(t_i, p_j^{i-1}, p_j^i) = \left(p_j^i - p_j^{i-1}\right)^+ \alpha_j f_{1j}(t_i) + \left(p_j^{i-1} - p_j^i\right)^+ \beta_j f_{2j}(t_i), \tag{35}$$

for some weighting functions $f_{1j}(t_i)$ and $f_{2j}(t_i)$. For given t_i, changes are linearly penalized, and the slopes are time-dependent.

Proposition 7. *Let $f_{1j}(t_i)$ and $f_{2j}(t_i)$ be twice-differentiable real functions that are defined on the same real interval. If $g_j(t_i, p_j^{i-1}, p_j^i)$ as in (35) is convex, then $\alpha_j = 0$ or $\alpha_j = \infty$ or $f_{1j}(t_i)$ is constant. Furthermore, $\beta_j = 0$ or $\beta_j = \infty$ or $f_{2j}(t_i)$ is constant.*

Proof. Let $\bar{g}_j(t_i, y)$ be the restriction of $g_j(t_i, p_j^{i-1}, p_j^i)$ to $y = p_j^i - p_j^{i-1}$. That is, $\bar{g}_j(t_i, y) = y^+ \alpha_j f_{1j}(t_i) + (-y)^+ \beta_j f_{2j}(t_i)$.

If $\alpha_j < \infty$, then $\bar{g}_j$ is twice differentiable for all t_i in the domain of $f_{1j}(t_i)$ and $f_{2j}(t_i)$ and all $y > 0$. For any such point, we have

$$\nabla^2 \bar{g}_j(t_i, y) = \begin{bmatrix} y\alpha_j f_{1j}''(t_i) & \alpha_j f_{1j}'(t_i) \\ \alpha_j f_{1j}'(t_i) & 0 \end{bmatrix}.$$

If $\alpha_j \neq 0$ then $\nabla^2 \bar{g}_j(t_i, y)$ can only be positive semi-definite if $f_{1j}'(t_i) = 0$. This implies that $\bar{g}_j(t_i, y)$ cannot be convex, unless $f_{1j}(t_i)$ is a constant. As $\bar{g}_j$ is a restriction of g_j to a linear subspace, it follows that $g_j(t_i, p_j^{i-1}, p_j^i)$ cannot be convex either. If $\beta_j < \infty$ and $\beta_j \neq 0$ we can similarly show that $\bar{g}_j(t_i, y)$ cannot be convex, unless $f_{2j}(t_i)$ is a constant. □

Proposition 7 demonstrates that using time-dependent linear penalties results in $g_j(t_i, p_j^{i-1}, p_j^i)$ not being convex, except in pathological cases. As a result, the conditions of Proposition 2 and Proposition 4 are no longer satisfied. It follows that it is not clear if the deterministic equivalent mathematical program can be solved efficiently for even a single scenario. Furthermore, if the F_{i+1} functions are not convex, dual decomposition methods can no longer be used. We conclude that introducing time-dependent linear penalties considerably complicates the simple DTWAP, although the discretization approach remains an option.

6 Effect of discretization

In this section, we analyze the effect of discretizing the possible adjustments and the voluntary waiting times. Discretization was proposed in Section 3.4 to allow for solving the stochastic dynamic programming recursions directly. First, we show the negative result that discretization may lead to suboptimal solutions in general, even for instances with integer parameters and discrete travel time distributions with finite support. On the other hand, we prove in the same setting that the discretization approach is exact for the simple DTWAP introduced in Section 5.

Specifically, we consider the following setting. We assume that the vehicle starting time and all initial time windows are integer. Furthermore, we assume that the travel time distributions are discrete with finite support, and all possible realizations are integer. If the sets of possible adjustments are bounded, we assume that they are bounded by integers. That is, $\min_{p\in\mathcal{P}_i}\{p\} \in \mathbb{Z}$ and $\max_{p\in\mathcal{P}_i}\{p\} \in \mathbb{Z}$ for all $i \in V'$, whenever the minimum and maximum are defined.

We do not force the actions to be integral: both the adjustments and the voluntary waiting times are assumed to be real. Given these assumptions, we are interested in whether there exist optimal time window adjustments and voluntary waiting times that are integer.

6.1 Discretization for general instances

We now construct a DTWAP instance that satisfies the assumptions above, but does not admit optimal actions that are integer. Consider an $n = 1$ customer instance with dissatisfaction functions $g_1(t_0, 0, p_1^0) = (p_1^0)^2$ and $h_1(x_1, p_1^0) = (1 - p_1^0)^2$. Let the vehicle depart the depot a time $x_0 = t_0 = 0$, and let the initial time window of customer 1 be given by $[a_i, b_i] = [0, 0]$. The travel time to customer 1 is deterministic and equal to 0. The adjustment type (EXT or POS) and the waiting behavior (NW, AW, or VW) can be chosen arbitrarily. The set of possible postponements is given by $\mathcal{P}_1 = [0, 1]$.

For the described instance, the DTWAP reduces to the following problem:

$$\min_{p_1^0\in[0,1]} \left\{(p_1^0)^2 + (1 - p_1^0)^2\right\}. \tag{36}$$

It can be seen that the only possible integer actions, $p_1^0 = 0$ and $p_1^0 = 1$, result in strictly higher dissatisfaction than the action $p_1^0 = \frac{1}{2}$. Hence, this instance proves that discretization may lead to suboptimal solutions in general, even under the current assumptions.

6.2 Discretization for the simple DTWAP

Under the same assumptions, we now prove that the simple DTWAP admits optimal actions that are integer, regardless of the adjustment type (EXT or POS) and the waiting behavior (NW, AW, or VW). It follows that discretizing the actions into integers still yields an optimal solution.

In Proposition 8, we prove our claim for the cases EXT-NW, POS-NW, and EXT-AW. As shown in Section 5.2, the simple DTWAP is decomposable in these cases. This significantly simplifies our proof, compared to the cases POS-AW, EXT-VW, and POS-VW.

Proposition 8. *The simple DTWAP with discretized time allows for optimal integer decisions in the cases of EXT-NW, POS-NW, and EXT-AW.*

Proof. See Appendix C. □

Next, we consider the non-decomposable cases POS-AW, EXT-VW, and POS-VW. In these cases, the optimal action p^i_{i+1} additionally depends on $\bar{c}_{i+1}$. In Appendix D, we prove several lemmas before we prove Proposition 9.

Proposition 9. *The simple DTWAP with discretized time allows for optimal integer decisions in the cases of POS-AW, EXT-VW, and POS-VW.*

Proof. See Appendix D. □

Propositions 8 and 9 together prove that for the simple DTWAP, under the assumptions stated at the beginning of this section, the optimal actions may be assumed to be integer. Hence, in these cases, the discretization approach as introduced in Section 3.4 provides a straightforward and exact algorithm to solve the simple DTWAP.

7 Illustrative example

In this section, we provide an illustrative example of how the results of this paper can be used to improve customer satisfaction in a practical application. In Section 7.1, we present an attended home delivery problem with adjustment type POS and waiting behavior AW. To model dissatisfaction, we use the customer dissatisfaction functions of the simple DTWAP (see Section 5) as a basis, and we modify them to better fit this specific setting.

In Section 7.2, we make use of Sections 3 and 4 to construct a solution method. Our computational results are presented in Section 7.3. Finally, we present three heuristics in Section 7.4, which use ideas from Section 5 and Section 6.

7.1 DTWAP instance

For our example, we consider an *attended home delivery* setting. That is, the customers have to be present at the time of delivery. There can be various reasons to require the customer to be present, including security reasons (e.g., electronics), because the products are perishable (e.g., groceries), or because the products are large (e.g., furniture) (Agatz et al., 2008).

We assume that the vehicle leaves the depot at 7.00 am and performs a route with ten customers (see Figure 1). That is, $t_0 = x_0 = 420$ and $n = 10$. Note that time is measured in minutes since midnight, i.e., 420 corresponds to 7.00 am.

The travel times t_{ii+1} for all $i \in V' \cup \{0\}$ are assumed to be independent and discrete uniformly distributed between 50 and 70 minutes, and the service times are assumed to be negligible. The initial time windows of the customers are 20 minutes in width, and are centered around the time of arrival when all travel times are equal to their expected values, and no waiting is necessary. For example, the initial time window of customer two is given by [8.50 am, 9.10 am], which is centered around 9.00 am. For each customer $j \in V'$, this is modeled as $a_j = 420 + 60j - 10$ and $b_j = 420 + 60j + 10$.

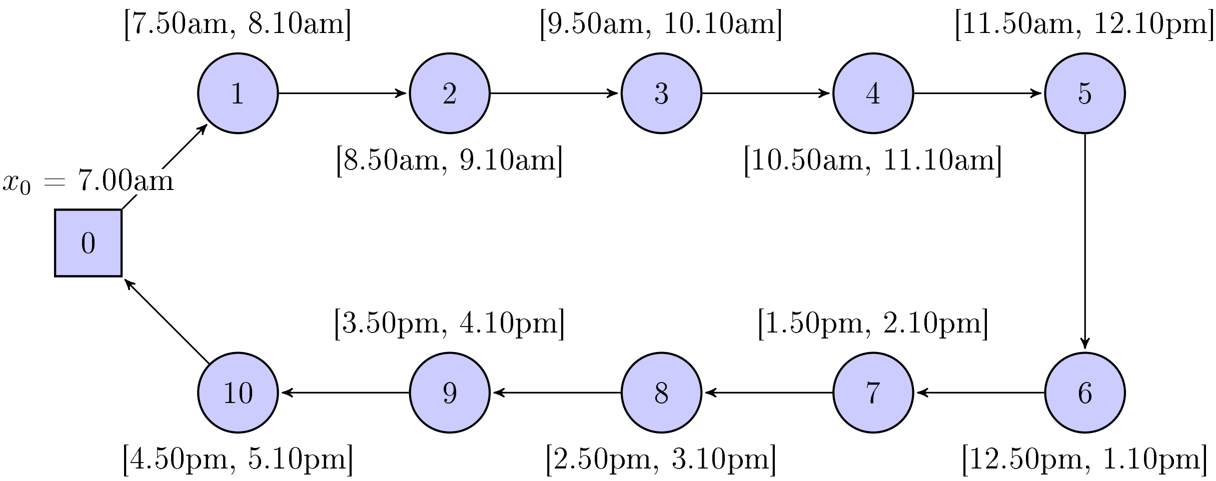

Figure 1: Example instance.

In our attended home delivery setting, the customers are present from the start of the current time window onwards, but are not available earlier. In the case of late delivery, customers stay at home until the delivery is made. Because the customers are not available earlier, we use waiting behavior AW to always wait until the time window opens.

We allow the time windows to be postponed by 0, 5, 10, 15, 20, 25 or 30 minutes, i.e., we use adjustment type POS and $\mathcal{P}_j = \{0, 5, 10, 15, 20, 25, 30\}$ for all $j \in V'$. By postponing the time windows in steps of five minutes, we obtain time windows that are easy to remember for the customers. To prevent that customers have to rush back home due to time window adjustments, we only allow time windows to be moved forward in time, which will be incorporated in our customer dissatisfaction functions.

Our customers dislike adjustments, but are not influenced by the time of communication until three hours before the current deadline. For example, consider customer 5. If the current time window is given by [11.50 am, 12.10 pm], then customer 5 does not care at what time a postponement is communicated, as long as it is before 9.10 am. After 9.10 am, it becomes increasingly more difficult for customer 5 to change their plans to be at home during the time window. As such, the dissatisfaction is larger when the postponement is communicated later. After the current deadline, the delivery is considered to be late and the time window can no longer be adjusted.

7.1.1 Customer dissatisfaction functions

We use the customer dissatisfaction functions of the simple DTWAP as a basis, and we modify them to fit our setting. The simple DTWAP has the advantage that we can perform the analysis in Section 5.2. However, without modification, the associated customer dissatisfaction functions are not appropriate for our practical application.

First, we consider the functions $g_j(t_i, p_j^{i-1}, p_j^i)$, which model the dissatisfaction of customer j due to changing the postponement from p_j^{i-1} to p_j^i at time t_i. The function g_j that is used in the simple DTWAP, Equation (20), is not appropriate for our example. First, Equation (20) assumes that customers are indifferent about the timing of information, which is not the case here. Second, Equation (20) allows for adjusting the time window of customer j when t_i is past the current deadline of customer j, which we do not allow in our example.

Based on Section 5.3, we propose the following dissatisfaction function:

$$g_j\left(t_i, p_j^{i-1}, p_j^i\right) = \left(p_j^i - p_j^{i-1}\right)^+ \alpha_j f_j(t_i), \tag{37}$$

on domain $p_j^{i-1} \leq p_j^i$ and $t_i \leq b_j + p_j^{i-1}$, with

$$f_j(t_i) = 1 + \nu_j\left(t_i - \left(b_j + p_j^{i-1} - L_j\right)\right)^+, \tag{38}$$

and non-negative parameters α_j, ν_j, and L_j.

Equation (37) provides a time-dependent linear penalty for changing the adjustment, as introduced in Section 5.3. The domain restrictions correspond to our assumptions: the condition $p_j^{i-1} \leq p_j^i$ ensures that the adjustments can only be increased, and $t_i \leq b_j + p_j^{i-1}$ prevents adjustments to be made after the deadline has passed. For convenience, we consider g_j to evaluate to ∞ outside of its domain.

The function $f_j(t_i)$ models how the dissatisfaction due to time window adjustments is affected by time. If the time at which the adjustment is communicated is at least L time units before the current deadline $b_j + p_j^{i-1}$, then $f_j(t_i) = 1$, and the time of communication does not affect the dissatisfaction. In our example, $L_j = 180$ for every customer, which corresponds to three hours. After time $b_j + p_j^{i-1} - L_j$, the parameter ν_j indicates how strongly customer j is affected by the timing of information.

Next, we consider the functions $h_j(x_j, p_j^{j-1})$, which model the dissatisfaction due to missing the deadline. We modify the function h_j used for the simple DTWAP, Equation (22), by including a fixed dissatisfaction κ_j for missing the deadline. We obtain the following function:

$$h_j(t_j, p_j^{j-1}) = \left(t_j - (b_j + p_j^{j-1})\right)^+ \gamma_j + I(t_j > b_j + p_j^{j-1})\kappa_j, \tag{39}$$

with $I(.)$ the indicator function. Note that we do not define a penalty for early delivery, because the vehicle always waits until the time window opens.

For our computational experiments, we assume that the parameters for customer $j \in V'$ are given by $\alpha_j = 0.1$, $\nu_j = 0.1$, $L_j = 180$, $\gamma_j = 1$, and $\kappa_j = 100$. Note that these values are chosen for demonstrative purposes, and may differ in practical applications.

Finally, we verify that our DTWAP instance satisfies the two general assumptions stated in Section 3.1. The dissatisfaction functions g_j and h_j are all non-negative, which implies that the DTWAP instance is sufficiently expensive. By definition, only finite postponements are allowed, which implies that the optimal actions are attainable. It follows that the assumptions are satisfied.

7.2 Solution method

In this section, we describe a solution method to solve the DTWAP constructed in Section 7.1. Recall that Section 3 presents three potential solution methods: solving the deterministic equivalent mathematical program, using dual decomposition methods, and solving the discretized stochastic dynamic program. In Section 4, we discuss the properties of the DTWAP, and we state conditions such that the methods in Section 3 can be applied.

For the current example, we have that the g_j functions are not convex. This follows from the use of time-dependent linear penalties, as shown in Proposition 7. As a result, Proposition 2 and Proposition 4 in Section 4 are not applicable.

It follows that it is not clear if the deterministic equivalent mathematical program can be solved efficiently for even a single scenario. Furthermore, it is unclear if dual decomposition methods can be applied. For these reasons, we apply the third method: solving the discretized stochastic dynamic program. As discussed in Section 3, this method does not require convexity assumptions.

Section 3.4 details how the DTWAP instance can be modified such that the stochastic dynamic program (8)-(10) can be solved directly. In our case, we discretize time in minutes, and we observe that all relevant parameters already have integer values. The same holds true for the travel time distributions, which have finite support and all realizations are integer. It follows that the stochastic dynamic program (8)-(10) can be solved directly by backward or forward recursion, as explained in Section 3.4.

In the attended home delivery setting, solving the stochastic dynamic program can be accelerated by making an observation similar to Observation 6, which is used in Section 5 to speed up the solution methods for the simple DTWAP. This observation states that if the customer dissatisfaction is independent of the time of communication, then it is optimal to delay adjustment decisions.

In our setting, customer j is indifferent about the timing of the information, given that it is at least three hours before the current deadline. Hence, if the next opportunity to change the adjustments is, *with certainty*, at least three hours before the deadline, then we can delay the adjustment decision. Mathematically speaking, if the arrival time at customer i is t_i, and the probability that $t_{i+1} \leq b_j - L_j$ is one, then $p^i_j = 0$ is optimal.

7.3 Computational results

We solve the stochastic dynamic program (8)-(10) by forward recursion, as described in Section 3.4, including the acceleration strategy discussed in Section 7.2. The algorithm is coded in C++, and our computational results are obtained with an Intel Core i7-8550U processor.

In Table 1, we compare the results of the DTWAP with two benchmarks: *no adjustment* and *a priori*. For the no adjustment benchmark, we do not make any time window adjustments. That is, the action at customer i is given by $p^i = 0$ for every state. For the a priori benchmark, we only allow adjusting the time windows before leaving the depot. That is, all adjustments are made a priori, and the time windows cannot be updated dynamically.

Recall that the set of possible adjustments is given by $\mathcal{P}_j = \{0, 5, 10, 15, 20, 25, 30\}$ for all $j \in V'$. To obtain a fair comparison with the DTWAP, we use the same possible adjustments for the a priori benchmark.

The a priori benchmark is calculated by enumerating all possible adjustment vectors. For each adjustment vector, we obtain a DTWAP instance by fixing the adjustment options accordingly. For example, if $p^0 = (5, 10, \dots, 10)$, then we set $\mathcal{P}_1 = \{5\}$, $\mathcal{P}_2 = \{10\}, \dots,$ $\mathcal{P}_{10} = \{10\}$. We then use our DTWAP algorithm to obtain the expected total dissatisfaction for the given adjustment vector. By enumeration, the best a priori adjustment vector is found.

The reported statistics are the expected values over all $21^{10} \approx 17$ trillion scenarios. To calculate these statistics efficiently, we first decide on the action for every state. We then obtain the statistics recursively by solving stochastic dynamic programs that are similar

	No adjustment	A priori	DTWAP
Total expected dissatisfaction	235.3	37.6	27.9
Expected dissatisfaction due to missed deadlines (% of total)	100.0%	33.6%	23.7%
Expected dissatisfaction due to adjustments (% of total)	0.0%	66.4%	76.3%
Expected percentage of deadlines missed	21.5%	1.2%	0.6%
Expected average lateness per customer	120 seconds	3 seconds	2 seconds
Expected average postponement time per customer	0 minutes	20 minutes	14 minutes
Expected average number of postponements per customer	0	0.9	1.2
Solution time	0 seconds	9962 seconds	86 seconds

Table 1: Comparison between no adjustment, a priori adjustment, and dynamic time window adjustment.

to (8)-(10), but with the minimization over p^i replaced by the action chosen earlier.

Before discussing the results in Table 1, we want to emphasize that we only consider a single DTWAP instance. To make general statements, a more extensive computational study is required. Instead, we use our illustrative example to demonstrate the potential benefit of dynamic time window adjustment in a practical application.

If no adjustments are made, in expectation 21.5% of the customers is served after the communicated deadline. Even though the expected average lateness per customer is only 2.0 minutes, the dissatisfaction is high because the customers really dislike late deliveries, even if they are just a couple of minutes late. This characteristic corresponds to the parameter $\kappa_j = 100$ in the h_j function (Equation (39)).

If we compare the no adjustment benchmark to the DTWAP, we see that using dynamic time window adjustments can significantly reduce the expected total dissatisfaction from 235.3 to 27.9. In expectation, on average, the time window of a customer is adjusted 1.2 times, and the final average postponement is 14 minutes. The expected percentage of deadlines missed decreases from 21.5% to only 0.6%, and the expected average lateness per customer decreases from 2 minutes to 2 seconds.

Next, we compare the a priori benchmark to the DTWAP. For the a priori benchmark, the expected probability of missing a deadline is 1.2%, which is twice the probability of 0.6% for the DTWAP. The average postponement time per customer is also higher for the a priori benchmark, with 20 minutes versus 14 minutes. We conclude that dynamic time window adjustments may simultaneously reduce the probability of missing a deadline, and reduce the average time window postponement.

There is a large difference in solution time between calculating the a priori benchmark and solving the DTWAP: 9962 seconds versus 86 seconds. This difference is due to the large number of adjustment vectors that are enumerated in the a priori case. Ten customers with seven adjustment options each results in $7^{10} \approx 282$ million possible vectors. For the DTWAP, this number is significantly smaller, because many decisions can be delayed (see Section 7.2). For example, if we can delay all decisions, except for the postponements of the next four customers, then only $7^4 = 2401$ adjustment vectors need to be considered at the current state.

To gain more insight into the optimal DTWAP solution, Figure 2 presents the final postponements for the scenario in which all travel times are equal to their expected value. That is, all travel times turn out to be 60 minutes. If we would have known in advance that this scenario would occur, no time window adjustment would have been necessary. Without foresight, we use time windows adjustments to anticipate the uncertainty in the travel times.

In Figure 2, the postponements are larger for later customers, even though all travel times are equal to their expected value. This is because the vehicle always waits until the time window opens, and because time windows can only be postponed further.

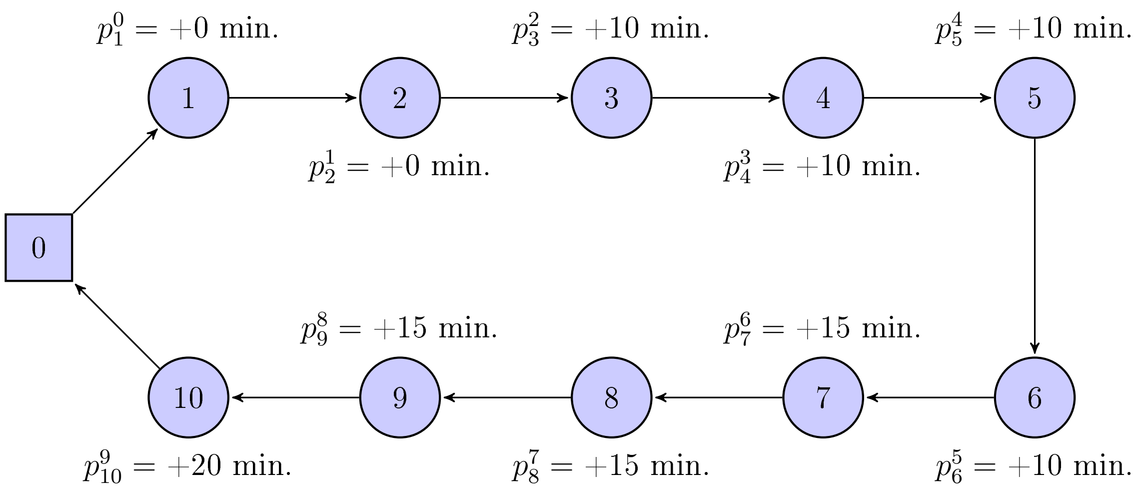

Figure 2: Optimal adjustments for the scenario with expected travel times.

As a result, postponements that are used to anticipate randomness cause delays themselves, prompting larger time window adjustments for the later customers. The attended home delivery example shows that, even if such a snowball effect occurs, time window adjustment can be beneficial.

7.4 Heuristic solution methods

Based on the analysis in this paper, we can construct various heuristic solution methods. We present three such heuristics, and we apply them to our attended home delivery example. The heuristics are compared on solution time and on expected total dissatisfaction.

In Section 5, we have analyzed the simple DTWAP. For this problem, by Observation 6, it is optimal to only decide on the adjustment for the next customer. Based on this fact, we have explained how to speed up the solution methods. We can use the same idea as a heuristic: only consider the adjustments for the next given number of customers. We implement this heuristic by making straightforward changes to our DTWAP algorithm.

In Section 6, we have considered the effect of discretization. For the simple DTWAP, we have shown that continuous adjustments can be replaced by integer adjustments without loss of solution quality. As a heuristic, we can use a more coarse discretization for the possible adjustments. For our example, we will consider replacing $\mathcal{P}_j = \{0, 5, 10, 15, 20, 25, 30\}$ by $\mathcal{P}_j = \{0, 10, 20, 30\}$.

For our third heuristic, we solve the stochastic dynamic program (8)-(10) heuristically by assuming that all travel times are deterministic and equal to their expected value. That is, given the current state, we solve a DTWAP in which each travel time distribution is replaced by a degenerate distribution that is always equal to the expected travel time. Note that the difference in dissatisfaction between applying this heuristic and solving the DTWAP can be considered as the *value of stochastic information*.

In Table 2, we compare the two benchmarks, the DTWAP, and the three heuristics. For the first heuristic, we present two variants: only consider the next customer, and consider the next three customers.

	Seconds	Dissatisfaction
No adjustment	0.0	235.3
A priori	9962.4	37.6
DTWAP	86.3	27.9
Only consider the next customer	0.0	81.7
Only consider the next three customers	5.1	31.2
Coarse discretization	1.7	32.0
No stochastic information	0.0	134.6

Table 2: Comparison of heuristics on solution time and total expected dissatisfaction.

The first heuristic is very fast, but does not produce a good solution. Only considering the next customer results in communicating each time window adjustment at the latest possible time. Because customers appreciate being informed timely, the resulting solution has a relatively high expected total dissatisfaction.

This is different for considering the next three customers. In our example, customers do not care about the exact time at which time window adjustments are communicated, given that they are communicated at least three hours before the current deadline. Note that the expected travel time is 60 minutes, such that three hours roughly corresponds to visiting the next three customers. We see that this heuristic is able to find a solution that is relatively close to the optimal solution, in a fraction of the time that we need to solve the DTWAP.

For our example, the coarse discretization heuristic is the second best heuristic in terms of solution quality. By limiting the number of adjustment options, the number of possible adjustment vectors decreases exponentially. This explains the relatively fast solution time. We also want to point out that the coarse discretization heuristic provides a better solution than the a priori benchmark. This demonstrates that, in our example, allowing adjustments to be dynamic is more important than providing more adjustment options.

Finally, we consider the heuristic in which we do not use any stochastic information. This heuristic is fast, but the solution quality is bad. Our attended home delivery example thus demonstrates that not accounting for random travel times can be harmful to the satisfaction of the customers.

8 Conclusion

In this paper, we introduce dynamic time window adjustment to the literature. To the best of our knowledge, this topic has not yet attracted the attention of researchers. On the other hand, dynamic time window adjustments are highly relevant in practice, and are often used to improve customer satisfaction.

We provide a general model and we introduce the DTWAP to optimize the dynamic time window adjustments. We consider three different solution methods for the DTWAP, and we discuss their advantages and disadvantages.

We also introduce the simple DTWAP, which we analyze in more detail. We explain why the simple DTWAP can be solved more efficient than in general, and for three variants we show that the simple DTWAP decomposes into independent problems per customer. Furthermore, under certain assumptions, we prove that discretizing time still yields an optimal solution for the simple DTWAP.

In our illustrative example on attended home delivery, we demonstrate how the results of this paper can be used to solve the DTWAP in a practical application. Based on our analysis, we choose the solve our problem by discretization and forward recursion. Ideas developed throughout the paper are used to construct and test three different heuristics.

Our computational results show that dynamic time window adjustment has the potential to improve customer satisfaction. If we compare a priori (non-dynamic) time window adjustment to dynamic time window adjustment, we see for our example that using the latter results in missing less deadlines, while the time windows are adjusted by smaller amounts.

For future work, it can be interesting to use dynamic time window adjustments in specific settings, including parcel delivery, attended home delivery, and retailer distribution. This includes the construction of suitable customer dissatisfaction functions and metrics to measure the actual improvement. Another direction for further research is to search for special cases that allow the DTWAP to be solved relatively efficiently, as is the case for the simple DTWAP, and to develop specialized algorithms.

Finally, we may integrate the DTWAP with other problems. The initial time windows, for example, are now assumed to be fixed. Integrating the initial time window assignment with the DTWAP is a relevant practical problem that has not yet been considered in this paper.

Another possibility is combining the DTWAP with stochastic vehicle routing. Our computational experiments indicate that after making a priori decisions, dynamic time window adjustments can improve customer satisfaction further. As such, we see an interesting opportunity in improving customer satisfaction by combining dynamic time window adjustment and stochastic vehicle routing.

References

Niels Agatz, Ann Campbell, Moritz Fleischmann, and Martin W. P. Savelsbergh. Challenges and Opportunities in Attended Home Delivery. In Bruce L. Golden, S. Raghavan, and Edward Wasil, editors, *The Vehicle Routing Problem: Latest Advances and New Challenges*, pages 379–396. Springer, 2008.

Niels Agatz, Ann Campbell, Moritz Fleischmann, and Martin W. P. Savelsbergh. Time Slot Management in Attended Home Delivery. *Transportation Science*, 45(3):435–449, 2011.

Robert J. Aumann and Roberto Serrano. An Economic Index of Riskiness. *Journal of Political Economy*, 116(5):810–836, 2008.

Jacques F. Benders. Partitioning procedures for solving mixed-variables programming problems. *Numerische Mathematik*, 4(1):238–252, 1962.

Dimitri P. Bertsekas. *Dynamic Programming and Optimal Control, Vol. I.* Athena Scientific, 3rd edition, 2005.

Endre Boros and Peter L. Hammer. The max-cut problem and quadratic 0–1 optimization; polyhedral aspects, relaxations and bounds. *Annals of Operations Research*, 33 (3):151–180, 1991.

Stephen Boyd and Lieven Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.

Fausto Errico, Guy Desaulniers, Michel Gendreau, Walter Rei, and Louis-Martin Rousseau. A priori optimization with recourse for the vehicle routing problem with hard time windows and stochastic service times. *European Journal of Operational Research*, 249(1):55–66, 2016.

Michael R. Garey and David S. Johnson. *Computers and Intractability: A Guide to the Theory of Np-Completeness*. W. H. Freeman and Company, 1979.

Michel Gendreau, Gilbert Laporte, and René Séguin. Stochastic vehicle routing. *European Journal of Operational Research*, 88(1):3–12, 1996.

Chris Groër, Bruce L. Golden, and Edward Wasil. The Consistent Vehicle Routing Problem. *Manufacturing & Service Operations Management*, 11(4):630–643, 2009.

Shuihua Han, Ling Zhao, Kui Chen, Zong-wei Luo, and Deepa Mishra. Appointment scheduling and routing optimization of attended home delivery system with random customer behavior. *European Journal of Operational Research*, 262(3):966–980, 2017.

Soumia Ichoua, Michel Gendreau, and Jean-Yves Potvin. Vehicle dispatching with time-dependent travel times. *European Journal of Operational Research*, 144(2):379–396, 2003.

Ola Jabali, Roel Leus, Tom van Woensel, and Ton G. de Kok. Self-imposed time windows in vehicle routing problems. *OR Spectrum*, 37(2):331–352, 2015.

Patrick Jaillet, Jin Qi, and Melvyn Sim. Routing optimization under uncertainty. *Operations Research*, 64(1):186–200, 2016.

Peter Kall and János Mayer. *Stochastic Linear Programming: Models, Theory, and Computation*. Springer, 2$^{\text{nd}}$ edition, 2011.

Anton J. Kleywegt, Alexander Shapiro, and Tito Homem de Mello. The Sample Average Approximation Method for Stochastic Discrete Optimization. *SIAM Journal on Optimization*, 12(2):479–502, 2002.

Mario V. F. Pereira and Leontina M. V. G. Pinto. Multi-stage stochastic optimization applied to energy planning. *Mathematical Programming*, 52(1-3):359–375, 1991.

Alexander Shapiro. Analysis of stochastic dual dynamic programming method. *European Journal of Operational Research*, 209(1):63–72, 2011.

Alexander Shapiro and Arkadi Nemirovski. On complexity of stochastic programming problems. In Vaithilingam Jeyakumar and Alexander Rubinov, editors, *Continuous Optimization*, pages 111–146. Springer, 2005.

Alexander Shapiro, Darinka Dentcheva, and Andrzej Ruszczyński. *Lectures on stochastic programming: modeling and theory*. SIAM, 2009.

Duygu Taş, Michel Gendreau, Nico Dellaert, Tom van Woensel, and Ton G. de Kok. Vehicle routing with soft time windows and stochastic travel times: A column generation and branch-and-price solution approach. *European Journal of Operational Research*, 236(3):789–799, 2014.

Marlin W. Ulmer and Barrett W. Thomas. Enough Waiting for the Cable Guy–Estimating Arrival Times for Service Vehicle Routing. *Transportation Science*, page forthcoming, 2019.

R. M. van Slyke and Roger Wets. L-Shaped Linear Programs with Applications to Optimal Control and Stochastic Programming. *SIAM Journal on Applied Mathematics*, 17(4):638–663, 1969.

Junlong Zhang, William H. K. Lam, and Bi Yu Chen. A Stochastic Vehicle Routing Problem with Travel Time Uncertainty: Trade-Off Between Cost and Customer Service. *Networks and Spatial Economics*, 13(4):471–496, 2013.

Junlong Zhang, William H. K. Lam, and Bi Yu Chen. On-time delivery probabilistic models for the vehicle routing problem with stochastic demands and time windows. *European Journal of Operational Research*, 249(1):144–154, 2016.

Yu Zhang, Roberto Baldacci, Melvyn Sim, and Jiafu Tang. Routing optimization with time windows under uncertainty. *Mathematical Programming*, page forthcoming, 2018.

A Proof Proposition 1

Proposition 1. *The DTWAP is strongly* NP*-hard, even for deterministic travel times, and for dissatisfaction functions that can be evaluated in polynomial time.*

Proof. We first observe the following. If $n = 1$, $t_0 = 0$, $\mathcal{P}_1 = \mathbb{N}_{\geq 0}$, and $h_1(x_1, p_1^0) = 0$, then the DTWAP simplifies to

$$\min_{p_1^0 \in \mathbb{N}_{\geq 0}} g_1(0, 0, p_1^0). \tag{40}$$

That is, solving the DTWAP is at least as hard as minimizing a general one-dimensional function over $\mathbb{N}_{\geq 0}$. Note that this fact is independent of the adjustment type (EXT or POS) and the waiting behavior (NW, AW, or VW).

Next, we present a polynomial-time reduction from the MAX-CUT problem to Problem (40). For our purpose, it is sufficient to note the following two facts. First, MAX-CUT is defined on a graph with m vertices and is strongly NP-hard (Garey and Johnson, 1979). Second, MAX-CUT can be stated as $\min_{y \in \{0,1\}^m} f(y)$, for some quadratic function f (Boros and Hammer, 1991). The values of the coefficients of f are polynomial in the input values.

Now define the function $g_1(0, 0, p_1^0)$, as follows. Without loss of generality, we assume that p_1^0 is encoded as a binary number. If p_1^0 has more than m digits, then $g_1(0, 0, p_1^0) = \infty$. If p_1^0 has at most m digits, then pad the left with zeros to obtain a binary number with exactly m digits. Interpret this binary number as a binary vector y and return $f(y)$. Clearly, every binary vector of length m can be constructed in this way.

Hence, we have shown that MAX-CUT reduces to Problem (40), which is a special case of the DTWAP. It can be verified that this is a polynomial-time reduction. Furthermore, it can be seen that our reduction does not introduce large numerical parameters. It follows that the DTWAP is strongly NP-hard. $\square$

B Properties of the simple DTWAP - Voluntarily Wait

In the VW case, the voluntary waiting time $w_i \in W_i = \mathbb{R}_{\geq 0}$ is a decision variable. The departure time $d_{i+1}(t_{i+1}, w_{i+1}, p^i_{i+1}) = t_{i+1} + w_{i+1}$ is the sum of the arrival time and the voluntary waiting time.

Let $w^*_i(t_i, p^{i-1}_i)$ be an optimal waiting time for when the vehicle arrives at customer i, at time t_i, with time window adjustment p^{i-1}_i. That is, we define

$$w^*_i(t_i, p^{i-1}_i) = \arg\min_{w_i \geq 0} \left\{ \bar{c}_i(t_i + w_i) + h_i(t_i + w_i, p^{i-1}_i) \right\}. \tag{41}$$

In Equation (29), we may then replace the minimization over w_{i+1} with an optimal value w^*_{i+1}. We obtain

$$\bar{c}_i(x_i) = \min_{p^i_{i+1} \in \mathcal{P}_{i+1}} \left\{ g_{i+1}(0, p^i_{i+1}) + \mathbb{E}\left[\bar{c}_{i+1}(x_{i+1}) + h_{i+1}(x_{i+1}, p^i_{i+1}) \right] \right\}, \tag{42}$$

with $x_{i+1} = t_{i+1} + w^*_{i+1}(t_{i+1}, p^i_{i+1})$ and $t_{i+1} = x_i + t_{ii+1}$ in the right-hand side.

Proposition 10. *Let $X^*_i = \arg\min_{X_i \in \overline{\mathbb{R}}} \{\bar{c}_i(X_i) - X_i \delta_i\}$ be the preferred departure time from customer i. Then $w^*_i(t_i, p^{i-1}_i)$ can be stated as follows.*

- *For EXT-VW, $w^*_i(t_i, p^{i-1}_i) = (\min\{X^*_i, a_i\} - t_i)^+$.*
- *For POS-VW, $w^*_i(t_i, p^{i-1}_i) = \left(\min\{X^*_i, a_i + p^{i-1}_i\} - t_i\right)^+$.*

Proof. Recall that the value function $\bar{c}_i$ is a convex function. Additionally, under VW, the value function is non-decreasing in the departure time x_i. This follows from the fact that the vehicle can always depart later by waiting.

The function $h_i(x_i, p^{i-1}_i)$ is non-increasing in the departure time x_i until the start of the (adjusted) time window, zero within the time window, and non-decreasing after the end of the time window. As the value function $\bar{c}_i$ is non-decreasing in the departure time, it is optimal to not wait further than the beginning of the time window. That is, it is optimal to assume that $w^*_i(t_i, p^{i-1}_i) \leq (a_i - t_i)^+$ for EXT-VW and $w^*_i(t_i, p^{i-1}_i) \leq (a_i + p^{i-1}_i - t_i)^+$ for POS-VW.

In the case of EXT-VW, we show that

$$\begin{aligned} w^*_i(t_i, p^{i-1}_i) &= \arg\min_{w_i \geq 0} \left\{ \bar{c}_i(t_i + w_i) + h_i(t_i + w_i, p^{i-1}_i) \right\} && (43) \\ &= \arg\min_{0 \leq w_i \leq (a_i - t_i)^+} \left\{ \bar{c}_i(t_i + w_i) - w_i \delta_i \right\}. && (44) \end{aligned}$$

Note that (43) is simply the definition of $w^*_i(t_i, p^{i-1}_i)$.

To go from (43) to (44), we first add the upper bound $w_i \leq (a_i - t_i)^+$ as discussed above. If $t_i \geq a_i$, then we arrive at customer i after the start of the time window. It is then optimal not to wait, i.e., $w_i = 0$, which is forced by $0 \leq w_i \leq (a_i - t_i)^+$.

If $t_i < a_i$, then $w_i \leq (a_i - t_i)^+$ implies that $x_i = t_i + w_i \leq a_i$. That is, the vehicle departs from customer i before or at the time that the time window opens. By definition, the function h_i is non-increasing on this domain with a fixed slope of δ_i. Because we are only interested in the argument, and not in the value of the minimum, we may ignore the other variables and replace $h_i(t_i + w_i, p_i^{i-1})$ by $-w_i\delta_i$.

To solve the problem in (44), we first make a change of variables from w_i to $X_i = t_i + w_i$. Let X_i^* be the unconstrained minimum over the extended real number line, denoted by $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. That is,

$$\begin{aligned} X_i^* &= \arg\min_{X_i \in \overline{\mathbb{R}}} \{\bar{c}_i(X_i) - (X_i - t_i)\delta_i\} && (45) \\ &= \arg\min_{X_i \in \overline{\mathbb{R}}} \{\bar{c}_i(X_i) - X_i\delta_i\}. && (46) \end{aligned}$$

Note that we may remove $t_i\delta_i$ because this term is not dependent on X_i.

The value X_i^* is the preferred departure time from customer i. That is, if there exists a waiting time $0 \leq w_i \leq (a_i - t_i)^+$ such that the vehicle departs at X_i^*, then this is optimal. If no such waiting time w_i satisfies the constraints, it is optimal to select the closest feasible point. This is a property of one-dimensional convex minimization problems. It can be shown that this results in $w_i^*(t_i) = (\min\{X_i^*, a_i\} - t_i)^+$ for EXT-VW. For POS-VW, a similar argument can be given to demonstrate that $w_i^*(t_i, p_i^{i-1}) = \left(\min\{X_i^*, a_i + p_i^{i-1}\} - t_i\right)^+$. $\square$

By Proposition 10, we may simplify (42) further. For EXT-VW, we obtain

$$\bar{c}_i(x_i) = \min_{p_{i+1}^i \in \mathcal{P}_{i+1}} \left\{g_{i+1}(0, p_{i+1}^i) + \mathbb{E}\left[h_{i+1}(x_{i+1}, p_{i+1}^i)\right]\right\} + \mathbb{E}\left[\bar{c}_{i+1}\left(x_{i+1}\right)\right], \qquad (47)$$

with $x_{i+1} = t_{i+1} + \left(\min\{X_{i+1}^*, a_{i+1}\} - t_{i+1}\right)^+$ and $t_{i+1} = x_i + t_{ii+1}$ on the right-hand side.

Note that in this case, we *cannot* decompose the simple DTWAP in the same way as before. This is because the departure time x_{i+1} depends on the preferred departure time X_{i+1}^*, which depends on the following customers.

It is true, however, that the optimal adjustment vector p^i only depends on the future customers through X_{i+1}^*. It is possible that this fact can be exploited by solution methods, for example by estimating X_{i+1}^*, instead of using the value function $\bar{c}_{i+1}$.

For POS-VW, we are unable to simplify Equation (42) further. In this case, the departure time x_{i+1} depends on the optimal waiting time $w_i^*(t_i, p_i^{i-1})$, which depends on the adjustment p_i^{i-1}. As such, the $\mathbb{E}[\bar{c}_{i+1}]$ term cannot be taken out of the minimum.

C Proof Proposition 8

Proposition 8. *The simple DTWAP with discretized time allows for optimal integer decisions in the cases of EXT-NW, POS-NW, and EXT-AW.*

Proof. First consider EXT-NW and POS-NW. In Section 5.2, we have shown that in the cases of EXT-NW and POS-NW, the function $\bar{c}_i$ simplifies to

$$\bar{c}_i\left(x_i\right) = \min_{p^i_{i+1} \in \mathcal{P}_{i+1}} \left\{ g_{i+1}(0, p^i_{i+1}) + \mathbb{E}\left[h_{i+1}(x_i + t_{ii+1}, p^i_{i+1})\right] \right\} + \mathbb{E}\left[\bar{c}_{i+1}\left(x_i + t_{ii+1}\right)\right]. \quad (30)$$

For convenience, we define

$$f(p^i_{i+1}) = g_{i+1}(0, p^i_{i+1}) + \mathbb{E}\left[h_{i+1}(x_i + t_{ii+1}, p^i_{i+1})\right], \quad (48)$$

such that an optimal action can be found by minimizing $f(p^i_{i+1})$ over $p^i_{i+1} \in \mathcal{P}_{i+1}$.

Note that f is a piece-wise linear function: the functions g_{i+1} and h_{i+1} are polyhedral convex by definition, and by assumption, the expectation can be written as a finite sum. It follows that the sum is a polyhedral convex function, which is piece-wise linear by definition.

Now assume that $x_i \in \mathbb{Z}$. By piece-wise linearity, there exists an optimal solution p^i_{i+1} that is a breakpoint of f, or a boundary point of $\mathcal{P}_{i+1}$. If p^i_{i+1} is a breakpoint of f, then p^i_{i+1} must also be a breakpoint of one of its summands. It is easily verified that for $x_i \in \mathbb{Z}$ and integer travel times, the functions g_{i+1} and h_{i+1} only have breakpoints for integer p^i_{i+1}. This follows immediately from the definitions (20) and (21)-(22), respectively, and from the integrality of the parameters. If p^i_{i+1} is a boundary point of $\mathcal{P}_{i+1}$, then p^i_{i+1} is integer by assumption.

The departure time from the depot, x_0, is integer by assumption. By the argument above, there exists an optimal integer action p^0_1. By assumption, the time windows and the travel times are integer. It follows that the arrival and departure time at customer 1 are again integer, which leads to an optimal integer action p^1_2, etc. By induction, we have proven that the simple DTWAP allows for optimal integer decisions in the cases of EXT-NW and POS-NW.

The proof for EXT-AW is similar to the proof for EXT-NW and POS-NW, but with a different function f. In this case, we use Equation (32) to obtain

$$f(p^i_{i+1}) = g_{i+1}(0, p^i_{i+1}) + \mathbb{E}\left[h_{i+1}(\max\{x_i + t_{ii+1}, a_{i+1}\}, p^i_{i+1})\right]. \quad (49)$$

Note that the function f remains piece-wise linear in p^i_{i+1}. It is again easily verified that for $x_i \in \mathbb{Z}$, integer travel times, and integer time windows, the functions g_{i+1} and h_{i+1} only have breakpoints for integer p^i_{i+1}. The other parts of the proof are the same as in the NW case. □

D Proof Proposition 9

Before proving Proposition 9, we prove Lemmas 11 to 16. For convenience, we define functions $f(p^i_{i+1})$, similar as in the proof of Proposition 8 (see Appendix C). Using Equation (27), it is straightforward to show that all f are polyhedral convex functions. For POS-AW, we use Equation (34) to obtain

$$f(p^i_{i+1}) = g_{i+1}(0, p^i_{i+1}) + \mathbb{E}\left[h_{i+1}(\max\{x_i + t_{ii+1}, a_{i+1} + p^i_{i+1}\}, p^i_{i+1})\right] + \mathbb{E}\left[\bar{c}_{i+1}\left(\max\{x_i + t_{ii+1}, a_{i+1} + p^i_{i+1}\}\right)\right]. \tag{50}$$

For EXT-VW and POS-VW, we use Equation (47) to obtain

$$f(p^i_{i+1}) = g_{i+1}(0, p^i_{i+1}) + \mathbb{E}\left[h_{i+1}(x_{i+1}, p^i_{i+1})\right] + \mathbb{E}\left[\bar{c}_{i+1}\left(x_{i+1}\right)\right], \tag{51}$$

with in the right-hand side

- $x_{i+1} = t_{i+1} + \left(\min\{X^*_{i+1}, a_{i+1}\} - t_{i+1}\right)^+$ and $t_{i+1} = x_i + t_{ii+1}$ for EXT-VW,
- $x_{i+1} = t_{i+1} + \left(\min\{X^*_{i+1}, a_{i+1} + p^i_{i+1}\} - t_{i+1}\right)^+$ and $t_{i+1} = x_i + t_{ii+1}$ for POS-VW.

For brevity, we define Assumption 1, which we refer to in the lemmas.

Assumption 1. *Assume that the combination of adjustment type and waiting behavior is POS-AW, EXT-VW, or POS-VW. Let $i \in \{0, \ldots, n-1\}$ and assume that $\bar{c}_{i+1}$ is piece-wise linear and all breakpoints are integer.*

Lemma 11. *Under Assumption 1, every breakpoint p^i_{i+1} of f satisfies at least one of the equations as marked in the table below:*

Description	***Equation***	***POS-AW***	***EXT-VW***	***POS-VW***
No adjustment:	$p^i_{i+1} = 0$	×	×	×
Arrive at start time window POS for some t_{ii+1}:	$x_i + t_{ii+1} = a_{i+1} + p^i_{i+1}$	×		×
Arrive at end time window for some t_{ii+1}:	$x_i + t_{ii+1} = b_{i+1} + p^i_{i+1}$	×	×	×
Start time window POS is integer:	$a_{i+1} + p^i_{i+1} \in \mathbb{Z}$	×		×
Time window POS start at preferred departure time:	$X^*_{i+1} = a_{i+1} + p^i_{i+1}$			×
Time window ends at preferred departure time:	$X^*_{i+1} = b_{i+1} + p^i_{i+1}$		×	×
Time window has width zero:	$a_{i+1} = b_{i+1} + p^i_{i+1}$		×	

Proof. If p^i_{i+1} is a breakpoint of f, then p^i_{i+1} must also be a breakpoint of one of its summands. Recall that by assumption, the expectation can be written as a finite sum.

Consider the case POS-AW. If p^i_{i+1} is a breakpoint of g_{i+1}, then it follows from the definition of g_{i+1}, Equation (20), that $p^i_{i+1} = 0$. Next, consider the h_{i+1} term. Note that h_{i+1} is defined by Equation (22). Without loss of generality, we may assume that $\delta_{i+1} = 0$, i.e., there is no penalty for early delivery. This is possible because we always wait until the time window opens. It follows that the h_{i+1} term within the expectation can be written as $\left((\max\{x_i + t_{ii+1}, a_{i+1} + p^i_{i+1}\} - (b_{i+1} + p^i_{i+1})\right)^+ \gamma_{i+1}$.

We observe two potential breakpoints for each h_{i+1} term. First, we may have a breakpoint when the two arguments of the maximum are equal, i.e., $x_i + t_{ii+1} = a_{i+1} + p^i_{i+1}$. Second, if $x_i + t_{ii+1} > a_{i+1} + p^i_{i+1}$ then $x_i + t_{ii+1} = b_{i+1} + p^i_{i+1}$ is a potential breakpoint. Note that if $x_i + t_{ii+1} < a_{i+1} + p^i_{i+1}$, then the h_{i+1} term is equal to $\left(a_{i+1} + p^i_{i+1} - (b_{i+1} + p^i_{i+1})\right)^+ \gamma_{i+1} = (a_{i+1} - b_{i+1})^+ \gamma_{i+1}$, which is constant in p^i_{i+1} and does not result in any breakpoints.

Next, we consider breakpoints due to the $\bar{c}_{i+1}$ term. Again, we can have a breakpoint if the arguments of the minimum are equal. By Assumption 1, the function $\bar{c}_{i+1}$ only has integer breakpoints. Hence, if $x_i + t_{ii+1} < a_{i+1} + p^i_{i+1}$, we have a potential breakpoint for $a_{i+1} + p^i_{i+1} \in \mathbb{Z}$. Note that if $x_i + t_{ii+1} > a_{i+1} + p^i_{i+1}$, then the $\bar{c}_{i+1}$ term is constant in p^i_{i+1}. This concludes the proof for POS-AW.

Next, consider the case EXT-VW. For the g_{i+1} term, we again obtain $p^i_{i+1} = 0$. In the definition of f, Equation (51), we have

$$x_{i+1} = \begin{cases} x_i + t_{ii+1} & \text{if } x_i + t_{ii+1} \geq \min\{X^*_{i+1}, a_{i+1}\} \\ \min\{X^*_{i+1}, a_{i+1}\} & \text{if } x_i + t_{ii+1} \leq \min\{X^*_{i+1}, a_{i+1}\}. \end{cases} \tag{52}$$

That is, if the vehicle arrives at customer $i+1$ before $\min\{X^*_{i+1}, a_{i+1}\}$, then it waits until that time. If the vehicle arrives later, customer $i+1$ is served immediately.

Now consider the h_{i+1} terms. From the definition, Equation (21), it follows that these terms are given by $\left(x_{i+1} - (b_{i+1} + p^i_{i+1})\right)^+ \gamma_j + (a_{i+1} - x_{i+1})^+ \delta_j$. If $x_{i+1} = x_i + t_{ii+1}$, we have a potential breakpoint for $x_i + t_{ii+1} = b_{i+1} + p^i_{i+1}$. Note that $(a_{i+1} - (x_i + t_{ii+1}))^+ \delta_j$ is constant in p^i_{i+1}, and thus does not yield breakpoints. If $x_{i+1} = \min\{X^*_{i+1}, a_{i+1}\}$, then depending on whether X^*_{i+1} or a_{i+1} is the minimum, we obtain potential breakpoints for $X^*_{i+1} = b_{i+1} + p^i_{i+1}$ and $a_{i+1} = b_{i+1} + p^i_{i+1}$. Finally, we consider $x_i + t_{ii+1} = \min\{X^*_{i+1}, a_{i+1}\}$, i.e., the value for which (52) switches cases. Again, we find potential breakpoints for $X^*_{i+1} = b_{i+1} + p^i_{i+1}$ and $a_{i+1} = b_{i+1} + p^i_{i+1}$.

Next, we consider breakpoints due to the $\bar{c}_{i+1}$ term. It follows from the definition of x_{i+1}, Equation (52), that $\bar{c}_{i+1}(x_{i+1})$ is constant in p^i_{i+1}. Hence, the $\bar{c}_{i+1}$ term does not provide additional potential breakpoints. This completes the proof for EXT-VW.

Finally, we consider POS-VW. For the g_{i+1} term, we again obtain $p^i_{i+1} = 0$. In the definition of f, Equation (51), we have

$$x_{i+1} = \begin{cases} x_i + t_{ii+1} & \text{if } x_i + t_{ii+1} \geq \min\{X^*_{i+1}, a_{i+1} + p^i_{i+1}\} \\ \min\{X^*_{i+1}, a_{i+1} + p^i_{i+1}\} & \text{if } x_i + t_{ii+1} \leq \min\{X^*_{i+1}, a_{i+1} + p^i_{i+1}\}. \end{cases} \tag{53}$$

That is, if the vehicle arrives at customer $i+1$ before $\min\{X^*_{i+1}, a_{i+1} + p^i_{i+1}\}$, then it waits until that time. If the vehicle arrives later, customer $i+1$ is served immediately.

The h_{i+1} terms are defined by Equation (22). As in the POS-AW case, we assume without loss of generality that $\delta_{i+1} = 0$. It follows that the h_{i+1} terms are given by $\left(x_{i+1} - (b_{i+1} + p^i_{i+1})\right)^+ \gamma_j$. If $x_{i+1} = x_i + t_{ii+1}$, we obtain a potential breakpoint for $x_i + t_{ii+1} = b_{i+1} + p^i_{i+1}$. Similarly, if $x_{i+1} = X^*_{i+1}$, we obtain a potential breakpoint for $X^*_{i+1} = b_{i+1} + p^i_{i+1}$. For $x_{i+1} = a_{i+1} + p^i_{i+1}$ we have $\left(x_{i+1} - (b_{i+1} + p^i_{i+1})\right)^+ \gamma_j = (a_{i+1} - b_{i+1})^+ \gamma_j$, which is constant in p^i_{i+1}.

Next, we consider the points where x_{i+1} switches cases. From $x_i + t_{ii+1} = \min\{X^*_{i+1}, a_{i+1} + p^i_{i+1}\}$ we obtain a potential breakpoint for $x_{i+1} + t_{ii+1} = a_{i+1} + p^i_{i+1}$. Note that if $X^*_{i+1} < a_{i+1} + p^i_{i+1}$, then the minimum is constant in p^i_{i+1} and does not yield potential breakpoints. Finally, there can be a breakpoint if the arguments of the minimum are equal, i.e., $X^*_{i+1} = a_{i+1} + p^i_{i+1}$.

Now consider the breakpoints due to the $\bar{c}_{i+1}$ term. By Assumption 1, we have a potential breakpoint due to $\bar{c}_{i+1}(x_{i+1})$ is x_{i+1} is not constant in p^i_{i+1}, and x_{i+1} is integer. It follows that we can only get a breakpoint if $x_{i+1} = a_{i+1} + p^i_{i+1}$. The potential breakpoints that we find are given by $a_{i+1} + p^i_{i+1} \in \mathbb{Z}$. This completes the proof of POS-VW. After proving the cases POS-AW, EXT-VW, and POS-VW, we have proven the lemma. □

Lemma 12. *Under Assumption 1, the preferred departure time X^*_{i+1} from customer $i+1$ (see Proposition 10, Appendix B) can be chosen such that $X^*_{i+1} \in \mathbb{Z} \cup \{-\infty, \infty\}$.*

Proof. By definition, $X^*_{i+1} = \arg\min_{X_{i+1} \in \overline{\mathbb{R}}} \{\bar{c}_{i+1}(X_{i+1}) - X_{i+1}\delta_{i+1}\}$. Under Assumption 1, the function $\bar{c}_{i+1}(X_{i+1}) - X_{i+1}\delta_{i+1}$ is piece-wise linear and only has breakpoints on the integers. As X^*_{i+1} is the arg min of this function, we may assume that $X^*_{i+1} \in \mathbb{Z} \cup \{-\infty, \infty\}$. □

Lemma 13. *Based on the parameter x_i, let $p^*(x_i) = \arg\min_{p^i_{i+1} \in \mathcal{P}_{i+1}} f(p^i_{i+1})$ be an optimal adjustment. Under Assumption 1, if $\bar{x}_i \notin \mathbb{Z}$, then $p^*(x_i) = C$ or $p^*(x_i) = x_i + C$ on the domain $x_i \in (\lfloor \bar{x}_i \rfloor, \lceil \bar{x}_i \rceil)$, for some integer value C.*

Proof. For convenience, we define $\bar{p}^i_{i+1} = p^*(\bar{x}_i)$ to be the optimal adjustment that corresponds to the departure time $\bar{x}_i$. Throughout this proof, we assume that $x_i \in (\lfloor \bar{x}_i \rfloor, \lceil \bar{x}_i \rceil)$. We refer to this set as the *neighborhood of* $\bar{x}_i$. We use $\bar{f}$ to denote the function f in which x_i is replaced by $\bar{x}_i$.

We define Q to be the set of *potential breakpoints* of f. That is, for a given value of x_i, the set Q contains all values of p^i_{i+1} that satisfy at least one of the relevant equations in Lemma 11. Similarly, we define $\bar{Q}$ to be the set of potential breakpoints of $\bar{f}$. By definition, there is a bijection between Q and $\bar{Q}$.

Consider $q \in Q$ and $\bar{q} \in \bar{Q}$, both defined by the same equation. By going over the equations in Lemma 11, and using that $\bar{x}_i \notin \mathbb{Z}$ and $x_i \in (\lfloor \bar{x}_i \rfloor, \lceil \bar{x}_i \rceil)$, the following three implications can be verified.

1. If $\bar{q} \in \mathbb{Z}$ then $q - \bar{q} = 0$, i.e., the difference between q and $\bar{q}$ is zero.
2. If $\bar{q} \notin \mathbb{Z}$ then $\bar{q} - \bar{x}_i \in \mathbb{Z}$, i.e., $\bar{q}$ and $\bar{x}_i$ have the same fractional part.
3. If $\bar{q} \notin \mathbb{Z}$ then $q - \bar{q} = x_i - \bar{x}_i$, i.e., the difference between q and $\bar{q}$ is equal to the difference between x_i and $\bar{x}_i$.

For example, consider the third equation in Lemma 11. We obtain $x_i + t_{ii+1} = b_{i+1} + q$ and $\bar{x}_i + t_{ii+1} = b_{i+1} + \bar{q}$. We first verify the first statement. If $\bar{q} \in \mathbb{Z}$, then it follows from the integrality of t_{ii+1} and b_{i+1} that $\bar{x}_i \in \mathbb{Z}$. This contradicts the assumption that $\bar{x}_i \notin \mathbb{Z}$, and the implication trivially holds. Next, we consider $\bar{q} \notin \mathbb{Z}$. Rewriting the equations yields $q - x_i = t_{ii+1} - b_{i+1}$ and $\bar{q} - \bar{x}_i = t_{ii+1} - b_{i+1}$, which shows $\bar{q} - \bar{x}_i \in \mathbb{Z}$ and $q - \bar{q} = x_i - \bar{x}_i$. Hence, both implications hold. The verification of the three implications for the other equations in Lemma 11 is similar.

It can be seen that the subdifferential $\partial f(p^i_{i+1})$ is uniquely determined by the set of potential breakpoints q such that $q < p^i_{i+1}$, and whether p^i_{i+1} is a potential breakpoint itself. This follows from the fact that the slopes of the summands of f can only change at the potential breakpoints, and that replacing x_i by $\bar{x}_i$ only changes the locations of the potential breakpoints, and not the slopes before and after the potential breakpoints.

Next, we show that the ordering of the potential breakpoints is the same for f and $\bar{f}$. Earlier, we have shown that if $\bar{q} \in \mathbb{Z}$, then $q - \bar{q} = 0$, i.e., the potential breakpoints are equal (Implication 1). In the case of $\bar{q} \notin \mathbb{Z}$ we have shown that $q - \bar{q} = x_i - \bar{x}_i$ (Implication 3) and that $\bar{q}$ and $\bar{x}_i$ have the same fractional part (Implication 2). By assumption, $\bar{x} \notin \mathbb{Z}$, and $x_i \in (\lfloor \bar{x}_i \rfloor, \lceil \bar{x}_i \rceil)$. It follows immediately that $q \in (\lfloor \bar{q} \rfloor, \lceil \bar{q} \rceil)$. In summary, f and $\bar{f}$ have the same integer breakpoints, and all fractional breakpoints change by the same amount. The fractional breakpoints remain fractional, which implies that the ordering of the potential breakpoints is the same for f and $\bar{f}$.

As an immediate consequence, we have $\partial f(q) = \partial \bar{f}(\bar{q})$ for corresponding potential breakpoints q and $\bar{q}$. Furthermore, the subdifferential of f at a point between two sequential potential breakpoints q and q' is equal to the subdifferential of $\bar{f}$ at a point between $\bar{q}$ and $\bar{q}'$. A similar argument can be made for a point between a boundary point of $\mathcal{P}_{i+1}$ and the closest potential breakpoint. Note that the lemma is trivially satisfied if $\mathcal{P}_{i+1}$ is a singleton.

We are now ready to prove that if $\bar{x}_i \notin \mathbb{Z}$, then $p^*(x_i) = C$ or $p^*(x_i) = x_i + C$, for x_i in the neighborhood of $\bar{x}_i$, for an integer value C. Consider an optimal adjustment

$\bar{p}^i_{i+1} \in \mathbb{R}$ for a given departure time $\bar{x}_i \notin \mathbb{Z}$. Because $\bar{f}$ is a polyhedral convex function, we may assume that $\bar{p}^i_{i+1}$ is a breakpoint of $\bar{f}$ with $0 \in \partial \bar{f}(\bar{p}^i_{i+1})$, or $\bar{p}^i_{i+1}$ is a boundary point of $\mathcal{P}_{i+1}$.

We first consider the breakpoints of $\bar{f}$. If $\bar{p}^i_{i+1}$ is a breakpoint of $\bar{f}$ and $\bar{p}^i_{i+1} \in \mathbb{Z}$, then $p^i_{i+1} = \bar{p}^i_{i+1}$ is the corresponding breakpoint of f. It follows that $\partial \bar{f}(\bar{p}^i_{i+1}) = \partial f(p^i_{i+1}) \ni 0$, which implies by convexity that p^i_{i+1} is optimal for x_i. Hence, we have $p^*(x_i) = C$ with $C = \bar{p}^i_{i+1}$, which is integer by assumption. Similarly, if $\bar{p}^i_{i+1}$ is breakpoint of $\bar{f}$ and $\bar{p}^i_{i+1} \notin \mathbb{Z}$, then $p^i_{i+1} - \bar{p}^i_{i+1} = x_i - \bar{x}_i$ (Implication 3) $\Leftrightarrow p^i_{i+1} = x_i + \bar{p}^i_{i+1} - \bar{x}_i$ is optimal for x_i. That is, $p^*(x_i) = x_i + C$, with $C = \bar{p}^i_{i+1} - \bar{x}_i$. Note that C is integer by Implication 2.

Next, we consider the case that $\bar{p}^i_{i+1}$ is a boundary point of $\mathcal{P}_{i+1}$. By assumption, the boundary points of $\mathcal{P}_{i+1}$ are integral. It follows that $p^i_{i+1} = \bar{p}^i_{i+1}$. Hence, we can use $p^*(x_i) = C$, with $C = \bar{p}^i_{i+1}$ integer. This completes the proof. □

Lemma 14. *Under Assumption 1, for case POS-AW, we have that $\bar{c}_i$ is piece-wise linear and all breakpoints are integer.*

Proof. From the definition of $\bar{c}_i$, Equation (27), it is straightforward to show that $\bar{c}_i$ is a polyhedral convex function. Hence, $\bar{c}_i$ is piece-wise linear. It remains to prove that $\bar{c}_i$ has no breakpoints on the integers.

We prove this fact by contradiction. First, we assume that $x_i \notin \mathbb{Z}$ is a breakpoint of $\bar{c}_i(x_i)$. By Lemma 13, we have that the optimal adjustment in the neighborhood of x_i is given by $p^*(x_i) = C$ or $p^*(x_i) = x_i + C$ for some integer value C.

If we substitute the optimal actions into the definition of $\bar{c}_i(x_i)$, we obtain a straightforward expression for this function that is valid in the neighborhood of the current state x_i. By analyzing this expression, we show that it only has breakpoints on the integers. By contradiction, $x_i \notin \mathbb{Z}$ cannot be a breakpoint.

By definition, $\bar{c}_i(x_i) = \min_{p^i_{i+1} \in \mathcal{P}_{i+1}} f(p^i_{i+1})$. Instead of minimizing over p^i_{i+1}, we use the optimal adjustment $p^*(x_i)$. We then obtain $\bar{c}_i(x_i) = f(p^*(x_i))$.

First consider $p^*(x_i) = C$. Recall that $C \in \mathbb{Z}$ by Lemma 13. It follows from Equation (50) that

$$\begin{aligned} \bar{c}_i(x_i) = g_{i+1}(0, C) + \mathbb{E}\left[h_{i+1}(\max\{x_i + t_{ii+1}, a_{i+1} + C\}, C)\right] + \\ \mathbb{E}\left[\bar{c}_{i+1}\left(\max\{x_i + t_{ii+1}, a_{i+1} + C\}\right)\right]. \end{aligned} \tag{54}$$

Next, we consider the breakpoints of $\bar{c}_i(x_i)$. Note that the g_{i+1} term is constant in x_i, and does not result in any breakpoints. It is straightforward to verify that h_{i+1} only has breakpoints for integer x_i. This follows from the fact that all parameters are integer. By Assumption 1, $\bar{c}_{i+1}$ only has breakpoints for the integers. If $x_i + t_{ii+1} < a_{i+1} + C$, then the $\bar{c}_{i+1}$ term is constant in x_i and does not give any potential breakpoints. If $x_i + t_{ii+1} > a_{i+1} + C$, then potential breakpoints are given by $x_i + t_{ii+1} \in \mathbb{Z}$, which implies that the breakpoints x_i of $\bar{c}_i$ are integer. Finally, we may have a breakpoint for $x_i + t_{ii+1} = a_{i+1} + C$, which also corresponds to $x_i \in \mathbb{Z}$.

Hence, for $p^*(x_i) = C$, we have shown that if $x_i \notin \mathbb{Z}$ is a breakpoint of $\bar{c}_i$, then the function $\bar{c}_i$ defined in the neighborhood of x_i only has breakpoints on the integers. By contradiction, $x_i \notin \mathbb{Z}$ cannot be a breakpoint. It follows that $\bar{c}_i$ only has breakpoints on the integers.

We have to show the same result for $p^*(x_i) = x_i + C$. In this case, we have

$$\begin{aligned}\bar{c}_i(x_i) = g_{i+1}(0, x_i + C) + \mathbb{E}\left[h_{i+1}(\max\{x_i + t_{ii+1}, a_{i+1} + x_i + C\}, x_i + C)\right] +\\ \mathbb{E}\left[\bar{c}_{i+1}\left(\max\{x_i + t_{ii+1}, a_{i+1} + x_i + C\}\right)\right].\end{aligned} \tag{55}$$

Again, by combining Assumption 1 with the fact that all parameters are integer, it is straightforward to show that $\bar{c}_i$ only has breakpoints on the integers. Applying the same contradiction argument as for $p^*(x_i) = C$ completes the proof. □

Lemma 15. *Under Assumption 1, for case EXT-VW, we have that $\bar{c}_i$ is piece-wise linear and all breakpoints are integer.*

Proof. We use the same argument as in Lemma 14, but for a different function $\bar{c}_i$. Specifically, we show that the function $\bar{c}_i(x_i) = f(p^*(x_i))$ only has breakpoints for integer x_i, with f as in Equation (51). The other parts of the proof are identical.

In this case, we obtain

$$\begin{aligned}\bar{c}_i(x_i) = g_{i+1}(0, p^*(x_i)) +&\\ \mathbb{E}\left[h_{i+1}\left(x_i + t_{ii+1} + \left(\min\{X^*_{i+1}, a_{i+1}\} - (x_i + t_{ii+1})\right)^+, p^*(x_i)\right)\right] +&\\ \mathbb{E}\left[\bar{c}_{i+1}\left(x_i + t_{ii+1} + \left(\min\{X^*_{i+1}, a_{i+1}\} - (x_i + t_{ii+1})\right)^+\right)\right].&\end{aligned} \tag{56}$$

It then needs to be verified that $\bar{c}_i$ only has integer breakpoints when $p^*(x_i) = C$ and when $p^*(x_i) = x_i + C$, for some integer C. This can be shown using the integrality of the parameters, and using that $\bar{c}_{i+1}$ only has breakpoints on the integers (Assumption 1). The arguments are straightforward, and similar to those in Lemma 11 and Lemma 14. As such, we omit them here.

After it is shown that (56) only has integer breakpoints, the proof for the EXT-VW case is the same as the proof for the POS-AW case (Lemma 14). □

Lemma 16. *Under Assumption 1, for case POS-VW, we have that $\bar{c}_i$ is piece-wise linear and all breakpoints are integer.*

Proof. Similar to the proof of Lemma 15, but for a different function $\bar{c}_i$. From Equation (51) we obtain In this case, we obtain

$$\begin{aligned}\bar{c}_i(x_i) = {} & g_{i+1}(0, p^*(x_i)) + \\ & \mathbb{E}\left[h_{i+1}\left(x_i + t_{ii+1} + \left(\min\{X^*_{i+1}, a_{i+1} + p^*(x_i)\} - (x_i + t_{ii+1})\right)^+, p^*(x_i)\right)\right] + \\ & \mathbb{E}\left[\bar{c}_{i+1}\left(x_i + t_{ii+1} + \left(\min\{X^*_{i+1}, a_{i+1} + p^*(x_i)\} - (x_i + t_{ii+1})\right)^+\right)\right]. \end{aligned} \tag{57}$$

Similar as in Lemma 11, Lemma 14, and Lemma 15, it can be proven that $\bar{c}_i$ only has integer breakpoints. These steps are omitted here. Using the argument of Lemma 14 completes the proof. □

Proposition 9. *The simple DTWAP with discretized time allows for optimal integer decisions in the cases of POS-AW, EXT-VW, and POS-VW.*

Proof. By definition, $\bar{c}_n = 0$, which is linear and does not have breakpoints. Lemmas 14 to 16 show for all $i \in \{0, \ldots, n-1\}$ that if $\bar{c}_{i+1}$ is piece-wise linear and only has integer breakpoints, then the same is true for $\bar{c}_i$. By induction, it follows that $\bar{c}_i$ is piece-wise linear and only has integer breakpoints for all $i \in \{0, \ldots, n\}$.

It then follows from Lemma 11 that for a given $x \in \mathbb{Z}$, there exists an optimal adjustment $p^i_{i+1} \in \mathbb{Z}$. It follows from Lemma 12, and from the definition of the optimal waiting time, that there also exist optimal voluntary waiting times that are integer.

Hence, for a given integer state, there exist optimal integer actions. If integer actions are taken, the next state will again be integer. This can be seen from Equation (29), the definition of $\bar{c}_i$. It follows that the simple DTWAP with discretized time allows for optimal integer decisions in the cases of POS-AW, EXT-VW, and POS-VW. □