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Note

A simple proof of Liang's lower bound for on-line bin packing and the extension to the parametric case

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Abstract

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In this note we present a simplified proof of a lower bound for on-line bin packing. This proof also covers the well-known result given by Liang in Inform. Process Lett. 10 (1980) 76–79.

1. Introduction

The one-dimensional bin packing problem can be described as follows: given a list $L=(a_1,\dots,a_n)$ of items with size $0<s(a_i)\leq 1$ and an infinite supply of unit-capacity bins, place the items of the list L into the minimal number of bins so that the sum of the sizes in each bin is at most equal to one. Since this problem is NP-complete (cf. [3]) various heuristics have been studied. In this note we will only consider heuristics which pack the items one at a time in the order given by the list L

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and once packed, these items can never be moved again. These heuristics are called on-line and to measure their worst-case performance we introduce the following notations. Let V_r, $r \in \mathbb{N}_+$, denote the set of all lists L for which the maximal size of the elements is bounded from above by $1/r \leq 1$. If $\mathrm{OPT}(L)$ represents the optimal number of bins to pack list L and $A(L)$ the number of bins used by heuristic A to pack the same list then

$$R_A(k,r) := \max\left\{\frac{A(L)}{k} : L \in V_r \text{ and } \mathrm{OPT}(L) = k\right\}$$

for every $k \geq 1$. The (parametric) asymptotic worst-case ratio of the heuristic A is now given by

$$R_A^\infty(r) := \limsup_{k \uparrow \infty} R_A(k,r).$$

Liang (cf. [5]) presented an up to now best known lower bound for $R_A^\infty(1)$ for on-line heuristics which was generalized in [2] to the parametric case, i.e., $r \geq 1$. In the next section we give a simplified proof of both results.

2. A short proof

To review the result by Liang and Galambos we need the following sequence

$$\begin{aligned} &t_1(r) = r+1,\ t_2(r) = r+2, \\ &t_{i+1}(r) = t_i(r)(t_i(r)-1)+1, \quad i \geq 2 \end{aligned} \tag{1}$$

with $r \geq 1$ some fixed integer.

Observe this sequence also appears in the analysis of the worst-case behaviour of the Next Fit Decreasing heuristic (cf. [1]) and of the Harmonic Fit heuristic for $r=1$ (cf. [4]). In the paper by Liang ($r=1$) a different notation for this sequence is used. This author uses the sequence m_i, $i \geq 0$. However, it is easy to verify that $t_{i+1}(1) = m_i + 1$, $i \geq 0$. The main result proved by the above authors is listed in the next theorem.

Theorem 1 (cf. [2,5]). *If A is an on-line heuristic and $r \geq 1$ some integer then*

$$R_A^\infty(r) \geq \left(1 + \sum_{j=2}^\infty \frac{j}{t_j(r)-1}\right)\left(1 + \sum_{j=2}^\infty \frac{1}{t_j(r)-1}\right)^{-1}.$$

In order to start the elementary proof of Theorem 1 we consider for fixed k the sublists L_j^k, $1 \leq j \leq k$ with

$$L_1^k = \left\{(t_{k+1}(r)-1)r \text{ items of size } \varepsilon_{1,k} + \frac{1}{t_1(r)}\right\} \tag{2}$$

and

$$L_j^k = \left\{ t_{k+1}(r) - 1 \text{ items of size } \varepsilon_{j,k} + \frac{1}{t_j(r)} \right\}, \quad 2 \le j \le k \tag{3}$$

where

$$0 < \varepsilon_{1,k} < (k+r-1)^{-1}(t_{k+1}(r)-1)^{-1}$$

and

$$0 < \varepsilon_{j+1,k}(t_j(r)-1) < \varepsilon_{j,k}, \quad 1 \le j \le k-1.$$

Observe for $r = 1$ these sublists reduce to the sublists used by Liang and as in [5] an intuitive justification can be given for the above lists by generalizing and refining to the parametric case earlier results of Yao (cf. [6]).

For the lists displayed in (2) and (3) it follows (cf. [2]) by an easy argument that

$$\mathrm{OPT}(L_k \cdots L_j) = \begin{cases} \dfrac{t_{k+1}(r)-1}{t_j(r)-1}, & \text{if } j \ge 2, \\ t_{k+1}(r) - 1, & \text{if } j = 1. \end{cases} \tag{4}$$

In (4) the superscript k in L_j^k is deleted and this will also be done in the remainder of the proof. The first part of this simplified proof uses only simple counting arguments and does not depend on the specific structure of the chosen sublists. In this part the following notation is introduced for convenience. Let $\mathscr{B} = \{B_1, \ldots, B_{A(L_k \cdots L_1)}\}$ denote the packing of the concatenated list $L_k \cdots L_1$ generated by the on-line heuristic A.

A bin belonging to $\mathscr{B}$ is called a bin of type $\boldsymbol{i} = (i_1, \ldots, i_k)$ if it contains i_j items of list L_j, $1 \le j \le k$, whereas the number of bins in $\mathscr{B}$ of type $\boldsymbol{i}$ is denoted by $n(\boldsymbol{i})$. Moreover, the subset $\mathscr{B}_j \subseteq \mathscr{B}$, $1 \le j \le k$, contains only those bins that were used for the first time during the packing of the sublists L_j and

$$T_j := \{\boldsymbol{i}\colon \textit{there exists a bin of type } \boldsymbol{i} \textit{ in } \mathscr{B}_j\}.$$

Since A is on-line and hence only bins belonging to $\mathscr{B}_p$, $p \ge j$ might contain items from L_j and the number of items in L_j, $j \ge 2$, equals $t_{k+1}(r) - 1$ it must follow that

$$t_{k+1}(r) - 1 = \sum_{p=j}^{k} \sum_{\boldsymbol{i} \in T_p} i_j n(\boldsymbol{i}) \tag{5}$$

for every $2 \le j \le k$. Applying a similar argument to L_1 and $A(L_k \cdots L_j)$, $j \ge 1$ yields

$$(t_{k+1}(r) - 1)r = \sum_{p=1}^{k} \sum_{\boldsymbol{i} \in T_p} i_1 n(\boldsymbol{i}) \tag{6}$$

and

$$A(L_k \cdots L_j) = \sum_{p=j}^{k} \sum_{\boldsymbol{i} \in T_p} n(\boldsymbol{i}). \tag{7}$$

This concludes the first part of the proof. In the remainder we use the properties of the chosen list $L_k \cdots L_1$.

Observe, if

$$R_A^k(r) := \max\left\{\frac{A(L_k \cdots L_j)}{\mathrm{OPT}(L_k \cdots L_j)} : 1 \le j \le k\right\}$$

that by (4)

$$A(L_k \cdots L_j) \le R_A^k(r)\mathrm{OPT}(L_k \cdots L_j)$$

$$= \begin{cases} \dfrac{t_{k+1}(r)-1}{t_j(r)-1} R_A^k(r), & \text{if } 2 \le j \le k, \\ (t_{k+1}(r)-1)R_A^k(r), & \text{if } j=1. \end{cases} \tag{8}$$

Hence by summing (8) and applying (7) we obtain

$$(t_{k+1}(r)-1)\left(1+\sum_{j=2}^{k}\frac{1}{t_j(r)-1}\right)R_A^k(r) \ge \sum_{j=1}^{k} A(L_k \cdots L_j)$$

$$= \sum_{p=1}^{k} p \sum_{i \in T_p} n(i). \tag{9}$$

Applying again the properties of the chosen list $L_k \cdots L_1$ we show in the next lemma that for every $i \in T_p$, $1 \le p \le k$ the inequality

$$p \ge \sum_{j=1}^{p} \frac{j i_j}{t_j(r)-1} \tag{10}$$

must hold. Hence by (9) and (10)

$$(t_{k+1}(r)-1)\left(1+\sum_{j=2}^{k}\frac{1}{t_j(r)-1}\right)R_A^k(r) \ge \sum_{p=1}^{k}\sum_{j=1}^{p}\frac{j}{t_j(r)-1}\sum_{i \in T_p} i_j n(i)$$

$$= \sum_{j=1}^{k}\sum_{p=j}^{k}\frac{j}{t_j(r)-1}\sum_{i \in T_p} i_j n(i)$$

$$= (t_{k+1}(r)-1)\left(1+\sum_{j=2}^{k}\frac{j}{t_j(r)-1}\right)$$

where the last equality is justified by (6) and (5) or equivalently

$$R_A^k(r) \ge \left(1+\sum_{j=2}^{k}\frac{j}{t_j(r)-1}\right)\left(1+\sum_{j=2}^{k}\frac{1}{t_j(r)-1}\right)^{-1}. \tag{11}$$

Since $\limsup_{k\uparrow\infty} R_A^k(r) \le R_A^\infty(r)$ the desired result follows by (11) taking $k \uparrow \infty$. □

For the verification of Theorem 1 we made use of inequality (10). A similar inequality for $r=1$ also had to be verified by Liang (see [5, Section 5]). However, instead of introducing different subcases as done in [5] it is possible to simplify and shorten the proof of (10) considerably by means of the following argument. Consider an arbitrary bin B belonging to $\mathscr{B}_p$ and start to replace all the elements from

L_j, $j \le p-1$ by an appropriate number of elements from L_{j+1} in such a way that the new packing in the bin stays feasible. Notice that this is possible since elements of L_{j+1} are smaller than those of L_j. Repeating this argument we obtain a feasible packing in B with only elements from L_p and by showing that this procedure does not decrease the "weight" of the original packing and observing that the "weight" (10) holds trivially for a bin containing only elements from L_p the result follows. More formally this argument is proved in the following lemma.

Lemma 2. *For every $i \in T_p$, $1 \le p \le k$, it follows that $\sum_{j=1}^{p} j i_j/(t_j(r)-1) \le p$.*

Proof. Consider a bin of type $i \in T_p$ and so for a given $j \le p-1$ it follows that this bin contains i_j elements of list L_j. Let us replace each element from L_j by elements from L_{j+1}. If $j=1$ then each element of L_1 can be replaced by one element of L_2. This is possible since

$$\frac{1}{t_2(r)} + \varepsilon_{2,k} < \frac{1}{t_1(r)} + \varepsilon_{1,k}.$$

If $j \ge 2$ we replace each element of L_j by $t_j(r)-1$ elements of L_{j+1}. By (1) and the definition of $\varepsilon_{j,k}$ it is easy to see that

$$(t_j(r)-1)\left(\frac{1}{t_{j+1}(r)} + \varepsilon_{j+1,k}\right) < \frac{1}{t_j(r)} + \varepsilon_{j,k}$$

and hence the total sum of the new constructed packing is bounded from above by one. Moreover, since for $j \ge 2$,

$$(t_j(r)-1)\frac{j+1}{t_{j+1}(r)-1} = \frac{j+1}{t_j(r)} \ge \frac{j}{t_j(r)-1}$$

where we used $j+1 \le t_j(r)$ and for $j=1$,

$$\frac{2}{t_2(r)-1} = \frac{2}{r+1} \ge \frac{1}{r} = \frac{1}{t_1(r)-1}$$

it follows that the total weighted sum $\sum_{j=1}^{p} j i_j^{(\mathrm{new})}/(t_j(r)-1)$ of the new constructed packing in this bin does not decrease. Repeating this procedure for $j=1,\dots,p-1$ we finally obtain a feasible packing with only items from L_p and with no decreased weighted sum. Since every bin with only items from L_p can contain at most $t_p(r)-1$ items it follows that $i_p/(t_p(r)-1) \le 1$ and hence the desired result is proved. □

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