

# SVM-Maj: A Majorization Approach to Linear Support Vector Machines with Different Hinge Errors

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## Abstract

Support vector machines (SVM) are becoming increasingly popular for the prediction of a binary dependent variable. SVMs perform very well with respect to competing techniques. Often, the solution of an SVM is obtained by switching to the dual. In this paper, we stick to the primal support vector machine (SVM) problem, study its effective aspects, and propose varieties of convex loss functions such as the standard for SVM with the absolute hinge error as well as the quadratic hinge and the Huber hinge errors. We present an iterative majorization algorithm that minimizes each of the adaptations. In addition, we show that many of the features of an SVM are also obtained by an optimal scaling approach to regression. We illustrate this with an example from the literature and do a comparison of different methods on several empirical data sets.

Keywords: Support vector machines, Iterative majorization, I-Splines, Absolute hinge error, Quadratic hinge error, Huber hinge error, Optimal scaling.

## 1 Introduction

In recent years, support vector machines (SVMs) have become a popular technique to predict two groups from a set of predictor variables (Vapnik, 2000). This data analysis problem is not new and such data can also be analyzed

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through alternative techniques such as linear and quadratic discriminant analysis, neural networks, and logistic regression. However, SVMs seem to compare favorably in their prediction quality with respect to competing models. Also, their optimization problem is well defined and can be solved through a quadratic program. Furthermore, the classification rule derived from an SVM is relatively simple and it can be readily applied to new, unseen samples. At the downside, the interpretation in terms of the predictor variables in nonlinear SVM is not always possible. In addition, the usual dual formulation of an SVM may not be so easy to grasp.

In this paper, we offer a different way of looking at linear SVMs that makes the interpretation easier. First of all, we stick to the primal problem and formulate the SVM in terms of a loss function that is regularized by a penalty term. From this formulation, it can be seen that SVMs use robustified errors. Apart from the standard SVM loss function that uses the absolute hinge error, we introduce two other hinge errors, the Huber and quadratic hinge errors, and show the relation with ridge regression.

The second theme of this paper is to show the connection between optimal scaling regression and SVMs. The idea of optimally transforming a variable so that a criterion is being optimized has been around for more than 30 years (see, for example, Young, 1981; Gifi, 1990). We show that optimal scaling regression using an ordinal transformation with the primary approach to ties comes close to the objective of SVMs. We discuss the similarities between both approaches and give a formulation of SVM in terms of optimal scaling.

A third theme is to propose a new majorization algorithm that minimizes the loss for any of the hinge errors. The advantage of majorization is that each iteration is guaranteed to reduce the SVM loss function until convergence is reached. Finally, we provide numerical experiments on a suite of 14 empirical data sets to study the predictive performance of the different errors in SVMs and compare it to optimal scaling regression. We also compare the computational efficiency of the majorization approach for the SVM to several standard SVM solvers.

Note that this paper is a significantly extended version of Groenen, Nalban-tov, and Bioch (2007).

## 2 The SVM Loss Function

In many ways, an SVM resembles regression quite closely. Let us first introduce some notation. Let  $\mathbf{X}$  be the  $n \times m$  matrix of predictor variables of  $n$  objects and  $m$  variables. The  $n \times 1$  vector  $\mathbf{y}$  contains the grouping of the objects into two classes, that is,  $y_i = 1$  if object  $i$  belongs to class 1 and  $y_i = -1$  if object  $i$  belongs to class  $-1$ . Obviously, the labels  $-1$  and  $1$  to distinguish the classes are unimportant. Let  $\mathbf{w}$  be the  $m \times 1$  vector with weights used to make a linear combination of the predictor variables. Then, the predicted value  $q_i$  for object  $i$  is

$$q_i = c + \mathbf{x}_i' \mathbf{w}, \tag{1}$$

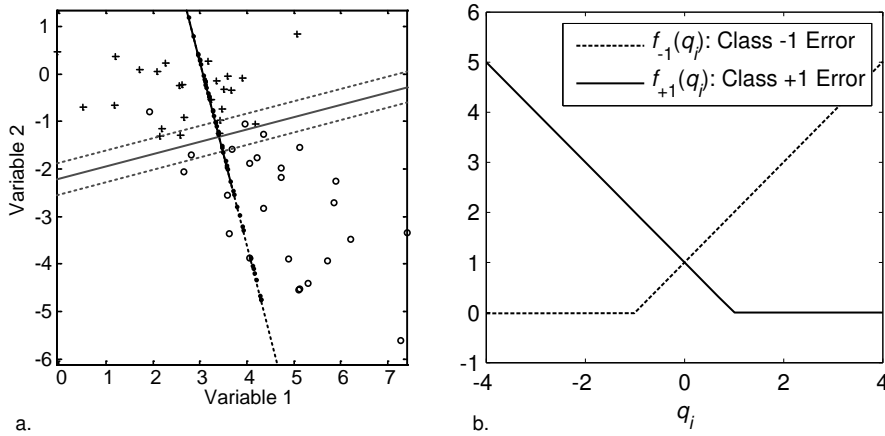


Figure 1: Panel a Projections of the observations in groups 1 (+) and -1 (o) onto the line given by  $w_1$  and  $w_2$ . Panel b shows the absolute hinge error function  $f_1(q_i)$  for class 1 objects (solid line) and  $f_{-1}(q_i)$  for class -1 objects (dashed line).

where  $\mathbf{x}'_i$  is row  $i$  of  $\mathbf{X}$  and  $c$  is an intercept. Consider the example in Figure 1a where for two predictor variables, each row  $i$  is represented by a point labelled '+' for the class 1 and 'o' for class -1. Every combination of  $w_1$  and  $w_2$  defines a direction in this scatter plot. Then, each point  $i$  can be projected onto this line. The idea of the SVM is to choose this line in such a way that the projections of the points of class 1 are well separated from those of class -1. The line of separation is orthogonal to the line with projections and the intercept  $c$  determines where exactly it occurs. Note that if  $\mathbf{w}$  has length 1, that is,  $\|\mathbf{w}\| = (\mathbf{w}'\mathbf{w})^{1/2} = 1$ , then Figure 1a explains fully the linear combination (1). If  $\mathbf{w}$  doesn't have length 1, then the scale values along the projection line should be multiplied by  $\|\mathbf{w}\|$ . The dotted lines in Figure 1a show all those points that project to the lines at  $q_i = -1$  and  $q_i = 1$ . These dotted lines are called the margin lines in SVMs. Note that if there are more than two variables the margin lines become hyperplanes. Summarizing, the SVM has three sets of parameters that determines its solution: (1) the weights normalized to have length 1, that is,  $\mathbf{w}/\|\mathbf{w}\|$ , (2) the length of  $\mathbf{w}$ , that is,  $\|\mathbf{w}\|$ , and (3) the intercept  $c$ .

SVMs count an error as follows. Every object  $i$  from class 1 that projects such that  $q_i \geq 1$  yields a zero error. However, if  $q_i < 1$ , then the error is linear with  $1 - q_i$ . Similarly, objects in class -1 with  $q_i \leq -1$  do not contribute to the error, but those with  $q_i > -1$  contribute linearly with  $q_i + 1$ . In other words, objects that project on the wrong side of their margin contribute to the error, whereas objects that project on the correct side of their margin yield zero error. Figure 1b shows the error functions for the two classes. Because of its hinge form, we call this error function the *absolute hinge* error.

As the length of  $\mathbf{w}$  controls how close the margin lines are to each other, it can be beneficial for the number of errors to choose the largest  $\|\mathbf{w}\|$  possible, so that fewer points contribute to the error. To control the  $\|\mathbf{w}\|$ , a penalty term that is dependent on  $\|\mathbf{w}\|$  is added to the loss function. The penalty term also avoids overfitting of the data.

Let  $G_1$  and  $G_{-1}$  respectively denote the sets of class 1 and  $-1$  objects. Then, the SVM loss function can be written as

$$\begin{aligned} L_{\text{SVM}}(c, \mathbf{w}) &= \sum_{i \in G_1} \max(0, 1 - q_i) + \sum_{i \in G_{-1}} \max(0, q_i + 1) + \lambda \mathbf{w}'\mathbf{w} \\ &= \sum_{i \in G_1} f_1(q_i) + \sum_{i \in G_{-1}} f_{-1}(q_i) + \lambda \mathbf{w}'\mathbf{w} \\ &= \text{Class 1 errors} + \text{Class } -1 \text{ errors} + \text{Penalty for nonzero } \mathbf{w}, \end{aligned} \quad (2)$$

where  $\lambda > 0$  determines the strength of the penalty term. For similar expressions, see Hastie, Tibshirani, and Friedman (2001) and Vapnik (2000). Note that (2) can also be expressed as

$$L_{\text{SVM}}(c, \mathbf{w}) = \sum_{i=1}^n \max(0, 1 - y_i q_i) + \lambda \mathbf{w}'\mathbf{w},$$

which is closer to the expressions used in the SVM literature.

Assume that we have found a  $c$  and  $\mathbf{w}$  that minimizes (2). All the objects  $i$  that project on the correct side of their margin, contribute with zero error to the loss. As a consequence, these objects could be removed from the analysis without changing the solution. Therefore, all the objects  $i$  that project at the wrong side of their margin and thus induce error or if an object falls exactly on the margin, then these objects determine the solution. Such objects are called support vectors as they form the fundament of the SVM solution. Note that these objects (the support vectors) are not known in advance and, therefore, the analysis needs to be carried out with all  $n$  objects present in the analysis.

What can be seen from (2) is that any error is punished linearly, not quadratically. Thus, SVMs are more robust against outliers than a least-squares loss function. The idea of introducing robustness by absolute errors is not new. For more information on robust multivariate analysis, we refer to Huber (1981), Vapnik (2000), and Rousseeuw and Leroy (2003). In the next section, we discuss two other error functions, one of which is robust.

The SVM literature usually presents the SVM loss function as follows (Burges, 1998):

$$L_{\text{SVMClas}}(c, \mathbf{w}, \xi) = C \sum_{i \in G_1} \xi_i + C \sum_{i \in G_2} \xi_i + \frac{1}{2} \mathbf{w}'\mathbf{w}, \quad (3)$$

$$\text{subject to} \quad 1 + (c + \mathbf{w}'\mathbf{x}_i) \leq \xi_i \text{ for } i \in G_{-1} \quad (4)$$

$$1 - (c + \mathbf{w}'\mathbf{x}_i) \leq \xi_i \text{ for } i \in G_1 \quad (5)$$

$$\xi_i \geq 0, \quad (6)$$

where  $C$  is a nonnegative parameter set by the user to weight the importance of the errors represented by the so-called slack variables  $\xi_i$ . Suppose that object  $i$

in  $G_1$  projects at the correct side of its margin, that is,  $q_i = c + \mathbf{w}'\mathbf{x}_i \geq 1$ . As a consequence,  $1 - (c + \mathbf{w}'\mathbf{x}_i) \leq 0$  so that the corresponding  $\xi_i$  can be chosen as 0. If  $i$  projects on the wrong side of its margin, then  $q_i = c + \mathbf{w}'\mathbf{x}_i < 1$  so that  $1 - (c + \mathbf{w}'\mathbf{x}_i) > 0$ . Choosing  $\xi_i = 1 - (c + \mathbf{w}'\mathbf{x}_i)$  gives the smallest  $\xi_i$  satisfying the restrictions in (4), (5), and (6). As a consequence,  $\xi_i = \max(0, 1 - q_i)$  and is a measure of error. A similar derivation can be made for class  $-1$  objects. Note that in the SVM literature (3) and (6) are often expressed more compactly as

$$\begin{aligned} L_{\text{SVMClas}}(c, \mathbf{w}, \xi) &= C \sum_{i=1}^n \xi_i + \frac{1}{2} \mathbf{w}'\mathbf{w}, \\ \text{subject to} & \quad y_i(c + \mathbf{w}'\mathbf{x}_i) \leq 1 - \xi_i \text{ for } i = 1, \dots, n \\ & \quad \xi_i \geq 0. \end{aligned}$$

Choosing  $C = (2\lambda)^{-1}$  gives

$$\begin{aligned} & L_{\text{SVMClas}}(c, \mathbf{w}, \xi) \\ &= (2\lambda)^{-1} \left( \sum_{i \in G_1} \xi_i + \sum_{i \in G_{-1}} \xi_i + 2\lambda \frac{1}{2} \mathbf{w}'\mathbf{w} \right) \\ &= (2\lambda)^{-1} \left( \sum_{i \in G_1} \max(0, 1 - q_i) + \sum_{i \in G_{-1}} \max(0, q_i + 1) + \lambda \mathbf{w}'\mathbf{w} \right) \\ &= (2\lambda)^{-1} L_{\text{SVM}}(c, \mathbf{w}). \end{aligned}$$

showing that the two formulations (2) and (3) are exactly the same up to a scaling factor  $(2\lambda)^{-1}$  and yield the same  $c$  and  $\mathbf{w}$ . However, the advantage of (2) is that it can be interpreted as a (robust) error function with a penalty. The quadratic penalty term is used for regularization much in the same way as in ridge regression, that is, to force the  $w_j$  to be close to zero. The penalty is particularly useful to avoid overfitting. Furthermore, it can be easily seen that  $L_{\text{SVM}}(c, \mathbf{w})$  is a convex function in  $c$  and  $\mathbf{w}$  because all three terms are convex in  $c$  and  $\mathbf{w}$ . As the function is also bounded below by zero and it is convex, the minimum of  $L_{\text{SVM}}(c, \mathbf{w})$  is a global one. In fact, (3) allows the problem to be treated as a quadratic program. However, in Section 5, we optimize (2) directly by the method of iterative majorization.

### 3 Other Error Functions

An advantage of clearly separating error from penalty is that it is easy to apply other error functions. Instead of the absolute hinge error in Figure 2a, we can use different definitions for the errors  $f_1(q_i)$  and  $f_{-1}(q_i)$ . A straightforward alternative for the absolute hinge error is the *quadratic hinge* error, see Figure 2b. This error simply squares the absolute hinge error, yielding the loss function

$$L_{\text{Q-SVM}}(c, \mathbf{w}) = \sum_{i \in G_1} \max(0, 1 - q_i)^2 + \sum_{i \in G_{-1}} \max(0, q_i + 1)^2 + \lambda \mathbf{w}'\mathbf{w}, \quad (7)$$

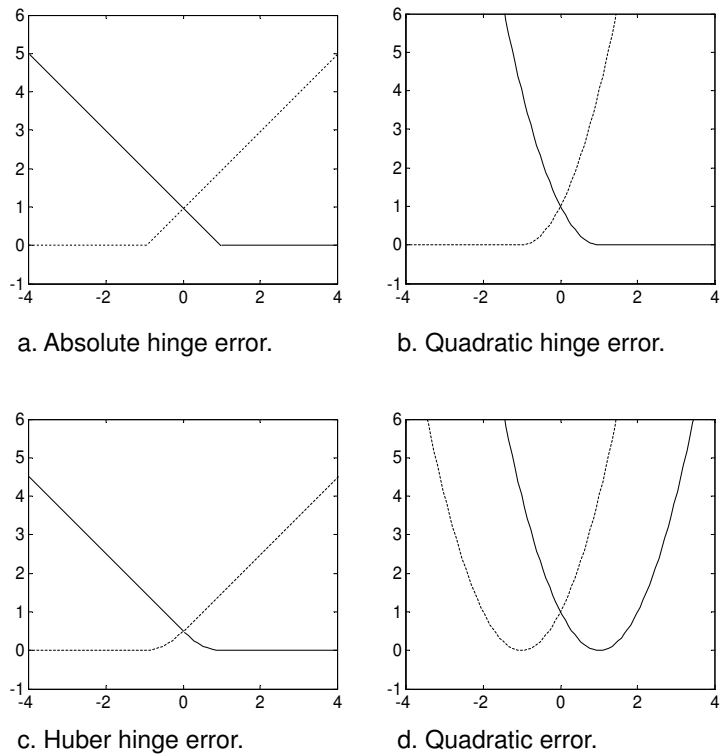


Figure 2: Four error functions: a. the absolute hinge error, b. the quadratic hinge error, c. the Huber hinge error, and d. the quadratic error.

see also, Vapnik (2000) and Cristianini and Shawe-Taylor (2000). It uses the quadratic error for objects that have prediction error and zero error for correctly predicted objects. An advantage of this loss function is that both error and penalty terms are quadratic. In Section 5, we see that the majorizing algorithm is very efficient because in each iteration a linear system is solved very efficiently. A disadvantage of the quadratic hinge error is that outliers can have a large influence on the solution.

Here, we propose a new smooth and robust alternative: the *Huber hinge* error, see Figure 2c. Its definition is found in Table 1 and the corresponding SVM problem is defined by

$$L_{\text{H-SVM}}(c, \mathbf{w}) = \sum_{i \in G_1} h_{+1}(q_i) + \sum_{i \in G_{-1}} h_{-1}(q_i) + \lambda \mathbf{w}' \mathbf{w}. \quad (8)$$

The Huber hinge error is characterized by a linearly increasing error if the error is large, a smooth quadratic error for errors between 0 and the linear part, and zero for objects that are correctly predicted. The smoothness is governed by a value

Table 1: Definition of error functions that can be used in the context of SVMs.

Error	$f_{-1}(q_i)$
Absolute hinge	$\max(0, q_i + 1)$
Quadratic hinge	$\max(0, q_i + 1)^2$
Huber hinge	$h_{-1}(q_i) = (1/2)(k + 1)^{-1} \max(0, q_i + 1)^2$ if $q_i \leq k$ $h_{-1}(q_i) = q_i + 1 - (k + 1)/2$ if $q_i > k$
Quadratic	$(q_i + 1)^2$
$f_{+1}(q_i)$	
Absolute hinge	$\max(0, 1 - q_i)$
Quadratic hinge	$\max(0, 1 - q_i)^2$
Huber hinge	$h_{+1}(q_i) = 1 - q_i - (k + 1)/2$ if $q_i \leq -k$ $h_{+1}(q_i) = (1/2)(k + 1)^{-1} \max(0, 1 - q_i)^2$ if $q_i > -k$
Quadratic	$(1 - q_i)^2$

$k \geq -1$ . The Huber hinge approaches the absolute hinge for  $k \downarrow -1$ , so that the Huber hinge SVM loss solution can approach the classical SVM solution. If  $k$  is chosen too large, then the Huber hinge error essentially approaches the quadratic hinge function. Thus, the Huber hinge error can be seen as a compromise between the absolute and quadratic hinge errors. As we will see in Section 5, it is advantageous to choose  $k$  sufficiently large, for example,  $k = 1$ , as is done in Figure 2c. A similar computational efficiency as for the quadratic hinge error is also available for the Huber hinge error.

In principle, any robust error can be used. To inherit as much of the nice properties of the standard SVM it is advantageous that the error function has two properties: (1) if the error function is convex in  $q_i$  (and hence in  $\mathbf{w}$ ), then the total loss function is also convex and hence has a global minimum that can be reached, (2) the error function should be asymmetric and have the form of a hinge so that objects that are predicted correctly induce zero error.

In Figure 2d the quadratic error is used, defined in Table 1. The quadratic error alone simply equals a multiple regression problem with a dependent variable  $y_i = -1$  if  $i \in G_{-1}$  and  $y_i = 1$  if  $i \in G_1$ , that is,

$$\begin{aligned}
 L_{\text{MReg}}(c, \mathbf{w}) &= \sum_{i \in G_1} (1 - q_i)^2 + \sum_{i \in G_{-1}} (1 + q_i)^2 + \lambda \mathbf{w}' \mathbf{w} \\
 &= \sum_{i \in G_1} (y_i - q_i)^2 + \sum_{i \in G_{-1}} (y_i - q_i)^2 + \lambda \mathbf{w}' \mathbf{w} \\
 &= \sum_i (y_i - c - \mathbf{x}'_i \mathbf{w})^2 + \lambda \mathbf{w}' \mathbf{w} \\
 &= \|\mathbf{y} - c\mathbf{1} - \mathbf{X}\mathbf{w}\|^2 + \lambda \mathbf{w}' \mathbf{w}. \tag{9}
 \end{aligned}$$

Note that for  $i \in G_{-1}$  we have the equality  $(1 + q_i)^2 = ((-1)(1 + q_i))^2 = (-1 - q_i)^2 = (y_i - q_i)^2$ .  $L_{\text{MReg}}(c, \mathbf{w})$  has been extensively discussed in Suykens,

Van Gestel, De Brabanter, De Moor, and Vandewalle (2002). To show that (9) is equivalent to ridge regression, we column center  $\mathbf{X}$  and use  $\mathbf{JX}$  with  $\mathbf{J} = \mathbf{I} - n^{-1}\mathbf{1}\mathbf{1}'$  being the centering matrix. Then (9) is equivalent to

$$\begin{aligned} L_{\text{MRreg}}(c, \mathbf{w}) &= \|\mathbf{y} - c\mathbf{1} - \mathbf{JX}\mathbf{w}\|_{n^{-1}\mathbf{1}\mathbf{1}'}^2 + \|\mathbf{y} - c\mathbf{1} - \mathbf{JX}\mathbf{w}\|_{\mathbf{J}}^2 + \lambda\mathbf{w}'\mathbf{w} \\ &= \|\mathbf{y} - c\mathbf{1}\|_{n^{-1}\mathbf{1}\mathbf{1}'}^2 + \|\mathbf{Jy} - \mathbf{JX}\mathbf{w}\|^2 + \lambda\mathbf{w}'\mathbf{w}, \end{aligned} \quad (10)$$

where the norm notation is defined as  $\|\mathbf{Z}\|_{\mathbf{A}}^2 = \text{tr } \mathbf{Z}'\mathbf{AZ} = \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^K a_{ijk} z_{ik} z_{jk}$ . Note that (10) a decomposition in three terms with the intercept  $c$  appearing alone in the first term so that it can be estimated independently of  $\mathbf{w}$ . The optimal  $c$  in (10) equals  $n^{-1}\mathbf{1}'\mathbf{y}$ . The remaining optimization of (10) in  $\mathbf{w}$  simplifies into a standard ridge regression problem. Hence, the SVM with quadratic errors is equivalent to ridge regression. As the quadratic error has no hinge, even properly predicted objects with  $q_i < -1$  for  $i \in G_{-1}$  or  $q_i > 1$  for  $i \in G_1$  can receive high error. In addition, the quadratic error is nonrobust, hence can be sensitive to outliers. Therefore, ridge regression is more restrictive than the quadratic hinge error and expected to give worse predictions in general.

## 4 Optimal Scaling and SVM

Several ideas that are used in SVMs are not entirely new. In this section, we show that the application of optimal scaling known since the 1970s has almost the same aim as the SVM. Optimal scaling in a regression context goes back to the models MONANOVA (Kruskal, 1965), ADDALS (Young, De Leeuw, & Takane, 1976a), MORALS (Young, De Leeuw, & Takane, 1976b), and, more recently, CatREG (Van der Kooij, Meulman, & Heiser, 2006; Van der Kooij, 2007). The main idea of optimal scaling regression (OS-Reg) is that a variable  $\mathbf{y}$  is replaced by an optimally transformed variable  $\hat{\mathbf{y}}$ . The regression loss function is not only optimized over the usual weights, but also over the optimally scaled variable  $\hat{\mathbf{y}}$ . Many transformations are possible, see, for example, Gifi (1990). However, to make OS-Reg suitable for the binary classification problem, we use the so-called ordinal transformation with the primary approach to ties. This transformation was proposed in the context of multidimensional scaling to optimally scale the ordinal dissimilarities. As we are dealing with two groups only, this means that the only requirement is to constrain all  $\hat{y}_i$  in  $G_{-1}$  to be smaller than or equal to all  $\hat{y}_j$  in  $G_1$ . An example of such a transformation is given in Figure 3a.

OS-Reg can be formalized by minimizing

$$L_{\text{OS-Reg}}(\hat{\mathbf{y}}, \mathbf{w}) = \sum_{i=1}^n (\hat{y}_i - \mathbf{x}_i'\mathbf{w})^2 + \lambda\mathbf{w}'\mathbf{w} = \|\hat{\mathbf{y}} - \mathbf{X}\mathbf{w}\|^2 + \lambda\mathbf{w}'\mathbf{w} \quad (11)$$

subject to  $\hat{y}_i \leq \hat{y}_j$  for all combinations of  $i \in G_{-1}$  and  $j \in G_1$  and  $\hat{\mathbf{y}}'\hat{\mathbf{y}} = n$ . The latter requirement is necessary to avoid the degenerate zero-loss solution of



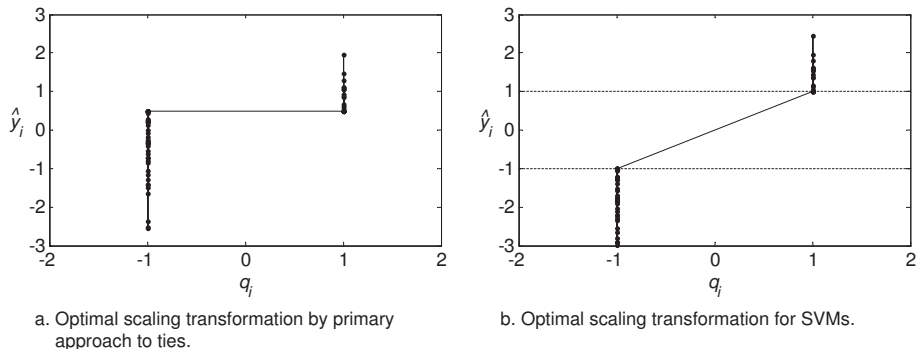


Figure 3: Optimal scaling transformation  $\hat{\mathbf{y}}$  of the dependent variable  $\mathbf{y}$ . Panel a shows an example transformation for the OS-Reg, Panel b for SVM.

$\hat{\mathbf{y}} = \mathbf{0}$  and  $\mathbf{w} = \mathbf{0}$ . In the usual formulation, no penalty term is present in (11), but here we add it because of ease of comparison with SVMs.

The error part of an SVM can also be expressed in terms of an optimally scaled variable  $\hat{\mathbf{y}}$ . Then, the SVM loss becomes

$$L_{\text{SVM-Abs}}(\hat{\mathbf{y}}, \mathbf{w}, c) = \sum_{i=1}^n |\hat{y}_i - \mathbf{x}'_i \mathbf{w} - c| + \lambda \mathbf{w}' \mathbf{w} \quad (12)$$

subject to  $\hat{y}_i \leq -1$  if  $i \in G_{-1}$  and  $\hat{y}_i \geq 1$  if  $i \in G_1$ . Clearly, for  $i \in G_{-1}$  a zero error is obtained if  $\mathbf{x}'_i \mathbf{w} + c \leq -1$  by choosing  $\hat{y}_i = \mathbf{x}'_i \mathbf{w} + c$ . If  $\mathbf{x}'_i \mathbf{w} + c > -1$ , then the restriction  $\hat{y}_i \leq -1$  becomes active so that  $\hat{y}_i$  must be chosen as  $-1$ . Similar reasoning holds for  $i \in G_1$ , where  $\hat{y}_i = \mathbf{x}'_i \mathbf{w} + c$  if  $\mathbf{x}'_i \mathbf{w} + c \geq 1$  (yielding zero error) and  $\hat{y}_i = 1$  if  $\mathbf{x}'_i \mathbf{w} + c < 1$ .

Just as the SVM, OS-Reg also has a limited number of support vectors. All objects  $i$  that are below or above the horizontal line yield zero error. All objects  $i$  that have a value  $\hat{y}_i$  that is on the horizontal line generally give error, hence are support vectors.

The resemblances of SVM and OS-Reg is that both can be used for the binary classification problem, both solutions only use the support vectors, and both can be expressed in terms of an optimal scaled variable  $\hat{\mathbf{y}}$ . Although, the SVM estimates the intercept  $c$ , OS-Reg implicitly estimates  $c$  by leaving the position free where the horizontal line occurs, whereas the SVM attains this freedom by estimating  $c$ . One of the main differences is that OS-Reg uses squared error whereas SVM uses the absolute error. Also, in its standard form  $\lambda = 0$  so that OS-Reg does not have a penalty term. A final difference is that OS-Reg solves the degenerate zero loss solution of  $\hat{\mathbf{y}} = \mathbf{0}$  and  $\mathbf{w} = \mathbf{0}$  by imposing the length constraint  $\hat{\mathbf{y}}' \hat{\mathbf{y}} = n$  whereas the SVM does this through setting having a minimum difference of 2 between  $\hat{y}_i$  and  $\hat{y}_j$  if  $i$  and  $j$  are from different groups.

In some cases with  $\lambda = 0$ , we found occasionally OS-Reg solutions where one of the groups collapsed at the horizontal line and the some objects of the other

group were split into two points: one also at the horizontal line, the other at a distinctly different location. In this way, the length constraint is satisfied, but it is hardly possible to distinguish the groups. Fortunately, these solutions do not occur often and they never occurred with an active penalty term ( $\lambda > 0$ ).

## 5 SVM-Maj: A Majorizing Algorithm for SVM with Robust Hinge Errors

In the SVM literature, the dual of (3) is expressed as a quadratic program and is solved by special quadratic program solvers. A disadvantage of these solvers is that they may become computationally slow for large number of objects  $n$  (although fast specialized solvers exist). Here we derive an iterative majorization (IM) algorithm for the primal SVM problem. An advantage of IM algorithms is that each iteration reduces the SVM loss function. Each of the three loss functions discussed is convex. Because IM is a guaranteed descent algorithm, the IM algorithm will stop when the estimates are sufficiently close to the global minimum. The combination of these properties forms the main strength of the majorization algorithm. In principle, a majorization algorithm can be derived for any error function that has a bounded second derivative as most robust errors have.

Let  $f(\mathbf{q})$  be the function to be minimized. Iterative majorization operates on an auxiliary function, called the majorizing function  $g(\mathbf{q}, \bar{\mathbf{q}})$ , that is dependent on  $\mathbf{q}$  and the previous (known) estimate  $\bar{\mathbf{q}}$ . The majorizing function  $g(\mathbf{q}, \bar{\mathbf{q}})$  has to fulfill several requirements: (1) it should touch  $f$  at the supporting point  $\mathbf{y}$ , that is,  $f(\bar{\mathbf{q}}) = g(\bar{\mathbf{q}}, \bar{\mathbf{q}})$ , (2) it should never be below  $f$ , that is,  $f(\mathbf{q}) \leq g(\mathbf{q}, \bar{\mathbf{q}})$ , and (3)  $g(\mathbf{q}, \bar{\mathbf{q}})$  should be simple, preferably linear or quadratic in  $\mathbf{q}$ . Let  $\mathbf{q}^*$  be such that  $g(\mathbf{q}^*, \bar{\mathbf{q}}) \leq g(\bar{\mathbf{q}}, \bar{\mathbf{q}})$ , for example, by finding the minimum of  $g(\mathbf{q}, \bar{\mathbf{q}})$ . Because the majorizing function is never below the original function, we obtain the so called sandwich inequality

$$f(\mathbf{q}^*) \leq g(\mathbf{q}^*, \bar{\mathbf{q}}) \leq g(\bar{\mathbf{q}}, \bar{\mathbf{q}}) = f(\bar{\mathbf{q}})$$

showing that the update  $\mathbf{q}^*$  obtained by minimizing the majorizing function never increases  $f$  and usually decreases it. This constitutes a single iteration. By repeating these iterations, a monotonically nonincreasing (usually a decreasing) series of loss function values  $f$  is obtained. For convex  $f$  and after a sufficient number of iterations, the IM algorithm stops at a global minimum. More information on iterative majorization can be found in De Leeuw (1994), Heiser (1995), Lange, Hunter, and Yang (2000), Kiers (2002), and Hunter and Lange (2004) and an introduction in Borg and Groenen (2005).

An additional property of IM is useful for developing the algorithm. Suppose we have two functions,  $f_1(\mathbf{q})$  and  $f_2(\mathbf{q})$ , and each of these functions can be majorized, that is,  $f_1(\mathbf{q}) \leq g_1(\mathbf{q}, \bar{\mathbf{q}})$  and  $f_2(\mathbf{q}) \leq g_2(\mathbf{q}, \bar{\mathbf{q}})$ . Then, the function  $f(\mathbf{q} = f_1(\mathbf{q}) + f_2(\mathbf{q}))$  can be majorized by  $g(\mathbf{q} = g_1(\mathbf{q}, \bar{\mathbf{q}}) + g_2(\mathbf{q}, \bar{\mathbf{q}}))$

$$f(\mathbf{q} = f_1(\mathbf{q}) + f_2(\mathbf{q})) \leq g_1(\mathbf{q}, \bar{\mathbf{q}}) + g_2(\mathbf{q}, \bar{\mathbf{q}}) = g(\mathbf{q}, \bar{\mathbf{q}})$$

For notational convenience, we refer in the sequel to the majorizing function as  $g(\mathbf{q})$  without the implicit argument  $\bar{\mathbf{q}}$ .

To find an algorithm, we need to find a majorizing function for (2). For the moment, we assume that a quadratic majorizing function exists for each individual error term of the form

$$f_{-1}(q_i) \leq a_{-1i}q_i^2 - 2b_{-1i}q_i + c_{-1i} = g_{-1}(q_i) \quad (13)$$

$$f_1(q_i) \leq a_{1i}q_i^2 - 2b_{1i}q_i + c_i = g_1(q_i). \quad (14)$$

Then, we combine the results for all terms and come up with the total majorizing function that is quadratic in  $c$  and  $\mathbf{w}$  so that an update can be readily derived. In the next subsection, we derive the SVM-Maj algorithm for general hinge errors assuming that (13) and (14) are known for the specific hinge error. In the appendix, we derive  $g_{-1}(q_i)$  and  $g_1(q_i)$  for the absolute, quadratic, and Huber hinge error SVM.

## 5.1 SVM-Maj

We interpret (2) for use with  $f_{-1}(q)$  and  $f_1(q)$  any of the three hinge errors discussed above. For deriving the SVM-Maj algorithm, we assume that (13) and (14) are known for these hinge losses. Figure 4 shows that this is the case indeed. Then, let

$$a_i = \begin{cases} \max(\delta, a_{-1i}) & \text{if } i \in G_{-1}, \\ \max(\delta, a_{1i}) & \text{if } i \in G_1, \end{cases} \quad (15)$$

$$b_i = \begin{cases} b_{-1i} & \text{if } i \in G_{-1}, \\ b_{1i} & \text{if } i \in G_1, \end{cases} \quad (16)$$

$$c_i = \begin{cases} c_{-1i} & \text{if } i \in G_{-1}, \\ c_{1i} & \text{if } i \in G_1. \end{cases} \quad (17)$$

Summing all the individual terms leads to the majorization inequality

$$L_{\text{SVM}}(c, \mathbf{w}) \leq \sum_{i=1}^n a_i q_i^2 - 2 \sum_{i=1}^n b_i q_i + \sum_{i=1}^n c_i + \lambda \sum_{j=1}^m w_j^2. \quad (18)$$

Because  $q_i = c + \mathbf{x}'_i \mathbf{w}_i$ , it is useful to add an extra column of ones as the first column of  $\mathbf{X}$  so that  $\mathbf{X}$  becomes  $n \times (m+1)$ . Let  $\mathbf{v}' = [c \ \mathbf{w}']$  so that  $\mathbf{q} = \mathbf{X}\mathbf{v}$ . Now, (2) can be majorized as

$$\begin{aligned} L_{\text{SVM}}(\mathbf{v}) &\leq \sum_{i=1}^n a_i (\mathbf{x}'_i \mathbf{v})^2 - 2 \sum_{i=1}^n b_i \mathbf{x}'_i \mathbf{v} + \sum_{i=1}^n c_i + \lambda \sum_{j=2}^{m+1} v_j^2 \\ &= \mathbf{v}' \mathbf{X}' \mathbf{A} \mathbf{X} \mathbf{v} - 2 \mathbf{v}' \mathbf{X}' \mathbf{b} + c_m + \lambda \mathbf{v}' \mathbf{P} \mathbf{v} \\ &= \mathbf{v}' (\mathbf{X}' \mathbf{A} \mathbf{X} + \lambda \mathbf{P}) \mathbf{v} - 2 \mathbf{v}' \mathbf{X}' \mathbf{b} + c_m, \end{aligned} \quad (19)$$

where  $\mathbf{A}$  is a diagonal matrix with elements  $a_i$  on the diagonal,  $\mathbf{b}$  is a vector with elements  $b_i$ , and  $c_m = \sum_{i=1}^n c_i$ , and  $\mathbf{P}$  is the identity matrix except for

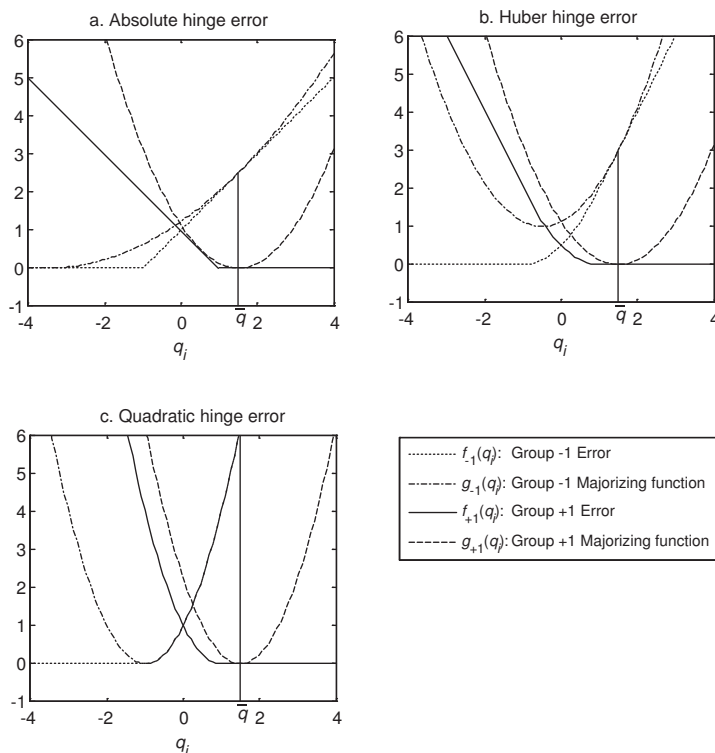


Figure 4: Quadratic majorization functions for (a) the absolute hinge error, (b) the Huber hinge error, and (c) the quadratic hinge error. The supporting point is  $\bar{q} = 1.5$  both for the Group  $-1$  and  $1$  error so that the majorizing functions touch at  $q = \bar{q} = 1.5$ .

element  $p_{11} = 0$ . If  $\mathbf{P}$  were  $\mathbf{I}$ , then the last line of (19) would be of the same form as a ridge regression. Differentiation the last line of (19) with respect to  $\mathbf{v}$  yields the system of equalities linear in  $\mathbf{v}$

$$(\mathbf{X}'\mathbf{A}\mathbf{X} + \lambda\mathbf{P})\mathbf{v} = \mathbf{X}'\mathbf{b}. \quad (20)$$

The update  $\mathbf{v}^+$  solves this set of linear equalities, for example, by Gaussian elimination, or, less efficiently, by

$$\mathbf{v}^+ = (\mathbf{X}'\mathbf{A}\mathbf{X} + \lambda\mathbf{P})^{-1}\mathbf{X}'\mathbf{b}. \quad (21)$$

Because of the substitution  $\mathbf{v}' = [c \ \mathbf{w}']$ , the update of the intercept is  $c^+ = v_1$  and  $w_j^+ = v_{j+1}^+$  for  $j = 1, \dots, m$ . The update  $\mathbf{v}^+$  forms the heart of the majorization algorithm for SVMs.

Extra computational efficiency can be obtained for the quadratic and Huber hinge errors for which  $a_{-1i} = a_{1i} = a$  for all  $i$  and this  $a$  does not depend on  $\bar{q}$ .

In these cases, (21) simplifies into

$$\mathbf{v}^+ = (a\mathbf{X}'\mathbf{X} + \lambda\mathbf{P})^{-1}\mathbf{X}'\mathbf{b}.$$

Thus, the  $m \times n$  matrix  $\mathbf{S} = (a\mathbf{X}'\mathbf{X} + \lambda\mathbf{P})^{-1}\mathbf{X}'$  can be computed once and stored in memory, so that the update (21) simply amounts to setting  $\mathbf{v}^+ = \mathbf{S}\mathbf{b}$ .

The majorizing algorithm for minimizing the standard SVM in (2) is summarized in Algorithm 1. This algorithm has several advantages. First, it iteratively approaches the global minimum closer in each iteration. In contrast, quadratic programming of the dual problem needs to solve the dual problem completely to have the global minimum of the original primal problem. Secondly, the progress can be monitored, for example, in terms of the changes in the number of misclassified objects. Thirdly, to reduce the computational time, smart initial estimates of  $c$  and  $\mathbf{w}$  can be given if they are available, for example, from a previous cross validation run. Note that in each majorization iteration a ridge regression problem is solved so that the SVM-Maj algorithm can be seen as a solution to the SVM problem via successive solutions of ridge regressions.

```

Algorithm:SVM-Maj
input :  $\mathbf{y}, \mathbf{X}, \lambda, \epsilon$ , Hinge
output:  $c_t, \mathbf{w}_t$ 

 $t = 0$ ;
Set  $\epsilon$  to a small positive value;
Set  $\mathbf{w}_0$  and  $c_0$  to random initial values;
if Hinge = Huber or Quadratic then
  if Hinge = Quadratic then  $a = 1$ ;
  if Hinge = Huber then  $a = (1/2)(k + 1)^{-1}$ ;
   $\mathbf{S} = (a\mathbf{X}'\mathbf{X} + \lambda\mathbf{P})^{-1}\mathbf{X}'$ ;
end
Compute  $L_{\text{SVM}}(c_0, \mathbf{w}_0)$  according to (2);
while  $t = 0$  or  $(L_{t-1} - L_{\text{SVM}}(c_t, \mathbf{w}_t))/L_{\text{SVM}}(c_t, \mathbf{w}_t) > \epsilon$  do
   $t = t + 1$ ;
   $L_{t-1} = L_{\text{SVM}}(c_{t-1}, \mathbf{w}_{t-1})$ ;
  Comment:Compute  $\mathbf{A}$  and  $\mathbf{b}$  for different hinge errors
  if Hinge = Absolute then
    Compute  $a_i$  by (22) if  $i \in G_{-1}$  and by (25) if  $i \in G_1$ ;
    Compute  $b_i$  by (23) if  $i \in G_{-1}$  and by (26) if  $i \in G_1$ ;
  else if Hinge = Quadratic then
    Compute  $b_i$  by (29) if  $i \in G_{-1}$  and by (32) if  $i \in G_1$ ;
  else if Hinge = Huber then
    Compute  $b_i$  by (35) if  $i \in G_{-1}$  and by (38) if  $i \in G_1$ ;
  end
  Make the diagonal matrix  $\mathbf{A}$  with elements  $a_i$ ;
  Comment:Compute update
  if Hinge = Absolute then
    Find  $\mathbf{v}$  by that solves (20):  $(\mathbf{X}'\mathbf{A}\mathbf{X} + \lambda\mathbf{P})\mathbf{v} = \mathbf{X}'\mathbf{b}$ ;
  else if Hinge = Huber or Quadratic then
     $\mathbf{v} = \mathbf{S}\mathbf{b}$ ;
  end
  Set  $c_t = v_1$  and  $w_{tj} = v_{j+1}$  for  $j = 1, \dots, m$ ;
end

```

**Algorithm 1:** The SVM majorization algorithm SVM-Maj.

An illustration of the iterative majorization algorithm is given in Figure 5 for the absolute hinge SVM. Here,  $c$  is fixed at its optimal value and the minimization is only over  $\mathbf{w}$ , that is, over  $w_1$  and  $w_2$ . Each point in the horizontal plane represents a combination of  $w_1$  and  $w_2$ . The majorization function is indeed located above the original function and touches it at the dotted line. The  $w_1$  and  $w_2$  where this majorization function finds its minimum,  $L_{\text{SVM}}(c, \mathbf{w})$  is lower than at the previous estimate, so  $L_{\text{SVM}}(c, \mathbf{w})$  has decreased. Note that the separation line and the margins corresponding to the current estimates of  $w_1$  and  $w_2$  are given together with the class 1 points represented as open circles and the class  $-1$  points as closed circles.

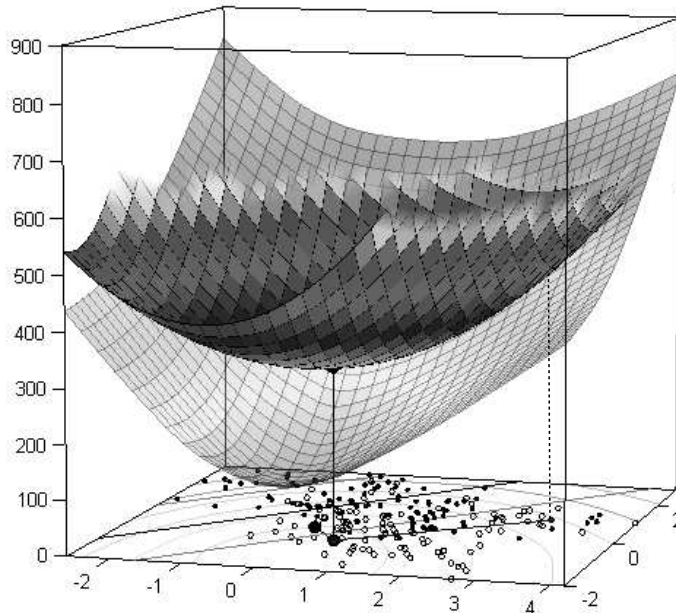


Figure 5: Example of the iterative majorization algorithm for SVMs in action where  $c$  is fixed and  $w_1$  and  $w_2$  are being optimized. The majorization function touches  $L_{\text{SVM}}(c, \mathbf{w})$  at the previous estimates of  $\mathbf{w}$  (the dotted line) and a solid line is lowered at the minimum of the majorizing function showing a decrease in  $L_{\text{SVM}}(c, \mathbf{w})$  as well.

## 6 Experiments

To investigate the performance of the various variants of SVM algorithms, we report experiments on several data sets from the UCI repository (Newman, Hettich, Blake, & Merz, 1998) and the homepage of LibSVM software (Chang & Lin, 2006). These data sets cover a wide range of characteristics such as extent of being unbalanced (one group is larger than the other), number of observations  $n$ , ratio of observations to attributes  $m/n$ , and sparsity (the percentage of nonzero attribute values  $x_{ij}$ ). More information on the data sets are given in Table 2.

In the experiments, we applied the standard absolute hinge ( $\epsilon$ -insensitive), the Huber hinge and quadratic hinge SVM loss functions. All experiments have been carried out in Matlab 7.2, on a 2.8Ghz Intel processor with 2GB of memory under Windows XP. The performance of the majorization algorithms is com-

Table 2: Information on the 14 datasets used in the experiments.  $n_1$  and  $n_{-1}$  are the number of observations with  $y_i = 1$  and  $y_i = -1$ , respectively. Sparsity equals the percentage of zeros in the dataset. The scaling for the scaled attributes is between +1 and 1.

Dataset	Source	$n$	$n_1$	$n_{-1}$	$m$	Sparsity	Notes
Australian	UCI	690	307	383	14	20.04	
Breast_cancer_w	UCI	699	458	241	9	0.00	
Heart_statlog	UCI	270	120	150	13	0.00	Standardized data
Hepatitis	UCI	155	123	32	19	39.86	
Sonar	UCI	208	97	111	60	0.07	
Voting	UCI	434	167	267	16	45.32	
Liver-disorders	LibSVM	345	200	145	6	0.00	
Liver-disorders2	LibSVM	345	200	145	6	0.92	Scaled attributes
Diabetes	LibSVM	768	500	268	8	0.00	
Diabetes2	LibSVM	768	500	268	8	0.15	Scaled attributes
Ionosphere	LibSVM	351	225	126	34	11.59	Scaled attributes
German.number	LibSVM	1000	300	700	24	0.00	
German.number2	LibSVM	1000	300	700	24	4.16	Scaled attributes
Splice	LibSVM	1000	517	483	60	0.00	

pared to those of the off-the-shelf programs LibSVM, BSVM (Hsu & Lin, 2006), SVM-Light (Joachims, 1999), and SVM-Perf (Joachims, 2006). Although these programs can handle nonlinearity of the predictor variables by using special kernels, we limit our experiments to the linear kernel. Note that not all of these SVM-solvers are optimized for the linear kernel. In addition, no comparison between majorization is possible for the Huber hinge loss function as it is not supported by these solvers.

The numerical experiments address several issues. First, how well are the different hinge losses capable of predicting the two groups? Second, we focus on the performance of the majorization algorithm with respect to its competitors. We would like to know how the time needed for the algorithm to converge scales with the number of observations  $n$ , the strictness of the stopping criterion, and with  $\lambda$ ; what is a suitable level for the stopping criterion.

To answer these questions, we consider the following measures. First, we define convergence between two steps as the relative decrease in loss between two subsequent steps, that is, by  $L_{\text{diff}} = (L_{t-1} - L_t)/L_t$ . The error rate in the training data set is defined as the number of misclassified cases. To measure how well a solution predicts, we define the accuracy as the percentage correctly predicted out-of-sample cases in 5-fold cross validation.

## 6.1 Predictive Performance for the Three Hinge Errors

It is interesting to compare the performance of the three hinge loss functions. Consider Table 3, which compares the 5-fold cross-validation accuracy for the three different loss function. For each data set, we tried a grid of  $\lambda$  values ( $\lambda = 2^p$  for  $p = -15, -14.5, -14, \dots, 7.5, 8$  where  $2^{-15} = 0.000030518$  and  $2^8 = 256$ ).



Table 3: Optimal  $\lambda = 2^p$  obtained by 5 fold cross validation, the CPU-time to convergence for the optimal  $\lambda$ , and the prediction accuracy (in %) for 14 different test data sets and four loss functions.

Data set	Optimal $p$				CPU time in sec.				5-fold CV accuracy			
	Abs.	Hub.	Quad.	OS	Abs.	Hub.	Quad.	OS	Abs.	Hub.	Quad.	OS
Australian	-0.5	2.0	3.0	-15.0	0.20	0.12	0.14	0.14	85.4	86.7	86.7	18.1
Breast_cancer_w	7.5	6.0	8.0	-15.0	0.13	0.11	0.04	0.49	96.7	96.6	96.7	97.7
Heart_statlog	0.0	5.5	7.0	5.5	0.03	0.01	0.01	0.04	84.4	84.4	84.4	9.3
Hepatitis	0.0	0.0	2.0	-8.5	0.04	0.02	0.01	0.41	85.8	87.1	86.5	69.0
Sonar	0.5	1.5	1.5	-2.0	0.06	0.02	0.02	0.15	77.4	76.9	78.4	13.5
Voting	-5.5	-1.5	-0.5	-0.5	0.46	0.11	0.10	0.35	97.0	96.8	97.0	6.2
Liver-disorders	3.0	8.0	2.5	8.0	0.05	0.02	0.01	0.07	68.7	68.1	66.1	24.3
Liver-disorders2	-7.0	-3.0	1.0	-2.5	0.07	0.02	0.01	0.07	68.4	68.1	66.4	24.6
Diabetes	1.0	0.5	3.5	7.5	0.24	0.04	0.01	0.07	77.3	78.0	77.6	79.0
Diabetes2	-2.0	-2.5	4.0	8.0	0.13	0.04	0.01	0.03	77.6	78.3	77.9	80.1
Ionosphere	-5.0	2.5	-0.5	-8.0	4.03	0.10	0.17	5.62	90.3	89.5	90.6	25.1
German.number	-0.5	-4.0	3.5	3.5	1.71	0.17	0.03	0.18	77.2	77.0	77.3	26.2
German.number2	2.5	2.0	-0.5	1.5	0.81	0.16	0.03	0.37	77.2	77.0	77.2	33.0
Splice	5.0	7.0	-0.5	3.5	3.21	0.34	0.14	0.61	80.7	81.4	81.1	13.0

Alongside are given the values of the optimal  $\lambda$ 's and times to convergence (stop whenever  $L_{\text{diff}} < 3 \times 10^{-7}$ ). From the accuracy, we see that there is no one best loss function that is suitable for all data sets. The absolute hinge is best in six of the cases, the Huber hinge is best in six of the cases, and the quadratic hinge is best in eight of the cases. The total number is greater than 14 due to equal accuracies. In terms of computational speed, the order invariably is: absolute hinge is the slowest, Huber hinge is faster, and the quadratic hinge is the fastest.

The implementation of optimal scaling regression was also done in MatLab, but the update in each iteration for  $\hat{\mathbf{y}}$  by monotone regression using the primary approach to ties was calculated by a compiled Fortran subroutine. Therefore, the CPU time is not comparable to those of the other SVM methods that were solely programmed in MatLab. Optimal scaling regression performs well on three data sets (Breast cancer, Diabetes and Diabetes2) where the accuracy is better than the three SVM methods. On the remaining data sets, the accuracy is worse or much worse when compared to the SVM methods. It seems that in some cases OS regression can predict well, but its poor performance for the majority of the data sets makes it hard to use it as a standard method for the binary classification problem. It seems that more study is needed to understand why this is so and, if possible, provide adaptations that make it work better for more data sets.

## 6.2 Computational Efficiency of SVM-Maj

To see how computationally efficient the majorization algorithms are, two types of experiments were done. In the first experiment, the majorization algorithm

is studied and tuned. In the second, the majorization algorithm SVM-Maj for the absolute hinge error is compared with several off-the-shelf programs that minimize the same loss function.

As the majorization algorithm is guaranteed to improve the  $L_{SVM}(c, \mathbf{w})$  in each iteration by taking a step closer to the final solution, the computational efficiency of SVM-Maj is determined by its stopping criterion. The iterations of SVM-Maj stop whenever  $L_{\text{diff}} < \epsilon$ . It is also known that majorization algorithms have a linear convergence rate (De Leeuw, 1994), which can be slow especially for very small  $\epsilon$ . Therefore, we study the relations between four measures as they change during the iterations: (a) the difference between present and final loss,  $L_t - L_{\text{final}}$ , (b) the convergence  $L_{\text{diff}}$ , (c) CPU time spent so far, and (d) the difference between current and final within sample error rate.

Figure 6 shows the relationships between these measures for three exemplary data sets: Liver disorders, Sonar and Australian. Note that Figures 6c and 6d the direction of the horizontal axis is reversed so that in all four panels the right side of the horizontal axis means more computational investment. Figure 6a draws the relationship between CPU-time on the one hand and  $L_t - L_{\text{final}}$ , with  $L_{\text{final}}$  the objective function values obtained at convergence with  $\epsilon = 3 \times 10^{-7}$ . Notice that in most of the cases the first few iterations are responsible for the bulk of the decreases in the objective function values and most of the CPU time is spent to obtain small decreases in loss function values. Figure 6b shows the relationship between  $L_t - L_{\text{final}}$  and the convergence  $L_{\text{diff}}$  that is used as a stopping criterion. The two lower panels show the development of the within sample error rate and CPU time (Figure 6c) and convergence  $L_{\text{diff}}$  (Figure 6d). To evaluate whether it is worthwhile using a looser stopping criterion, it is instructive to observe the path of the error rate over the iterations (the lower right panel). It seems that the error rate stabilizes for values of  $L_{\text{diff}}$  below  $10^{-6}$ . Nevertheless, late-time changes sometimes occur in other data sets. Therefore, it does not seem recommendable to stop the algorithm much earlier, hence our recommendation of using  $\epsilon = 3 \times 10^{-7}$ .

The analogues of Figures 6 and 7 were also produced for the Huber hinge and quadratic hinge loss functions. Overall, the same patterns as for the absolute hinge function can be distinguished, with several differences: the objective function decreases much faster (relative to CPU time), and the error rate stabilizes already at slightly greater values for the convergence criterion. In addition, the number of iterations until convergence by and large decline (vis-a-vis the absolute hinge function).

Figure 7 investigates how sensitive the speed of SVM-Maj is relative to changes in the values of  $\lambda$  for four illustrative datasets (Splice, German-number with scaled attributed, Ionosphere, and Sonar). As expected, the relationship appears to be decreasing. Thus, for large  $\lambda$  the penalty term dominates  $L_{SVM}$  and the SVM-MajAbs does not need too many iterations to converge. Note that the same phenomenon is in general observed for the other SVM-solvers as well so that, apparently, the case for large  $\lambda$  is an easier problem to solve.

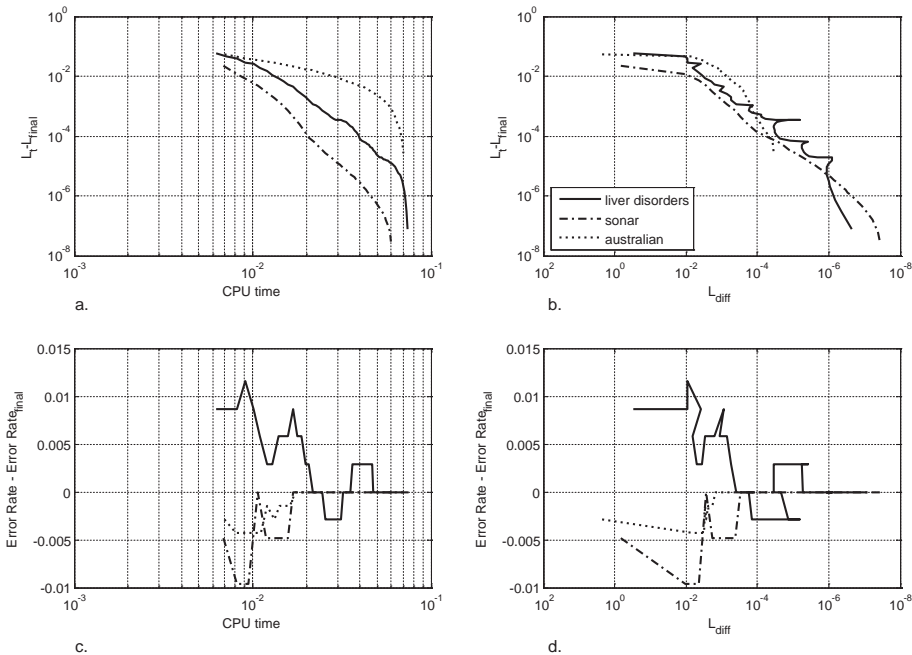


Figure 6: The evolution of several statistics (see text for details) of three datasets: Australian (dotted lines), Sonar (dash-dot lines), and Liver Disorders (scaled, solid lines). Values of  $\lambda$ 's are fixed at optimal levels for each dataset. Loss function used: absolute hinge.

### 6.3 Comparing Efficiency of SVM-Maj with Absolute Hinge

The efficiency of SVM-Maj can be compared with off-the-shelf programs for the absolute hinge error. As competitors of SVM-Maj, we use LibSVM, BSVM, SVM-Light, and SVM-Perf. We use the same 14 data sets as before. As all methods minimize exactly the same loss function  $L_{SVM}$  they all should have the same global minimum. In addition to  $L_{SVM}$ , the methods are compared on speed (CPU-time in seconds) at optimal levels of the  $\lambda = 2^p$  (or equivalent) parameter. Note that the optimal levels of  $\lambda$  could differ slightly between methods as the off-the-shelf programs perform their own grid search for determining the optimal  $\lambda$ , that could be slightly different from those reported in Table 3. We note that the relationship between the  $\lambda$  parameter in SVM-Maj and the  $C$  parameter in LibSVM and SVM-light is given by  $\lambda = 0.5/C$ . For SVM-Maj, we choose three stopping criteria, that is, the algorithm is stopped whenever  $L_{diff}$  is respectively smaller than  $10^{-4}$ ,  $10^{-5}$ , and  $10^{-6}$ .

For some data sets, it was not possible to run the off-the-shelf programs, sometimes because the memory requirements were too large, sometimes because no convergence was obtained. Such problems occurred for three data sets

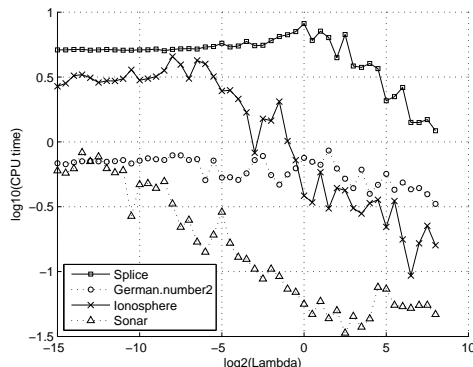


Figure 7: The effect of changing  $\lambda$  on CPU time taken to converge. Loss function used: absolute hinge.

with SVM-Perf and two data sets with SVM-Light. Table 4 shows the results. Especially for  $\epsilon = 10^{-6}$ , SVM-Maj gives solutions that are close to the best minimum found. Generally, Lib-SVM and SVM-Light obtain the lowest  $L_{SVM}$ . SVM-Maj performs well with  $\epsilon = 10^{-6}$ , but even better values can be obtained by a stronger convergence criterion. BSVM finds proper minima but is not able to handle all data sets. In terms of speed SVM-Maj is faster than its competitors in almost all cases. Of course, a smaller  $\epsilon$  increases the CPU-time of SVM-Maj. Nevertheless, even for  $\epsilon = .0001$  good solutions can be found in a short CPU-time.

These results are also summarized in Figure 8, where SVM-Maj is used with the default convergence criterion of  $\epsilon = 3 \times 10^{-7}$ . As far as speed is concerned (see Figure 8a), SVM-Maj ranks consistently amongst the fastest method. The quality of SVM-Maj is also consistently good as it has the same loss function as the global minimum with differences occurring less than 0.01. Note that BSVM and SVM-Perf find consistently much higher loss function values than SVM-Maj, LibSVM and SVM-Light. Generally, the best quality solutions are obtained by LibSVM and SVM-Light although they tend to use more CPU time reaching it.

## 7 Conclusions and Discussion

We have discussed how linear SVM can be viewed as a the minimization of a robust error function with a regularization penalty. The regularization is needed to avoid overfitting in the case when the number of predictor variables increases. We provided a new majorization algorithm for the minimization of the primal SVM problem. This algorithm handles the standard absolute hinge error, the quadratic hinge error, and the newly proposed Huber hinge

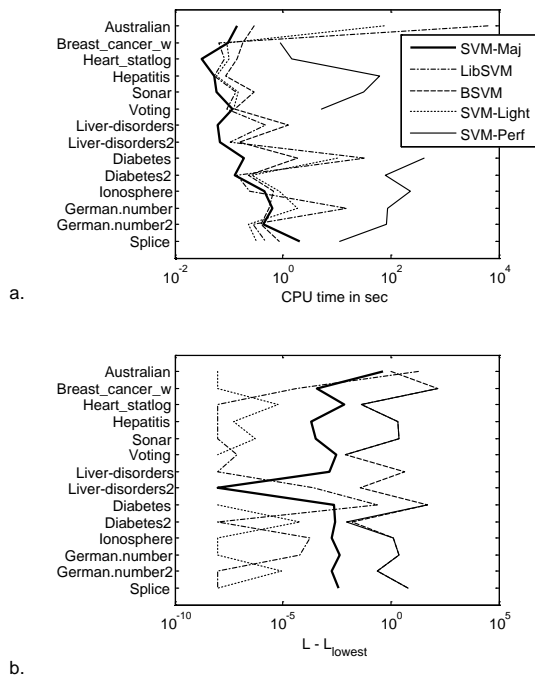


Figure 8: Difference in performance of SVM algorithms with absolute hinge and SVM-Maj using  $\epsilon = 3 \times 10^{-7}$ . Panel a shows the CPU time used in seconds and Panel b shows the difference of  $L$  and the lowest  $L$  amongst the methods.

error. The latter hinge is smooth everywhere yet is linear for large errors. The majorizing algorithm has the advantage that it operates on the primal, is easy to program, and can easily be adapted for robust hinge errors. We also showed that optimal scaling regression has several features in common with SVMs. Numerical experiments on fourteen empirical data sets showed that there is no clear difference between the three hinge errors in terms of cross validated accuracy. The speed of SVM-Maj for the absolute hinge error is similar or compares favorably to the off-the-shelf programs for solving linear SVMs.

There are several open issues and possible extensions. First, the SVM-Maj algorithm is good for situations where the number of objects  $n$  is (much) larger than the number of variables  $m$ . The reason is that each iteration solves an  $(m + 1) \times (m + 1)$  linear system. As  $m$  grows, each iteration becomes slower. Other majorization inequalities can be used to solve this problem yielding fast iterations at the cost of making (much) smaller steps in each iteration. A second limitation is the size of  $n$ . Eventually, when  $n$  gets large, then iterations will become slow. The good thing about SVM-Abs is that each iteration is guaranteed to improve the SVM-Loss. The bad thing is that at most linear convergence can be reached so that for large  $n$  one has to be satisfied with an approximate

solution only.

Second, this paper has focussed on linear SVMs. Nonlinearity can be brought in in two ways. In (Groenen et al., 2007), we proposed to use optimal scaling for the transformation of the predictor variables. Instead of using kernels, we propose to use I-splines to accommodate nonlinearity in the predictor space. The advantage of this approach is that it can be readily applied in any linear SVM algorithm. The standard way of introducing nonlinearity in SVMs is by using kernels. We believe that this is also possible for SVM-Maj and intend to study this possibility in future publications.

SVMs can be extended to problems with more than two classes in several ways. If the extension has error terms of the form  $f_1(q)$  or  $f_{-1}(q)$  then the present majorization results can be readily applied for an algorithm. We believe that applying majorization to SVMs is a fruitful idea that opens new applications and extensions to this area of research.

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## A Majorizing the Hinge Errors

Here we derive the quadratic majorizing functions for the three hinge functions.

### A.1 Majorizing the Absolute Hinge Error

Consider the term  $f_{-1}(q) = \max(0, q + 1)$ . For notational convenience, we drop the subscript  $i$  for the moment. The solid line in Figure 2a shows  $f_{-1}(q)$ . Because of its shape of a hinge, we have called this function the absolute hinge function. Let  $\bar{q}$  be the known error  $q$  of the previous iteration. Then, a majorizing function for  $f_{-1}(q)$  is given by  $g_{-1}(q, \bar{q})$  at the supporting point  $\bar{q} = 2$ . We want  $g_{-1}(q)$  to be quadratic so that it is of the form  $g_{-1}(q) = a_{-1}q^2 - 2b_{-1}q + c_{-1}$ . To find  $a_{-1}$ ,  $b_{-1}$ , and  $c_{-1}$ , we impose two supporting points, one at  $\bar{q}$  and the other at  $-2 - \bar{q}$ . These two supporting points are located symmetrically around  $-1$ . Note that the hinge function is linear at both supporting points, albeit with different gradients. Because  $g_{-1}(q)$  is quadratic, the additional requirement that  $f_{-1}(q) \leq g_{-1}(q)$  is satisfied if  $a_{-1} > 0$  and the derivatives at the two supporting points of  $f_{-1}(q)$  and  $g_{-1}(q)$  are the same. More formally, the requirements are that

$$\begin{aligned} f_{-1}(\bar{q}) &= g_{-1}(\bar{q}), \\ f'_{-1}(\bar{q}) &= g'_{-1}(\bar{q}), \\ f_{-1}(-2 - \bar{q}) &= g_{-1}(-2 - \bar{q}), \\ f'_{-1}(-2 - \bar{q}) &= g'_{-1}(-2 - \bar{q}), \\ f_{-1}(q) &\leq g_{-1}(q). \end{aligned}$$

It can be verified that the choice of

$$a_{-1} = \frac{1}{4}|\bar{q} + 1|^{-1}, \quad (22)$$

$$b_{-1} = -a_{-1} - \frac{1}{4}, \quad (23)$$

$$c_{-1} = a_{-1} + \frac{1}{2} + \frac{1}{4}|\bar{q} + 1|, \quad (24)$$

satisfies all these requirements. Figure 4a shows the majorizing function  $g_{-1}(q)$  with supporting points  $\bar{q} = 1.5$  as the dotted line.



For Class 1, a similar majorizing function can be found for  $f_1(q) = \max(0, 1-q)$ . However, in this case, we require equal function values and first derivative at  $\bar{q}$  and at  $2 - \bar{q}$ , that is, symmetric around 1. The requirements are

$$\begin{aligned} f_1(\bar{q}) &= g_1(\bar{q}), \\ f_1'( \bar{q} ) &= g_1'( \bar{q} ), \\ f_1(2 - \bar{q}) &= g_1(2 - \bar{q}), \\ f_1'(2 - \bar{q}) &= g_1'(2 - \bar{q}), \\ f_1(q) &\leq g_1(q). \end{aligned}$$

Choosing

$$a_1 = \frac{1}{4}|1 - \bar{q}|^{-1} \quad (25)$$

$$b_1 = a_1 + \frac{1}{4} \quad (26)$$

$$c_1 = a_1 + \frac{1}{2} + \frac{1}{4}|1 - \bar{q}| \quad (27)$$

satisfies these requirements. The functions  $f_1(q)$  and  $g_1(q)$  with supporting points  $\bar{q} = 2$  or  $\bar{q} = 0$  are plotted in Figure 4a.

Note that  $a_{-1}$  is not defined if  $\bar{q} = -1$ . In that case, we choose  $a_{-1}$  as a small positive constant  $\delta$  that is smaller than the convergence criterion  $\epsilon$  (introduced below). Strictly speaking, the majorization requirements are violated. However, by choosing  $\delta$  small enough, the monotone convergence of the sequence of  $L_{\text{SVM}}(\mathbf{w})$  will be no problem. The same holds for  $a_1$  if  $\bar{q} = 1$ .

## A.2 Majorizing the Quadratic Hinge Error

The majorizing algorithm for the SVM with the quadratic hinge function is developed along the same lines as for the absolute hinge function. However, because of its structure, each iteration boils down to a matrix multiplication of an fixed  $m \times n$  matrix with an  $n \times 1$  vector that changes over the iterations. Therefore, the computation of the update is of order  $O(nm)$  which is more efficient than the majorizing algorithm for the absolute hinge error.

To majorize the term  $f_{-1}(q) = \max(0, q+1)^2$  is relatively easy. For  $\bar{q} > -1$ ,  $f_{-1}(q)$  coincides with  $(q+1)^2$ . Therefore, if  $\bar{q} > -1$ ,  $(q+1)^2$  can be used to majorize  $\max(0, q+1)^2$ . Note that  $(q+1)^2 \geq 0$  so that  $(q+1)^2$  also satisfies the majorizing requirements for  $q < 1$ . For the case  $\bar{q} \leq -1$ , we want a majorizing function that has the same curvature as  $(q+1)^2$  but touches at  $\bar{q}$ , which is obtained by the majorizing function  $(q+1 - (\bar{q}+1))^2 = (q - \bar{q})^2$ . Therefore, the majorizing function  $g_{-1} = a_{-1}q^2 - 2b_{-1}q + c_{-1}$  has coefficients

$$a_{-1} = 1, \quad (28)$$

$$b_{-1} = \begin{cases} \bar{q} & \text{if } \bar{q} \leq -1 \\ -1 & \text{if } \bar{q} > -1 \end{cases}, \quad (29)$$

$$c_{-1} = \begin{cases} 1 - 2(\bar{q}+1) + (\bar{q}+1)^2 & \text{if } \bar{q} \leq -1 \\ 1 & \text{if } \bar{q} > -1 \end{cases}. \quad (30)$$

Similar reasoning can be held for  $f_1(q) = \max(0, 1 - q)^2$  which has majorizing function  $g_1 = a_1q^2 - 2b_1q + c_1$  and coefficients

$$a_1 = 1, \quad (31)$$

$$b_1 = \begin{cases} 1 & \text{if } \bar{q} \leq 1 \\ \bar{q} & \text{if } \bar{q} > 1 \end{cases}, \quad (32)$$

$$c_1 = \begin{cases} 1 & \text{if } \bar{q} \leq 1 \\ 1 - 2(1 - \bar{q}) + (1 - \bar{q})^2 & \text{if } \bar{q} > 1 \end{cases}. \quad (33)$$

Again,  $a_i, b_i,$  and  $c_i$  are defined as in (15), (16), and (17), except that  $\delta$  in (15) can be set to 0, so that  $a_i = 1 = a$  for all  $i$ .

### A.3 Majorizing the Huber Hinge Error

The majorizing algorithm of the Huber hinge error function shares a similar efficiency as for the quadratic hinge: the coefficients  $a_1$  and  $a_{-1}$  are the same for all  $i$ , so that again an update boils down to a matrix multiplication of a matrix of order  $m \times n$  with an  $n \times 1$  vector.

To majorize  $h_{-1}(q)$  we use the fact that the second derivative of  $h_{-1}(q)$  is bounded. For  $q \geq k$ ,  $h_{-1}(q)$  is linear with first derivative  $h'_{-1}(q) = 1$ , so that its second derivative  $h''_{-1}(q) = 0$ . For  $q \leq -1$ ,  $h_{-1}(q) = 0$ , so that here too  $h''_{-1}(q) = 0$ . Therefore,  $h''_{-1}(q) > 0$  only exists for  $-1 < q < k$ , where  $h''_{-1}(q) = 1$ . Therefore, for  $-1 < q < k$ , the quadratic majorizing function is equal to  $h_{-1}(q)$ , for  $q \leq -1$  and  $q \geq k$ , a quadratic majorizing function with the same second derivative of  $(1/2)(k + 1)^{-1}$  is produced that touches at the current estimate  $\bar{q}$ . Let the majorizing function  $g_{-1} = a_{-1}q^2 - 2b_{-1}q + c_{-1}$  has coefficients

$$a_{-1} = (1/2)(k + 1)^{-1}, \quad (34)$$

$$b_{-1} = \begin{cases} a_{-1}\bar{q} & \text{if } \bar{q} \leq -1 \\ -a_{-1} & \text{if } -1 < \bar{q} < k \\ a_{-1}\bar{q} - 1/2 & \text{if } \bar{q} \geq k \end{cases}, \quad (35)$$

$$c_{-1} = \begin{cases} a_{-1}\bar{q}^2 & \text{if } \bar{q} \leq -1 \\ a_{-1} & \text{if } -1 < \bar{q} < k \\ 1 - (k + 1)/2 + a_{-1}\bar{q}^2 & \text{if } \bar{q} \geq -k \end{cases}. \quad (36)$$

It may be verified for any  $\bar{q}$  from the three intervals that  $h_{-1}(\bar{q}) = g_{-1}(\bar{q})$  and  $h'_{-1}(\bar{q}) = g'_{-1}(\bar{q})$  hold. In addition,  $g''_{-1}(q) = (1/2)(k + 1)^{-1} \geq h''_{-1}(q)$  for all  $q$  (as long as  $k > -1$ ) so that the second derivative  $d''_{-1}(q)$  of the difference function  $d_{-1}(q) = g_{-1}(q) - h_{-1}(q)$  equals  $g''_{-1}(q) - h''_{-1}(q) \geq 0$  indicating that  $d_{-1}(q)$  is convex. As  $g_{-1}(q)$  touches  $h_{-1}(q)$  at  $\bar{q}$ ,  $d_{-1}(\bar{q}) = 0$ , so that, combined with convexity of  $d_{-1}(q)$  the inequality  $d_{-1}(q) \geq 0$  must hold implying the majorizing inequality  $h_{-1}(q) \leq g_{-1}(q)$  for all  $q$  with equality at  $\bar{q}$ .

For  $h_1(q)$  similar reasoning can be held. Let the majorizing function  $g_1 = a_{-1}q^2 - 2b_{-1}q + c_{-1}$  has coefficients

$$a_1 = (1/2)(k + 1)^{-1}, \quad (37)$$

$$b_1 = \begin{cases} 1/2 + a_1 \bar{q} & \text{if } \bar{q} \leq -k \\ a_1 & \text{if } -k < \bar{q} < 1 \\ a_1 \bar{q} & \text{if } \bar{q} \geq 1 \end{cases}, \quad (38)$$

$$c_1 = \begin{cases} 1 - (k+1)/2 + a_{-1} \bar{q}^2 & \text{if } \bar{q} \leq -k \\ a_1 & \text{if } -k < \bar{q} < 1 \\ a_1 \bar{q}^2 & \text{if } \bar{q} \geq 1 \end{cases}. \quad (39)$$

Note that  $a_{-1}$  and  $a_1$  are exactly the same and both independent of  $\bar{q}$ . Therefore, the curvature of the majorizing functions for all Huber hinge errors is the same. This property is exploited in the simple update derived from (22).

Table 4: Comparisons between SVM solvers: time to convergence in CPU sec. and objective values. The values of  $\lambda = 2^p$ 's are fixed at levels close to the optimal ones of Table 3.

Dataset	$p$	Time to converge, in CPU sec.										$L_{SVM}$					
		SVM					Lib-					SVM					
		$10^{-4}$	$10^{-5}$	$10^{-6}$	Light	Perf	$10^{-4}$	$10^{-5}$	$10^{-6}$	Light	Perf	$10^{-4}$	$10^{-5}$	$10^{-6}$	Light	Perf	
Australian	0	0.07	0.08	0.11	6395.27	0.30	76.80	—	—	—	202.67	202.69	202.66	207.32	220.81	202.20	—
Breast_cancer_w	6	0.03	0.06	0.10	0.06	0.18	0.09	0.89	0.89	0.89	58.21	58.05	58.02	58.03	205.03	58.03	205.02
Heart_statlog	0	0.02	0.02	0.02	0.08	0.14	0.10	1.43	1.43	1.43	91.50	91.49	91.49	91.48	91.52	91.48	91.52
Hepatitis	0	0.01	0.02	0.03	0.05	0.09	0.06	62.14	62.14	62.14	46.31	46.29	46.28	46.28	48.43	46.28	48.43
Sonar	0	0.02	0.04	0.05	0.13	0.30	0.15	31.08	31.08	31.08	114.54	114.51	114.51	114.51	116.83	114.51	116.83
Voting	-5	0.06	0.11	0.11	0.09	0.13	0.12	5.15	5.15	5.15	26.46	25.76	25.76	25.76	25.76	25.76	25.76
Liver-disorders	3	0.04	0.06	0.06	0.49	1.25	—	—	—	—	248.56	248.43	248.42	248.42	253.03	—	—
Liver-disorders2	-6	0.03	0.04	0.07	0.11	0.15	—	—	—	—	249.67	249.62	249.59	249.59	249.63	—	—
Diabetes	1	0.06	0.09	0.15	33.05	1.88	10.30	407.81	407.81	407.81	396.73	396.60	396.57	396.81	444.18	396.57	444.18
Diabetes2	-2	0.05	0.07	0.12	0.14	0.23	0.26	77.35	77.35	77.35	399.81	399.69	399.66	399.66	399.67	399.66	399.66
Ionosphere	-5	0.12	0.15	0.38	0.24	0.67	0.88	233.19	233.19	233.19	55.63	55.51	55.33	55.32	56.56	55.32	56.56
German.number	0	0.13	0.35	0.55	15.21	0.54	1.87	87.73	87.73	87.73	522.37	521.94	521.89	521.88	524.33	521.88	524.33
German.number2	3	0.14	0.22	0.38	0.27	0.39	0.23	82.49	82.49	82.49	539.51	539.44	539.40	539.39	539.63	539.39	539.63
Splice	5	0.55	1.01	1.55	0.47	0.87	0.33	11.50	11.50	11.50	427.37	427.21	427.19	427.18	433.68	427.18	433.68