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The all-pay auction with complete information*

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Summary. In a (first price) all-pay auction, bidders simultaneously submit bids for an item. All players forfeit their bids, and the high bidder receives the item. This auction is widely used in economics to model rent seeking, R&D races, political contests, and job promotion tournaments. We fully characterize equilibrium for this class of games, and show that the set of equilibria is much larger than has been recognized in the literature. When there are more than two players, for instance, we show that even when the auction is symmetric there exists a continuum of asymmetric equilibria. Moreover, for economically important configurations of valuations, there is no revenue equivalence across the equilibria; asymmetric equilibria imply higher expected revenues than the symmetric equilibrium.

JEL Classification Numbers: D44, D72.

I Introduction

In a (first price) all-pay auction, each bidder ($i = 1, 2, \dots, n$) submits a (non-negative) sealed bid, x_i, for an item valued by player i at v_i. All players forfeit their bids, but the high bidder wins the item. (Ties are broken randomly). When there is complete

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information, the payoff to player i is given by

$$u_i(x_1, \ldots, x_n) = \begin{cases} -x_i & \text{if } \exists j \text{ such that } x_j > x_i \\ \dfrac{v_i}{m} - x_i & \text{if } i \text{ ties for the high bid with } m-1 \text{ others} \\ v_i - x_i & \text{if } x_i > x_j \ \forall j \neq i \end{cases} \tag{1}$$

The all-pay auction is similar to a standard (winner pay) first-price auction, except that losers must also pay the auctioneer their bids.[1]

In an all-pay auction, one can interpret differences in the v_i's as arising from differences in abilities. To see this, suppose the utility to player i of winning a prize of W by putting forth effort x_i is $u_i^* = U_i(W) - \beta_i x_i$, where x_i is effort, and β_i is the marginal cost to player i of effort. Since behavior is invariant to affine transformations, we may just as well write the utility function as $u_i \equiv u_i^*/\beta_i = v_i - x_i$, where $v_i \equiv U_i(W)/\beta_i$. Thus, differences in the v_i's may be due to differences in valuations or differences in the abilities of players to convert an entry into a prize: players with higher v_i's can be thought of as stronger players.

The all pay auction is widely used in economics because it captures the essential elements of contests. It has been used to model (1) the lobbying for rents in regulated and trade protected industries [cf. Moulin (1986a, b); Hillman and Riley (1989); Hillman and Samet (1987); Hillman (1988) and Baye et al. (1993)], (2) technological competition and R&D races [cf. Dasgupta (1986)], and (3) a host of other situations including political campaigns, tournaments and job promotion.[2] Essentially, these economic problems boil down to a contest that is an all-pay auction in effort; the player putting forth the greatest effort wins the prize, while the efforts of other contestants go unrewarded.[3]

Section II of this paper completely characterizes the set of Nash equilibria in the first price all-pay auction with complete information. Our characterization shows that for $n > 2$ the set of equilibria is larger than recognized in the existing literature, and critically depends on the configurations of player valuations. We show that with homogeneous valuations ($v_1 = v_2 = v_3 \cdots = v_n$) there exists a unique symmetric equilibrium *and a continuum of asymmetric equilibria.* All of these equilibria are payoff equivalent, as is the expected sum of the bids (revenue to the auctioneer).

[1] The war of attrition is a second-price all pay auction: all players forfeit their bids except the winner, who pays the second-highest bid. Hendricks et al. (1988) characterize the set of equilibria for the war of attrition with complete information in continuous time and with general payoff functions.

[2] For instance, in the literature on rent seeking (Tullock, 1980), political campaigns (Snyder, 1989), job promotions (Rosen, 1986), and commitment (Dixit, 1987), the probability player i wins a contest by putting forth an effort of x_i is modeled as $x_i^\gamma/\Sigma_j x_j^\gamma$, where $\gamma > 0$. As γ goes to infinity, the player putting forth the greatest effort is certain to win the contest, and thus the limit of these models is the all-pay auction. The all-pay auction may also be interpreted as the limit of many games with uncertainty or incomplete information, including the models of Lazear and Rosen (1981), Nalebuff and Stiglitz (1983), Weber (1985), and Bull, Schotter and Weigelt (1987). As the incomplete information or uncertainty vanishes, these models converge to the complete information all-pay auction.

[3] Many other games with discontinuous payoffs (and in which only mixed-strategy equilibria exist) have a structure that is isomorphic to the all-pay auction, including Varian (1980), Narasimhan (1988), Broecker (1990) Raju et al. (1990), Baye and de Vries (1992), Baye, Kovenock and de Vries (1992), Deneckere et al. (1992), and Dennert (1993). The characterization results presented in this paper are thus pertinent to a wide body of literature in economics.

When $v_1 > v_2 = v_3 \geq v_4 \geq \cdots \geq v_n$, there is a unique "symmetric equilibrium" (symmetric in the sense that all agents with identical values use the same strategy), *as well as a continuum of asymmetric equilibria.* The expected sum of the bids (revenue to the auctioneer) varies across the continuum of equilibria; there is *not* "revenue equivalence." The case where $v_1 > v_2 > v_3 \geq \cdots \geq v_n$ is known to have a unique equilibrium (Hillman and Riley, 1989).[4]

Our theoretical results have important implications for economic applications of the all-pay auction. To highlight these implications, Section III reconsiders the regulation game analyzed by Wenders (1987) and Ellingsen (1991).

II Characterization of equilibria

The all-pay auction with complete information does not have a Nash equilibrium in pure strategies, but does have a Nash equilibrium in mixed-strategies. Accordingly, let $G_i(x_i)$ denote the cumulative distribution function (cdf) representing the equilibrium mixed-strategy of player i. Player i is said to *randomize continuously on* $A \subseteq \boldsymbol{R}$ if he plays a mixed strategy that is atomless (i.e., contains no mass points) and has a strictly increasing cdf almost everywhere on A.

Our first theorem characterizes equilibrium for the case when $m > 2$ players have the highest valuation of the prize. For this case, Hillman and Samet (1987) have shown that there exists a symmetric equilibrium and a finite number of asymmetric equilibria where some agents with the highest valuation bid zero with probability one, and claim this exhausts all equilibria. Our Theorem 1 shows, however, that there actually exists a continuum of asymmetric equilibria when three or more players have the highest valuation of the prize. Nonetheless, we show that all of the equilibria imply the same expected payoff (zero) for each player, and yield the auctioneer the same expected revenue.

Theorem 1: *When $v_1 = \cdots = v_m > v_{m+1} \geq \cdots \geq v_n$ and $m \geq 2$:*

(A) If $m = 2$, the Nash equilibrium is unique and symmetric. If $3 \leq m \leq n$, there is a unique symmetric Nash equilibrium, as well as a continuum of asymmetric Nash equilibria. In any equilibrium players $m + 1$ through n bid zero with probability one, and at least two players randomize continuously on $[0, v_1]$. Each other player $i \in \{1, \ldots, m\}$ randomizes continuously on $[b_i, v_1]$, where $b_i \geq 0$ is a free parameter, and bids 0 with positive probability if $b_i > 0$.[5] *When two or more players randomize continuously on a common interval, their corresponding cdf's are identical over that interval.*[6]

(B) In any equilibrium, the expected payoff to each player is zero.

(C) All equilibria are revenue equivalent: the expected sum of the bids in any equilibrium equals v_1.

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[4] In this unique equilibrium, only players one and two actively bid (players 3 through n bid zero with probability one).

[5] If $b_i \geq v_1$, player i bids 0 with probability one.

[6] Equation 2 below summarizes the algebraic form of the complete set of equilibria.

The formal proof of Theorem 1 is similar to the proof contained in the Appendix for our Theorem 2 below, and is thus omitted (our 1990 working paper contains a complete proof). However, it is useful to highlight some of the features of equilibrium, as well as some intuition for the existence of a continuum of equilibria. The basic issues can be illustrated in the case where $m=n=3$, so that $v_1=v_2=v_3(\equiv v$, say). Theorem 1 implies, in this case, that in every equilibrium two players randomize continuously on the interval $[0, v]$, while the third player randomizes continuously on the interval $[b, v]$ and concentrates all remaining mass at zero (this mass is $(b/v)^{1/2}$, and is thus zero if $b=0$). (Note that $b \geq 0$ is an arbitrary constant). Since two players randomize continuously on $[0, v]$, and any atoms in the third player's mixed strategy (player 3's, say) are isolated at 0, the highest bid is positive and unique with probability one. Furthermore, since zero is contained in the support of all three players' mixed strategies and at least two players use mixed strategies that do not put mass at zero, each player earns an expected payoff of zero.

Given the characterization of the support of each player's mixed strategy, we know that all three players randomize continuously on $[b, v]$, and hence, all three are capable of generating a winning bid in the interval $[b, v]$. Equilibrium requires that, for any bid in $[b, v]$, each player earns an expected payoff of zero, given the mixed strategies used by the other two players. Three non-degenerate mixed strategies over $[b, v]$ are uniquely determined as the solution to three equations that set the expected payoff of each player i to be zero for bids in $[b, v]$:

$$\text{For } i \neq j, k\text{:} \quad u_i(x)=G_j(x)G_k(x)[v-x]-[1-G_j(x)G_k(x)]x=0 \quad \forall x \in [b, v].$$

The solution to this system of equations is symmetric and given by

$$G_1=G_2=G_3=(x/v)^{1/2} \text{ for } x \in [b, v].$$

The probability player 3 submits a winning bid in the interval $[0, b]$ is zero, since the characterization of player 3's support requires that (remaining) mass of $G_3(b) \equiv (b/v)^{1/2}$ be isolated at 0 if $b>0$. Given $G_3(b)$, and the fact that only players 1 and 2 can submit a winning bid in the interval $[0, b]$ with positive probability, the mixed strategies for players 1 and 2 must satisfy

$$\text{For } i \neq j, 3\text{:} \quad u_i(x)=G_j(x)G_3(b)[v-x]-[1-G_j(x)G_3(b)]x=0 \quad \forall x \in [b, v].$$

For a given b, the solution to this system of equations is symmetric:

$$G_1=G_2=(x/v)[G_3(b)]^{-1}=(x/v)(b/v)^{-1/2} \text{ for } x \in [0, b].$$

These mixed-strategies for players 1 and 2 are sufficiently aggressive on the interval $[0, b]$ to ensure that player 3 will not find it profitable to deviate by submitting a bid in the open interval $(0, b)$.

Thus, for a given b, we have constructed Nash equilibrium mixed-strategies for the three players. On the interval $[b, v]$, all players randomize continuously according to the three-player symmetric equilibrium. On the interval $[0, b]$, player 3's mixed strategy concentrates all mass at zero (unless $b=0$), while players 1 and 2 randomize continuously according to mixed-strategies that are proportional to the two-player symmetric equilibrium. But since b is arbitrary, by varying b from 0 to

v one generates a continuum of equilibria, ranging from the unique symmetric equilibrium (when $b=0$) to the extremely asymmetric one in which only players 1 and 2 actively compete (when $b=v$, player 3 bids zero with probability one).[7]

More generally, Theorem 1 allows us to explicitly characterize the algebraic form of the family of equilibrium mixed strategies for the case where $v_1=v_2=v_3=\cdots=v_m>v_{m+1}\geq\cdots\geq v_n$. Let $v=v_1=v_2=v_3=\cdots=v_m$. By the theorem, players $m+1$ through n bid zero with probability one, so suppose without loss of generality that players $i=1,2,\ldots,h$, $m\geq h\geq 2$, randomize continuously over $[0,v]$ with players $i=h+1,\ldots,m$ randomizing continuously over $[b_i,v]$, with $b_{h+1}\leq b_{h+2}\leq\cdots\leq b_m\leq v$. (The b_i's are arbitrary, and varying the b_i's generates the continuum of equilibria). Players $m+1$ through n bid zero with probability one. One can easily verify that the following family of cdf's are equilibrium strategies for the players:

$$
\begin{array}{lll}
\forall x\in[b_m,v]: & G_i(x)=\left[\dfrac{x}{v}\right]^{1/(m-1)} & i=1,\ldots,m; \\[2ex]
\forall x\in[b_j,b_{j+1}): & G_i(x)=\left[\dfrac{x}{v}\right]^{1/(j-1)}\left[\prod_{k>j}G_k(b_k)\right]^{1/(j-1)} & i=1,\ldots,j; \\[2ex]
j\in\{h+1,\ldots,m-1) & & \\[1ex]
 & G_k(x)=G_k(b_k) & k=j+1,\ldots,m; \\[2ex]
\forall x\in[0,b_{h+1}): & G_i(x)=\left[\dfrac{x}{v}\right]^{1/(h-1)}\left[\prod_{k>h}G_k(b_k)\right]^{-1/(h-1)} & i=1,\ldots,h; \\[2ex]
 & G_k(x)=G_k(b_k) & k=h+1,\ldots,m.
\end{array}
\tag{2}
$$

By Theorem 1, these are all the possible equilibrium cdf's.[8]

Our next Theorem shows that revenue equivalence breaks down when one "strong" player competes against several weaker, but equal, players. This case is economically interesting, because in the literature on regulation (cf. Rogerson, 1982 and Ellingsen, 1991), R&D races (Dasgupta, 1986), or political contests (cf. Snyder, 1989), one player (often the incumbent) is modeled as having an advantage over a number of identical challengers. Hillman (1988) uses the case $v_1>v_2=v_3\cdots=v_n$ to model protectionism, and erroneously claims that only two agents actively partici-

[7] For instance, if $v_1=v_2=v_3=1$ and $b=1$, then player 3 bids zero with probability one while players 1 and 2 randomize according to a uniform distribution on $[0,1]$. If $b=0$, all three players randomize according to the distribution function $x^{1/2}$ on $[0,1]$. If $b=1/4$, then players one and two randomize using $G_1=G_2=2x$ on $[0,1/4]$, while player 3 uses $G_3=1/2$ on $[0,1/4]$. All three players use $G_i=x^{1/2}$ on the interval $[1/4,1]$.

[8] The symmetric equilibrium ($h=m$) is used in Moulin (1986b) and Dasgupta (1986). Somewhat more general is the case $b_{h+1}=v$ and $2\leq h\leq m$, i.e. some agents may be inactive. This is discussed in Hillman and Samet (1987, p. 72), Hillman (1988, p. 66) and Hillman and Riley (1989, fn. 12). Hillman and Samet (1987, p. 72) claim there are no other equilibria. Also, Proposition 1c in Hillman and Riley which claims that at most one active agent bids zero with positive probability is erroneous, as up to $m-2$ active agents can do so.

pate. Theorem 2 shows, however, that there actually exists a continuum of equilibria with up to n active participants.[9]

Theorem 2: *When $v_1 > v_2 = \cdots = v_m > v_{m+1} \geq \cdots \geq v_n$, and $3 \leq m \leq n$:*

(A) There exists a continuum of Nash equilibria. In any equilibrium, player 1 randomizes continuously on the interval $[0, v_2]$ and players $m+1$ through n bid zero with probability one. Each player $i, i \in \{2, \ldots, m\}$, employs a strategy G_i with support contained in $[0, v_2]$ that has an atom $\alpha_i(0)$ at 0. The size of the atom may differ across players, but $\Pi_{i=2}^{m} \alpha_i(0) = (v_1 - v_2)/v_1$. Each G_i is characterized by a number $b_i \geq 0$, where $b_i = 0$ for at least one $i \neq 1$, such that $G_i(x) = G_i(0) = \alpha_i(0)\ \forall x \in [0, b_i]$ and player i randomizes continuously on $(b_i, v_2]$.[10] Furthermore, when two or more players in the set $\{2, \ldots, m\}$ randomize continuously on a common interval, their cdf's are identical on that interval.[11]

(B) In any equilibrium player one earns an expected payoff of $v_1 - v_2$, while each of the players two through n earns an expected payoff of zero.

(C) There is not revenue equivalence. In particular, the expected sum of the bids is

$$\sum Ex_i = \frac{v_2}{v_1} v_2 + \left(1 - \frac{v_2}{v_1}\right) Ex_1, \tag{3}$$

where Ex_1 varies across the continuum of equilibria, is minimized when symmetric players use symmetric strategies, and is maximized when only one of the players 2 through m is active (i.e., submits positive bids with positive probability).

This theorem, which is proved in the Appendix, allows us to construct the family of equilibrium mixed-strategies for the case where $v_1 > v_2 = \cdots = v_m > v_{m+1} \geq \cdots \geq v_n$. By the theorem, players $m+1$ through n bid zero with probability one, so suppose without loss of generality that of the players $\{2, \ldots, m\}$ players $i = 2, \ldots, h, h \geq 2$ randomize continuously over $(0, v_2]$, with players $i = h+1, \ldots, m$ randomizing continuously over $(b_i, v_2]$, (where $b_i = v_2$ implies $\alpha_i(0) = 1$) with $b_{h+1} \leq b_{h+2} \leq \cdots \leq b_m \leq v_2$. (Again, the b_i's are arbitrary, and varying the b_i's generates the continuum of equilibria). Players $m+1$ through n bid zero with probability one. In light of Theorem 2, the family of cdf's below constitute the entire set of Nash equilibrium strategies:

$$\forall x \in [b_m, v_2]: \quad G_i(x) = \left[\frac{v_1 - v_2 + x}{v_1}\right]^{1/(m-1)} \qquad i = 2, \ldots, m$$

$$G_1(x) = \frac{x}{v_2}\left[\frac{v_1 - v_2 + x}{v_1}\right]^{(2-m)/(m-1)};$$

$$\forall x \in [b_j, b_{j+1}): \quad G_i(x) = \left[\frac{v_1 - v_2 + x}{v_1}\right]^{1/(j-1)} \left[\prod_{k>j} G_k(b_k)\right]^{-1/(j-1)} \qquad i = 2, \ldots, j$$

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[9] This serves as a caveat to the claim by Magee, Brock and Young (1989, p. 217) that two-ness is a general property of political contests.

[10] If $b_i \geq v_2$, $\alpha_i(0) = 1$.

[11] Equation 4 below summarizes the algebraic form of the complete set of equilibria.

$j \in \{h+1, \dots, m-1\}$

$$G_k(x) = G_k(b_k) \qquad k = j+1, \dots, m;$$

$$G_1(x) = \frac{x}{v_2}\left[\frac{v_1 - v_2 + x}{v_1}\right]^{(2-j)/(j-1)}\left[\prod_{k>j} G_k(b_k)\right]^{-1/(j-1)}$$

$$\forall x \in [0, b_{h+1}): \qquad G_i(x) = \left[\frac{v_1 - v_2 + x}{v_1}\right]^{1/(h-1)}\left[\prod_{k>h} G_k(b_k)\right]^{-1/(h-1)} \qquad i = 2, \dots, h$$

$$G_k(x) = G_k(b_k) \qquad k = h+1, \dots, m$$

$$G_1(x) = \frac{x}{v_2}\left[\frac{v_1 - v_2 + x}{v_1}\right]^{(2-h)/(h-1)}\left[\prod_{k>h} G_k(b_k)\right]^{-1/(h-1)}. \tag{4}$$

In addition to the multiplicity of equilibria, the key implication of Theorem 2 is part C: expected revenue varies across the continuum of equilibria. Note that the theorem states that expected revenue is maximized in the equilibrium that maximizes the expected bid of the player with the highest valuation. Given the form of the mixed strategies in equation (4), this occurs in the asymmetric equilibrium where player 1 and exactly one other player submit a positive bid with positive probability.[12]

To complete the characterization, we need the following result originally formulated by Hillman (1988) and Hillman and Riley (1989) (a rigorous proof is contained in our 1990 CentER working paper):

Theorem 3 (Hillman and Riley): *If $v_1 > v_2 > v_3 \geq \cdots \geq v_n$, the Nash equilibrium is unique. In equilibrium, player 1 randomizes continuously on $[0, v_2]$. Player 2 randomizes continuously on $(0, v_2]$, placing an atom of size $\alpha_2(0) = (v_1 - v_2)/v_1$ at zero. Players 3 through n bid zero with probability one. Player 1's equilibrium payoff is $u_1^* = v_1 - v_2$, while players 2 through n earn payoffs of zero.*

The algebraic form of the equilibrium mixed strategies for the case when $v_1 > v_2 > v_3 \geq \cdots \geq v_n$ are as follows. Players 3 through n bid zero with probability one. Players one and two randomize according to $G_1(x) = x/v_2$ and $G_2(x) = (v_1 - v_2 + x)/v_1$ for $x \in [0, v_2]$.

III A concluding example

We conclude with an example that highlights our results in the context of the regulatory contest discussed by Wenders (1987) and Ellingsen (1991).[13] Suppose

[12] Milgrom (1981) and Bikhchandani and Riley (1991) examine a similar issue in standard (winner pay) auctions, and find the opposite result often holds for classes of standard auctions: symmetric strategies may yield higher expected revenues.

[13] Of course, there are numerous other applications, as noted in the introduction. For example, the following analysis is analogous to the case of an incumbent versus a number of potential entrants as discussed in Rogerson (1982) and Dasgupta (1986).

$M \geq 2$ potential producers compete for the monopoly right to run a public utility. They face opposition from a consumer organization. The regulatory body decides to reward the organization which exerts the highest effort in the lobbying process. If this turns out to be one of the producers, the monopoly solution is implemented. If the consumer organization wins, the marginal cost pricing solution is implemented.

If the consumer group wins, it earns a payoff equal to the sum of the would-be monopoly profits (call this amount "T" for "Tullock square") and the would-be deadweight loss (H, for "Harberger triangle"). If one of the producers wins, it earns the monopoly profits, T. Thus $v_1 = T + H$ and $v_2 = v_3 = \cdots = v_{M+1} = T$. By Theorem 2A, there exists a continuum of equilibria to this game, and by 2C the equilibria are not revenue equivalent. In particular, the expected revenue to the regulator is

$$E\sum x_i = \frac{v_2}{v_1} v_2 + \left(1 - \frac{v_2}{v_1}\right) Ex_1 = \frac{T^2}{T+H} + \frac{H}{T+H} Ex_1. \tag{5}$$

Since Ex_1 varies depending upon which equilibrium is played, when the regulator receives the lobbying expenditures as "bribes" she is not indifferent to the equilibrium that is played. By Theorem 2C, Ex_1 is maximized when only one of the firms participates in the lobbying process.[14] The selfish regulator does best in the equilibrium where only the consumer group and one of the M firms engage in lobbying.

It turns out that the expected social waste due to lobbying also depends on which equilibrium is played. Suppose that only a proportion $\lambda, 0 \leq \lambda \leq 1$, of the lobbying expenditures is socially wasteful (see, e.g., Fudenberg and Tirole, 1977; Brooks and Heydra, 1990; and Dougan, 1991). Expected social waste, W, equals the expected deadweight loss plus a fraction λ of the expected lobbying expenditures. If P_1 is the probability the consumer group wins, then the expected social waste is

$$W = (1 - P_1)H + \lambda E\sum x_i. \tag{6}$$

Using equation (5) and the fact[15] that $Ex_1 = P_1 v_1 + v_2 - v_1$, this can be written as

$$W = \lambda T + (1 - \lambda)\frac{H}{H+T}(T - E[x_1]). \tag{7}$$

If $\lambda = 1$ (all of the lobbying is socially wasteful) the expected social waste is T. Notice that this result is independent of which equilibrium is played.[16] In contrast, when

[14] As an example, consider the three player case with $v_1 = 2$, and $v_2 = v_3 = 1$. In the most asymmetric equilibrium, player 3 bids zero with probability one, while $G_1 = x$ and $G_2 = (1 + x)/2$ on $[0, 1]$. In this equilibrium, $Ex_1 = 1/2$, and by equation (5), $\Sigma Ex_i = 3/4$. In the "symmetric equilibrium", player one randomizes with $G_1(x) = x[(1 + x)/2]^{-1/2}$, while players two and three use $G_2 = G_3 = [(1 + x)/2]^{1/2}$. In this case,

$$\sum Ex_i = [5 - 2\sqrt{2}]/3,$$

which is less than 3/4, as of course it must be by Theorem 2C.

[15] Theorem 2B implies that, in equilibrium, $Eu_1 = P_1 v_1 - Ex_1 = v_1 - v_2$.

[16] Ellingsen (Proposition 1) considers the case where $\lambda = 1$ and a finite number of possible equilibria. Equation 7, however, reveals that Ellingsen's result is valid across the entire continuum of other equilibria.

$0 \leq \lambda < 1$ the expected social waste is a decreasing function of Ex_1, which in turn depends on which equilibrium is played. By Theorem 2C, it follows that the more symmetric the bidding strategies of the producers, the greater the expected social waste, W. This holds irrespective of λ, except when lobbying is completely wasteful (in which case, $\lambda = 1$ and hence $W = T$). When $\lambda \in [0, 1)$, different equilibria imply different expected social wastes, and society prefers fewer firms lobbying for monopoly rights to more.

Appendix: Proof of Theorem 2

Proof of 2A and 2B: The proof of parts A and B of Theorem 2 are contained in the following lemmas. Before proceeding, note that if $\bar{s}_i$ and $\underline{s}_i$ are the upper and lower bounds of the support of player i's mixed strategy, then $\forall i, v_i \geq \bar{s}_i \geq \underline{s}_i \geq 0$. Also, recall that $\alpha_i(x)$ is the mass placed at x by player i's mixed strategy.

The first lemma is used in Lemma 2 to show that the lower bound of the support of each player's mixed strategy is zero.

Lemma 1: If $\exists i$ such that $\underline{s}_i \geq \underline{s}_j$ and $\alpha_i(\underline{s}_j) = 0$, then $\underline{s}_j = 0$ and $G_j(0) = \lim_{x \uparrow \underline{s}_i} G_j(x)$. If, in addition, $\alpha_i(\underline{s}_i) = 0$, then $G_j(0) = G_j(\underline{s}_i)$.

Proof: Let $u_j(x_j, G_{-j})$ denote j's payoff to bidding x_j when strategies G_{-j} are employed by the other $n-1$ players. Now $u_j(\underline{s}_j, G_{-j}) = -\underline{s}_j < 0$ for $\underline{s}_j > 0$. Since the same holds for $u_j(x_j, G_{-j})$ for $x_j < \underline{s}_i$, and also for $x_j = \underline{s}_i$ if $\alpha_i(\underline{s}_i) = 0$, the claim follows. □

Lemma 2: $\underline{s}_i = 0 \ \forall i$.

Proof: Clearly, $v_i \geq \underline{s}_i \geq 0 \ \forall i$, so it is sufficient to show that no player employs a mixed strategy that has a support with a strictly positive lower bound. By way of contradiction, suppose $S \equiv \{i | \underline{s}_i > 0\}$ is nonempty, i.e., $\underline{s}_i > 0$ for at least one i.

If S consists of a single player i, then $\underline{s}_i > \underline{s}_j = 0 \ \forall j \neq i$. In this case, if $\alpha_i(\underline{s}_i) = 0$, Lemma 1 implies that $G_j(0) = G_j(\underline{s}_i) \ \forall j \neq i$, which in turn implies that $u_i(\underline{s}_i, G_{-i}) < \lim_{x_i \downarrow 0} u_i(x_i, G_{-i})$. This contradicts the hypothesis that $\underline{s}_i > 0$. If $\alpha_i(\underline{s}_i) > 0$, then $\forall j \neq i$, $\alpha_j(\underline{s}_i) = 0$, so $G_j(0) = \lim_{x_j \uparrow \underline{s}_i} G_j(x_j)$ leads to a similar contradiction.

If S contains more than one player, then an argument similar to that just made implies $\underline{s}_i = \underline{s}_j > 0 \ \forall i, j \in S$. At least one player $i \in S$ must employ a mixed strategy with $\alpha_i(\underline{s}_i) = 0$, for otherwise any $j \in S$ could gain by increasing $\underline{s}_j$ by a small $\varepsilon > 0$ (unless $\underline{s}_j = v_j$, in which case j has incentive to reduce the bid v_j to 0). But this means that there exist $i, j \in S$ such that $\underline{s}_i = \underline{s}_j > 0$ and $\alpha_i(\underline{s}_i) = 0$, a contradiction to Lemma 1.

Thus, we conclude that $\underline{s}_i = 0$ for all i. □

The next lemma shows that, in any mixed-strategy equilibrium, each player 2 through n must employ a strategy that places an atom at 0, while player 1 cannot employ a strategy that places an atom at 0. This, in conjunction with Lemma 2, implies that players 2 through n earn equilibrium expected payoffs, u_i^*, of zero.

Lemma 3: (a) $\alpha_1(0)=0$;
(b) $\forall i \neq 1, \alpha_i(0)>0$.
(c) $u_i^* = 0 \; \forall i \neq 1$.

Proof: (a) Since player i would never use a strategy that puts mass on (v_i, ∞) (setting the bid equal to zero strictly dominates such a strategy), player 1 clearly has no incentive to use a strategy that puts mass in the interval $(v_2, v_1]$. Hence, $\forall i, \bar{s}_i \leq v_2 < v_1$, which guarantees that player 1 must have an equilibrium payoff u_1^* of at least $v_1 - v_2 > 0$. This, and the fact that not all players can use mixed strategies that have an atom at 0, implies that player 1's mixed strategy cannot place an atom at 0. (b) From part (a), $u_1^* > 0$ in every neighborhood above 0, so player 1 must outbid every other player with a probability that is bounded away from zero. Thus, every player but player 1 must use a strategy that has an atom at 0. (c) Since player 1's mixed strategy does not have an atom at 0, it follows from part (b) that $\forall i \neq 1$, $u_i^* = u_i(0, G_{-i}) = 0$ □

We have now established that zero is the lower bound of the support of each player's equilibrium mixed strategy, that all players but player 1 must employ equilibrium strategies that contain an atom at 0, and that the equilibrium payoffs for players $\{2, 3, \ldots, n\}$ are zero. The next lemma establishes that at least two players have v_2 as the upper bound of the support of their mixed strategies.

Lemma 4: $\bar{s}_i \leq v_2 \forall i$, with strict equality for at least two players.

Proof: From the proof of Lemma 3, $\bar{s}_i \leq v_2 \forall i$. By way of contradiction, suppose $\bar{s}_i < v_2$ for all i. By bidding above $\bar{s} \equiv \max_k\{\bar{s}_k\}$ by an arbitrarily small amount, player 2 can earn arbitrarily close to $v_2 - \bar{s} > 0 = u_2^*$, which contradicts Lemma 3. Thus, $\bar{s}_i = v_2$ for at least one i. Another player $j \neq i$ must also have $\bar{s}_j = v_2$, for otherwise player i could gain by reducing $\bar{s}_i$ by a small $\varepsilon > 0$. □

The next five lemmas provide the rough characterization of the equilibrium strategies of players $\{2, 3, \ldots, n\}$ stated in Theorem 2A. For notational convenience, we define $A_i(x) \equiv \Pi_{j\neq i} G_j(x)$, $A_{ik}(x) \equiv \Pi_{j\neq i,k} G_j(x)$, and $A_{ikh}(x) \equiv \Pi_{j\neq i,k,h} G_j(x)$.

Lemma 5: For all $j \in \{1, 2, \ldots, n\}$, G_j contains no atoms in the half open interval $(0, v_2]$.

Proof: Suppose one of the cdf's, say G_i, has an atom at $x_i \in (0, v_2]$. Lemma 2 implies that $\forall x \in (0, v_2]$, $A_{ij} G_i > 0$, and hence $A_{ij} G_i$ has an upward jump at x_i, $\forall j \neq i$. This follows directly from the monotonicity of the cdf's. For $x_i < v_j$ this implies that player j can gain by transferring mass from an ε-neighborhood below x_i to some δ neighborhood above x_i. At $x_i = v_j$ it pays for j to transfer mass from an ε-neighborhood below x_i to zero. Thus, there would be an ε-neighborhood below x_i in which no other player's mixed strategy puts mass. But then it is not an equilibrium strategy for player i to put mass at x_i. □

Define $B_i(x_i) \equiv (v_i - x_i) A_i(x_i) - x_i(1 - A_i(x_i)) = v_i A_i(x_i) - x_i$.

Lemma 6: $B_i(x_i)$ is constant and equal to u_i^* at the points of increase of G_i on $(0, v_2]$ $\forall i$. $B_i(x_i) \leq u_i^*$ if x_i is not a point of increase of G_i on $(0, v_2]$.

Proof: By Lemma 5 there are no atoms in $(0, v_2]$. Thus, $B_i(x_i)$ is the expected payoff to player i from bidding $x_i \in (0, v_2]$. If x_i is a point of increase of G_i, player i must make his equilibrium payoff at x_i. □

Lemma 7: $\forall x \in (0, v_2], \exists i_1, i_2$ such that $\forall \varepsilon > 0$: $G_i(x+\varepsilon) - G_i(x-\varepsilon) > 0$, $i = i_1, i_2$. □

Proof: Immediate.

Lemma 8: $\bar{s}_i = 0 \; \forall i > m$.

Proof: Without loss of generality assume $\bar{s}_{m+1} = \max_{i>m}\{\bar{s}_i\}$. Suppose $\bar{s}_{m+1} \neq 0$. Then there exists an interval $(\bar{s}_{m+1} - \varepsilon, \bar{s}_{m+1}]$ in which G_{m+1} increases and in which $B_{m+1}(x) = u^*_{m+1} = 0 = v_{m+1}A_{m+1}(x) - x$. Thus, $v_{m+1} = x/A_{m+1}(x) \forall x \in (\bar{s}_{m+1} - \varepsilon, \bar{s}_{m+1}]$. From Lemma 7, $\forall x \in (\bar{s}_{m+1}, v_2]$ $\exists i \in \{2, \ldots, m\}$ such that G_i is increasing at x. Since there are no atoms in $(\bar{s}_{m+1}, v_2]$, for each $x \in (\bar{s}_{m+1}, v_2]$ there is a player $i \in \{2, \ldots, m\}$ such that $v_i = x/A_i(x)$. This implies that for any $x > \bar{s}_{m+1}$, but arbitrarily close to $\bar{s}_{m+1}$, there exists an $i \in \{2, \ldots, m\}$ such that $A_i(x) = \Pi_{j \neq i} G_j(x) > \Pi_{j \neq i} G_j(\bar{s}_{m+1}) > \Pi_{j \neq m+1} G_j(\bar{s}_{m+1}) = A_{m+1}(\bar{s}_{m+1})$, a contradiction to the fact that $v_{m+1} < v_i$. Thus, $\bar{s}_{m+1} = 0$.

Lemma 8 demonstrates that when $n > m$, players $m+1$ through n bid zero with probability one. We now proceed to characterize the equilibrium strategies of players 1 through m.

Lemma 9: Suppose $x \in (0, v_2]$ is a point of increase in G_i and G_j for $i, j \in \{2, 3, \ldots, m\}$. Then $G_i = G_j$ at x.

Proof: By Lemmas 3c and 6, $B_i(x) = B_j(x) = 0$, which may be written as $(v_2 - x)G_j(x)A_{ij}(x) - x[1 - G_j(x)A_{ij}(x)] = 0$. This implies that $G_j(x)A_{ij}(x) = x/v_2 = G_i(x)A_{ji}(x)$. Division by $A_{ij} = A_{ji} > 0$ gives $G_j(x) = G_i(x)$. □

Lemma 10: If G_i, $i \in \{2, \ldots, m\}$ is strictly increasing on some open subset (a, b), where $0 < a < b < v_2$, then G_i is strictly increasing on the entire interval, $(a, v_2]$. Furthermore, at least one of players $\{2, \ldots, m\}$ randomizes continuously on the interval $(0, v_2]$.

Proof: Suppose to the contrary that G_i were constant on (b, c), $b < c \leq v_2$. Then from Lemma 5, $G_i(b) = G_i(c)$. By Lemma 7, there exists an $\varepsilon > 0$ such that on the interval $(b, b+\varepsilon)$ there exist at least two players, h and k, with strictly increasing cdf's over the interval. At least one of these players, say h, must be an element of $\{2, \ldots, m\}$. Since the mixed strategies contain no atoms in the interval $(0, v_2]$, from Lemma 9 $G_h(b) = G_i(b) > 0$. But from Lemmas 3c and 6, $B_i(b) = B_h(b) = B_h(x) \; \forall x \in (b, b+\varepsilon)$. Hence, $B_i(x) \leq B_h(x) \; \forall x \in (b, b+\varepsilon)$, since such values of x do not lie in i's support. But this implies that $A_i(x) \leq A_h(x)$, and hence $G_h(x) \leq G_i(x)$, a contradiction to the fact that $G_i(b) = G_h(b)$, G_h is increasing on $(b, b+\varepsilon)$, and G_i is constant on $(b, b+\varepsilon)$. The second statement follows from the first part of Lemma 10 and from Lemmas 4 and 7. □

Lemma 10 thus shows that, in equilibrium, at least one of the players $\{2, 3, \ldots, m\}$ randomizes continuously on $(0, v_2]$. Notice that, by Lemma 3, the mixed strategies of each of the players $\{2, 3, \ldots, m\}$ contain an atom at 0 but, by Lemma 5, no player's mixed strategy places an atom in the half-open interval $(0, v_2]$. Lemma 10 thus implies that if $G_i, i \in \{2, 3, \ldots, m\}$ is increasing over any interval (a, b), $0 < a < b < v_2$, then G_i must be strictly increasing on the interval $(a, v_2]$. Hence, any gap in the support of player i's mixed strategy must be of the form $(0, b_i]$ for some $b_i > 0$. Furthermore, from Lemma 9, for any point of increase $x \in (0, v_2]$ of G_i and G_j, $i, j \in \{2, 3, \ldots, m\}$, these distribution functions must take identical values.

In order to provide the complete characterization of the equilibrium distributions provided in Equation (4) of the text, we need to say more about player 1's equilibrium strategy and payoffs. We continue with

Lemma 11: (a) $\bar{s}_1 = v_2$. Furthermore, for every bid $0 < x < v_2$ in the support of G_1, $G_1(x) < G_i(x)$, $i \in \{2, \ldots, m\}$.
(b) $u_1^* = v_1 - v_2$.

Proof: (a) From Lemma 10, at least one $j \in \{2, 3, \ldots, m\}$ randomizes continuously on $(0, v_2]$. Without loss of generality suppose player two is such a player. From Lemma 3, player 1's mixed strategy does not have an atom at 0, and from Lemma 5, no player's mixed strategy has an atom in $(0, v_2]$. Thus, there exists some point $x \in (0, v_2)$ at which $G_1(x)$ is increasing. At any such point, $B_1(x) \geq v_1 - v_2$, since the right-hand-side represents what player 1 can obtain by bidding v_2 with probability one. Rearranging this expression we obtain $A_1(x) \geq (v_1 - v_2 + x)/v_1$. From Lemmas 3 and 6, $A_2(x) = x/v_2$. Subtracting A_1 from A_2 gives

$$A_2(x) - A_1(x) \leq [(v_2 - x)(1 - v_1/v_2)]/v_1 < 0,$$

where the strict right-hand inequality follows from the assumption that $v_2 > x$ and $v_1 > v_2$. Thus, at any point of increase of G_1 in $(0, v_2)$, $A_1 > A_2$. This directly implies that $G_2 > G_1$ for any such point. But since G_2 has support $[0, v_2]$ and G_1 has no atoms, this implies $\bar{s}_1 = v_2$. Furthermore, since for any other player $i \in \{2, \ldots, m\}$ and for any $x \in [0, v_2]$, $G_2(x) \leq G_i(x)$, we have the second claim. (b) Part (a), together with Lemma 6, implies that player 1's equilibrium payoff is $u_1^* = v_1 - v_2$. □

This completes the proof to part B of Theorem 2. To complete the proof to part A we must show:

Lemma 12: (a) Player 1 randomizes continuously on support $[0, v_2]$.
(b) $\Pi_{i=2}^{m} \alpha_i(0) = (v_1 - v_2)/v_1$.

Proof: (a) We know that $\bar{s}_1 = v_2$ and $\underline{s}_1 = 0$. Suppose there is a gap (a, b) in which $G_1(x)$ is constant, $0 < a < b < v_2$. By Lemmas 6, 7, and 8, we know that at $x = a$ there are at least two players $i, k \in \{2, \ldots, m\}$ such that $A_i(x) = A_k(x) = x/v_2$. At $x = b$ this holds as well. In addition, since a and b are in the support of $G_1, A_1(x) = (v_1 - v_2 + x)/v_1, x = a, b$. Thus we have

$$G_1(x)G_k(x)A_{ik1}(x) = x/v_2, \qquad x = a, b \tag{A1}$$

$$G_i(x)G_k(x)A_{ik1}(x) = (v_1 - v_2 + x)/v_1, \quad x = a, b. \tag{A2}$$

Since $G_1(a) = G_1(b)$ by assumption, and by Lemma 9 $G_i(x) = G_k(x)$ for $x \in [a, b]$, equation A1 implies

$$[G_k(a)A_{ik1}(a)]/[G_k(b)A_{ik1}(b)] = a/b,$$

while equation A2 implies

$$[[G_k(a)]^2 A_{ik1}(a)]/[[G_k(b)]^2 A_{ik1}(b)] = (v_1 - v_2 + a)/(v_1 - v_2 + b).$$

Combining these gives $G_k(a) = G_k(b)[b(\theta + a)/a(\theta + b)]$, where $\theta \equiv v_1 - v_2 > 0$. Since $b/a > (b + \theta)/(a + \theta)$, this implies $G_k(a) > G_k(b)$, which contradicts the fact that $b > a$. Thus, player 1's mixed strategy distributes positive mass to every open interval in $[0, v_2]$. This, along with Lemmas 3a and 5, implies that player 1's mixed strategy contains no atoms and has a strictly increasing cdf on its support, $[0, v_2]$. Part (b) follows from part (a), Lemma 6, and Lemma 11b. □

We now know that in any equilibrium: (1) player 1 earns an expected payoff of $v_1 - v_2$, while all other players earn expected payoffs of zero; (2) player 1's mixed strategy contains no atoms or gaps in its support, and thus G_1 is strictly increasing on its support, $[0, v_2]$; (3) players $m + 1$ through n bid zero with probability one; and (4) all other players $j \in \{2, \ldots, m\}$ play a mixed-strategy that has an atom at zero and a strictly increasing cdf on some interval of the form $(b_j, v_2]$, where $b_j \geq 0$ for all j, with strict equality for at least one j. Lemma 9 guarantees that in subintervals of $(0, v_2]$ where the mixed strategies of any subset of the players $\{2, 3, \ldots, m\}$ apply a positive mass, the players have the same value of their cdf's. The system of equations given by $B_i(x) = u_i^*$ for $i \in \{1, 2, \ldots, m\}$ in Lemma 6 thus determines the equilibrium mixed-strategies, $G_i(x)$, for any given nonnegative vector $(b_2, b_3, \ldots, b_m)$ for which at least one $b_i = 0$. These are given in equation (4) in the text. Recursive application of Lemma 9 for given b_i's implies that these constitute all the equilibria.

Proof of 2C: Theorem 1 in Baye, Kovenock and De Vries (1993) establishes that, in any Nash equilibrium, $\Sigma Ex_i = (v_2/v_1)v_2 + (1 - v_2/v_1)Ex_1$. Hence it is sufficient to establish that (a) Ex_1 is maximized in an equilibrium in which all but one of players 2 through m bid zero with probability one, and (b) Ex_1 is minimized when players 2 through m play symmetric strategies. Our proof makes use of the fact that if cdf F stochastically dominates cdf G, then $E_F[x] > E_G[x]$.

(a) By Lemma 10, in any equilibrium at least one of the players $2, 3, \ldots, m$, randomizes continuously on the interval $(0, v_2]$. Suppose player i is such a player. By Lemma 6, $B_i(x) = 0 \; \forall x \in (0, v_2]$. Isolating the cdf of player 1, G_1, in the expression for A_i yields $G_1(x) = [x/(v_2 \Pi_{j \neq 1,i} G_j(x))]$. Hence, across all equilibria, $G_1(x)$ is minimized for each $x \in (0, v_2]$ when the denominator is maximized (note that every equilibrium must have an $i \in \{2, \ldots, m\}$ randomizing continuously over $(0, v_2]$). This implies that $G_1(x)$ is minimized when $\Pi_{j \neq 1,i} G_j(x) = 1$ (that is, in the equilibrium where only players 1 and i actively bid.) But this means that G_1 in this asymmetric equilibrium stochastically dominates the corresponding G_1's that arise in the other equilibria, which implies Ex_1 is maximized in this equilibrium.

(b) Similarly, suppose player $i \in \{2, \ldots, m\}$ randomizes continuously on $(0, v_2]$. Then $G_1(x)$ is maximized for each $x \in [0, v_2]$ across equilibria when $\Pi_{j \neq 1,i} G_j(x)$ is minimized. By Lemma 12, in any equilibrium player 1 randomizes continuously over $(0, v_2]$. This implies by Lemma 6 that in any equilibrium $A_1(x) = (v_1 - v_2 + x)/v_1, \forall x \in (0, v_2]$. Since $A_1(x) = \Pi_{j \neq 1} G_j(x)$ is constant across equilibria, $\Pi_{j \neq 1, i} G_j(x)$ is minimized in an equilibrium in which $G_i(x)$ is maximized. But by Lemmas 5, 9, and 10, in any equilibrium and for every $j \in \{2, \ldots, m\}$, $j \neq i$, $G_i(x) \leq G_j(x)$ $\forall x \in [0, v_2]$. Hence maximizing $G_1(x)$ across equilibria requires maximizing the minimum of the $G_k(x)$'s, $k \in \{2, \ldots, m\}$. Since for each $x \in (0, v_2]$, $A_1(x)$ is constant across equilibria, this is done by setting $G_k(x) = G_j(x)$ for all $k, j \in \{2, \ldots, m\}$ on $[0, v_2]$. $\square$

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