A One-Line Proof for Complementary Symmetry

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ABSTRACT. Complementary symmetry was derived before under particular theories, and used to test those. Progressively general results were published. This paper proves the condition in complete generality, providing a one-line proof.

KEYWORDS: complementary symmetry, buying price, selling price, WTP, WTA

Birnbaum et al. (2016) introduced a complementary symmetry preference condition for binary monetary prospects. Their Theorem 1 showed that it holds for the version of prospect theory of Schmidt, Starmer, & Sugden (2008), considered before by Birnbaum & Zimmermann (1998), under some popular parametric assumptions. Those included power utility with the same power for gains and loseses. Before, Birnbaum & Zimmermann (1998, Eq. 22) had obtained that result under prospect theory for fifty-fifty binary prospects. Lewandowski (2018) extended the result to any strictly increasing continuous utility function u with $u(0) = 0$, both for regular prospect theory and for the theory of Birnbaum & Zimmermann (1998) and Schmidt, Starmer, & Sugden (2008). Finally, Chudziak (2020) extended the result to any preference functional that gives unique buying and selling prices. Birnbaum (2018) discussed the empirical performance of complementary symmetry, in particular its violations.

All aforementioned results concerned the domain of all binary prospects and assumed a preference functional, implying weak ordering, on that domain. We generalize the result to any binary relation on any subset of binary prospects. Our proof takes only one line.

Let $x_p y$ denote a *prospect* yielding *outcome* x with probability $0 \le p \le 1$ and outcome y with probability $1-p$. Outcomes are real-valued, designating money. The prospect $0_1 0$ is identified with the outcome 0. By ~ we denote a binary relation on binary prospects. The aforementioned papers assumed that ~ is the indifference part of a transitive complete preference relation, but we will not impose any restriction on ~.

B is a *buying price* of $x_p y$ if

$$0 \sim (x-B)_p(y-B) . \tag{1}$$

S is a *selling price* of $x_{1-p}y$ $(=y_p x)$, or a *complementary selling price* of $x_p y$, if

$$0 \sim (S-y)_p(S-x) . \tag{2}$$

These definitions are the most common ones. Several alternative definitions have been considered (Bateman et al, 2005, §3; Lewandowski 2018, appendix). We use the ones that Birnbaum et al. (2016) adopted in their definition of complementary symmetry, given below. In economics, the terms willingness to pay and willingness to accept are often used instead of buying and selling prices.

Substituting $S = x + y - B$, Equations (1) and (2) are identical:

[B = buying price of x_py] $\Longleftrightarrow$ [S = x+y–B is complementary selling price of x_py] (3)

Eq. 3 is called *complementary symmetry for* x_py, and provides a one-line proof of the following theorem, generalizing the results cited above.

THEOREM 1.[1] For each x_py, complementary symmetry holds. Hence, a buying price B exists if and only if a complementary selling price S exists. B is unique if and only if S is unique. If B is unique, then $S = x + y - B$. □

Because we consider complementary symmetry only for one x_py, our result can be applied to any subset of binary prospects. Our main contribution is the simplified proof. An empirical implication is that the violations of complementary symmetry, surveyed by Birnbaum (2018), concern more fundamental problems than thought before.

References

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[1] Further, under existence and uniqueness: if one of the three $[0 \sim (x-B)_p(y-B)]$, $[0 \sim (S-y)_p(S-x)]$, and $[B + S = x + y]$ holds, then the other two are equivalent (Chudziak 2020 Theorem 2.2).

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