Convolutions of heavy tailed random variables and applications to portfolio diversification and MA(1) time series

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Abstract

Suppose X_1, X_2 are independent random variables satisfying a second order regular variation condition on the tail-sum and a balance condition on the tails. In this paper we give a description of the asymptotic behavior as $t \to \infty$ for $P(X_1 + X_2 > t)$.

The result is applied to the problem of risk diversification in portfolio analysis and to the estimation of the parameter in a MA(1) model.

1 Introduction

Assume X_1, X_2 are independent random variables whose distribution function tail sums

$$\overline{F}_i(t) := P(|X_i| \geq t),\ i = 1, 2$$

are regularly varying with index $-\alpha < 0$, i.e.,

$$\lim_{t\to\infty} \frac{\overline{F}_i(tx)}{\overline{F}_i(t)} = x^{-\alpha} \text{ for all } x > 0. \tag{1.1}$$

For a general overview of regular variation theory the reader is referred to Bingham et al.(1987). The asymptotic behavior of the tail sum of the convolution of X_1 and X_2 is studied in Feller (1971). More recently Datta and McCormick (1998) gave the behavior of the tail sum for linear processes $\sum_{i=1}^{\infty} c_i X_{t-i}$ under suitable conditions on the coefficients c_i.

In this paper we investigate the behavior of convolutions in case a refinement of (1.1), called second order regular variation, on the tails of the distribution functions of the random variables holds. Specifically, we assume that the tail sums satisfy

$$\lim_{t\to\infty} \frac{\frac{\overline{F}_i(tx)}{\overline{F}_i(t)} - x^{-\alpha}}{a_i(t)} = x^{-\alpha}\frac{x^{\rho}-1}{\rho} \tag{1.2}$$

for $x > 0, i = 1, 2$, where a_i is a function satisfying $a_i(t) \to 0$ $(t \to \infty)$. It follows that the functions $|a_i|$ are regularly varying with index $\rho \leq 0$. In case $\rho = 0$ read $\frac{x^{\rho}-1}{\rho} = \log x$.

For positive random variables satisfying (1.2) the tail behavior under convolution was studied in Geluk, de Haan, Resnick and Starica (1997) and Geluk and Peng (1999). In this paper we do not make the assumption of positive random variables and provide more precise estimates for a number of cases. We replace the positivity condition with the following tail balance condition

$$\lim_{t\to\infty} \frac{\frac{1-F_i(t)}{\overline{F}_i(t)} - p_i}{a_i(t)} = r_i \in (-\infty, \infty), \tag{1.3}$$

where $p_i \in [0, 1]$, $i = 1, 2$.

Under the assumptions (1.2) and (1.3) we give an asymptotic expansion for $P(X_1 + X_2 > t)$. It turns out that the number of terms in the expansion depends on the value of the parameter α whereas the type of the terms in the expansion depends on α, p_1, p_2, ρ and the convergence of the $\alpha-$th moment.

The first application of the main result is on portfolio management. The performance of a portfolio is measured in terms of the returns on investment, i.e. the percentage gain or loss on initial capital. The structure of the portfolio influences the portfolio performance. Some assets have low expected returns and others have high expected returns, but at the cost of higher risk. Perhaps the most important rule of thumb in finance prescribes how one should not structure one's portfolio: 'Don't put all your eggs in one basket'. The idea behind this rule of thumb is that through diversification one can reduce the portfolio risk, measured as the variance of the portfolio return, by virtue of the law of large numbers, since returns on individual assets are imperfectly correlated with each other. The effects of diversification on the mean and variance of a portfolio is well understood and can be found in all elementary textbooks on finance. In this section we elaborate on the virtues of diversification regarding tail risk, about which much less is known.

The tail risk is the probability that there is a very 'large' loss on a portfolio. It depends on the economic context what constitutes a 'large' loss. For a pension fund this constitutes a loss so large that it is unable to pay out the pensions, and for a commercial bank a loss is large if it is unable to meet the cash demand by deposit holders, which may trigger a bank run. It suffices for our purposes to identify the meaning of large by a typical quantile in the left tail of the return distribution. Financial institutions measure the downside risk or tail risk of their proprietary trading portfolio on a daily basis, both for the purpose of internal risk management, and because this is a regulatory requirement (external risk management imposed by public agencies to ensure prudence in the financial sector). This downside risk measurement operation is now commonly known as the Value-at-Risk (VaR) exercise, see e.g. Jorion (1997), Dowd (1998), Danielsson and De Vries (1997, 1998) and Longin (1997). Hence it is important to study how the downside risk or VaR is affected by diversification. In section 3 we first give a brief review of first order tail effects and then provide a number of new results on second order refinements based on the result in section 2.

Our second application is in time series analysis. Suppose we have observations $Y_1, \cdots, Y_n$ from the MA(1) model, i.e.,

$$Y_i = \epsilon_i - \theta\epsilon_{i-1}, \tag{1.4}$$

where $\{\epsilon_i\}$ is a sequence of independent and identically distributed random variables with mean zero and finite variance. In case $|\theta| < 1$, the maximum likelihood estimator $\hat{\theta}_{MLE}$ for θ has the following asymptotic limit:

$$\sqrt{n}(\hat{\theta}_{MLE} - \theta) \xrightarrow{d} N(0, 1 - \theta^2) \tag{1.5}$$

(see Brockwell and Davis (1991)). However, in case $|\theta| = 1$, the standard asymptotic normal distribution theory does not apply (see Brockwell and Davis (1991)). On the other hand, the normal limit in (1.5) provides a particularly inaccurate approximation for values of $|\theta|$ close to one. For the applications in which inference about $|\theta|$ at or close to one, we refer to Davis, Chen and Dunsmuir (1995) and Davis and Dunsmuir (1996). Moreover, Davis and Dunsmuir (1996) proposed the local maximum estimator $\hat{\theta}_{LM}$, defined as the largest of the local maximizers of the likelihood, and derived the asymptotic limit of $\hat{\theta}_{LM}$ too.

Recently, Davis and Mikosch (1998) obtained the limit behaviour of the local maximizer closest to 1 of the Gaussian likelihood and the corresponding likelihood ratio statistic, used in Davis and Dunsmuir (1996), when $\{\epsilon_i\}$ is an i.i.d. sequence with symmetric stable law with index $\alpha \in (0, 2)$. However the limit in Davis and Mikosch (1998) is complicated. For the estimation of $\theta \in (-1, 1)$ we refer to Lii and Rosenblatt (1982,1992).

In this paper we give a semi-parametric estimator for $\theta \in [-1, 1]$ under the assumption that ϵ_i satisfies

$$\begin{cases} P(\epsilon_i > t) = cpt^{-\alpha}(1 + o(1)) \\ P(\epsilon_i < -t) = c(1 - p)t^{-\alpha}(1 + o(1)), \end{cases} \tag{1.6}$$

as $t \to \infty$, where $\alpha > 0$, $c > 0$ and $p \in [0, 1]$. Thus, ϵ_i is in the domain of attraction of a stable law with index α in case $\alpha < 2$, and ϵ_i is in the domain of attraction of a normal distribution if $\alpha \geq 2$.

First we give the intuitive derivation of our new estimator. For simplicity we assume to have $n + 1$ observations and define $Z_i = |Y_{i+1} + Y_i|$ and $W_i = |Y_{i+1} - Y_i|$ $(i = 1, \cdots, n)$. Let $Z_{n,1} \leq \cdots \leq Z_{n,n}$ and $W_{n,1} \leq \cdots \leq W_{n,n}$ denote the order statistics of $Z_1, \cdots, Z_n$ and $W_1, \cdots, W_n$, respectively.

It follows from (1.1), (1.3) and Feller (1971) that as $t \to \infty$

$$\begin{aligned} & P(Z_i > t) \\ = \;& P(Y_{i+1} + Y_i > t) + P(Y_{i+1} + Y_i < -t) \\ = \;& \{P(\epsilon_{i+1} > t) + P((1 - \theta)\epsilon_i > t) + P(-\theta\epsilon_{i-1} > t)\}(1 + o(1)) \\ & + \{P(\epsilon_{i+1} < -t) + P((1 - \theta)\epsilon_i < -t) + P(-\theta\epsilon_{i-1} < -t)\}(1 + o(1)) \\ = \;& c\{1 + (1 - \theta)^\alpha + |\theta|^\alpha\}t^{-\alpha}(1 + o(1)). \end{aligned}$$

Similarly,

$$P(W_i > t) = c\{1 + (1+\theta)^\alpha + |\theta|^\alpha\} t^{-\alpha}(1+o(1)).$$

Hence we can estimate $c\{1 + (1-\theta)^\alpha + |\theta|^\alpha\}$ and $c\{1 + (1+\theta)^\alpha + |\theta|^\alpha\}$ by $\frac{k}{n} Z_{n,n-k}^{\hat\alpha_Z(k)}$ and $\frac{m}{n} W_{n,n-m}^{\hat\alpha_W(m)}$, respectively, where $k = k(n) \to \infty$, $k/n \to 0$, $m = m(n) \to \infty$, $m/n \to 0$ and

$$\begin{cases} \hat\alpha_Z(k) := \{\frac{1}{k}\sum_{i=1}^k \log Z_{n,n-i+1} - \log Z_{n,n-k}\}^{-1} \\ \hat\alpha_W(m) := \{\frac{1}{m}\sum_{i=1}^m \log W_{n,n-i+1} - \log W_{n,n-m}\}^{-1}. \end{cases}$$

Note that $\hat\alpha_Z(k)$ and $\hat\alpha_W(m)$ are Hill estimators of the tail index α (see Hill (1975)). Since $\hat\alpha_Z(k)$ and $\hat\alpha_W(m)$ are consistent estimators of α (see e.g. Mason (1982)), it follows that $\frac{k}{m} Z_{n,n-k}^{\hat\alpha_Z(k)} W_{n,n-m}^{-\hat\alpha_W(m)}$ is a consistent estimator of

$$f(\theta) := \frac{1 + |\theta|^\alpha + (1-\theta)^\alpha}{1 + |\theta|^\alpha + (1+\theta)^\alpha}.$$

Define

$$f_n(\theta) := \frac{1 + |\theta|^{\hat\alpha_W(m)} + (1-\theta)^{\hat\alpha_W(m)}}{1 + |\theta|^{\hat\alpha_W(m)} + (1+\theta)^{\hat\alpha_W(m)}}.$$

It is easy to check that $f_n(\theta)$ is a decreasing function of θ on the interval $[-1,1]$. Let $f_n^-(\theta)$ denote the inverse function of $f_n(\theta)$. It follows that

$$\hat\theta_n := f_n^-(\frac{k}{m} Z_{n,n-k}^{\hat\alpha_Z(k)} W_{n,n-m}^{-\hat\alpha_W(m)}) \tag{1.7}$$

is a consistent estimator of θ.

In order to prove asymptotic normality of $\hat\theta_n$, we need second order regular variation conditions for both $P(Z_i > t)$ and $P(W_i > t)$. In section 4 these conditions are obtained from assumptions on the innovations using theorem 2.1.

2 Second order behaviour of convolutions

Theorem 2.1. *Let X_i, $i = 1, 2$ be independent random variables with distribution functions F_i satisfying (1.2) and (1.3). Define for $j \ge 0$ integer and $\alpha > 0$,*

$$c_{\alpha,j} = \frac{\Gamma(\alpha + j)}{j!\Gamma(\alpha)}. \tag{2.1}$$

Consider the following cases

A. If $0<\alpha<1$ *then as* $t\to\infty$

$$P(X_1+X_2>t)=\sum_{i=1}^{2}(r_i+o(1))\overline{F}_i(t)a_i(t)$$
$$+\sum_{i=1}^{2}p_i\overline{F}_i(t)+(d+o(1))\overline{F}_1(t)\overline{F}_2(t), \tag{2.2}$$

where

$$d=p_1p_2\Big\{-\frac{\Gamma(1-\alpha)^2}{\Gamma(1-2\alpha)}+\frac{2\Gamma(1-\alpha)\Gamma(2\alpha)}{\Gamma(\alpha)}\Big\}-(p_1+p_2)\frac{\Gamma(1-\alpha)\Gamma(2\alpha)}{\Gamma(\alpha)}.$$

B. If $\alpha\geq 1$ *and* $E|X_i|^\alpha<\infty$ $(i=1,2)$, *then as* $t\to\infty$

$$P(X_1+X_2>t)=\sum_{i=1}^{2}(r_i+o(1))\overline{F}_i(t)a_i(t)$$
$$+\sum_{i=1}^{2}p_i\overline{F}_i(t)\Big\{\sum_{j=0}^{[\alpha]-1}c_{\alpha,j}\frac{EX_{3-i}^j}{t^j}+(c_{\alpha,[\alpha]}+o(1))\frac{EX_{3-i}^{[\alpha]}}{t^{[\alpha]}}\Big\},$$

where $[\alpha]$ *is the greatest integer less than or equal* α.

C. If $E|X_i|^\alpha=\infty$ $(i=1,2)$ *and one of the following holds:*

1. $\alpha\geq 1$ *is even*
2. $\alpha\geq 1$ *is odd and* $p_i\neq\frac{1}{2}$ $(i=1,2)$
3. $\alpha\geq 1$ *is odd,* $p_i=\frac{1}{2},\rho=0$ *and* $r_i\neq 0$ $(i=1,2)$
4. $\alpha\geq 1$ *is odd,* $\rho<0$ *and* $r_i\neq 0$ $(i=1,2)$,

then as $t\to\infty$

$$\begin{aligned}&P\left(X_1+X_2>t\right)=\\ =\ &\sum_{i=1}^{2}(r_i+o(1))\overline{F}_i(t)a_i(t)+\sum_{i=1}^{2}p_i\overline{F}_i(t)\Big\{\sum_{j=0}^{\alpha-1}c_{\alpha,j}\frac{EX_{3-i}^j}{t^j}+\\ +\ &(\alpha c_{\alpha,\alpha}+o(1))\frac{1}{t^\alpha}\int_0^t(1-F_{3-i}(y)+(-1)^\alpha F_{3-i}(-y))y^{\alpha-1}dy\Big\}.\end{aligned}$$

D. If $E|X_i|^\alpha = \infty$ $(i = 1, 2)$ and $\alpha > 1$ is non-integer, then as $t \to \infty$

$$P(X_1 + X_2 > t) = \sum_{i=1}^{2}(r_i + o(1))\overline{F}_i(t)a_i(t) + $$
$$+ \sum_{i=1}^{2} p_i\overline{F}_i(t) \sum_{j=0}^{[\alpha]-1} c_{\alpha,j}\frac{EX_{3-i}^j}{t^j} + (h_\alpha + o(1))\overline{F}_1(t)\overline{F}_2(t),$$

where $[\alpha]$ is the greatest integer less than or equal α and h_α is a constant.

The following result (see de Haan and Pereira (1999)) is needed for the proof of the theorem.

Lemma 2.1. *Let f be a measurable function and for some function $a_1(t) > 0$ we have*

$$\lim_{t\to\infty}\frac{f(tx) - f(t)}{a_1(t)} = \frac{x^\gamma - 1}{\gamma}$$

for all $x > 0$ where γ is a real parameter. Then there exists a positive function a with $a(t) \sim a_1(t)$ $(t \to \infty)$ with the property that for every $\varepsilon, \varepsilon' > 0$ there exists a $t_0 > 0$ such that for $t \ge t_0, tx \ge t_0$

$$x^{-\gamma}e^{-\varepsilon'|\log x|}\left|\frac{f(tx) - f(t)}{a(t)} - \frac{x^\gamma - 1}{\gamma}\right| < \varepsilon.$$

Proof of Theorem 2.1. Note that

$$P(X_1 + X_2 > t) - P(X_1 > \frac{t}{2})P(X_2 > \frac{t}{2}) =$$
$$= \sum_{i=1}^{2}\int_{-\infty}^{t/2} P(X_i > t - y)dF_{3-i}(y)$$
$$= \sum_{i=1}^{2}\overline{F}_i(t)a_i(t)\int_{-\infty}^{t/2}\frac{\frac{P(X_i > t-y)}{\overline{F}_i(t)} - p_i(1-\frac{y}{t})^{-\alpha}}{a_i(t)}dF_{3-i}(y)$$
$$+\sum_{i=1}^{2} p_i\overline{F}_i(t)\int_{-\infty}^{t/2}(1-\frac{y}{t})^{-\alpha}dF_{3-i}(y). \quad (2.3)$$

Denote the integrals on the right hand side with I_i and J_i respectively.

Substituting the balance condition $P(X_i > t-y) = p_i\overline{F}_i(t-y) + (r_i + o(1))\overline{F}_i(t-y)a_i(t-y)$ as $t \to \infty$ (valid uniformly for $y \in (-\infty, t/2)$) we have

$$
\begin{aligned}
I_i &= p_i \int_{-\infty}^{t/2} \frac{\frac{\overline{F}_i(t-y)}{\overline{F}_i(t)} - (1-\frac{y}{t})^{-\alpha}}{a_i(t)} dF_{3-i}(y) + \\
&+ (r_i + o(1)) \int_{-\infty}^{t/2} \frac{\overline{F}_i(t-y)}{\overline{F}_i(t)} \frac{a_i(t-y)}{a_i(t)} dF_{3-i}(y) =: I_{i1} + I_{i2}. \quad (2.4)
\end{aligned}
$$

Now $I_{i1} \to 0$ as $t \to \infty$ follows using dominated convergence. The dominating function is provided with the above Lemma. Similarly Potter's inequality (see e.g. Bingham et al.(1987)) gives the dominating function which is necessary to apply dominated convergence in I_{i2}. This gives $I_{i2} \to r_i$ as $t \to \infty$.

The method of estimation of the integral J_i depends on the value of α.

A. Suppose $\alpha < 1$. Integration by parts gives

$$
\begin{aligned}
&\int_0^{t/2} \Big\{(1-\frac{y}{t})^{-\alpha} - 1\Big\} dF_i(y) = \\
&\quad = -(2^\alpha - 1)P(X_i > \frac{t}{2}) + \alpha \int_0^{1/2} P(X_i > ty)(1-y)^{-\alpha-1} dy.
\end{aligned}
$$

Using $P(X_i > \frac{t}{2}) \sim p_i \overline{F}_i(\frac{t}{2}) \sim p_i 2^\alpha \overline{F}_i(t)$ and regular variation of $\overline{F}_i$ we find

$$
\begin{aligned}
&\int_0^{t/2} \Big\{(1-\frac{y}{t})^{-\alpha} - 1\Big\} dF_i(y) \\
&\sim \quad p_i \Big\{ -(2^\alpha - 1)2^\alpha + \alpha \int_0^{\frac{1}{2}} y^{-\alpha}(1-y)^{-\alpha-1} dy \Big\} \overline{F}_i(t).
\end{aligned}
$$

Another integration by parts gives

$$
\begin{aligned}
&\int_{-\infty}^0 \Big\{(1-\frac{y}{t})^{-\alpha} - 1\Big\} dF_i(y) \\
&= \quad -\alpha \int_0^\infty F_i(-ty)(1+y)^{-\alpha-1} dy,
\end{aligned}
$$

hence

$$\begin{aligned}
&\textstyle\int_{-\infty}^{0} \Big\{(1-\frac{y}{t})^{-\alpha}-1\Big\}dF_i(y)\\
&\sim \quad -\alpha(1-p_i)\overline{F}_i(t)\int_0^\infty y^{-\alpha}(1+y)^{-\alpha-1}dy,
\end{aligned}$$

Combination of the estimates now gives

$$\begin{aligned}
J_i &= F_{3-i}(\frac{t}{2})+\Big\{-(2^\alpha-1)2^\alpha p_{3-i}+\alpha p_{3-i}\int_0^{\frac{1}{2}} y^{-\alpha}(1-y)^{-\alpha-1}dy\\
&\quad -\alpha(1-p_{3-i})\int_0^\infty y^{-\alpha}(1+y)^{-\alpha-1}dy+o(1)\Big\}\overline{F}_{3-i}(t)\\
&= 1+\Big\{-p_{3-i}2^{2\alpha}+\alpha p_{3-i}\int_0^{\frac{1}{2}} y^{-\alpha}(1-y)^{-\alpha-1}dy\\
&\quad -\alpha(1-p_{3-i})\int_0^\infty y^{-\alpha}(1+y)^{-\alpha-1}dy+o(1)\Big\}\overline{F}_{3-i}(t). \qquad (2.5)
\end{aligned}$$

Note that

$$\prod_{i=1}^{2} P(X_i>\frac{t}{2})\sim\prod_{i=1}^{2} p_i 2^\alpha\overline{F}_i(t). \qquad (2.6)$$

Substitution of (2.4), (2.5) and (2.6) in (2.3) gives (2.2), where

$$\begin{aligned}
d &= p_1p_2\Big\{-2^{2\alpha}+2\alpha\int_0^{\frac{1}{2}} y^{-\alpha}(1-y)^{-\alpha-1}dy+2\alpha\int_0^\infty y^{-\alpha}(1+y)^{-\alpha-1}dy\Big\}\\
&\quad -\alpha(p_1+p_2)\int_0^\infty y^{-\alpha}(1+y)^{-\alpha-1}dy.
\end{aligned}$$

The stated representation now follows since

$$2\alpha\int_0^{\frac{1}{2}} y^{-\alpha}(1-y)^{-\alpha-1}dy=2^{2\alpha}-\frac{\Gamma(1-\alpha)^2}{\Gamma(1-2\alpha)}$$

and

$$\int_0^\infty y^{-\alpha}(1+y)^{-\alpha-1}dy=\frac{\Gamma(1-\alpha)\Gamma(2\alpha)}{\Gamma(1+\alpha)}.$$

B. Suppose $\alpha\geq 1$ **and** $E|X_i|^\alpha<\infty$ $(i=1,2)$. Since

$$\frac{(1+x)^{-\alpha}-\sum_{j=0}^{[\alpha]-1}\binom{-\alpha}{j}x^j}{x^{[\alpha]}}\to\frac{(-\alpha)(-\alpha-1)\dots(-\alpha-[\alpha]+1)}{[\alpha]!},\quad (x\to 0)$$

the above ratio is bounded for $x \in (-\infty, \frac{1}{2})$. ($[\alpha]$ is the greatest integer less than or equal to α.) Hence we may use dominated convergence in order to find

$$\int_{-\infty}^{t/2} y^{[\alpha]} \frac{(1-\frac{y}{t})^{-\alpha} - \sum_{j=0}^{[\alpha]-1}\binom{-\alpha}{j}(-\frac{y}{t})^j}{(-\frac{y}{t})^{[\alpha]}} dF_{3-i}(y) \to$$
$$\to \frac{(-\alpha)(-\alpha-1)\ldots(-\alpha-[\alpha]+1)}{[\alpha]!}\int_{-\infty}^{\infty} y^{[\alpha]} dF_{3-i}(y) \quad (2.7)$$

as $t \to \infty$.

Since $\overline{F_i}$ is regularly varying, we have for $j = 0, \ldots, [\alpha]-1$ $\int_{t/2}^{\infty} y^j dF_i(y) = O(t^j\overline{F_i}(t))$ by Karamata's theorem (see e.g. Bingham et al.(1987)). Combination of this observation with (2.7) shows that for the case under consideration we have as $t \to \infty$

$$J_i = \sum_{j=0}^{[\alpha]-1} c_{\alpha,j}\frac{EX_{3-i}^j}{t^j} + (c_{\alpha,[\alpha]} + o(1))\frac{EX_{3-i}^{[\alpha]}}{t^{[\alpha]}} + O(\overline{F}_{3-i}(t)), \qquad (2.8)$$

where $c_{\alpha,j}$ is as in (2.1).

Substituting (2.7) and (2.8) in (2.3) gives

$$P(X_1+X_2>t) = P(X_1>\frac{t}{2})P(X_2>\frac{t}{2}) + \sum_{i=1}^{2}(r_i+o(1))\overline{F}_i(t)a_i(t) +$$
$$+\sum_{i=1}^{2} p_i\overline{F}_i(t)\Big\{\sum_{j=0}^{[\alpha]-1} c_{\alpha,j}\frac{EX_{3-i}^j}{t^j} + (c_{\alpha,[\alpha]}+o(1))\frac{EX_{3-i}^{[\alpha]}}{t^{[\alpha]}} + O(\overline{F}_{3-i}(t))\Big\}. \qquad (2.9)$$

The result follows since $E|X_i|^{[\alpha]} < \infty$ implies

$$t^{[\alpha]}\overline{F}_i(t) \to 0 \text{ as } t \to \infty,$$

hence $\prod_{i=1}^2 P(X_i > \frac{t}{2}) = O(\prod_{i=1}^2 \overline{F}_i(t)) = o(t^{-[\alpha]} \wedge_{i=1}^2 \overline{F}_i(t))$.

C1,C2 Suppose $E|X_i|^\alpha = \infty$ $(i=1,2)$, α is integer and one of the following holds: 1. $\alpha \geq 1$ is even 2. $\alpha \geq 1$ is odd and $p_i \neq \frac{1}{2}$.

Write $J_i = \int_{-\infty}^{t/2}(1-\frac{y}{t})^{-\alpha}dF_{3-i}(y) =: J_{1i}+J_{2i}$, where J_{1i} and J_{2i} are the integrals over $(-\infty, -\frac{t}{2})$ and $(-\frac{t}{2}, \frac{t}{2})$ respectively. Then $|J_{1i}| \leq 2^\alpha F_{3-i}(-\frac{t}{2}) \leq$

$2^\alpha \overline{F}_{3-i}(\frac{t}{2}) = O(\overline{F}_{3-i}(t))$, using regular variation of $\overline{F}_{3-i}$. In order to estimate $|J_{2i}|$ note that for $|y| \le t/2$

$$(1-\frac{y}{t})^{-\alpha} = 1 + \sum_{j=1}^{\alpha} c_{\alpha,j}(\frac{y}{t})^j + c_{\alpha,\alpha+1}(\frac{\theta y}{t})^{\alpha+1}, \tag{2.10}$$

where $|\theta| = |\theta(y,t)| \le 1$.

For $j = 0, \ldots, \alpha-1$ Karamata's theorem shows that as $t \to \infty$

$$\frac{1}{t^j}\int_{-t/2}^{t/2} y^j dF_{3-i}(y) = \frac{EX_{3-i}^j}{t^j} + O(\overline{F}_{3-i}(t)). \tag{2.11}$$

Integration by parts gives

$$\begin{aligned}\frac{1}{t^\alpha}\int_{-t/2}^{t/2} y^\alpha dF_{3-i}(y) &= -\frac{1}{2^\alpha}(1-F_{3-i}(\frac{t}{2}) + (-1)^\alpha F_{3-i}(-\frac{t}{2}))\\ &+ \frac{\alpha}{t^\alpha}\int_0^{t/2}(1-F_{3-i}(y)+(-1)^\alpha F_{3-i}(-y))y^{\alpha-1}dy\\ &\sim \frac{\alpha}{t^\alpha}\int_0^{t}(1-F_{3-i}(y)+(-1)^\alpha F_{3-i}(-y))y^{\alpha-1}dy =: K_i(t), \end{aligned}\tag{2.12}$$

where the asymptotic equality follows by Karamata's theorem.

In case $j = \alpha+1$ we have similarly

$$\frac{1}{t^{\alpha+1}}\int_{-t/2}^{t/2} y^{\alpha+1} dF_{3-i}(y) = O(\overline{F}_{3-i}(t)). \tag{2.13}$$

Collecting the above estimates it follows that

$$\begin{aligned} J_i &= F_{3-i}(\frac{t}{2}) - F_{3-i}(-\frac{t}{2}) + O(\overline{F}_{3-i}(t)) + \sum_{j=1}^{\alpha-1} c_{\alpha,j}\frac{EX_{3-i}^j}{t^j}\\ &\quad +(c_{\alpha,\alpha}+o(1))\frac{\alpha}{t^\alpha}\int_0^t (1-F_{3-i}(y)+(-1)^\alpha F_{3-i}(-y))y^{\alpha-1}dy\\ &= 1 + \sum_{j=1}^{\alpha-1} c_{\alpha,j}\frac{EX_{3-i}^j}{t^j} +\\ &\quad +(c_{\alpha,\alpha}+o(1))\frac{\alpha}{t^\alpha}\int_0^t (1-F_{3-i}(y)+(-1)^\alpha F_{3-i}(-y))y^{\alpha-1}dy. \end{aligned}$$

Note that since $\overline{F}_{3-i}$ is regularly varying with index $-\alpha$, we have $\overline{F}_{3-i}(t) = o(K_i(t))$, hence as $t \to \infty$ $\prod_{i=1}^2 P(X_i > t/2) = o(\sum_{i=1}^2 K_i(t)\overline{F}_i(t))$.

The result now follows if we combine the estimates for I_i and J_i.

C3, C4 Suppose α odd, $p_i = \frac{1}{2}, r_i \neq 0$ for $i = 1, 2$. Using (1.3) it follows that $1 - F_i(t) + (-1)^\alpha F_i(-t) \sim 2r_i a_i(t) F_i(t)$, hence this function is (in absolute value) regularly varying with index $\rho - \alpha$. It follows that, if $\rho = 0$, (2.12) holds again and we have the same estimate as in C1 and C2.

The case C4 is similar.

D Suppose $E|X_i|^\alpha = \infty$ $(i = 1, 2)$ and $\alpha > 1$ non-integer

In this case we need a more precise estimate for J_i. An integration by parts shows that

$$\begin{aligned} J_{1i} &= \int_{-\infty}^{-t/2} (1 - \frac{y}{t})^{-\alpha} dF_{3-i}(y) \\ &= (\frac{3}{2})^{-\alpha} F_{3-i}(-\frac{t}{2}) - \alpha \int_{-\infty}^{-1/2} F_{3-i}(ty)(1-y)^{-\alpha-1} dy \\ &\sim (1-p)\Big[(\frac{3}{2})^{-\alpha} \overline{F}_{3-i}(\frac{t}{2}) - \alpha \int_{-\infty}^{-1/2} \overline{F}_{3-i}(-ty)(1-y)^{-\alpha-1} dy\Big] \\ &\sim (1-p)\overline{F}_{3-i}(t)\Big[(\frac{4}{3})^{\alpha} - \alpha \int_{-\infty}^{-1/2} (-y)^{-\alpha}(1-y)^{-\alpha-1} dy\Big]. \end{aligned}$$

Now (2.11) is replaced with the estimate (valid for $j = 0, \ldots, [\alpha]$)

$$\frac{1}{t^j} \int_{-t/2}^{t/2} y^j dF_{3-i}(y) = \frac{EX_{3-i}^j}{t^j} + \frac{\alpha}{\alpha - j} \overline{F}_{3-i}(t) + o(\overline{F}_{3-i}(t)).$$

In case $j = [\alpha] + 1$ we have as $\to \infty$

$$\frac{1}{t^{[\alpha]+1}} \int_{-t/2}^{t/2} y^{[\alpha]+1} dF_{3-i}(y) \sim \frac{\alpha}{[\alpha] + 1 - \alpha} 2^{\alpha - [\alpha] - 1} \overline{F}_{3-i}(t).$$

The rest of the proof follows as before.

3 Portfolio diversification effects

3.1 First order effects

It is a stylized fact that financial asset returns are heavy tailed distributed as in (1.1). Then Feller's (1971, VIII.8) classical result can be used to investigate the benefits from cross-sectional portfolio diversification.

To this end we first briefly have to review the standard finance model for (relative) risk pricing. In the so called Capital Asset Pricing Model (CAPM), see e.g. Fama and Miller (1972) and Copeland and Weston (1983, ch.7), the return R_i on an individual asset i is related to the return r_f on the riskfree asset (government bond) and the return R on the market portfolio (measured as the return on an index like the S&P500). Suppose that

$$R_i - r_f = \beta_i(R - r_f) + Q_i,$$

where Q_i is the idiosyncratic or unsystematic risk factor of the return R_i on asset i, and β_i is the correlation coefficient in a regression of $R_i - r_f$ on $R - r_f$. The typical assumption is that Q_i, Q_j and R are cross-sectionally independently distributed. Thus β_i reflects how R_i covaries with the market. The CAPM holds that market forces determine what happens in expectation

$$E[R_i - r_f] = \beta_i E[R - r_f].$$

Since for what follows the risk free rate r_f plays no role, we economize on notation and set $r_f = 0$.

Consider a portfolio of m assets with weights w_i, $w_i > 0$, $\sum_1^m w_i = 1$. We focus on equally weighted portfolios $w_i = 1/m$. Let $\overline{\beta} = \frac{1}{m}\sum_1^m \beta_i$. Dacorogna et al. (1998) report the following diversification result:

Lemma 3.1 (diversification benefits). *Suppose the Q_i are cross sectionally i.i.d. distributed and satisfy (1.1). For large loss levels the conditional tail diversification benefits from the equally weighted portfolio are larger if the returns have finite variance but are heavy tailed, than if they are normally distributed. Specifically we find as $x \to -\infty$ that*

$$P\{\frac{1}{m}\sum_1^m R_i < x | R = r\} \sim P\{\frac{1}{m^{1-1/\alpha}} Q_i + \overline{\beta} R < x | R = r\}. \qquad (3.1)$$

Diversification against tail risk is more effective if returns are heavy tailed distributed with $\alpha > 2$ than if the underlying distribution is normal. Recall that under normality risk is reduced by the square root of m. It has been noted in the economics literature that the effect of diversification is less pronounced if $\alpha < 2$ in comparison with the normal distribution. Fama and Miller (1972, p. 270) discuss the case of sum stable distributions. They note that for $\alpha < 1$ diversification actually increases the dispersion, and hence putting all eggs in the same basket is advisable in such cases. But financial data do not display such heavy tails, rather $\alpha > 2$. We are not aware of a discussion of downside risk diversification in case $\alpha > 2$.

Note that the above result is conditional. The motivation for stating it this way is that through diversification one can reduce the contribution of the unsystematic risk factors Q_i to the total risk, but one cannot get rid of the contribution of the systematic risk factor R. Nevertheless the following is straightforward.

Lemma 3.2. *Suppose the R and the Q_i are i.i.d. distributed and satisfy (1.1) with the same scale coefficient and tail index. Then*

$$P\{\frac{1}{m}\sum_{1}^{m} R_i < x\} \sim P\{(m^{1-\alpha} + (\overline{\beta})^{\alpha})^{1/\alpha} Q_i < x\} \text{ as } x \to -\infty.$$

3.2 Second order diversification benefits

Since the asset returns can be positive or negative, we need a result on convolutions with heavy tails on both sides. Under continuous compounding the whole *real* line is the support of the return distribution for assets like equity and foreign exchange. Assume furthermore that the tail index $\alpha > 2$ for both tails. Financial data usually indicate that the mean and the variance are finite. We specialize the general result of the paper to two special cases which are of interest from an economic point of view. To restrict the number of different combinations that will arise, we assume that the tails are symmetric. This is a reasonable assumption for e.g. foreign currency return data when the exchange rate is left freely floating. Other possibilities are left to the reader.

Corollary 1 (Similar tail behavior). *Suppose that as $x \to \infty$,*

$$\begin{aligned} P\{X > x\} &= ax^{-\alpha}(1 + bx^{\rho} + o(x^{\rho})) \quad (a > 0, b \neq 0), \\ P\{X \leq -x\} &= ax^{-\alpha}(1 + bx^{\rho} + o(x^{\rho})) \quad (a > 0, b \neq 0). \end{aligned} \tag{3.2}$$

Moreover, assume that $\alpha > 2, \rho < 0$, so that $E[X]$ and $E[X^2]$ are finite. Suppose X_1 and X_2 are i.i.d. and satisfy (3.2). Then application of theorem

2.1 gives

$$P\ \{\ X_1 + X_2 > x\} = P\{X_1 + X_2 \leq -x\} \tag{3.3}$$
$$= \ 2ax^{-\alpha}\left(1 + bx^{\rho} + \alpha E[X]x^{-1} + \frac{\alpha(\alpha+1)}{2}E[X^2]x^{-2})\right) + \tag{3.4}$$
$$+o(x^{-\alpha-2}) + o(x^{-\alpha+\rho}) \tag{3.5}$$

as $x \to \infty$.

We find that because the distribution of asset returns is two-sided, vis a vis the case of positive random variables considered in Geluk et al. (1997), a new factor depending on $E[X^2]$ enters as the second order term if $E[X] = 0$ and $\rho \leq -2$. Again, for the case of freely floating exchange rates one typically finds that the mean is zero. But for other assets like equity a positive mean is more reasonable case since the mean equity returns reflect the positive long run growth rate of the economy. For the purpose of diversification we iterate further and find:

Corollary 2. *Under the conditions of Corollary 1 if* $\rho < -1$ *and* $E[X] > 0$ *we have as* $x \to \infty$

$$P\{\frac{1}{m}\sum_{1}^{m} X_i \leq -x\} = m^{1-\alpha}as^{-\alpha}\left(1 + \frac{m-1}{m}\alpha E[X]x^{-1} + o(x^{-1})\right); \tag{3.6}$$

while if $E[X] = 0$ *and* $\rho < -2$

$$P\{\frac{1}{m}\sum_{1}^{m} X_i \leq -x\} = m^{1-\alpha}as^{-\alpha}\left(1 + \frac{m-1}{m^2}\frac{\alpha\,(\alpha+1)}{2}E[X^2]x^{-2} + o(x^{-2})\right). \tag{3.7}$$

We return to the question of diversification. A more precise evaluation of the diversification benefits for equally weighted portfolios is given in the next result.

Proposition 1. *Under the conditions of Lemma 3.1 and Corollary 2, for the case of equation (3.6) and when* $m > 1$

$$P\{\frac{1}{m}\sum_{1}^{m} R_i \ \leq \ -x|R = r\} \tag{3.8}$$
$$= \ m^{1-\alpha}ax^{-\alpha}\left(1 + \{\frac{m-1}{m}\alpha E[Q_i] + \alpha\overline{\beta}r\}x^{-1} + o(x^{-1})\right);$$

while if equation (3.7) applies we get for $m > 1$

$$P\{\frac{1}{m}\sum_{1}^{m} R_i \leq -x | R = r\} \tag{3.9}$$

$$= m^{1-\alpha} a x^{-\alpha}\left(1 + \alpha\overline{\beta} r x^{-1} + \frac{\alpha(\alpha+1)}{2}\{\frac{m-1}{m^2}E[Q_i^2] + (\overline{\beta} r)^2\}x^{-2} + o(x^{-2})\right).$$

Proof. Combine Lemma 3.1 and Corollary 2 repeatedly. Finally calculate the shift due to the translation of $\frac{1}{m}\sum_1^m Q_i$ by $\overline{\beta} r$. □

Remark 1. *If R also satisfies (3.2) but with a first order tail index $\alpha_R > \alpha_Q + 2$, or if the cdf of R has light tails, then we can replace r in the Proposition (1) by the expectation $E[R]$.*

Remark 2. *If $\rho = -1$ then (3.8) becomes*

$$P\{\frac{1}{m}\sum_{1}^{m} R_i \leq -x | R = r\} = m^{1-\alpha} a x^{-\alpha}\left(1 + (\frac{m-1}{m}\alpha E[Q_i] + \alpha\overline{\beta} r + \frac{b}{m})x^{-1} + o(x^{-1})\right);$$

while if $\rho = -2$ then (3.9) becomes

$$P\{\frac{1}{m}\sum_{1}^{m} R_i \leq -x | R = r\}$$

$$= m^{1-\alpha} a x^{-\alpha}\left(1 + \alpha\overline{\beta} r x^{-1} + \{\frac{\alpha(\alpha+1)}{2}(\frac{m-1}{m^2}E[Q_i^2] + (\overline{\beta} r)^2) + \frac{b}{m^2}\}x^{-2} + o(x^{-2})\right).$$

Example 1 (Student). *An example may help to clarify what the propositions actually imply. Suppose we can take an open position in one or two currencies and suppose that the interest rates across the three countries are equal. In that case investing abroad is just a fair gamble. Furthermore assume that the two exchange rates are i.i.d. Empirically the Student-t with say 3 degrees of freedom is known to give a decent fit to foreign currency return data. The density reads*

$$f(x) = 2\pi^{-1}3^{-1/2}\left[1 + x^2/3\right]^{-2}. \tag{3.10}$$

It follows that (3.2) holds with $\alpha = 3, \rho = -2, a = 2\sqrt{3}/\pi, b = -18/5$. From (3.10) we compute the effect of diversification. Putting all money in a single currency gives downside risk equal to

$$P\{X \le -x\} = \frac{2\sqrt{3}}{\pi} x^{-3}(1 - \frac{18}{5} x^{-2} + o(x^{-2})).$$

Application of theorem 2.1 shows that since $E[X] = 0$ and $\rho = -2$, the second order term consists of two parts

$$\left(b + \frac{\alpha(\alpha+1)}{2} E[X^2]\right) x^{-2} = \left(b + \frac{1}{2}\frac{\alpha^2(\alpha+1)}{\alpha - 2}\right) x^{-2} = \left(-\frac{18}{5} + \frac{36}{2}\right) x^{-2}.$$

Hence diversification by buying equal shares into the two currencies gives downside risk equal to

$$\begin{aligned} P\{\frac{X_1 + X_2}{2} &\le -x\} \\ &= 2^{1-3}\frac{2\sqrt{3}}{\pi} x^{-3}\left(1 - \frac{18}{5} 2^{-2} x^{-2} + \frac{3(3+1)}{2} E[X^2] 2^{-2} x^{-2} + o(x^{-2})\right) \\ &= \frac{1}{4}\frac{2\sqrt{3}}{\pi} x^{-3}\left(1 + \frac{18}{5} x^{-2} + o(x^{-2})\right). \end{aligned}$$

Since the first order scale coefficient of the diversified portfolio is only one-fourth of the first order scale coefficient of the undiversified portfolio, diversification is an important help for reducing the tail risk. Nevertheless, due to the switch in sign of the second order scale coefficient, diversification does not always reduce the tail risk. Let $Y = 7^{1/3} X_2$. Hence

$$P\{Y \le -x\} = 7\frac{2\sqrt{3}}{\pi} x^{-3}(1 - 7^{2/3}\frac{18}{5} x^{-2} + o(x^{-2})).$$

For the diversified portfolio consisting of X_1 and Y, the tail risk is

$$P\{\frac{X_1 + Y}{2} \le -x\} = \frac{2\sqrt{3}}{\pi} x^{-3}\left(1 + \frac{153 - 9.7^{2/3}}{40} x^{-2} + o(x^{-2})\right). \quad (3.11)$$

It follows that for large threshold levels x one is better off by putting all money in X_1, rather than to diversify and put half of the investment into Y. The reason is that while the portfolio $\frac{X_1+Y}{2}$ and X_1 have identical first order coefficients $\frac{2\sqrt{3}}{\pi} x^{-3}$, the second order scale coefficient of the portfolio $\frac{X_1+Y}{2}$ is positive and adds to the tail risk, and the opposite holds for the second order scale coefficient of X_1. The example thus shows the relevance of the second order terms for portfolio selection problems.

4 Asymptotic normality of $\hat{\theta}_n$

In order to obtain the limiting behavior of $\hat{\theta}_n$, we need a stricter condition than (1.1). Assume as $t \to \infty$

$$\begin{cases} P(\epsilon_i > t) = cpt^{-\alpha}\{1 + bt^{-\beta} + o(t^{-\beta})\} \\ P(\epsilon_i < -t) = c(1-p)t^{-\alpha}\{1 + dt^{-\beta} + o(t^{-\beta})\}, \end{cases} \tag{4.1}$$

where $c > 0, \alpha > 0, p \in [0, 1]$, $b \neq 0, d \neq 0$ and $\beta > 0$.

From Theorem 2.1 we have as $t \to \infty$

$$\begin{cases} P(Z_i > t) = c\{1 + (1-\theta)^\alpha + |\theta|^\alpha\}t^{-\alpha}\{1 + k_1 A(t) + o(A(t))\} \\ P(W_i > t) = c\{1 + (1+\theta)^\alpha + |\theta|^\alpha\}t^{-\alpha}\{1 + k_2 A(t) + o(A(t))\}, \end{cases} \tag{4.2}$$

where $k_i = k_i(\alpha, \beta, \theta, c, b, d, p)$ $(i = 1, 2)$ and

$$A(t) = \begin{cases} t^{-\alpha} \vee t^{-\beta} & \text{if } \alpha < 1 \\ \frac{\log t}{t} \vee t^{-\beta} & \text{if } \alpha = 1 \\ t^{-\alpha} \vee t^{-\beta} & \text{if } 1 < \alpha < 2 \\ t^{-1} \vee t^{-\beta} & \text{if } \alpha \geq 2. \end{cases} \tag{4.3}$$

Note that it is possible, but tedious to give explicit expressions for k_1 and k_2. We omit the details.

Hence, similar to the prooof of Proposition 2.1 of Resnick and Starica (1997), we have

$$\begin{cases} \sqrt{k}(\hat{\alpha}_Z(k) - \alpha) \xrightarrow{d} N_1(\lambda_1, \alpha^2(1 + 2\frac{1\wedge(1-\theta)^\alpha + |\theta|^\alpha + (1-\theta)^\alpha \wedge |\theta|^\alpha}{1+(1-\theta)^\alpha + |\theta|^\alpha})) \\ \sqrt{m}(\hat{\alpha}_W(m) - \alpha) \xrightarrow{d} N_2(\lambda_2, \alpha^2(1 + 2\frac{1\wedge(1+\theta)^\alpha + |\theta|^\alpha + (1+\theta)^\alpha \wedge |\theta|^\alpha}{1+(1+\theta)^\alpha + |\theta|^\alpha})) \end{cases}$$

provided that as $n \to \infty$

$$\begin{cases} k \to \infty, \quad k/n \to 0 \\ m \to \infty, \quad m/n \to 0 \\ \sqrt{k}A_1(n/k) \to \lambda_1 \in R, \quad \sqrt{m}A_2(n/m) \to \lambda_2 \in R \\ \text{either} \quad \limsup_{n\to\infty} n/k^{3/2} < \infty \quad \text{or} \quad \liminf_{n\to\infty} n/k^{3/2} > 0 \\ \text{either} \quad \limsup_{n\to\infty} n/m^{3/2} < \infty \quad \text{or} \quad \liminf_{n\to\infty} n/m^{3/2} > 0. \end{cases} \tag{4.4}$$

Let $U(t)$ denote the inverse function of $\frac{1}{P(Z_i>t)}$. Note that

$$\begin{aligned} & \tfrac{k}{n} Z_{n,n-k}^{\hat\alpha_Z(k)} - c(1+(1-\theta)^\alpha + |\theta|^\alpha) \\ = \; & \{\tfrac{k}{n} Z_{n,n-k}^{\hat\alpha_Z(k)} - \tfrac{k}{n} Z_{n,n-k}^{\alpha}\} + \tfrac{k}{n} U^\alpha(n/k)\{\tfrac{Z_{n,n-k}^\alpha}{U^\alpha(n/k)} - 1\} \\ & + \{\tfrac{k}{n} U^\alpha(n/k) - c(1+(1-\theta)^\alpha + |\theta|^\alpha)\} \\ = \; & \{\tfrac{k}{n} Z_{n,n-k}^{\alpha}(\hat\alpha_Z(k)-\alpha)\log Z_{n,n-k}\}(1+o_p(1)) \\ & + \tfrac{k}{n} U^\alpha(n/k)\{\tfrac{Z_{n,n-k}^\alpha}{U^\alpha(n/k)} - 1\} \\ & + \{\tfrac{k}{n} U^\alpha(n/k) - c(1+(1-\theta)^\alpha + |\theta|^\alpha)\}. \end{aligned}$$

Using the fact that $Z_{n,n-k}/U(n/k) = 1 + O_p(1/\sqrt{k})$, we have

$$\begin{aligned} & \tfrac{\sqrt{k}}{\log(n/k)}\{\tfrac{k}{n} Z_{n,n-k}^{\hat\alpha_Z(k)} - c(1+(1-\theta)^\alpha + |\theta|^\alpha)\} \\ \stackrel{d}{\to} \; & N_1(\alpha\lambda_1, \alpha^4(1+2\tfrac{1\wedge(1-\theta)^\alpha + |\theta|^\alpha + (1-\theta)^\alpha\wedge|\theta|^\alpha}{1+(1-\theta)^\alpha+|\theta|^\alpha})). \end{aligned}$$

Similarly,

$$\begin{aligned} & \tfrac{\sqrt{m}}{\log(n/m)}\{\tfrac{m}{n} W_{n,n-m}^{\hat\alpha_W(m)} - c(1+(1+\theta)^\alpha + |\theta|^\alpha)\} \\ \stackrel{d}{\to} \; & N_2(\alpha\lambda_2, \alpha^4(1+2\tfrac{1\wedge(1+\theta)^\alpha + |\theta|^\alpha + (1+\theta)^\alpha\wedge|\theta|^\alpha}{1+(1+\theta)^\alpha+|\theta|^\alpha})). \end{aligned}$$

Note that the above two normal limits are dependent and the dependence structure is unknown. However, by requiring $k/m \to 0$ which implies $\lambda_1 = 0$, we have

$$\frac{\sqrt{k}}{\log(n/k)}\{\frac{k}{m} Z_{n,n-k}^{\hat\alpha_Z(k)} W_{n,n-m}^{-\hat\alpha_W(m)} - f_n(\theta)\} \stackrel{d}{\to} N(0,\sigma^2),$$

where

$$\sigma^2 = c^2(1+(1+\theta)^\alpha + |\theta|^\alpha)^2\alpha^4\{1+2\frac{1\wedge(1-\theta)^\alpha + |\theta|^\alpha + (1-\theta)^\alpha\wedge|\theta|^\alpha}{1+(1-\theta)^\alpha+|\theta|^\alpha}\}.$$

Hence, it follows that the estimator $\hat\theta_n$ defined in (1.7) satisfies the following

Theorem 4.1. *Suppose (1.4), (4.1) and (4.4) hold and $k/m \to 0$ as $n \to \infty$. Then*

$$\frac{\sqrt{k}}{\log(n/k)}\{\hat\theta_n - \theta\} \stackrel{d}{\to} N(0, \sigma^2(\frac{d}{d\theta} f^-(\theta))^2).$$

Remark 3. *If we can find the dependence structure between the two normal limits N_1 and N_2, then we may be able to take $m = k$ and choose the sample fraction k in an optimal way as in the tail index estimation and extreme tail probability estimation (see Drees and Kaufmann (1998) and Hall and Weissman (1997)). This will be a part of our future work.*

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