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A new lower bound approach for single-machine multicriteria scheduling

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The concept of maximum potential improvement has played an important role in computing lower bounds for single-machine scheduling problems with composite objective functions that are linear in the job completion times. We introduce a new method for lower bound computation: objective splitting. We show that it dominates the maximum potential improvement method in terms of speed and quality.

single-machine scheduling; multicriteria scheduling; maximum potential improvement; objective splitting

1. Introduction

A single-machine job shop can be described as follows. A set of n independent jobs has to be scheduled on a single machine that is continuously available from time zero onwards and that can process no more than one job at a time. Each job $J_i (i = 1, \ldots, n)$ requires processing during a positive time p_i. In addition, it has a due data d_i, at which it should ideally be completed. A *schedule* defines for each job J_i its completion time C_i such that no two jobs overlap in their execution. A *performance measure* or *scheduling criterion* associates a value $f(\sigma)$ with each feasible schedule σ. Some well-known measures are the sum of the job completion times ΣC_i, the maximum job lateness $L_{\max} = \max_{1 \leqslant i \leqslant n}(C_i - d_i)$, and the maximum job earliness $E_{\max} = \max_{1 \leqslant i \leqslant n}(d_i - C_i)$.

In this paper, we adopt the terminology of Graham, Lawler, Lenstra and Rinnooy Kan (1979) to classify scheduling problems. Scheduling problems are classified according to a three-field notation $\alpha \mid \beta \mid \gamma$, where α specifies the machine environment, β the job characteristics, and γ the objective function. For instance, $1 \mid \text{nmit} \mid E_{\max}$ denotes the single-machine problem of minimizing maximum earliness, where nmit denotes that no machine idle time is allowed.

Most research has been concerned with a single criterion. In real life scheduling, however, it is necessary to take several performance measures into account. There are basically two approaches to cope with multiple criteria. If the scheduling criteria are subject to a welldefined hierarchy, they can be considered *sequentially* in order to relevance. An example is the problem of minimizing maximum lateness subject to the minimum number of tardy jobs, for which Shanthikumar (1983) presents a branch-and-bound algorithm.

The second approach is *simultaneous* optimization of several criteria. The K performance measures specified by the functions $f_k (k = 1, \ldots, K)$ are then transformed into one single *composite objective function* $F: \Omega \to \mathbb{R}$, where Ω denotes the set of all feasible schedules. We restrict ourselves to the case that F is a linear composition of the individual performance measures. This leads to the problem class (P) that contains all problems that can be formulated as

$$\min_{\sigma \in \Omega} \sum_{k=1}^{K} \alpha_k f_k(\sigma) \tag{P}$$

where $\alpha = (\alpha_1, \ldots, \alpha_K)$ is a given vector of real nonnegative weights. The problem of minimizing a linear function of the number of tardy jobs and maximum lateness, denoted as $1 \| \Sigma U_i + L_{\max}$, is a member of this class. Nelson, Sarin and Daniels (1986) present a branch-and-bound algorithm for its solution.

In addition to solving some problem in (P) for a given $\alpha \geqslant 0$, it may be of interest to determine the extreme set. The *extreme set* for given functions $f_1, \ldots, f_K$ is defined as the minimum cardinality set that contains an optimal schedule for *any* weight vector $\alpha \geqslant 0$. The elements of this set are the *extreme schedules*. If this set has been identified, then we can solve any problem for these functions by computing the function value for each extreme schedule and choosing the best. Hence, if the cardinality of the extreme set is polynomially bounded in n, the number of jobs, and if each extreme schedule can be found in polynomial time, then any problem in (P) with respect to these functions $f_1, \ldots, f_K$ can be solved in polynomial time.

Suppose that some problem in (P) is NP-hard and that one wishes to design a branch-and-bound method for its solution. In that case, good lower bounds are required. Unil now, virtuallly all lower bound computations for problems in (P) are based upon the so-called *maximum potential improvement* method. We prove in Section 2 that these bounds are dominated in terms of quality and computational effort by a much simpler method that we name *objective splitting*. In Section 3, we refine the basis objective splitting method.

The problem $1 \| \Sigma C_i + L_{\max} + E_{\max}$ is our benchmark in comparing the two lower bound approaches. It is still an open question whether this problem is NP-hard. Sen, Raiszadeh and Dileepan (1988) develop a branch-and-bound algorithm and derive lower bounds by means of the maximum potential improvement method. There is an optimal schedule for this problem without machine idle time, although $E_{\max}$ is nonincreasing in the job completion times. It is not meaningful to insert idle time, as the gain for $E_{\max}$ will at least be compensated by the increase of ΣC_i. We recall the following fundamental algorithms for the three embedded subproblems.

Theorem 1 (Smith, 1956). *The $1 \| \Sigma C_i$ problem is minimized by sequencing the jobs according to the shortest-processing-time (SPT) rule, that is, in order of nondecreasing p_i.*

Theorem 2 (Jackson, 1955). *The $1 \| L_{\max}$ problem is minimized by sequencing the jobs according to the earliest-due-date (EDD) rule, that is, in order of nondecreasing d_i.*

Theorem 3. *The $1 | \mathrm{nmit} | E_{\max}$ problem is solved by sequencing the jobs according to the minimum-slack-time (MST) rule, that is, in order of nondecreasing $d_i - p_i$.*

The proof of each of these algorithms proceeds by a straightforward interchange argument. Note that each of these problems is solved by arranging the jobs in a certain *priority order* that can be specified in terms of the parameters of the problem type.

The optimal solution values for these single-machine scheduling problems will be denoted by ΣC_i^*, $L_{\max}^*$ and $E_{\max}^*$, respectively. Furthermore, $\Sigma C_i(\sigma)$, $L_{\max}(\sigma)$, and $E_{\max}(\sigma)$ are the objective values for the schedule σ. In analogy, $C_i(\sigma)$, $L_i(\sigma)$, and $E_i(\sigma)$ denote the respective measures for job $J_i (i = 1, \ldots, n)$. Whenever (σ) is omitted, we are considering the performance measure in a generic sense, or there is no confusion possible as to the schedule we are referring to. The schedules that minimize ΣC_i, $L_{\max}$, and $E_{\max}$ are referred to as SPT, EDD, and MST respectively. In addition, $\nu(\cdot)$ denotes the optimal objective value for problem $\cdot$.

2. Maximum potential improvement versus objective splitting

Townsend (1978) proposed the maximum potential improvement method to compute lower bounds for minimizing a *quadratic* function of the job completion times. Since then, the method has been extended to problems in (P), including $1 | \Sigma C_i + L_{\max}$ (Sen and Gupta, 1983), $1 | \mathrm{nmit} |$

$L_{\max} + E_{\max}$ (Gupta and Sen, 1984), and $1 | | \Sigma C_i + L_{\max} + E_{\max}$ (Sen, Raiszadeh, and Dileepan, 1988). To our knowledge, there is only one publication on objective splitting *avant la lettre*: Tegze and Vlach (1988) obtained an extremely simple, but provably stronger lower bound for $1 | \mathrm{nmit} | L_{\max} + E_{\max}$.

Meanwhile, Hoogeveen (1990) and Hoogeveen and Van de Velde (1990) have found polynomial-time algorithms for $1 | \mathrm{nmit} | \alpha_1 L_{\max} + \alpha_2 E_{\max}$ and $1 | | \alpha_1 \Sigma C_i \mathrm{i} + \alpha_2 L_{\max}$. The former problem has $O(n)$ extreme schedules, each of which is found in $O(n \log n)$ time. The latter problem has $O(n^2)$ extreme schedules, each of which is determined in $O(n)$ time after appropriate preprocessing. However, it is an interesting issue how to derive lower bounds for NP-hard problems in (P). The maximum potential improvement method is a cumberstone procedure. However, by viewing it from a different angle, we derive a closed expression for the resulting lower bound. It is then immediately clear that the maximum potential improvement method is completely dominated by the much simpler objective splitting method.

Objective splitting is based upon the observation that

$$\min_{\sigma \in \Omega}\left[\sum_{k=1}^{K} \alpha_k f_k(\sigma)\right] \geqslant \sum_{k=1}^{K} \alpha_k\left[\min_{\sigma \in \Omega} f_k(\sigma)\right],$$

if $\alpha_k \geqslant 0$ for $k = 1, \ldots, K$. The application of this idea to $1 | | \Sigma C_i + L_{\max} + E_{\max}$ yields the problems $1 | | \Sigma C_i$, $1 | | L_{\max}$, and $1 | \mathrm{nmit} | E_{\max}$. Each problem is polynomially solvable, and we obtain the $LB^{\mathrm{OS}} = \Sigma C_i^* + L_{\max}^* + E_{\max}^*$. This bound is computed in $O(n)$ time in each node of the search tree, provided that the SPT, EDD, and MST sequences have been stored and that we employ a convenient branching strategy.

It is relatively easy to apply the *maximum potential improvement* method to problems in (P) for which each embedded single-machine problem has a priority order. The $1 | | \Sigma C_i + L_{\max} + E_{\max}$ problem has three: the SPT order for ΣC_i, the EDD order for $L_{\max}$, and the MST order for $E_{\max}$. Clearly, we have solved an instance of this problem in case these orders concur; in general though, the priority orders are conflicting.

Suppose we start with the MST schedule, which we refer to as the *primary* priority order. The scheduling cost induced by the MST schedule is $\Sigma C_i(\mathrm{MST}) + E_{\max}^* + L_{\max}(\mathrm{MST})$; this is obviously an upper bound on the optimal solution value. In addition, we know that any optimal schedule σ^* must have $E_{\max}(\sigma^*) \geqslant E_{\max}^*$, and $\Sigma C_i(\sigma^*) + L_{\max}(\sigma^*) \leqslant \Sigma C_i(\mathrm{MST}) + L_{\max}(\mathrm{MST})$. The maximum potential improvement method assesses the current schedule with respect to the maximum improvement that can be obtained for each of the performance measure *separately*. Accordingly, we get a lower bound by subtracting the total maximum potential improvement from the upper bound.

First, consider the maximum lateness criterion, which is the *secondary* priority order. If we interchange every pair of adjacent jobs J_i and J_j for which $d_i > d_j$ and $C_i < C_j$, then we need to conduct $O(n^2)$ interchanges before we have transformed the MST schedule into an EDD schedule. The actual effect on the objective value by one particular interchange depends on the interchanges that have been conducted thusfar. It might have no effect whatsoever on the performance of the schedule; this is true if both the maximum lateness and the maximum earliness remain unchanged. The *maximum possible decrease* of the scheduling cost, however, is $d_i - d_j$; if σ and π denote the schedule before and after the interchange, respectively, then the maximum decrease is realized if $L_{\max}(\sigma) = L_j(\sigma)$, $L_{\max}(\pi) = L_i(\pi)$ and $E_{\max}(\pi) = E_{\max}(\sigma)$. The effect that the interchange might have on the sum of the job completion times is not considered here and dealt with separately. Any interchange conducted to transform the MST schedule into the EDD schedule may improve the maximum lateness by the corresponding maximum possible decrease. The sum of these is the *maximum potential improvement* with respect to the initial lateness $L_{\max}(\mathrm{MST})$. It is given by

$$\mathrm{MPI}_2 = \sum_{i,j: d_i > d_j, C_i < C_j} (d_i - d_j).$$

Note that the maximum potential improvement does not depend on the order in which the interchanges are conducted.

Second, the sum of the job completion times, which is the *tertiary* priority order, is reduced by interchanging two adjacent jobs J_i and J_j with

$p_i > p_j$ and $C_i < C_j$. The maximum potential improvement is then $p_i - p_j$, which is also the *true* improvement. The maximum potential improvement with respect to $\Sigma C_j(\text{MST})$ is then

$$\text{MPI}_3 = \sum_{i,j:\, p_i > p_j, C_i < C_j} (p_i - p_j).$$

The lower bound LB^{MPI} suggested by Sen, Raiszadeh and Dileepan (1988) for $1 | | \Sigma C_i + L_{\max} + E_{\max}$ is then

$$\text{LB}^{\text{MPI}} = E^*_{\max} + L_{\max}(\text{MST}) - \text{MPI}_2 + \Sigma C_i(\text{MST}) - \text{MPI}_3.$$

Since $\Sigma C_i(\text{MST}) - \text{MPI}_3 = \Sigma C_i(\text{SPT}) = \Sigma C_i^*$ and $L_{\max}(\text{MST}) - \text{MPI}_3 \leqslant L^*_{\max}$, as we have systematically overestimated the reduction in maximum lateness, we conclude that

$$\text{LB}^{\text{MPI}} = E^*_{\max} + \Sigma C_i^* + L_{\max}(\text{MST}) - \text{MPI}_2 \leqslant \text{LB}^{\text{OS}}.$$

The maximum potential improvement method can be generalized to problems in (P) as follows. Let σ_k^* denote an optimal schedule for the k-th individual objective. Furthermore, let the optimal sequence that goes with the k-th objective be the k-th preference order. The first step is then to sequence the jobs according to the primary preference order, which gives the upper bound $\alpha_1 f_1(\sigma_1^*) + \Sigma_{k=2}^K \alpha_k f_k(\sigma_1^*)$. We then have to transform the primary preference order into the k-th preference order for $k = 2, \ldots, K$, and determine the corresponding maximum potential improvement MPI_k. The lower bound is then given by

$$\text{LB}^{\text{MPI}} = \alpha_1 f_1(\sigma_1^*) + \sum_{k=2}^{K} \alpha_k (f_k(\sigma_1^*) - \text{MPI}_k).$$

Note that this procedure requires $O(n^2)$ time for fixed K in addition to the time required to determine σ_k^*, for $k = 1, \ldots, K$. Since $f_k(\sigma_1^*) - \text{MPI}_k \leqslant f_k(\sigma_k^*)$ for each $k = 1, \ldots, K$, we have the following theorem.

Theorem 4. *For any problem in* (P), *the lower bound obtained by the maximum potential improvement method is dominated in terms of both quality and speed by the lower bound obtained by the objective splitting method.*

Consider the following example shown in Table 1 that is taken from Sen, Raiszadeh and Dileepan (1988) for the problem $1 | | q\Sigma C_i + (1-q)(L_{\max} + E_{\max})$ with $0 \leqslant q \leqslant 1$. By means of the maximum potential improvement method, we obtain the lower bound $\text{LB}^{\text{MPI}} = 64q + 9$. It is easy to verify that $\Sigma C_j^* = 73$, $L^*_{\max} = 14$, and $E^*_{\max} = 6$. This gives the bound $\text{LB}^{\text{OS}} = 53q + 20$. Note that $53q + 20 \geqslant 64q + 9$ for all q with $0 \leqslant q \leqslant 1$.

Table 1

	J_i			
	J_1	J_2	J_2	J_4
p_i	14	7	6	7
d_i	20	14	15	17
$d_i - p_i$	6	7	9	10

3. Improving the objective splitting procedure

The objective splitting procedure above was given in its simplest form: we separated the composite objective function into K single-criterion scheduling problems. We now propose a refinement that gives us a lower bound that is at least as good, but requires more time. Our more general approach allows combinations of objective functions. Let $(T_1, \ldots, T_H)$ be a partition of the set $\{1, \ldots, K\}$, i.e., the sets T_h are mutually disjoint and $\cup_{h=1}^H T_h = \{1, \ldots, K\}$. For any problem A in the class (P) we clearly have

$$\nu(A) \geqslant \sum_{h=1}^{H} \left[\min_{\sigma \in \Omega} \sum_{k \in T_h} \alpha_k f_k(\sigma_k) \right] \geqslant \sum_{k=1}^{K} \alpha_k [f_k(\sigma_k^*)] = \text{LB}^{\text{OS}}.$$

This idea can be refined even further, since it is not obligatory to match each performance criterion f_k with only one set T_h. Hence, let us relax the assumption that $(T_1, \ldots, T_H)$ is a partition of $\{1, \ldots, K\}$, and let α_{kh} denote the *fraction* of f_k that is assigned to T_h. We must have that $\Sigma_h \alpha_{kh} = \alpha_k$ for each $k = 1, \ldots, K$, and also that $\alpha_{kh} \geqslant 0$, since the composite objective function associated with the set S_h has to be nondecreasing in each of its arguments, for $h = 1, \ldots,$. We can compute the lower bound for *given* values of α_{kh} as

$$(\text{OS}) \quad \nu(\text{OS}) = \sum_{h=1}^{H} \left[\min_{\sigma \in \Omega} \sum_{k \in T_h} \alpha_{kh} f_k(\sigma) \right].$$

An interesting question is how to determine the values of α_{kh} that maximize the lower bound ν(OS). This problem, referred to as problem (D), is to

(D) maximize $\nu(\mathrm{OS})$

subject to

$$\sum_{h=1}^{H} \alpha_{kh} = \alpha_k \quad \text{for } k = 1, \ldots, K,$$

$$\alpha_{kh} \geqslant 0 \quad \text{for } k = 1, \ldots, K,\ h = 1, \ldots, H.$$

A sufficient condition for solving problem (D) in polynomial time (for fixed K) is that the extreme set for each problem induced by T_h $(h = 1, \ldots, H)$ can be determined in polynomial time. In that case, there is only a polynomial number of extreme schedules involved, and problem (D) can then be formulated as a linear programming problem with a polynomial number of constraints and variables. Let $N(h)$ be the number of extreme schedules for the problem associated with T_h $(h = 1, \ldots, H)$, and let $\sigma_{j(h)}$ denote the j-th extreme schedule for the problem associated with T_h. There are at most $2^K - 2$ sets T_h $(|T_h| < K$ and $T_h \neq \emptyset)$. The linear program is then to

maximize w

subject to

$$w \leqslant \sum_{h=1}^{H} \sum_{k \in T_h} \alpha_{kh} f_k(\sigma_{j(j)})$$

$$\text{for } j(h) = 1, \ldots, N(h),\ h = 1, \ldots, H,$$

$$\sum_{h=1}^{H} \alpha_{kh} = \alpha_k \quad \text{for } k = 1, \ldots, K,$$

$$\alpha_{kh} \geqslant 0 \quad \text{for } k = 1, \ldots, K,\ h = 1, \ldots, H.$$

In general, it would be unreasonable to presume that each of the possible $2^K - 2$ sets T_h would result into a polynomially solvable problem; it may be a formidable challenge to identify those that will. If we touch upon a problem that appears to be hard to solve, then we may relax the assumptions by allowing preemption (I.e., the processing of the jobs may be interrupted and resumed to the computational complexity, but also with respect to the lower bound quality. The latter follows particularly from the following theorem.

Theorem 6. *The optimal objective value of* $1|\mathrm{pmtn}|\Sigma_{k=1}^{K}\alpha_k f_k$ *is greater than or equal to* $\Sigma\alpha_k f_k(\sigma_k^*)$, *where* σ_k^* *is the optimal value for* $1||f_k (k = 1, \ldots, K)$.

Proof. The proof follows from the observation that σ_k^* also solves $1|\mathrm{pmtn}|f_k$, if f_k is either monotonically nondecreasing or monotonically nonincreasing in the job completion times. □

If we apply the refined objective splitting procedure to $1||\Sigma C_i + L_{\max} + E_{\max}$, then, except for the obvious single-criterion problems, we have to consider three problems: $1||\alpha_1\Sigma C_i + \alpha_2 L_{\max}$, $1|\mathrm{nmit}|\alpha_1\Sigma C_i + \alpha_2 E_{\max}$, and $1|\mathrm{nmit}|\alpha_1 L_{\max} + \alpha_2 E_{\max}$. Hoogeveen (1990) presents and $O(n^2 \log n)$ time algorithm for $1|\mathrm{nmit}|\alpha_1 L_{\max} + \alpha_2 E_{\max}$ to find the $O(n)$ extreme schedules, and Hoogeveen and Van de Velde (1990) present and $O(n^3)$ time algorithm for $1||\alpha_1\Sigma C_i + \alpha_2 L_{\max}$, which has $O(n^2)$ extreme schedules. For the problem (Hoogeveen and Van de Velde, 1990). The complexity of the case $\alpha_1 < \alpha_2$ is unknown. However, $1|\mathrm{nmit, pmtn}|\alpha_1\Sigma C_i + \alpha_2 E_{\max}$ is solvable in $O(n^4)$ time and has $O(n^2)$ extreme schedules.

If we reconsider the example, we find that there is one extreme schedule for ΣC_i and $L_{\max}$ with $\Sigma C_i = 73$ and $L_{\max} = 14$; there are two extreme schedules for $L_{\max}$ and $E_{\max}$ with values $L_{\max} = 14$ and $E_{\max} = 7$, and $L_{\max} = 17$ and $E_{\max} = 6$; there are three extreme schedules for $E_{\max}$ and ΣC_i if we allow preemption with values $E_{\max} = 6$ and $\Sigma C_i = 96$, $E_{\max} = 7$ and $\Sigma C_i = 74$, and $E_{\max} = 9$ and $\Sigma C_i = 73$, respectively.

The lower bound that is obtained by the improved objective splitting method depends on the parameter q. Suppose $q = \frac{1}{2}$. Then we obtain $\mathrm{LB}^{\mathrm{MPI}} = 41$ and $\mathrm{LB}^{\mathrm{OS}} = 46\frac{1}{2}$. It is easy to verify that the improved objective splitting method gives $47\frac{1}{2}$ as a lower bound. This bound is tight, since the optimal sequence (J_2, J_3, J_4, J_1) has the same value.

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References

R.L. Graham, E.L. Lawler, J.K. Lenstra and A.H.G. Rinnooy Kan (1979), "Optimization and approximation in deterministic sequencing and scheduling: A survey", *Ann. Discrete Math.* **5**, 287–326.

S.K. Gupta and T. Sen (1984), "Minimizing the range of lateness on a single machine", *J. Oper. Res. Soc.* **35**, 853–857.

J.A. Hoogeveen (1990), "Minimizing maximum earliness and maximum lateness on a single machine," Report BS-R9001, Centre for Mathematics and Computer Science, Amsterdam.

J.A. Hoogeveen and S.L. van de Velde (1990), "Polynomial-time algorithms for multicriteria scheduling problems", Report BS-R9008, Centre for Mathematics and Computer Science, Amsterdam.

J.R. Jackson (1955), "Scheduling a production line to minimize maximum tardiness", Research Report 43, Management Sciences Research Project, UCLA, Los Angeles, CA.

R.T. Nelson, R.K. Sarin and R.L. Daniels (1986), "Scheduling with multiple performance measures: The one-machine case", *Management Sci.* **32**, 464–479.

T. Sen and S.K. Gupta (1983), A branch-and-bound procedure to solve a bricriterion scheduling problem", *IIE Trans.* **15**, 84–88.

T. Sen, F.M.E. Raiszadeh and P. Dileepan (1988), "A branch-and-bound approach to the bicriterion scheduling problem involving total flowtime and range of lateness", *Management Sci.* **34**, 254–260.

J.G. Shanthikumar (1983), "Scheduling *n* jobs on one machine to minimize the maximum tardiness with minimum number tardy", *Comput. Oper. Res.* **10**, 255–266.

W.E. Smith (1956), "Various optimizers for single-stage production. *Naval Res. Logist. Quart.* **1**, 59–66.

M. Tegze and M. Vlach (1988), "Improved bounds for the range of lateness on a single machine", *J. Oper. Res. Soc.* **39**, 675–680.

W. Townsend (1978), "The single machine problem with quadratic penalty function of completion times: a branch-and-bound solution", *Management Sci.* **24**, 530–534.