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Optimal claim behaviour for third-party liability insurances or To claim or not to claim: that is the question

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It is proved that the optimal decision rule to claim or not to claim for damage is of the form: 'to claim for damage only if its amount exceeds a certain limit'. Optimal critical claim sizes are derived, and a sensitivity analysis is given with respect to changes in (the parameters of) the distributions of the number of claims and the claim size.

Keywords: Optimal critical claim size, Markov process, Sensitivity analysis.

1. Introduction

An important feature of premium rating systems for vehicle insurance is the no-claim or bonus–malus principle. This principle is meant to reward policy holders for not making claims during a year; that is, to grant a bonus to a careful driver. A bonus principle effects the policy holder's decision whether or not to claim in a particular instance. No claim will be made for some of the accidents where there is only slight damage. Philipson (1960) called this phenomenon 'hunger for bonus'.

Little is known about the real behaviour of policy holders with respect to the no-claim principle. It can, however, be expected that an individual policy holder will only claim for damage when its amount exceeds a certain limit, which is assumed by most authors. We may assume that the higher the number of claim-free years or the higher the class of the bonus–malus principle, the higher this limit will be. The policy holder must realize that

- if he claims for damage, the insurance company will pay for the damage, but he will pay a higher premium for a number of years;
- if he does not claim, he has to pay for the damage himself, but the premium he must pay for the next insurance year does not increase.

The premium for insuring a vehicle must be paid at the beginning of an insurance year. The amount of the premium depends on the bonus–malus class the policy holder at present belongs to. At the end of each year the policy holder moves to another class, according to certain rules, depending only on the fact whether he filed any claim during that year. We notice that the amount of the claim does not matter, only the number of claims determines the bonus–malus class to which the policy holder will belong during the next year. An example of a bonus–malus scheme operative in the Netherlands is given in Table 1.

In this paper we restrict our analysis to a third-party liability insurance, where the total amount of damage is covered, i.e., there is no deductible. The analysis is based on the definition of a state presented

Table 1

Percentages of the basic premium by bonus–malus class.

Bonus–malus class	Premium as of the basic premium	New bonus–malus class after … claims			
		0	1	2	≥ 3
14	30.0	14	9	5	1
13	32.5	14	8	4	1
12	35.0	13	8	4	1
11	37.5	12	7	3	1
10	40.0	11	7	2	1
9	45.0	10	6	1	1
8	50.0	9	5	1	1
7	55.0	8	4	1	1
6	60.0	7	3	1	1
5	70.0	6	2	1	1
4	80.0	5	1	1	1
3	90.0	4	1	1	1
2	100.0	3	1	1	1
1	120.0	2	1	1	1

by Norman and Shearn (1980). So we assume a discrete time axis. This assumption is not very restrictive in practical situations, for a policy holder is generally allowed some time (at least 24 hours) to decide to claim or not to claim for damage. [1] We will prove that – given some assumptions – the optimal decision rule to claim or not to claim, in such a way that the expected discounted costs are minimal, is of the form: 'to claim for damage only if its amount exceeds a certain limit'. We derive a system of recurrent equations, the solution of which gives the optimal critical claim sizes. These critical claim sizes are derived for both the case of a finite and that of an infinite horizon. It will appear that the optimal critical claim size depends on the probability distributions of the number of claims and the claim size. Therefore, we also analyse the sensitivity of the optimal critical claim size with respect to changes in (the values of the parameters of) the applied probability distributions. We finish our analysis with an example based on a Dutch bonus–malus premium rating system.

2. Review of the literature

No-claim problems have been discussed since the fifties. Several ASTIN meetings paid attention to this subject. Derron (1965) states 'that a subsequent adjustment of premiums according to the past claim record may well be a suitable way of obtaining a fair premium'. Gürtler (1960, 1961, 1962) introduced a standard for evaluating the fairness of a premium. Derron (1965) extends and complements the results obtained by Gürtler.

Welten (1969) points out that the bonus a policy holder obtains usually consists of at least three components which depend on the length of time preceding the current insurance period: a component concerning the individual claim frequency, an individual random factor and a collective random factor. The last two components tend to zero for increasing length of time. The sum of these last two components, called 'unearned bonus', should be taken into account by insurance companies in the short run, and would lead to a bonus reserve.

Alting von Geusau (1968) investigates 'to what extent it is possible to develop a theoretical framework to test (…) that a no-claim-discount-system will prevent the insured from submitting small claims to the

[1] Recently, some Dutch insurance companies allow an insurant to decide *at the end* of the insurance year whether he will file all claims made during that year or to take for his own account the costs of one or more claims.

insurance company', and 'that the insured who has just lost his no-claim discount will use every possibility for submitting claims with in his mind the idea that in this way he will earn back his higher non-reduced premium'.

Loimaranta (1972) develops formulas for some asymptotic properties of bonus systems, where bonus systems are understood as Markov chains. He introduces the quantities: efficiency of a bonus system, discriminatory power of bonus rules and minimum variance bonus scale. The last one gives an asymptotic solution for the problem to find locally 'best' bonus scales for given bonus rules. Bonus systems used in Denmark, Norway, Sweden, Finland, Switzerland and West Germany are studied by Vepsäläinen (1972), on the basis of the method given by Loimaranta. Lemaire (1976) defines an efficiency concept for a bonus–malus system, which differs from the concept given by Loimaranta (1972). De Pril (1978) presents a more general concept of efficiency, which includes both earlier ones as special cases.

Grenander (1957) derives equations to determine a rule of the form 'pay the damage if its amount is smaller than a critical value and claim it otherwise'. However, the equations are generally difficult to solve, and it is not proved that they really determine an optimal policy in the sense that the total expected discounted cost of premiums and payments during a long future planning period is minimized.

Haehling von Lanzenauer (1969, 1972a,b) analyses the problem on the assumption that a policy holder can cause at most one accident per year.

De Leve and Weeda (1968) develop a mathematical model, called generalized Markov programming, that yields an optimal strategy, which is a function $s(t)$ that minimizes the expected costs for the policy holder. This function $s(t)$ is such that 'if at any time t an accident occurs with damage s and no damages have been claimed since the last payment of premium, then s should be claimed if $s > s(t)$'. In this approach the decision depends on the point of time during the year and the premium paid at the beginning of that year. De Leve and Weeda allow more than one accident per year, but restrict their analysis to the case where, after making one claim during an insurance period, the policy holder is placed in the class with the highest premium. Weeda (1975) extends the analysis of the same model to the case where the damage distribution is given by an arbitrary distribution and focuses on the theoretical aspects of the derived iteration scheme. However, although the model is continuous with respect to the time axis, he discretizes time for computational purposes.

Martin-Löf (1973) shows that a decision rule of the form formulated by Grenander is optimal in the sense that it minimizes the total expected costs. The decision rule is derived by applying the general theory of Markov decision processes, which uses dynamic programming to find an optimal control iteratively. Martin-Löf, however, restricts the analysis to the case where the policy holder takes a decision only at the end of an insurance period for the total amount of damage sustained during that insurance period.

Haehling von Lanzenauer and Lundberg (1973) develop a model which can be used in deriving the distribution of the number of claims for insurances with merit-rating structures. The problem is formulated and solved as a regular Markov process with the claim behaviour integrated in the analysis.

Haehling von Lanzenauer (1974) develops an optimal decision rule – for situations where the policy holder takes a decision more than once a year – which is valid for any merit-rating system. He splits up a year into a number of periods, which results in a discrete model in which the optimal critical claim size can be determined by dynamic programming. However, his derivation of an optimal critical claim size is obscure.

Lemaire (1977) derives an algorithm for obtaining the optimal strategy for a policy holder. In his model the policy holder remains always insured (the so-called infinite horizon model) which leads to a critical claim size which is independent of the year in which the accident takes place. Also, in order to compute the optimal policy, he uses policy iteration, which is very time-consuming, whenever the state space is large. Lemaire (1976) applies this algorithm to compare bonus systems used in Denmark, Norway, Sweden, Finland, Switzerland and West Germany.

Hastings (1976) presents a simple model based on a typical British policy, assuming that the number of accidents is Poisson and the amount of damage is negative exponentially distributed. He assumes an optimal critical claim size, which is constant throughout the year, irrespective of the number of claims already made during the year, and irrespective of the time until the next premium payment. He determines optimal critical claim sizes, which minimize the long-run average costs of premiums and repairs. The problem is formulated as a Markov decision problem and is solved by dynamic programming.

Almost all studies mentioned above have in common that they assume a discrete time axis. De Pril (1979) gives a formulation based on a continuous time axis, where the optimal critical claim size can be determined by solving a set of recurrent differential equations. However, for solving these equations, a discretization is needed, giving rise to the same results as in Haehling von Lanzenauer (1974). Norman and Shearn (1980) build on Hastings' model, where they drop the restriction of a constant optimal critical claim size. Moreover, they present a much simpler state description than the one used by Haehling von Lanzenauer (1974). The optimal decision rule has been compared with rules of thumb that appear to produce remarkably good results. Tijms (1986, pp. 196–200) gives a model that is equal to that presented by Norman and Shearn as an illustration.

Kolderman and Volgenant (1985) present a continuous model based on generalized Markov programming, applicable to bonus–malus systems used by Dutch motor insurance companies. However, in the computational part of their study they discretize time for numerical reasons.

Lemaire (1985, Chapter 18) describes a simple model with the assumption that all claims are reported in the middle of the insurance period.

Menist and Volgenant (1986) compute the optimal critical claim size by considering the difference between the expected costs in case of claiming and that of not claiming damage. They restrict the analysis to a finite horizon.

3. The model

In this section we derive a discrete model for the decision problem to claim or not to claim damage. Therefore, we need three spaces: the state space, the decision space and the set of decision moments.

We divide an insurance period, generally a year, in N equal periods. We assume that the decision to claim or not to claim damage has to be taken at the end of a period. Let the decision process continue T years ($T \leq \infty$) and start at the beginning of a certain year. Then the set of decision moments is defined as

$$\mathscr{T} := \{ \tau = 1, 2, \ldots, H \},$$

where $H = NT$.

Next, we consider the probability distributions of the number of accidents and the amount of damage. We assume that the size of N, the number of periods in which the total insurance year is divided, is such that the probability of two or more accidents in any period is negligible. Further, we assume that the probability of one accident in a certain period varies from one period to another. We define the probability

$$p_n = \Pr(\text{'1 accident in period } n\text{'}), \quad n = 1, \ldots, N,$$

and a random variable $\underline{Z}_n$, where

$$\underline{Z}_n = i \ (i = 0, 1) \quad \text{if} \quad i \text{ accidents occur in period } n.$$

Then we have

$$\Pr[\underline{Z}_n = 1] = p_n \quad \text{and} \quad \Pr[\underline{Z}_n = 0] = 1 - p_n.$$

We assume $\underline{Z}_n$ ($n = 1, \ldots, N$) to be mutually stochastically independent.

Accidents generally imply damage. We define the random variable $\underline{Y}_n$ as the amount of damage resulting from an accident in period n. We assume $\underline{Y}_n$ ($n = 1, \ldots, N$) to be stochastically independently distributed. The Y_n's have the same probability distribution, but the parameters of their probability distributions depend on the period in which the accident occurs. Let $F_n(\cdot)$ and $f_n(\cdot)$ be the probability distribution function and the probability density function of $\underline{Y}_n$. Finally, we assume that the damage resulting from any accident is positive, so

$$f_n(y_n) = 0, \quad y_n \leq 0, \quad n = 1, \ldots, N$$

Finally, if we denote the total amount of damage in period n by the random variable $\underline{X}_n$, then clearly

$$\underline{X}_n = \underline{Y}_n \quad \text{if} \quad \underline{Z}_n = 1,$$
$$= 0 \quad \text{if} \quad \underline{Z}_n = 0.$$

Hence the probability distribution function $G_n(\cdot)$ belonging to the random variable $\underline{X}_n$ equals

$$G_n(x) = \Pr[\underline{X}_n \le x \mid \underline{Z}_n = 0]\ \Pr[\underline{Z}_n = 0] + \Pr[\underline{X}_n \le x \mid Z_n = 1]\ \Pr[\underline{Z}_n = 1]$$
$$= 1 - p_n + p_n F_n(x)$$

for every $x \ge 0$.

Moreover, the corresponding probability density function $g_n(\cdot)$ is given by

$$g_n(x) = 1 - p_n \quad \text{if} \quad x = 0,$$
$$= p_n f_n(x) \quad \text{if} \quad x > 0.$$

When a policy holder is involved in an accident during a certain period, he has to decide to claim or not to claim the resulting damage. Consequently, the decision space is defined as

$$\mathscr{B} := \{ b \mid b = 0, 1 \},$$

where 1 denotes to claim and 0 denotes not to claim. It is self-evident that when no accident occurs during a certain period, the policy holder does not claim at the end of that period. Finally, we must define a state E_γ at each decision moment. Haehling von Lanzenauer (1974) defines the state space on the basis of four components:

i = the policy holder's bonus–malus class,
k = the number of claims already filed during the current year,
n = the index defining a period,
x = the amount of damage resulting from an accident occurring in period n, where $x = 0$ when no accident occurs.

A state is then given by

$$E_\gamma = (i, k, n, x)$$

and the state space is defined as the set of states E_γ. Norman and Shearn (1980) simplify the state to three components:

$$E_\gamma := (j, n, x),$$

where the components n and x are defined as before. The component j is determined by

(1) the premium to be paid next year by the policy holder when he does not claim damage during the rest of the current year (π_0),
(2) the premium to be paid next year by the policy holder when he claims damage exactly once during the rest of the current year (π_1),

As will appear below, the premium, that he would pay when he would claim two or more times during the rest of the current year, does not play any role.

We can elicit every possible combination of π_0 and π_1 from the transition table of the bonus–malus scheme; let there be J such combinations. We denote a particular combination with j, where $j = 1, \ldots, J$.

Now we consider the two cases where j changes.

(1) The policy holder files a claim. The combination (π_0, π_1) does not change, when the policy holder does not file a claim. When he files a claim, he moves to a new combination (π_0^*, π_1^*), where $\pi_0^* = \pi_1$. We define $a(j)$ as the number of the new combination when the policy holder files a claim, and when the

Table 2

Norman and Shearn's definition of the state space applied to Table 1. [a]

j	$\rho_0(j)$	$\rho_1(j)$	$a(j)$	$b(j)$
1	120.0	120.0	1	2
2	100.0	120.0	1	3
3	90.0	120.0	1	4
4	80.0	120.0	1	5
5	70.0	120.0	1	7
6	60.0	120.0	1	9
7	60.0	100.0	2	9
8	55.0	100.0	2	10
9	55.0	90.0	3	10
10	50.0	80.0	4	11
11	45.0	70.0	5	12
12	40.0	60.0	6	13
13	37.5	55.0	8	14
14	35.0	55.0	9	15
15	32.5	50.0	10	16
16	30.0	50.0	10	17
17	30.0	45.0	11	17

[a] The functions $\rho_0(j)$ and $\rho_1(j)$ equal $\pi_0(j)$ and $\pi_1(j)$ as a percentage of the basic premium.

previous combination has number j. This can be represented schematically by

$$j \to \begin{cases} j & \text{no claims} \\ a(j) & \text{one claim} \end{cases}.$$

It follows that it is not necessary to keep how many claims per year are filed, because the next year premium is adapted immediately after a claim is filed.

(2) A new insurance year takes effect. The policy holder pays premium at the beginning of each insurance year. This premium is equal to $\pi_0(a(j))$ or $\pi_0(j)$, depending on whether he has or has not filed a claim in state (j, N, x). When he does not file a claim during a year, he moves to a higher bonus–malus class (unless he is already in the highest class), and he will pay a lower premium next year than he did in the current year. This corresponds to a different combination (π_0, π_1). Consequently, concerning j there are two transitions between the points of time $n = N$ of any year and $n = 1$ of the next year:

$$\begin{cases} j \to j \to b(j) & \text{when no claim is filed,} \\ j \to a(j) \to b(a(j)) & \text{when a claim is filed,} \end{cases}$$

where $b(j)$ is defined as the number of the new combination as the result of the transition to the new year. Given a transition mechanism, as presented in Table 1, we can determine the values of j, $a(j)$, $b(j)$ and of the corresponding premiums. Table 2 contains these values. We notice that different values will result, when a different bonus–malus system is effective.

The state space as formulated by Norman and Shearn contains less components and has less elements than the state space defined by Haehling von Lanzenauer, but is otherwise identical. The number of elements in the state space is important when the set of functional equations must be solved. Therefore, we prefer the state space definition of Norman and Shearn.

Finally, we must define the transition probabilities. That is, we must determine the probability density function corresponding to the state E_δ, given that the policy holder was in state E_γ at the previous point of time and has taken decision b. This function is denoted by $h(E_\delta \mid E_\gamma, b)$. Furthermore, we define $C(E_\gamma, b)$ as the costs in the next period, when decision b has been taken in the current period and the policy holder is in state E_γ. Table 3 contains the values of these functions.

Table 3

$C(E_\gamma, b)$ and $h(E_\delta \mid E_\gamma, b)$ for all E_γ, E_δ, b and n.

E_γ	n	b	E_δ	$C(E_\gamma, b)$	$h(E_\delta \mid E_\gamma, b)$
(j, n, x)	$1,\dots, N-1$	0	$(j, n+1, y)$	x	$g_{n+1}(y)$
		1	$(a(j), n+1, y)$	0	$g_{n+1}(y)$
(j, n, x)	N	0	$(b(j), 1, y)$	$x+\pi_0(j)$	$g_1(y)$
		1	$(b(a(j)), 1, y)$	$\pi_0(a(j))$	$g_1(y)$

For the other states the transition probabilities equal zero.

Given the definitions presented above, we are now able to formulate the above decision problems as Markov decision processes. [2]

Starting with a discounted finite model with an horizon of NT periods, we introduce, for every $1 \le i \le T < \infty$ and $1 \le n \le N$,

$V_i(j, n, x) :=$ minimal expected total discounted costs from period $N(T-i)+n$ through period NT when the policy holder is in state (j, n, x) at the end of period $N(T-i)+n$.

Obviously, the policy holder files all claims during the last year of the insurance. Therefore, for every state (j, n, x), $1 \le n \le N$, we get

$$V_1(j, n, x) = 0. \tag{1}$$

Moreover, by the specific form of the one-period cost functions (cf. Table 3) and Bellman's principle of optimality [cf. Bertsekas (1976), Ross (1983)] we obtain the following set of backward equations:

For every $2 \le i \le T$, $1 \le n \le N-1$ and $x > 0$ we have

$$V_i(j, n, x) = \min\begin{cases} x + \beta\mathbb{E}(V_i(j, n+1, \underline{X}_{n+1})) \\ \beta\mathbb{E}(V_i(a(j), n+1, \underline{X}_{n+1})) \end{cases}. \tag{2}$$

For every $2 \le i \le T$, $1 \le n \le N-1$ and $x = 0$ we get

$$V_i(j, n, 0) = \beta\mathbb{E}(V_i(j, n+1, \underline{X}_{n+1})). \tag{3}$$

For every $2 \le i \le T$, $n = N$ and $x > 0$ it holds that

$$V_i(j, N, x) = \min\begin{cases} x + \pi_0(j) + \beta\mathbb{E}(V_{i-1}(b(j), 1, \underline{X}_1)) \\ \pi_0(a(j)) + \beta\mathbb{E}(V_{i-1}(b(a(j)), 1, \underline{X}_1)) \end{cases}. \tag{4}$$

Finally, for every $2 \le i \le T$, $n = N$ and $x = 0$ we have

$$V_i(j, N, 0) = \pi_0(j) + \beta\mathbb{E}(V_{i-1}(b(j), 1, \underline{X}_1)), \tag{5}$$

where $\mathbb{E}$ denotes the mathematical expectation.

In order to simplify expressions we define, for every $1 \le i \le T$,

$$V_i(j, n) := \mathbb{E}(V_i(j, n, \underline{X}_n)), \quad 1 \le n \le N, \tag{6}$$

$$D_i(j, n) := \beta(V_i(a(j), n+1) - V_i(j, n+1)), \quad 1 \le n \le N-1, \tag{7}$$

and, for every $2 \le i \le T$,

$$D_i(j, N) := \beta(V_{i-1}(b(a(j)), 1) - V_{i-1}(b(j), 1)) + \pi_0(a(j)) - \pi_0(j). \tag{8}$$

Clearly, $V_i(j, n)$ and $D_i(j, n)$ are non-negative for every (j, n).

[2] For an overview on the theory of Markov decision processes the interested reader is referred to Van der Wal and Wessels (1985).

By (2), (3), (4) and (5) we observe, for every $2 \le i \le T$,

$$V_i(j, n, x) = \begin{cases} x + \beta V_i(j, n+1) & \text{if } 0 \le x \le D_i(j, n) \\ \beta V_i(a(j), n+1) & \text{if } x > D_i(j, n) \end{cases} \quad \text{whenever} \quad 1 \le n \le N-1, \tag{9}$$

and

$$V_i(j, N, x) = \begin{cases} x + \pi_0(j) + \beta V_{i-1}(b(j), 1) & \text{if } x \le D_i(j, N) \\ \pi_0(a(j)) + \beta V_{i-1}(b(a(j)), 1) & \text{if } x > D_i(j, N) \end{cases} \quad \text{for} \quad n = N. \tag{10}$$

From (1), (9) and (10) it is obvious that the policy holder, observing state (j, n, x) at the end of period $N(T-i)+n$, $1 \le i \le T$, $1 \le n \le N$, will claim if and only if the amount of damage exceeds a certain limit. After having identified the *form* of the optimal policy we are interested in the *computation* of this so-called threshold policy, i.e., the computation of the optimal critical claim sizes.

Clearly by (1) the optimal critical claim sizes at the end of period $n + N(T-1)$, $1 \le n \le N$ are zero. For the computation of the other critical claim sizes we need the equations (9) and (10).

By these relations we immediately obtain, for every $2 \le i \le T$,

$$V_i(j, n) := \mathbb{E}(V_i(j, n, \underline{X}_n))$$

$$= \begin{cases} K_n(D_i(j, n)) + \beta\big(V_i(j, n+1)G_n(D_i(j, n)) + V_i(a(j), n+1)(1 - G_n(D_i(j, n)))\big) \\ \quad \text{if } 1 \le n \le N-1, & (11) \\ K_N(D_i(j, N)) + \pi_0(j)G_N(D_i(j, N)) + \pi_0(a(j))(1 - G_N(D_i(j, N))) \\ \quad + \beta\big(V_{i-1}(b(j), 1)G_N(D_i(j, N)) + V_{i-1}(b(a(j)), 1)(1 - G_N(D_i(j, N)))\big) \\ \quad \text{if } n = N, & (12) \end{cases}$$

where $K_n(d)$ is the expected amount of damage to be paid by the policy holder having critical claim size d during period n;

$$K_n(d) := \mathbb{E}\big(\underline{X}_n 1_{\{\underline{X}_n \le d\}}\big) = p_n \int_0^d x \, \mathrm{d}F_n(x), \quad 1 \le n \le N,$$

and

$$G_n(d) := P\{\underline{X}_n \le d\} = \begin{cases} 1 - p_n & \text{if } d = 0 \\ 1 - p_n + p_n F_n(d) & \text{if } d > 0 \end{cases}.$$

By (11) and (12) the optimal critical claim sizes can easily be found in the following recurrent way.

Clearly by (1) and (8) it holds that

$$D_2(j, N) = \pi_0(a(j)) - \pi_0(j) \quad \text{for every } j.$$

Hence by (12), $V_2(j, N)$ is known for every j and this yields by (7) the value of $D_2(j, N-1)$ for every j. Suppose now we have computed for some $2 \le i \le T$ and $1 \le n \le N$ the values of $D_2(j, N)$, $V_2(j, N), \dots, D_i(j, n)$ for every j.

By (11) (if $1 \le n \le N-1$) or (12) (if $n = N$) this yields $V_i(j, n)$ and hence by (7) and (8) $D_i(j, n-1)$ (if $n \ge 2$) or $D_{i+1}(j, N)$ (if $n = 1$) is known.

The above iterative procedure for computing every critical claim size needs $NJ(T-1)$ recurrent steps involving $O(1)$ computations and this implies that the complexity of the above algorithm is $O(NJT)$, assuming it takes a constant time to evaluate the functions in (11) or (12). This concludes for the time being our discussion of the finite NT-horizon model. In the next section we will discuss some sensitivity results for this model with respect to variations in the input parameters F_n and p_n, $n = 1, \dots, N$. The remainder of this section is devoted to the analysis of the infinite horizon model.

Define

$V(j, n, x) :=$ minimal expected total discounted costs if the policy holder observes state (j, n, x) at the end of period n.

As for the discounted finite horizon model we obtain the following set of recurrence relations. For every $n = 1, \ldots, N-1$ and $x > 0$ we get

$$V(j, n, x) = \min \begin{cases} x + \beta \mathbb{E}\left(V(j, n+1, \underline{X}_{n+1})\right) \\ \beta \mathbb{E}\left(V(a(j), n+1, \underline{X}_{n+1})\right) \end{cases}. \tag{13}$$

For every $n = 1, \ldots, N-1$ and $x = 0$ we obtain

$$V(j, n, 0) = \beta \mathbb{E}\left(V(j, n+1, \underline{X}_{n+1})\right). \tag{14}$$

For every $n = N$ and $x > 0$ it holds that

$$V(j, n, x) = \min \begin{cases} x + \pi_0(j) + \beta \mathbb{E}\left(V(b(j), 1, \underline{X}_1)\right) \\ \pi_0(a(j)) + \beta \mathbb{E}\left(V(b(a(j)), 1, \underline{X}_1)\right) \end{cases}. \tag{15}$$

For $n = N$ and $x = 0$ it yields

$$V(j, N, 0) = \pi_0(j) + \beta \mathbb{E}\left(V(b(j), 1, \underline{X}_1)\right). \tag{16}$$

Similarly as for the finite horizon model we obtain that a policy holder, observing (j, n, x) at some decision point, will claim if and only if $x > D(j, n)$, where

$$V(j, n) := \mathbb{E}\left(V(j, n, \underline{X}_n)\right), \tag{17}$$

$$D(j, n) := \beta\left(V(a(j), n+1) - V(j, n+1)\right) \quad \text{if} \quad 1 \le n \le N-1 \tag{18}$$

and

$$D(j, N) := \pi_0(a(j)) - \pi_0(j) + \beta\left(V(b(a(j)), 1) - V(b(j), 1)\right). \tag{19}$$

Clearly $D(j, n)$ and $V(j, n)$ are non-negative for every (j, n).

Notice that the optimal policy for the discounted infinite horizon model is stationary as follows directly from the theory of Markov decision processes [cf. Ross (1983)].

In order to compute this threshold policy, i.e., the optimal critical claim sizes, we can proceed in the same way as for the finite horizon model. This yields the following set of equations:

$$V(j, n) := \mathbb{E}\left(V(j, n, \underline{X}_n)\right)$$

$$= \begin{cases} K_n(D(j, n)) + \beta\left(V(j, n+1) G_n(D(j, n)) + V(a(j), n+1)(1 - G_n(D(j, n)))\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad 1 \le n \le N-1, \quad (20) \\ K_N(D(j, N)) + \pi_0(j) G_n(D(j, N)) + \pi_0(a(j))(1 - G_N(D(j, N))) \\ \quad + \beta\left(V(b(j), 1) G_N(D(j, N)) + V(b(a(j)), 1)(1 - G_N(D(j, N)))\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad n = N. \end{cases} \tag{21}$$

The above system of equations can be solved by the well-known method of successive approximations [cf. Ross (1983)]. However, before proving this we have to rewrite (20) and (21) in a suitable form.

First of all, we apply Lemma A.1 (see the appendix) to these equations. This yields [3]

$$V(j, n) := \mathbb{E}\left(V(j, n, \underline{X}_n)\right)$$

$$= \begin{cases} \min_{d \ge 0} \left[K_n(d) + \beta\left(V(j, n+1) G_n(d) + V(a(j), n+1)(1 - G_n(d))\right)\right] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad 1 \le n \le N-1, \quad (22) \\ \min_{d \ge 0} \left[K_N(d) + \pi_0(j) G_N(d) + \pi_0(a(j))(1 - G_N(d)) \right. \\ \quad \left. + \beta\left(V(b(j), 1) G_N(d) + V(b(a(j)), 1)(1 - G_N(d))\right)\right] \quad \text{if} \quad n = N. \quad (23) \end{cases}$$

[3] A similar set of equations derived intuitively is also discussed by Norman and Shearn (1980).

In order to rewrite the above system in a compact matrix notation we introduce the following conventions. Let $\mathscr{F}$ denote the set of elements (j, n), i.e.,

$$\mathscr{F}=\{(2,1),\dots,(J,1),(1,2),\dots,(J,2),\dots,(1,N),\dots,(J,N)\}$$

and let

$$\boldsymbol{f}:=(f(j,n))_{(j,n)\in\mathscr{F}}=(f(2,1),\dots,f(J,1),\dots,f(1,N),\dots,f(J,N))$$

be an arbitrary vector on $\mathscr{F}$, i.e. $\boldsymbol{f}\in\mathbb{R}^{\mathscr{F}}$.

Define

$$\boldsymbol{v}:=(V(j,n))_{(j,n)\in\mathscr{F}},$$

$$\boldsymbol{d}:=(D(j,n))_{(j,n)\in\mathscr{F}},$$

$$r(\boldsymbol{f}):=(R(\boldsymbol{f})(j,n))_{(j,n)\in\mathscr{F}},\quad \boldsymbol{f}\geq 0,$$

with

$$R(\boldsymbol{f})(j,n) = \begin{cases} K_n(f(j,n)), & 1\leq n\leq N-1,\\ K_N(f(j,N))+G_N(f(j,N))\pi_0(j)+\pi_0(a(j))(1-G_N(f(j,N))) & \text{if}\quad n=N, \end{cases} \tag{24}$$

and $D(j, n)$ given by (18) and (19).

Moreover, let $Q(\boldsymbol{f})$ be the Markov matrix of the underlying Markov chain if the policy holder uses a threshold policy $\boldsymbol{f}\in\mathbb{R}^{\mathscr{F}}$, $\boldsymbol{f}\geq 0$. Hence

$$Q(\boldsymbol{f})=\begin{pmatrix} \mathbf{0} & Q_1(\boldsymbol{f}) & \cdots & \mathbf{0}\\ \vdots & & & \vdots\\ \vdots & & & Q_{N-1}(\boldsymbol{f})\\ Q_N(\boldsymbol{f}) & \cdots & \cdots & \mathbf{0} \end{pmatrix}.$$

This matrix consists of submatrices $Q_n(\boldsymbol{f})=(q_n(\boldsymbol{f})((j,n),(k,m)))$, $1\leq n\leq N$, satisfying

$$q_n(\boldsymbol{f})((j,n),(k,m))=\begin{cases} G_n(f(j,n)) & \text{if}\quad k=j;\ m=n+1;\ 1\leq n\leq N-1,\\ G_N(f(j,n)) & \text{if}\quad k=b(j);\ m=1;\ n=N,\\ 1-G_n(f(j,n)) & \text{if}\quad k=a(j);\ m=n+1;\ 1\leq n\leq N-1,\\ 1-G_n(f(j,n)) & \text{if}\quad k=b(a(j));\ m=1;\ n=N,\\ 0 & \text{elsewhere}. \end{cases}$$

Finally, we introduce the set of operators $L(\boldsymbol{f})$, $\boldsymbol{f}\geq 0$ and U: $\mathbb{R}^{\mathscr{F}}\to\mathbb{R}^{\mathscr{F}}$ defined by

$$L(\boldsymbol{f})\boldsymbol{w}=r(\boldsymbol{f})+\beta Q(\boldsymbol{f})\boldsymbol{w}$$

and

$$U\boldsymbol{w}=\min_{\boldsymbol{f}\geq 0}L(\boldsymbol{f})\boldsymbol{w}, \tag{25}$$

where the minimization is taken component-wise.

It is not difficult to verify that the pair of relationships (20), (21) and (22), (23) can be written as

$$\boldsymbol{v}=L(\boldsymbol{d})\boldsymbol{v} \tag{26}$$

and

$$\boldsymbol{v}=U\boldsymbol{v}. \tag{27}$$

Also, by Lemma A.1 we obtain, for every $\boldsymbol{w}\in\mathbb{R}^{\mathscr{F}}$,

$$U\boldsymbol{w}=L(\boldsymbol{f}_w)\boldsymbol{w}\quad\text{where}\quad \boldsymbol{f}_w:=(f_w(j,n))_{(j,n)\in\mathscr{F}} \tag{28}$$

with

$$f_w(j, n) := \begin{cases} \max(\beta(w(a(j), n+1) - w(j, n+1)), 0) & \text{if } 1 \le n \le N-1, \\ \max(\pi_0(a(j)) - \pi_0(j) + \beta(w(b(a(j)), 1) - w(b(j), 1)), 0) & \text{if } n = N. \end{cases}$$

Hence using well-known arguments from the theory of Markov Decision Processes [cf. Ross (1970)] we obtain the next result. Before discussing this result we introduce the so-called maxnorm $\|\bullet\|$ on $\mathbb{R}^{\mathscr{F}}$, i.e., $\|w\| := \max_{(j,n)} |w(j, n)|$.

Lemma 3.1. (a). *The operator* U: $\mathbb{R}^{\mathscr{F}} \to \mathbb{R}^{\mathscr{F}}$ *is a* β*-contraction mapping, i.,e.* $\|Uw_1 - Uw_2\| \le \beta \|w_1 - w_2\|$ *for every bounded* $w_1, w_2 \in \mathbb{R}^{\mathscr{F}}$.
(b) *The equation* $Uw = w$ *has a unique solution and this solution equals* v..

Proof. Part (a) can be proved by using relationship (28) and Theorem 6.5 of Ross (1970). The result in (b) is an immediate consequence of (a) and (27). □

By Lemma 3.1 and the result discussed in the appendix, it is clear that $\lim_{n\to\infty} \|v_n - v\| = 0$ where $v_n := Uv_{n-1} = L(f_{v_{n-1}})v_{n-1}$, $n \ge 1$ and v_0 is some bounded vector belonging to $\mathbb{R}^{\mathscr{F}}$.

The method of successive approximations is based upon this observation. However, before introducing this method and the corresponding algorithm, we have to define a stopping rule. The form of this stopping rule can be derived from the following well-known result.

Lemma 3.2. *Suppose* $v_n = Uv_{n-1}$, $n \ge 1$ *and* v_0 *is an arbitrary bounded vector on* $\mathscr{F}$. *If* e *denotes the vector on* $\mathscr{F}$ *with all components equal to* 1, *then for every* $n \ge 1$ *the next inequality holds*:

$$v_n + \frac{\beta}{1-\beta} \min_{(j,n)} ((v_n - v_{n-1})(j, n)) e \le v \le v_n + \frac{\beta}{1-\beta} \max_{(j,n)} ((v_n - v_{n-1})(j, n)) e.$$

Moreover, the lowerbound (upperbound) converges monotonically increasing (decreasing) to v *as* $n \to \infty$.

Proof. In order to prove these so-called McQueen bounds we use relation (26) and copy the proof of the above result for Markov Decision Processes with finite action spaces [cf. Hendrikx, Van Nunen and Wessels (1984)]. □

Defining $\text{span}(v) := \max_{(j,n)}(v(j, n)) - \min_{(j,n)}(v(j, n))$ we are now able to state the method of successive approximations.

Algorithm
Step 1. Choose v_0 some arbitrary bounded vector on $\mathscr{F}$.
Step 2. Compute, for $n \ge 1$,

$$v_n = Uv_{n-1} = L(f_{v_{n-1}})v_{n-1}.$$

Step 3. If span $(v_n - v_{n-1}) < \epsilon$ for some given $\epsilon > 0$, stop.
Otherwise, return to step 2 with $n := n + 1$.

Output: $$\hat{v} := v_n + \frac{1}{2}\frac{\beta}{1-\beta}\left(\max_{(j,n)}((v_n - v_{n-1})(j, n)) + \min_{(j,n)}((v_n - v_{n-1})(j,n))\right),$$

$$\hat{d} := f_{\hat{v}}.$$

By Lemma 3.2 and the definition of $\hat{d}$ is is easy to derive the next result.

Theorem 3.3. *After completion of the above algorithm the following holds*:

$$\| \hat{d} - d \| \leq \beta^2 \epsilon/(1-\beta).$$

Proof. By Lemma 3.2 we easily obtain

$$\| \hat{v} - v \| \leq \frac{1}{2} \cdot \frac{\beta}{1-\beta} \operatorname{span}(v_n - v_{n-1}).$$

Moreover, by the definition of $\hat{d}$ and (18) and (19), it is also not difficult to verify that

$$\| \hat{d} - d \| \leq 2\beta \| \hat{v} - v \|$$

and hence combining the above inequalities yields the desired result. □

The proof of Theorem 3.3 concludes this section. In the next section we will discuss some sensitivity results.

4. Sensitivity analysis

In this section we will prove some sensitivity results for both models discussed. In particular we will compare the optimal value-functions associated with the vectors of input parameters $\eta_1 = (\beta, p_1^{\mathrm{T}}, F_1^{\mathrm{T}})^{\mathrm{T}}$ and $\eta_2 = (\beta, p_2^{\mathrm{T}}, F_2^{\mathrm{T}})^{\mathrm{T}}$ whenever $p_1 \neq p_2$ or $F_1 \neq F_2$ with p_i $(i = 1, 2)$ the vector of probabilities of accident occurrences in subperiods $n = 1, \ldots, N$ and F_i $(i = 1, 2)$ the vector of damage distribution in subperiods $n = 1, \ldots, N$.

By giving an upper bound on the difference of these value-functions in the maxnorm $\|\bullet\|$, we can easily obtain a similar type of result for the difference of the vectors of optimal critical claim sizes. It turns out that the models discussed are robust and this implies that the optimal critical claim sizes obtained after an unbiased estimation of the input parameters are close to the real optimal critical claim sizes if the number of observations will become large. Starting the analysis of the above problem for the finite horizon model we introduce the following notations. Let $\mathscr{F}_n$ denote the set of elements of $\mathscr{F}$ whose second component is n, i.e., $\mathscr{F}_n = (\ldots, (3, n), \ldots, (J, n)\}$ and define for the input vector η_m, $m = 1, 2$,

$$v_{m,Ni-n} := (V_i(j, n))^{\mathrm{T}}_{(j,n)\in\mathscr{F}_n}, \quad r_{m,n}(f) := (R(f)(j, n))^{\mathrm{T}}_{(j,n)\in\mathscr{F}_n},$$

where $R(f)(j, n)$ and $V_i(j, n)$ are introduced in (24) and (11).

Moreover, $L_{m,n}(f)$ and $U_{m,n}$ denote the operators $L(f)$, resp. U restricted to $\mathbb{R}^{\mathscr{F}_{n+1}}$ (if $1 \leq n \leq N-1$) or $\mathbb{R}_1^{\mathscr{F}}$ (if $n = N$) for the same input vector η_m, i.e.,

$$L_{m,n}(f)w := r_{m,n}(f) + \beta Q_{m,n}(f)w \quad \text{and} \quad U_{m,n}w := \min_f L_{m,n}(f)w$$

for all $w \in \mathbb{R}^{\mathscr{F}_{n+1}}$ (if $1 \leq n \leq N-1$) or $w \in \mathbb{R}^{\mathscr{F}_1}$ (if $n = N$). By Lemma A.1 it is clear that the relationships (11) and (12) can be written in the following compact matrix notation:

$$v_{m,Ni-n} = U_{m,n} v_{m,Ni-n-1}, \quad 2 \leq i \leq T, \quad 1 \leq n \leq N, \quad m = 1, 2. \tag{29}$$

This relationship will be the starting point of the proof of the next result.

Theorem 4.1 (*finite horizon*).
(a) *For every pair of vectors* $\eta_m = (\beta, p_m^{\mathrm{T}}, F_m^{\mathrm{T}})^{\mathrm{T}}$, $m = 1, 2$, *satisfying* $p_1 \neq p_2$ *the following inequality holds*:

$$\| v_{1,NT-1} - v_{2,NT-1} \| \leq \| p_1 - p_2 \| \frac{1-\beta^{N(T-1)}}{1-\beta^N} \sum_{k=0}^{N-1} \beta^k \mathbb{E}(\underline{X}_{k+1}).$$

(b) *For every pair of vectors* $\eta_m = (\beta, \boldsymbol{p}^T, \boldsymbol{F}_m^T)^T$, $m = 1, 2$, *satisfying* $\boldsymbol{F}_1 \neq \boldsymbol{F}_2$ *the following inequality holds*:

$$\| \boldsymbol{v}_{1,NT-1} - \boldsymbol{v}_{2,NT-1} \| \leq \frac{1-\beta^{N(T-1)}}{1-\beta^N} \| \boldsymbol{p} \| \sum_{k=0}^{N-1} \beta^k L(F_{k+1,1}, F_{k+1,2}),$$

where $F_{n,m}$ $(m = 1, 2)$ *denotes the nth component of the vector* $\boldsymbol{F}_m$ *and*

$$L(F_{n,1}, F_{n,2}) := \int_0^\infty | F_{n,1}(z) - F_{n,2}(z) | \, \mathrm{d}z.$$

Proof. Since (a) and (b) can be proved in a similar way we will only consider the proof of (a). By (29) we have, for every $2 \leq i \leq T$ and $1 \leq n \leq N$,

$$\begin{aligned} \| \boldsymbol{v}_{1,N_i-n} - \boldsymbol{v}_{2,Ni-n} \| &= | U_{1,n}\boldsymbol{v}_{1,Ni-n-1} - U_{2,n}\boldsymbol{v}_{2,Ni-n-1} \| \\ &\leq \| U_{1,n}\boldsymbol{v}_{1,Ni-n-1} - U_{1,n}\boldsymbol{v}_{2,Ni-n-1} \| + \| U_{1,n}\boldsymbol{v}_{2,Ni-n-1} - U_{2,n}\boldsymbol{v}_{2,Ni-n-1} \|. \end{aligned}$$

Since $U_{1,n}$ is a β-contraction mapping for every $1 \leq n \leq N$ clearly the first term in the above inequality is bounded by $\beta \| \boldsymbol{v}_{1,Ni-n-1} - \boldsymbol{v}_{2,Ni-n-1} \|$.

Moreover, by Lemma A.1 we obtain

$$U_{m,n}\boldsymbol{v}_{2,Ni-n-1} = L_{m,n}(\boldsymbol{d}_{2,i})\boldsymbol{v}_{2,Ni-n-1}$$

for every $m = 1, 2$ and $\boldsymbol{d}_{2,i} := \{D_i(j, n))_{(j,n) \in \mathscr{F}}$ with $D_i(j, n)$ given by (7) and (8) using the input vector η_2. This yields

$$\| U_{1,n}\boldsymbol{v}_{2,Ni-n-1} - U_{2,n}\boldsymbol{v}_{2,Ni-n-1} \| = \| L_{1,n}(\boldsymbol{d}_{2,i})\boldsymbol{v}_{2,Ni-n-1} - L_{2,n}(\boldsymbol{d}_{2,i})\boldsymbol{v}_{2,Ni-n-1} \|$$

and hence after some calculations, using the definition of $L_{m,n}(\boldsymbol{d}_{2,i})$ $(m = 1, 2)$ we obtain

$$\begin{aligned} \| U_{1,n}\boldsymbol{v}_{2,Ni-n-1} - U_{2,n}\boldsymbol{v}_{2,Ni-n-1} \| &= \max_{(j,n)} | p_{n,1} - p_{n,2} | \int_0^{D_i(j,n)} (1 - F_n(x)) \, \mathrm{d}x \\ &\leq \| \boldsymbol{p}_1 - \boldsymbol{p}_2 \| \mathbb{E}\underline{X}_n, \end{aligned}$$

where $p_{n,m}$ denotes the nth component of the vector $\boldsymbol{p}_m$, $m = 1, 2$.

Combining the above inequalities yields for every $2 \leq i \leq T$ and $1 \leq n \leq N$ that

$$\| \boldsymbol{v}_{1,Ni-n} - \boldsymbol{v}_{2,Ni-n} \| \leq \beta \| \boldsymbol{v}_{1,Ni-n-1} - \boldsymbol{v}_{2,Ni-n-1} \| + \| \boldsymbol{p}_1 - \boldsymbol{p}_2 \| \mathbb{E}\underline{X}_n. \tag{30}$$

Since the value-functions $\boldsymbol{v}_{1,n}$ and $\boldsymbol{v}_{2,n}$, $1 \leq n \leq N-1$ are equal to the vector consisting of zeros, we obtain by iterating relationship (30) that

$$\begin{aligned} \| \boldsymbol{v}_{1,NT-1} - \boldsymbol{v}_{2,NT-1} \| &\leq \beta \| \boldsymbol{v}_{1,NT-2} - \boldsymbol{v}_{2,NT-2} \| + \| \boldsymbol{p}_1 - \boldsymbol{p}_2 \| \mathbb{E}\underline{X}_1 \\ &\leq \ldots \leq \sum_{j=0}^{T-2} \beta^{Nj} \sum_{k=0}^{N-1} \beta^k \mathbb{E}(\underline{X}_{k+1}) \| \boldsymbol{p}_1 - \boldsymbol{p}_2 \| + \beta^{N(T-1)} \| \boldsymbol{v}_{1,N-1} - \boldsymbol{v}_{2,N-1} \| \\ &= \frac{1-\beta^{N(T-1)}}{1-\beta^N} \sum_{k=0}^{N-1} \beta^k \mathbb{E}(\underline{X}_{k+1}) \end{aligned}$$

and hence the desired result is proved. □

By the observation that the infinite horizon model can be approximated by a sequence of finite horizon models (let $T \to \infty$) we immediately obtain he following result.

Theorem 4.2. (*Infinite horizon*).
(a) *For every pair of vectors* $\eta_m = (\beta, \boldsymbol{p}_m^T, \boldsymbol{F}^T)^T$, $m = 1, 2$, *satisfying* $\boldsymbol{p}_1 \neq \boldsymbol{p}_2$ *the following inequality holds*:

$$\| \boldsymbol{v}_1 - \boldsymbol{v}_2 \| \leq \| \boldsymbol{p}_1 - \boldsymbol{p}_2 \| (1 - \beta^N)^{-1} \sum_{k=0}^{N-1} \beta^k \mathbb{E}(\underline{X}_{k+1}).$$

(b) *For every pair of vectors* $\eta_m = (\beta, \boldsymbol{p}^{\mathrm{T}}, \boldsymbol{F}_m^{\mathrm{T}})^{\mathrm{T}}$, $m = 1, 2$, *satisfying* $\boldsymbol{F}_1 \neq \boldsymbol{F}_2$ *the following inequality holds*:

$$\| \boldsymbol{v}_1 - \boldsymbol{v}_2 \| \leq \| \boldsymbol{p} \| (1 - \beta^N)^{-1} \sum_{k=0}^{N-1} \beta^k L(F_{k+1,1}, F_{k+1,2}).$$

Proof. Apply Theorem 4.1 and let $T \to \infty$. □

By the above theorems it is not difficult to obtain similar types of results for the optimal critical claim sizes associated with the input parameters η_m ($m = 1, 2$) using $\| \boldsymbol{d}_1 - \boldsymbol{d}_2 \| \leq 2\beta \| \boldsymbol{v}_1 - \boldsymbol{v}_2 \|$. This concludes the section on sensitivity. In the next section we will discuss some computations.

5. Results

The model presented in Section 3 will be applied to the bonus–malus system given in Table 1. We divide an insurance year in N equal periods, for instance weeks or months and we assume that the probability p_n to have an accident during period n equals λ/N. Usually the number of accidents during a year is assumed to be Poisson-distributed, but the above choice is only slightly different for relevant values of λ. We assume the amount of damage $\underline{Y}_n$ to be lognormally distributed with parameters μ and σ^2. For reasons of simplicity we assume the parameters λ, μ and σ^2 to be independent of n, hence we assume that the parameter values are constant throughout the entire duration of the decision process.

For a third-party liability insurance a value of $\lambda = 0.1$ accidents per year is reasonable. The values of μ and σ^2 are assumed to be $\mu = 6.98849$ and $\sigma^2 = 1.0213$, which corresponds to a mathematical expectation equal to Dfl. 1800 and a modus equal to Dfl. 389.

The basic premium is equated to Dfl. 1000, and the annual interest rate to 5 percent. Therefore, the annual discount rate equals $\beta_0 = 0.95238$ and the discount rate per period equals $\beta = 0.99594$ for $N = 12$. As stop criterion for the infinite horizon model we take $\epsilon = 0.00001$. Substituting the values of β and ϵ in the formula given by Theorem 3.3, we obtain in this case

$$\| \hat{\boldsymbol{d}} - \boldsymbol{d} \| \leq 0.0024.$$

This corresponds to a relative error for the critical claim sizes which is less than 0.0005 percent. We obtain similar results if we take $N = 52$ in combination with $\beta = 0.99906$.

Table 4

Increase of the premium when claiming once or twice.

j	$\Delta\rho_0$	$\Delta\rho_1$
1	0.0	0.0
2	20.0	0.0
3	30.0	0.0
4	40.0	0.0
5	50.0	0.0
6	60.0	0.0
7	40.0	20.0
8	45.0	20.0
9	35.0	30.0
10	30.0	40.0
11	25.0	50.0
12	20.0	60.0
13	17.5	45.0
14	20.0	35.0
15	17.5	30.0
16	20.0	30.0
17	15.0	25.0

Table 5

Optimal critical claim sizes for a horizon of 10 years, in Dfl.

j	n											
	1	2	3	4	5	6	7	8	9	10	11	12
1		0	0	0	0	0	0	0	0	0	0	0
2	586	593	600	607	614	621	628	635	642	650	657	665
3	1075	1086	1098	1109	1121	1132	1144	1156	1168	1180	1192	1204
4	1536	1551	1566	1581	1596	1611	1627	1642	1657	1673	1689	1705
5	1959	1976	1994	2012	2030	2048	2066	2084	2103	2121	2140	2159
6		2338	2358	2378	2399	2419	2439	2460	2480	2501	2522	2543
7	1746	1758	1770	1782	1794	1806	1818	1830	1842	1854	1866	1878
8		2038	2052	2065	2079	2093	2107	2121	2134	2148	2162	2177
9	1549	1557	1565	1573	1581	1589	1597	1605	1613	1621	1629	1637
10	1341	1346	1351	1356	1361	1365	1370	1375	1380	1384	1389	1394
11	1130	1132	1134	1136	1138	1140	1142	1143	1145	1147	1149	1151
12	934	934	933	933	933	932	931	931	930	929	929	928
13	755	754	753	751	750	748	746	745	743	741	740	738
14	827	827	828	828	829	829	830	830	830	831	831	831
15	642	641	640	640	639	639	638	637	636	636	635	634
16	677	677	677	676	676	676	675	675	675	674	674	674
17	480	478	477	475	474	472	471	469	467	466	464	462

To interpret the results it is useful to examine the influence of the filing of a claim on the premium to be paid by the policy holder, where we denote the premium as a percentage of the basic premium. Let

$$\Delta\rho_0(j) = \rho_1(j) - \rho_0(j), \quad \Delta\rho_1(j) = \rho_1(a(j)) - \rho_1(j),$$

where the definitions of $\rho_0(j)$ and $\rho_1(j)$ are given in the footnote under Table 2.

The value of $\Delta\rho_0(j)$ equals the increase of the premium when one additional claim, and $\Delta\rho_1(j)$ equals the additional increase of the premium when a second additional claim is filed during the current year. Table 4 contains the values of $\Delta\rho_0(j)$ and $\Delta\rho_1(j)$ for each j, $j = 1, \ldots, 17$, derived from Table 2.

Table 6

Optimal critical claim sizes for a horizon of 25 years, in Dfl.

j	n											
	1	2	3	4	5	6	7	8	9	10	11	12
1		0	0	0	0	0	0	0	0	0	0	0
2	668	675	683	691	699	707	715	723	731	739	747	756
3	1228	1241	1253	1266	1279	1292	1305	1318	1331	1344	1357	1371
4	1748	1764	1780	1797	1813	1830	1847	1863	1880	1898	1915	1932
5	2212	2231	2250	2270	2289	2309	2328	2348	2368	2388	2409	2429
6		2617	2639	2660	2682	2704	2726	2748	2771	2793	2816	2838
7	1942	1955	1967	1980	1993	2005	2018	2031	2044	2057	2070	2083
8		2252	2267	2281	2296	2310	2325	2340	2355	2369	2384	2399
9	1691	1700	1708	1716	1725	1733	1741	1750	1758	1767	1775	1784
10	1439	1444	1449	1453	1458	1463	1468	1473	1478	1483	1488	1493
11	1195	1197	1199	1201	1203	1205	1206	1208	1210	1212	1214	1216
12	980	980	979	979	978	978	977	976	976	975	974	973
13	787	785	784	782	781	779	777	775	774	772	770	768
14	860	860	860	861	861	862	862	862	862	863	863	863
15	662	662	661	660	660	659	658	657	656	656	655	654
16	698	698	698	697	697	697	696	696	695	695	694	694
17	492	491	489	488	486	484	483	481	479	477	476	474

Table 7

Optimal critical claim sizes for an infinite horizon, in Dfl.

j	*n*											
	1	2	3	4	5	6	7	8	9	10	11	12
1		0	0	0	0	0	0	0	0	0	0	0
2	668	676	683	691	699	707	715	723	731	739	747	756
3	1229	1241	1254	1267	1279	1292	1305	1318	1331	1345	1358	1371
4	1748	1764	1781	1797	1814	1830	1847	1864	1881	1898	1915	1933
5	2212	2232	2251	2270	2290	2309	2329	2349	2369	2389	2409	2430
6		2618	2640	2661	2683	2705	2727	2749	2771	2794	2816	2839
7	1942	1955	1968	1980	1993	2006	2019	2032	2044	2057	2070	2083
8		2253	2267	2282	2296	2311	2326	2340	2355	2370	2385	2400
9	1692	1700	1708	1717	1725	1733	1742	1750	1759	1767	1776	1784
10	1439	1444	1449	1454	1459	1464	1468	1473	1478	1483	1488	1493
11	1195	1197	1199	1201	1203	1205	1207	1208	1210	1212	1214	1216
12	980	980	980	979	978	978	977	977	976	975	974	973
13	787	786	784	782	781	779	777	775	774	772	770	768
14	860	860	861	861	861	862	862	862	862	863	863	863
15	663	662	661	660	660	659	658	657	656	656	655	654
16	699	698	698	697	697	697	696	696	695	695	694	694
17	492	491	489	488	486	484	483	481	479	477	476	474

We compute the optimal critical claim sizes for horizons of 10 years and 25 years and for an infinite horizon, both for $N = 12$ and $N = 52$. To avoid spatial problems, only the results for $N = 12$ are given in Tables 5–7.

We make the following remarks.

(1) From Table 5 it follows that the combinations $(j, m) = (1, 1)$, $(6, 1)$ and $(8, 1)$ cannot occur.

(2) The results obtained for a finite horizon of 25 years (Table 6) differ only little from the results for an infinite horizon (Table 7).

(3) The estimated optimal critical claim sizes show a pattern similar to $\Delta\rho_0$ for any fixed value of n. That is, when the premium increases relatively little ($\Delta\rho_0$ low), then $\hat{\boldsymbol{d}}$ will be relatively low, and the other way round.

(4) The values of $\hat{\boldsymbol{d}}$ increases with n for $n = 1, \ldots, 11$, and 14, decreases with n for $n = 12$, 13 and 15, 16, 17. This pattern changes slightly during the last years of the insurance, but since it originates from differences between expected costs in various states it is difficult to explain this particular pattern conclusively.

(5) The longer the process continues before it ends, the higher the values of $\hat{\boldsymbol{d}}$ are. For the higher the number of years before the process ends, the more important it is for a policy holder to be in a high bonus–malus class, and consequently the higher $\boldsymbol{d}$ is.

(6) The values of $\hat{\boldsymbol{d}}$ for $N = 52$ are almost linear interpolations between the values for $N = 12$. Therefore, and for spatial reasons, we do not present these values.

Appendix

In this appendix we prove the next result.

Lemma A.1. Let $\mathscr{F}$ denote the set of elements (j, n) and $\boldsymbol{w}$: $\mathscr{F} \to \mathbb{R}$ is an arbitrary vector on $\mathscr{F}$. If
$$\boldsymbol{f}_w := (f_w(j, n))_{(j,n) \in \mathscr{F}}$$
with
$$f_w(j, n) := \begin{cases} \max(\beta(w(a(j), n+1) - w(j, n+1)), 0) & \text{if } 1 \le n \le N-1, \\ \max(\pi_0(a(j)) - \pi_0(j) + \beta(w(b(a(j)), 1) - w(b(j), 1)), 0) & \text{if } n = N, \end{cases}$$

then for every threshold policy $f \in \mathbb{R}^{\mathcal{F}}$, $f \geq 0$ and $\eta = (\beta, p^T, F^T)^T$ we obtain

$$L(f_w)w \leq L(f)w.$$

Proof. By definition [cf. (22)] it holds true that

$$L(f)w = r(f) + \beta Q(f)w$$

for every threshold policy f.

In scalar notation this reads

$$L(f)w(j, n)$$
$$= \begin{cases} \int_{0^-}^{f(j,n)} (x + \beta w(j, n+1))\, dG_n(x) + \int_{f(j,n)^+}^{\infty} \beta w(a(j), n+1)\, dG_n(x) & \text{if} \quad 1 \leq n \leq N-1, \\ \int_{0^-}^{f(j,n)} (x + \pi_0(j) + \beta w(b(j), 1))\, dG_N(x) + \int_{f(j,n)}^{\infty} (\pi_0(a(j)) + \beta w(b\ (a(j)), 1))\, dG_N(x) & \\ & \text{if} \quad n = N, \end{cases} \tag{A.1}$$

We now consider the following two cases for every (j, n).

(i) $f_w(j, n) = 0$. By the definition of G_n relation (A.1) reduces to

$$L(f)w(j, n) = \begin{cases} \beta((1 - p_n)w(j, n+1) + p_n w(a(j), n+1)) & \text{if} \quad 1 \leq n \leq N-1, \\ p_N \pi_0(a(j)) + (1 - p_N)\pi_0(j) & \\ \quad + \beta(((1 - p_N)w(b(j), 1) + p_N w(b(a(j)), 1)) & \text{if} \quad n = N. \end{cases} \tag{A.2}$$

Since $f_w(j, n) = 0$ we obtain

$$w(a(j), n+1) \leq w(j, n+1) \quad \text{if} \quad 1 \leq n \leq N-1 \tag{A.3}$$

or

$$\pi_0(a(j)) + \beta w(b(a(j)), 1) \leq \pi_0(j) + \beta w(b(j), 1) \quad \text{if} \quad n = N. \tag{A.4}$$

Hence by (A.1) up to (A.4)

$$L(f_w)w(j, n) \leq L(f)w(j, n)$$

for every $f \geq 0$.

(ii) $f_w(j, n) > 0$. By relation (A.1) it follows immediately that

$$L(f_w)w(j, n) - L(f)w(j, n)$$
$$= \begin{cases} \int_{f(j,n)}^{f_w(j,n)} [x - \beta(w(a(j), n+1) - w(j, n+1))]\, dG_n(x) & \text{if} \quad 1 \leq n \leq N-1, \\ \int_{f(j,n)}^{f_w(j,n)} [x - (\pi_0(a(j)) - \pi_0(j)) - \beta(w(b(a(j)), 1) - w(b(j), 1))]\, dG_n(x) & \\ & \text{if} \quad n = N. \end{cases} \tag{A.5}$$

Since $f_w(j, n) > 0$ we obtain

$$f_w(j, n) = \begin{cases} \beta(w(a(j), n+1) - w(j, n+1)) & \text{if} \quad 1 \leq n \leq N-1, \\ \pi_0(a(j)) - \pi_0(j) + \beta(w(b(a(j), 1) - w(b(j)), 1)) & \text{if} \quad n = N, \end{cases} \tag{A.6}$$

and this implies, by (A.5)

$$L(f_w)w(j, n) \leq L(f)w(j, n). \quad \square$$

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