When Debit=Credit,

The balance constraint in bookkeeping, its causes and consequences for accounting.

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Abstract

This paper studies the balance constraint (debit=credit) in bookkeeping, its causes and its consequences for accounting. Balance in the ledger is shown to: 1) imply balance in journal entries and vice versa; 2) link the value definitions in the earnings statement and balance sheet; 3) have direct implications for valuation puzzles encountered in accounting, like accounting for OCI or stock-based compensation, and the difference between earnings or balance-sheet approaches to valuation. These system-wide effects on accounting highlight a design question: why do we have the balance constraint in bookkeeping? Backward-engineering shows 6 axioms that logically lead to double-entry bookkeeping. The balance constraint follows from the existence of a residual account: owner's equity. A class of equivalently powerful record keeping systems is shown to exist. These systems use double-entry bookkeeping without the monetary-unit assumption and can be used to record other outputs of the organization, like societal impact. These systems can be implemented in relational databases, a blockchain, or a different technology all together. The discussion covers links with other mathematical descriptions of bookkeeping and potential avenues for future research in the mathematics of bookkeeping.

Keywords: Axioms for bookkeeping, duality, bookkeeping system design, mathematics of record keeping.

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1 Introduction

Accounting for an economic event occurs in three steps, first the event (transaction) is translated into accounting definitions, second the value of this event is defined, third the event is recorded. The first two steps deal with the definitions and valuation of transactions, they are covered by the accounting definitions within IFRS or local GAAP and are often discussed in the accounting literature. The third step, the actual bookkeeping, seems to have received less attention in recent accounting research.

Bookkeeping is the primary technology for accounting and a primary source of data for all of economics. The importance of the bookkeeping process for economic record-keeping should not be underestimated. A significant part (if not most) of accounting data is recorded through double-entry bookkeeping. This data is used for decision making within companies, to decide on stock investments by investors, to supply tax authorities with information, to calculate national income accounts, etc. The better we understand bookkeeping, the better we understand the data that it generates. Furthermore, as the last step in the accounting process, it should be the first step in the improvement or optimization of this process.[1] This is why questions about the best design for bookkeeping processes recur every time new technologies for record keeping are introduced. They were asked around the advent of business operations research (see for instance Mattessich 1964), with new developments in Accounting Information Systems (see for instance McCarthy 1982; Harper 1985; Geerts et al. 2013) and now again with the rise of a new type of record keeping: the blockchain (Andersen 2016; Arnold 2018; ICAEW 2019).

As bookkeeping is the primary technology of accounting, questions about the design of bookkeeping are particularly relevant for this field. Providing answers increases both the academic relevance (e.g. Arya et al. 2003; Demski 2007; Fellingham 2007; Basu 2012) and the practical relevance of accounting research (e.g. McCarthy 2012; Ellerman 2014). Through the design of improved or alternative record-keeping systems, we can (re-)evaluate how our current system serves its goals of stewardship and decision-usefulness in response to changing circumstances (Olivier 2000; Ravenscroft and Williams 2005). This paper takes some steps in the suggested direction and studies the cause and effect of the requirement that $debit = credit$ in any valid financial statement (further the balance constraint).[2] The results of this analysis shows why we impose the balance con-

[1] In optimization, as in planning and budgeting, Bellman's principle requires us to work backwards. Planning for a company starts with the sales budget, optimizing the accounting process starts with bookkeeping and both move backwards in time.

[2] There are several interpretations of this equality and its link to double entry bookkeeping, that have led to several names being used to discuss this idea in different streams of literature. Most accounting texts appear to imply that this constraint defines double-entry, since it requires that at least 2 accounts are present, one with a debit balance and one with a credit balance. Therefore, this constraint could be called the double-entry constraint. In mathematical treatments, double-entry refers to the two entries –debit and credit –in each element, regardless of balance. This difference in definitions of double entry

straint, some issues it causes in accounting and what solutions exist to these issues, and how double-entry bookkeeping can be used in alternative record-keeping systems like blockchain-based systems, or societal impact accounting systems.

Bookkeeping has a strict set of rules that need to be followed. As long as these rules are followed, processing a transaction (journal entry) transforms one set of valid bookkeeping statements (the ledger) into another set of valid bookkeeping statements (e.g. updated ledger). An operation that takes two elements of a set and transform them into a third element from the same set is analogous to a group law operating on its associated group set. This paper expands the group-theoretical work of Ellerman (1982, 1985), it adds the balance constraint to the Abelian group used to describe bookkeeping. A one-to-one relation between this Abelian groups and the group of $\mathcal{R}^n$ with vector addition is shown. Because these groups are equivalent, we can use what we know about addition of real numbers to more easily understand the mathematical description of bookkeeping. The first proposition shows that the balance constraint as imposed on the entire set of accounts held by an economic entity, implies that every journal entry made by this entity has to be balanced. The reverse result is also shown, if we require every journal entry made to be balanced, the resulting ledger will be balanced.

The balance constraint does, however, act as a constraint. The logic that implies that the change caused by a journal entry is balanced, also implies that any change caused by new definitions of value has to be balanced. If we change the value definitions that determine earnings, for instance because earnings are manipulated or we switch from IFRS to local GAAP, both the pre- and post-change financial statement are still required to be balanced. This is only possible if we adjust the value definitions used in the balance sheet and the value definitions used in earnings simultaneously. The balance constraint thus ties together the value definitions used in the balance sheet and income statement, making it impossible to define these separately. The connection between the value definitions used for the balance sheet makes statements falsifiable and thus creates control options, but it also means we loose flexibility in the definitions of value. The loss in flexibility can cause a relevance-reliability trade-off. For instance, to achieve maximum relevance of the earnings statement, we would want to value and record performance as accurately as possible regardless of the organizational-valuation effects. Similarly, we want to report existing valuation in the organization as accurately as possible, regardless

can lead to confusion and some irritation, see e.g. Ellerman (2014). Furthermore the balance constraint does not require debit and credit entries, but can be equivalently imposed on a list of scalar accounts, see Lemma 2. In theoretical accounting research, like Mattessich (1964); Ijiri (1975), this requirement is captured in the concept of duality. However, duality of each value flow is closely linked to the classification (what account) and measurement or valuation (what amount) of the transactions, while here we only require balance in recording. Hence, in this paper we will call it the balance constraint. The balance sheet equality, i.e. the fact that $debit = credit$ on a well-formed balance sheet, is a specific example of this more general requirement.

whether it is due to performance or random shocks. However, the value definitions of the earnings statement are tied to the value definitions of the balance sheet, such that these independent value definitions are not possible. This has direct implications for some valuation and reporting puzzles in accounting and the examples of OCI and stock awards are discussed. However, as the discussion of OCI and stock awards show, extra accounts can be used to overcome the relevance–reliability trade-offs. These accounts can act as buffers that absorb the differences between optimal performance and valuation measures of value. This suggests that the difference between the earnings and balance-sheet approaches to value, which is often perceived to exist in standard setting (Benston et al. 2007; Dichev 2008, e.g.), could be resolved by adding accounts. Such extra accounts have statistical properties that allows us to learn more about the measurement problem they are used for. If, for instance, OCI is consistently negative, that can indicate a upward bias in earnings reporting. However, if OCI is very volatile but has an expected value near 0, this indicates that the balance constraint was a severe impediment to relevant and reliable reporting of performance through net change in equity.

The restrictions, and relevance-reliability trade-off caused by the balance constraint does pose a fundamental design question: if balance imposes these issues on record keeping, why do we require it in the first place? Double-entry bookkeeping, with the balance constraint, has survived the test of time. Several alternative bookkeeping systems existed at the time of Pacioli (1494), which seems to indicate that this system works better than alternatives. To show what design choices caused double-entry bookkeeping to thrive, one first has to find the choices made. These choices are shown through a proposed axiomatic basis for double-entry bookkeeping. The six axioms are: 1, there is an entity that experiences events that require bookkeeping; 2, value is defined in the form of signed, real measures; 3, there is a list of $n < \infty$ values to keep track of; 4, all value is measured in some common (monetary) unit; 5, the event 'no transaction occurred' has zero value; 6, one of the n elements absorbs the residual value of the other $n-1$ elements. The first 4 of these stem from earlier work in accounting theory, most directly from Mattessich (1964) and Ijiri (1975). They combine the need for an administrative system (1,3) with the monetary-unit assumption in accounting (2,4). The fifth and sixth axioms form the reason for the balance constraint. The fifth axiom means that we do not have to keep track of events that do not impact valuation. The sixth axiom follows from the definition of owner's equity. The owners of a company receive anything that remains of the assets after the debt is paid, this automatically means that if the assets increase by more than the liabilities, owner's equity increases with the residual value. This account (and its sub-accounts) thus balances out the accounting system.

The results derived can be applied directly to the design of different record keeping systems. As long as all transactions are balanced, we can string them together into a balanced set of books. For blockchain-based record keeping, this means that any general ledger – when constructed from the record of individual, balanced transactions in the public ledger– will resemble the balance sheet of a central bank. Coins outstanding form one side of the derived ledger, the number of coins owned by private parties are posted at the other side of the derived ledger, as if the coins are deposited there. For relational databases, the equivalence between balanced journal entries and balanced ledgers can be used to aid efficient design. Verifying balance at the individual transactions implies that the balance verification of the entire system is redundant and thus could be skipped without any loss. The axiomatic design shows that as long as we can define a real measure for all relevant outputs and a marginal price for all these measures, double-entry bookkeeping can provide a measure of performance for the total impact these companies have on wider society. This performance number number can be calculated in a similar fashion as profits. Such a single-number performance measure allows the use of well-known performance contracts, like exist for profit maximization, to be applied around maximization of societal impact.

The bookkeeping process has been modeled mathematically before, for instance in terms of matrix algebra (e.g. Mattessich 1964; Butterworth 1972; Arya et al. 2000; Ijiri 1993). There are clear links between the matrix-description and the formalism used in this paper. The group theoretic work in this paper focuses more on fundamental properties of the information structure and how they influence the bookkeeping system, while the matrix-based description focuses more on the processes of updating the books and how this process can be modeled. As such they serve different, but complementary goals. Section 5 discusses the links between the mathematical descriptions in more detail.

In the following sections show the properties of the group used in double-entry bookkeeping and the crucial role played by the balance constraint. Section 2 derives the first proposition that relates the balance in the total system to balance in journal entries. Section 3 discusses how the balance constraint actually constraints the valuation choices and possibilities to record information in the accounting system and gives some important examples. Section 4 proposes an axiomatic basis underlying our current system. The axioms show how the definition of owner's equity leads to a bookkeeping system with the balance constraint. An extension of the basic axioms also shows that the monetary-unit assumption is without loss of generality in a much larger class of equivalent bookkeeping systems. Because part of the mathematical description in this paper is new, links to other mathematical descriptions of the paper are only discussed Section 5, after which the final section concludes the discussion. Formal proofs are relegated to the appendix whenever possible, while the main text focuses on accounting examples and intuition.

2 A mathematical description of bookkeeping

The first 2 lemma's in this section are due to Ellerman (1982) and are re-derived to keep this paper self-contained. To build an accounting system, we first define the smallest element used to keep track of some value. Since our bookkeeping system keeps track of many different values, we will build a larger system with more than one of these elements later.

Define an α-element as an ordered pair of non-negative numbers, $\alpha_i = [d_i \; // \; c_i]$, where $d_i \geq 0$ is a debit value and $c_i \geq 0$ is a credit value. The subscript i is used to index individual α-elements. The box, [ ], is used to separate α-elements, while the double slashes, //, are used to separate debit and credit entries in an element. Note that, the lay-out of a T-account achieves the same goal by having a title (for indexing), and a vertical line between the debit and credit column (to separate the entries). If we would only denote the column totals in the T-accounts, and not the entries of individual transactions, they would be equivalent to an α element. Denote the set off all α-elements as:

$$\mathcal{A} = \{[d \; // \; c] | d, c \geq 0, \; d, c \in \mathcal{R}\}$$

If we take two of these α-elements, $\alpha_1 = [d_1//c_1], \alpha_2 = [d_2//c_2]$, we can add them together by separately adding debit and credit values: $\alpha_1 +_{dc} \alpha_2 = [d_1 + d_2//c_1 + c_2] = a_3$. Here we denote by $+_{dc}$ the operation where we separately add the debit and credit values. Since a_3 is also an ordered pair of non-negative numbers, $a_3 \in \mathcal{A}$. Every account in the bookkeeping system can be associated with an α-element. For an inventory, the debit entry would equal the sum of all additions to the inventory, the credit entry the sum of all outflows out from the inventory. The value of the element (inventory) is found by taking the difference between debits (inflows) and credits (outflows) in the element (inventory). This property automatically implies that two of these α-elements have the same value if the difference between their debit and credit entries is the same. In our inventory example, debiting it for 60 and crediting it for 50, or debiting it for 100 and crediting it by 90, has the same result of 10 debit. That is, the value of two of these elements is equivalent (denoted by $\simeq$) if the difference between their debit and credit values is the same: $[60//50] \simeq [100//90]$).[3]

A binary operation that takes two elements from a set (the α-elements here) and yields a third element from the same set is what defines a group law. A set of elements combined with a group law defines a group, so that these α-elements together with the group law of separate addition of debit and credit values ($+_{dc}$) forms a group. We will refer to this

[3]Mathematically differences, subtraction and the equivalence relationship have to be defined with regards to the α-elements and the $+_{dc}$ operation. Since this leads to the intuitive definitions, this is left to the Appendix subsection A.1 and subsection B.3.

group as the $\mathcal{A}$-group, this group has been dubbed the Pacioli-group due to it's link with bookkeeping (Ellerman 1982, 1985, 2014).

Lemma 1. *the* $\mathcal{A}$*-group (or Pacioli-group),* $\{\mathcal{A}, +_{dc}\}$*, consisting of elements* $\alpha \in \mathcal{A}$ *and the group law of addition of debit and credit values,* $+_{dc}$*, forms a group.*

Proof. See appendix, a more general version of this result can be found in the appendix of Ellerman (1982). □

The appendix shows that the four necessary and sufficient properties for a group are met: closure; associativity; existence of an identity element in the set; and existence of an inverse element for every element in the set. We will briefly explain these requirements with the example of an inventory account and its associated α-element.
Closure is defined mathematically as $\forall \alpha_1, \alpha_2 \in \mathcal{A}, \quad \alpha_1 +_{dc} \alpha_2 \simeq \alpha_3 \in \mathcal{A}$. For an inventory, this means that if we (correctly) add the debits and the credits of two inventories together (α_1, α_2), the resulting combined inventory account (α_3) is also a member of this group-set, so a valid account in this system.
Associativity requires $(\alpha_1 +_{dc} \alpha_2) +_{dc} \alpha_3 \simeq \alpha_1 +_{dc} (\alpha_2 +_{dc} \alpha_3)$, such that the order of operations does not matter. This holds in our inventory example since the order of the transactions does not change the final result. If we first debit the inventory for 60 and then credit it for 50, the total debit value increases by 10. Similarly, if we first credit it for 50 and then debit it for 60, this also increases the debit value by 10.
The existence of an identity element, e, is defined as: $\exists e \ s.t. \ \alpha + e \simeq \alpha \ \forall \alpha \in \mathcal{A}$. There has to be some element in the group set that, when used with the group law, does not change the value of the other elements. Clearly, this element has to have zero-value: $e = [0//0] \simeq [x//x] \in \ \mathcal{A}$. If we debit and credit our inventory by the same amount, i.e. net zero, this does not change the inventory value as required.
Similarly, the presence of an inverse elements is easily shown. Formally the requirement is defined as $\forall \alpha \in \mathcal{A} \ \exists \alpha^{-1} \ s.t. \ \ \alpha +_{dc} \alpha^{-1} = e$. In our setting the element $e = [0 \ // \ 0]$ is an element without any value, like an empty inventory. If we want to set an inventory to zero value, we simply credit it for the existing debit value. We can construct the inverse elements for any account in this way, simply by exchanging the debit and credit values. The resulting inverse element is a pair of non-negative numbers, so the inverse of $\alpha_1 = [d_1//c_1]$ is simply $\alpha_1^{-1} = [c_1//d_1] \in \mathcal{A}$.
These 4 properties of the $\mathcal{A}$-group are necessary and sufficient to proof the group definition, and they provide a simple algebra to manipulate elements of the group and analyze the resulting expressions.

One interpretation of these elements with two non-negative values is as a trick to avoid using negative numbers (Ellerman 1985). We could have equivalently used negative numbers for credits and positive numbers for debits (or vice versa).[4] The relationship between the real numbers and the α-elements is in fact one-to-one, as the red line in Figure 1 shows. This makes the interpretation of these accounts and the operation of addition of debit and credit considerably easier, as it allows us to use normal addition and subtraction on real numbers to understand this group and group algebra.

Lemma 2. *There is a group-isomorphism from $\{\mathcal{A}, +_{dc}\}$ to the group formed by the real line and the group-law of addition, $\{\mathcal{R}, +\}$.*

Proof. See appendix, a similar result is derived in the appendix of Ellerman (1982). □

[FIGURE 1 SOMEWHERE HERE]

The bijection between the α-elements and the real line is from every possible real number to all α-elements with the same value $d_i - c_i$, i.e. an equivalence class of such elements. The bijection maps the number -10 on the real line to an element holding the value 10 credit, regardless of whether that 10 credit is caused by crediting the element by 10 and debiting it by 0, or by crediting it by $10,005$ and debiting it by 9995 since these operations yield equivalent results. If we have an element with a credit value of 10 and we credit it by 10, we get a credit value of 20. The equivalent operation on the real line looks like: $(-10) + (-10) = -20$. The value -20 is mapped to an element $[0//20]$, such that both the operations and the elements match, as required for a group bijection.

An accounting system has many accounts that keep track of different values, so the description of an accounting system requires more than one element. To make an accounting system, every account in the trial balance (or ledger) has to have an associated α-element. With $n < \infty$ accounts, we require a list of n individual α-elements. We will denote such lists with capital letters, such that $A = \{\alpha_1, ..., \alpha_n\} \in \mathcal{A}^n$, where $\mathcal{A}^n$ is the set of all length-n lists of α-elements. Within each list we use subscripts to index different elements, for instance, α_1 could be our cash account, α_2 the materials inventory, etc. Ledgers, trial balances, balance sheets, and journal entries all consist of such lists of accounts with associated α-elements. Collectively such lists will be referred to as bookkeeping statements, or statements for short. Appendix D shows that the set of lists of length n forms a group with the same binary operator, $+_{dc}$, applied over each indexed element i. Figure 2 shows how we can form a prototypical accounting system from this

[4] Historically, this system of notation was developed before the use of negative numbers took root. Since one can work with negative quantities of debit value – credit value – in this system without having to define negative numbers, it was a very convenient step in the development of number theory.

mathematical description.

[FIGURE 2 SOMEWHERE HERE]

If we update the balance sheet in Figure 2a through the journal entry in Figure 2b, we have to credit account 1 and debit account 2 by the amounts given in the journal entry. The resulting updated ledger-accounts, Figure 2c, have both debit and credit entries consisting of non-negative numbers, so they are also elements of the set $\mathcal{A}$. Since all elements have to come from the same set, A^5, the journal entry has three zero-value elements, a_3, a_4, a_5. In accordance with the property of the identity element in the group, the updated balance sheet shows these zero-elements do not affect the value of the accounts a_3, a_4, a_5. In practice, these zero accounts are usually suppressed in the notation of a journal entry, such that only α_1 and α_2 would be shown.

As with the individual accounts, we call two statements equivalent if each of their elements has the same value: $A \simeq A' \leftrightarrow \alpha_i \simeq \alpha'_i, \forall i, 1 \leq i \leq n$. The group satisfies closure and associativity in the same way as each of the elements in the lists does. The identity element is a list of elements with zero value. The inverse of any statement, is the statement where the debit and credit values are exchanged within each α-element. In the same way that each individual α-element can be mapped to $\mathcal{R}$ with a bijective mapping, all statements of length n can be mapped to $\mathcal{R}^n$ with a bijective mapping. So that the group $\{\mathcal{A}^n, +_{dc}\}$ is isomorphic to $\{\mathcal{R}^n, +\}$. Through this mapping the mathematical structure of the bookkeeping system can be compared to vectors and vector addition.

Note that the α-elements do not have a meaning independent of their context. They register the value of something in the entity. Therefore we have to associate every α-element to some part of the entity through the subscripts i. In mathematical notation it is easiest to work with i as a number between 1 and n and assign the same number to the corresponding account. It would, however, be just as correct to denote our cash account as q_{cash} rather than $i = 1$ and work with a list of names instead. Figure 2d shows an example indexing. For every index i it has a corresponding account in the accounting system, and thus some aspect of the entity that the element keeps track of. To model the entire accounting system every account in the entire chart of accounts should be associated to an element. Every element is uniquely connected to a single account, and all accounts are uniquely linked to a single element, such that we can refer to accounts and elements interchangeably without causing confusion. Through the association in Figure 2d, for instance, we see that the journal entry in Figure 2b corresponds to a cash purchase of materials for inventory.

The rules of bookkeeping have an additional important requirement: $debit = credit$. For each of the panels a,b, and c in Figure 2, we require that the sum of all debit values equals the sum of all credit values in the statement. Denote the value of a list A as: $Val(A) = \sum_{i=1}^{n} d_i - \sum_{i=1}^{n} c_i$ where d_i, c_i denote the debit and credit values of the ith-element in the list. Similarly let $dr(A)$, and $cr(A)$ denote the sum of all debit, and credit values in the list A respectively. Note that the value of an individual element i (a list of length 1) is equal to $d_i - c_i$, so this corresponds to the mapping in Lemma 2.

If any of the statements in Figure 2 is not balanced, this statement is not well-formed. Only those statements $A \in \mathcal{A}^n$ where $Val(A) = dr(A) - cr(A) = 0$ are valid statements. Define the set of all balanced statements with n elements as $\mathcal{B}^n = \{A \in \mathcal{A}^n | Val(A) = 0\}$. Denote item i on the list as b_i, these are α-elements indexed by i as before. By construction $\mathcal{B}^n \subset \mathcal{A}^n$, the elements in $\mathcal{B}^n$ are all in $\mathcal{A}^n$. However, all unbalanced statements of length n are not part of the strictly smaller set $\mathcal{B}^n$. In bookkeeping we keep track of all the values of inventories, liabilities, revenues, costs etc. A list of values, B, registered for some entity, describes the state of values in this entity. Since every element in the set $\mathcal{A}^n$ could describe a possible state of the entity, this implies that there are possible states in $\mathcal{A}^n$ that we cannot record in $\mathcal{B}^n$. In this way the balance constraint, constraints the record keeping system to a smaller set of potential recorded values.

The group structure gives us a formal syntax to analyze bookkeeping mathematically. There are also important benefits of this group-structure for the use of accounting as a communication tool. The shared understanding of the rules used in accounting, allows us to clearly communicate differences between different states of the world through the reported values. This makes a group-based accounting structure very well suited for the purpose of communicating financial, or value information (Stecher 2011). To ensure we keep both the formal syntax of the group-algebra, and use bookkeeping for effective communication of value, the next lemma shows that a balanced-constraint bookkeeping system together with the balance constraint also forms a group:

Lemma 3. *$\{\mathcal{B}^n, +_{dc}\}$, the set $\mathcal{B}^n$ with the group-law of itemwise addition of debit and credit values forms a group.*

Proof. see appendix. □

A bookkeeping statement B is a reasonable description of many bookkeeping tools. For instance, if we take the trial balance, or general ledger they contain a list of accounts in which we keep track of different values. Each account has a different value and name, so they are separated and we could link each different account to a different index i (for instance via the decimal schedule). To each account we can post both debit and credit values and we know that the sum of all credit values equals the sum of all debit values in the list, such that the general ledger and the trial balance satisfy the balance constraint.

With this notation and description of the balanced bookkeeping group, we can derive the first proposition. This proposition shows how the constraint on the entire accounting system, that if we add up all the debit values in the list it equals the total of the credit values in the list, implies that every entry made, for all individual transactions, has to be balanced:

Proposition 1. *The difference between two balanced lists of accounts is balanced:* $\forall B, B' \in \mathcal{B}^n \quad (B +_{dc} (B')^{-1}) = (B -_{dc} B') \simeq B'' \in \mathcal{B}^n$.

Proof. see appendix. □

Mathematically, the result is straightforward. From the group definition we know the inverse element of any element is also in the group, and thus balanced. Under addition on the real line, adding the inverse element is the same as taking differences. The same relation holds for the lists B, the difference between two lists is equivalent to adding the inverse of the second list to the first list with the $+_{dc}$ operation. Because B satisfies the group identity, this inverse list is an element of the group-set, and thus the difference must also be an element of the group-set. The difference between two balanced lists must thus be balanced. From an accounting point of view, the result is extremely relevant. The general ledger should always be balanced, and thus it is an element of $\mathcal{B}^n$ at any moment. Now take the ledger at two different moments, denoted as B^1, B^2. Then we know that $B^2 -_{dc} B^1 = \Delta B \in \mathcal{B}^n$, but the way we update the general ledger is via a journal entry like Figure 2b, so that this too has to be an element of $\mathcal{B}^n$. We require the general ledger to be balanced and Proposition 1 shows that this implies that the journal entry of every single transaction has to be balanced.

With a small addition, this same result also holds the other way round. If we start with a balanced statement and require all changes to be balanced, the resulting statement is also balanced.

Corollary 1. *If a statement is balanced (B_0) and all statements it is combined with (B_i) are also balanced, the result after $T < \infty$ manipulations (B^T) is balanced:* $B_0, B_i \in \mathcal{B}^n \rightarrow B_0 +_{dc} B_1 +_{dc} ... +_{dc} B_t = B^T \in \mathcal{B}^n \quad \forall 0 < T < \infty$

Proof. Since $Val(B_i) = 0 \; \forall i$, it is immediate that $Val(B^T) = 0 + 0 + .. + 0 = 0$. □

Before any transactions occur, an entity starts with a list of zero values –an empty statement– i.e. before investment by owners in the company, the company does not have any values to keep track of. If we restrict all transactions to be recorded via balanced entries, the resulting general ledger is always balanced over all. These two restrictions, a balance of the general ledger, or a balance of every journal entry, are thus mathematically the same. Therefore, we can verify the balance of the entire general ledger at the level of individual transactions. Checking all individual transactions is sufficient to guarantee the balance of the entire ledger by construction. Note that the result is much more general than just applying to the general ledger and journal entries. The result holds for any balanced set of elements. So it holds, for instance, for any set of accounts in our accounting system that we know to be balanced. At the moment we make a balance sheet, we make sure that the list of accounts it contains is balanced. Even though this is a much smaller set of accounts than the ledger, the result holds for balance sheets and differences between balance sheets. The result in Proposition 1 applies to the difference between two balance sheets, regardless of what causes the difference: changes over time, differences between economic entities, or differences between sets of valuation rules for the same entity at the same moment. All that is required is that both statements are of the same length, and that they are both balanced.

This result is quite useful in the design of accounting systems. If we look at the Accounting Information Systems based on relational databases, these consist of large tables that summarize the current state of the entity(Everest and Weber 1977). A consistency check that sums all debit values and credit values is easily implemented in such designs, similarly a restriction that allows only balanced transactions can be implemented to the same effect(Barra et al. 2010; Gentili and Giacomello 2017). Since both will have equivalent results, only the simplest one has to be implemented. In a blockchain system the public ledger stores a record of individual transactions.[5] In such a system, requiring that each individual transaction record is balanced is more natural.

In the discussion on the future of accounting systems, the blockchain is seen as a promising new technology to prevent falsification of data. If professional organizations and the popular press are anything to go by, this technology will have a deep and lasting impact on the accounting profession (Andersen 2016; Arnold 2018; ICAEW 2019). However, how to use the blockchain in accounting and what information should be encoded on the blockchain for accounting practice, are largely an unanswered question (Dai and Vasarhelyi 2017). Showing how the blockchain can implement a double-entry bookkeeping system, with a transaction level check on balance could be a useful step. Finding the fundamental design choices in our accounting system is an important further step towards implementing blockchain technology in double-entry accounting, and is taken up in Section 4. The next sections will discuss some examples of implications in valuation problems.

[5] A small note on terminology, the public ledger of a blockchain holds a history of all transactions on the chain. As such, it more closely resembles a journal that records all transactions, than the general ledger that shows the current state of the entity.

3 Valuation consequences of the balance constraint

The implications of the balance constraint in Proposition 1 go beyond the bookkeeping process per se. As bookkeeping is the final step in accounting for economic events, by the Bellman principle, we have to take bookkeeping into account in all earlier steps. Balance in bookkeeping statements can only be satisfied if we define and measure value in the accounts in a manner that preserves balance. That is, accounting valuation has to the bookkeeping balance constraint into account. For any given event, if we change valuation definitions of the transaction, the change immediately has to have the opposite effect on the recorded value elsewhere. For example, if we change our valuation of an receivable and thereby increase its net debit value in the books (i.e. reduce the allowance for doubtful accounts), we have to increase the credit value elsewhere (i.e. increase revenues).

In the notation used in this paper, valuation aspects have mostly been ignored. However, a lot of accounting definitions can still be found easily. Net income, for instance, is -1 times the value of all revenue and cost accounts in the trial balance (value is debit, equity is credit). To find it, first make a list of all accounts that are added together to form earnings, A_E. All other accounts form a second list A_E^C, so that each list is either in A_E or in its complement:

$$B = A_E \bigcup A_E^C, \quad A_E \bigcap A_E^C = \emptyset \quad \rightarrow Val(A_E) + Val(A_E^C) = 0$$

Net income can then be found as $NI = -Val(A_E)$. Similarly the balance total is found by defining a list of all items that are reported on the balance sheet (directly or as part of some aggregated account), A_{BST}, and then taking $dr(A_{BST}) = cr(A_{BST})$, etc. This type of definition via the value of a sublist of accounts is not part of the $\mathcal{B}$-group definition for two reasons. First, the resulting value (like net income) is not an element with two numbers, so it cannot be a member of $\mathcal{B}^n$. Second, the calculation does not change the recorded information in the accounts, so it is not an operation like the group law. Most reporting, as is done in the annual statements, can be done through such read-not-write operations on the set of accounts reported in the trial balance.

With this notation in place, we can look at the forced interaction between valuation in the balance sheet and the income statement:

Corollary 2. *The recording process through the $\mathcal{B}^n$-group binds the definitions in the balance sheet to the definitions of costs and revenues in the income statement.*

Proof. See appendix. □

To change reported net income, we have to change both an Earnings account (to change net income), and some account outside of the earnings list (to keep the balance in the ledger). At the level of aggregation in the annual statements, this can only be achieved by a change in value on the balance sheet (in the list $A_{BST} \bigcap A_E^C$). The gist of this corollary is quite familiar, we know that distorting measurement of assets and liabilities can be used to manage earnings. For instance, by managing the revenue booked for sales on accounts to increase earnings. However, this connection goes beyond the individual examples discussed in the literature. The connection is baked into to the bookkeeping system by design and no amount of clever tinkering can remove it. By definition, to find earnings all costs are subtracted from all revenues. To change the earnings number below the line and preserve the balance, it is therefore necessary to adjust an inventory or liability. It also holds regardless where the change in reported earnings comes from, whether it is caused by a change in accounting definitions of value, e.g. the change from local GAAP to IFRS, or because a value was altered to change earnings, e.g. earnings manipulation, or any other reason. Without the balance constraint, any manipulation of any account can be done without showing where the recorded value comes from. The balance constraint causes us to show the source and the destination of all value flows, what (Ijiri 1975) calls the causal duality of accounting. This makes it impossible to individually adjust earnings or asset numbers. It forces one to adjust at least two accounts. This requirement that both a stock and a flow have to change to alter the net recorded flow accounts, is part of the error-detection code build into the double-entry accounting system.

Corollary 2 is not without subtleties. We cannot manipulate the final earnings number without adjusting the balance total, but we can change the balance total without affecting reported earnings by increasing both assets and liabilities. One way of doing this would be to transfer IP-property to the company and paying for it in notes payable, or through the reduction of the value in these same accounts. Any such manipulation of balance-sheet items, changes the balance-sheet bottom line, without necessarily changing net earnings. Simultaneously, since all costs and revenues are netted out in earnings numbers, we can freely reclassify costs (or revenues) between accounts without necessarily affecting the balance sheet total or net income.

This corollary also shows that any earnings manipulation, for instance management of earnings through under- or over-estimation of some accrual, has to be reflected in the balance sheet at end of period. In models of earnings management this immediately means that manipulations should be 'parked' on the balance sheet as some sort of accrual. Managed earnings could therefore be as detectable in the balance sheet as they are in the earnings statement. Modeling earnings management as a pure mismatch of cash-flows to accruals can lead to ill-specified models. Without the additional step of parking the manipulation on the balance sheet, the models do not take into account the algebra used to record the manipulation.

The direct relation between the value definitions in the balance sheet and the income statement also means there should be no fundamental difference between the balance-sheet approach of measuring value that the IASB seems to adhere to, and an earnings-based approach that is proposed by some authors (Benston et al. 2007; Dichev 2008, e.g.). The balance constraint ties these value definitions together. Any change in how we measure a cost or benefit directly changes the way we define the balance sheet item these costs and benefits flow to and from. Through the definition of equity, the residual claim on the value in the company, such changes also immediately alter what we mean by equity. The same holds for the definition of the value of a stock, they imply the value of flows to and from and the stock.

3.1 Adding buffer accounts, the example of OCI

So far we assumed a fixed length of the list n. However, what if we add an extra account and start working with $n + 1$ elements? This is more than some academic discussion, it is a common solution to valuation problems. Examples include translation differences for foreign profits and possessions, variances in standard costing systems, as well as the introduction of Other Comprehensive Income (OCI) discussed below.

OCI is loosely defined as 'revenues and expenses, gains and losses that are not part of net income'. Common examples are gains or losses in the value of assets that are not held for trade, or not (yet) realized. Common examples are increases in the value of assets held in dollars by Euro companies caused by exchange fluctuations.[6] There are good reasons to want to keep track of OCI outside of net income. The accounting literature has so far focused on the empirical properties of OCI. Items reported in OCI are found to be more transitory, more volatile, and more likely to revert than other income componentsJones and Smith (2011). This means they do not predict future performance very well, leading to a lower value relevance of income including OCI then of income excluding OCI. Different empirical properties of the components, is clearly a reason to separate them for prediction and valuation goals.

There are additional, conceptual arguments for separately reporting OCI in the stewardship and governance goals of accounting. Net income is a performance measure of the company and its management. A good performance measure should closely follow the actions taken by the decision maker that impact performance. This means the measure has to be (a.o.) congruent and precise (Feltham and Xie 1994) indicators of actual performance. Reporting OCI separately from the rest of net income, creates a net income excluding OCI without the noisy, volatile and transitory OCI items. This allows the net income measure of performance to be a more precise measure of performance then net income with these OCI items included. Net income excluding OCI is also more congruent,

[6] For an overview of the research on this topic (Rees and Shane 2012; Bratten et al. 2016; Black 2016).

since it excludes items that partially reverse. If an item reverses, it is not a persistent addition to value, and thus not actual performance. Another, important principle in performance management is the principle of controllability. This principle says that performance measures used in incentive systems should reward good actions under control of the agent, not random shocks.[7] As a change in exchange rates is not something one can attribute to any individual entity, it can violate the controllability principle to include value changes due to the Dollar-Euro exchange rate in measured performance of the entity or manager. Reporting OCI separately can allow a better view of the performance of the entity and its top management, and thus aids in accountability.

Without the balance constraint, we could freely define earnings as the best possible measure for performance *and* report the current value of assets and liabilities as accurately as possible. For instance, by defining an earning number like net income excluding OCI, and use fair value accounting for all balance items. However, the total change in equity measured via the change in fair value of assets and liabilities does not correspond to the reported net income excluding OCI. Without separately reporting OCI, this combination of valuation measures violates the balance constraint. Hence, the need to report OCI separately, rather than simply report an optimal earnings number and fair value balance items, is caused by the interdependence of value definitions. It is forced by the balance constraint.

Adding accounts to our bookkeeping system is equivalent to changing from an n-element bookkeeping system, to an $n+1$ element system. Since $n+1<\infty$, this does not require a change in the mathematical description of the system. The resulting imbalance in the books is absorbed by the new account, OCI for instance. This seems to violate the idea of Corollary 2. If we would make the financial statements over some year in the n-element system, without OCI this would lead to a statement $B \in \mathcal{B}^n$. Then if we redo everything in an $n+1$-elements system incorporating OCI, this leads to a different statement $B' \in \mathcal{B}^{n+1}$. The difference between the first n elements in the reports will not be balanced, but this is only a seeming contradiction, since the introduction of the new account means the issue is avoided:

$$\sum_{i=1}^{n}(d_i - c_i) - (d_i' - c_i') = -(d_{n+1}' - c_{n+1}') = -Val(b_{n+1}') \neq 0$$

An alternative interpretation of the introduction of OCI is that the system always had a $n+1$ list of accounts, but this last account was set to 0 until the introduction. The difference between the two $n+1$ length lists, where the element $b_{n+1} = [0//0]$ until the introduction of OCI, we would find that the change is balanced. To start measuring

[7] For a general discussion on controllability see for instance Antle and Demski (1988). The same idea is presented in micro-economic theory under the name informativeness, e.g. Holmstrom et al. (1979).

OCI in b_{n+1}, we have to change the associated measure q_{n+1}, so that it yields non-zero values. As before, balance requires that the same value is then taken from some other measure. Just like a change in NI requires a non-earnings account to change as was shown in Corollary 2, booking value as OCI requires that some other account changes in the opposite direction. This makes the attention given to this type account quite warranted. The accounting system is used to compare entities over time, the introduction of OCI causes a change in value definitions of related income accounts. We have to take these changes in their definitions into account in later analysis.

3.2 Stock based compensation

The valuation of stock awards is a known issue in accounting. Stock grants, whether to the executives or to employees, are part of the payment to people working for the company. They differ from cash salaries because payment of cash salaries decreases a debit-stock (cash on hand) while the payment in stocks increases a credit stock (stocks outstanding). What is more, this credit stock is part of equity. Recording a cost decreases net income and equity, while the increase in the stock increases equity. Hence, payment in stocks appears to be equity-neutral. Stock based compensation cannot be classified as any other cost in the bookkeeping system because equity does not decrease. The payment is not, however, neutral to the existing owners. Existing owners see their stock dilute and loose the value awarded. This can create a problem in the assessment of the value of the firm and in measurement of the performance of the organization. When there are stock awards, the standard residual income and clean surplus models of investor valuation are mis-specified (Hess and Lüders 2001). The standard model assumes that only the owners of the firm will get the future residual income, but in fact employees that receive stock awards will claim part of these cash flows. The same holds for the performance as measured by net income. If the stocks would be sold on the market for cash, and consequently the same amount would have been awarded as salaries, the value of the salaries would be reported as part of total costs. Thus if we do not book the awarded treasury stock as costs, we decrease total cost and inflate performance as measured by net income.

Like with OCI, without the balance constraint the earnings definition could be adjusted without issue, with the constraint we need additional accounts to clean the performance measure. Adding an account of 'awarded stocks' as part of equity, next to an account of 'previous existing stock' would allow stock awards to be shown as a cost to the current stock holders. This does raise a question for longer-term record keeping: at what point does the 'awarded stocks' become part of normal equity? After receiving a stock award, any later awards also dilute the equity of the previously 'awarded stocks'. As such, it is questionable if keeping track of this 'awarded stocks' account in equity makes much sense after the first accounting cycle.

The bookkeeping system focuses on providing information on the entity, not information for the shareholders. In this transaction there is a big difference between the two. Note that this difference did not exist in OCI, which could explain why a different solution was found in these cases. Similar problems will occur with the related issue of convertible debt, where the conversion means fewer debts for the entity (higher equity) but dilution and loss in value for the existing owners.

4 A set of Axioms for bookkeeping

The last section described two examples of issues caused by the balance constraint in bookkeeping, more such issues were already hinted at in the discussion. Given the system-wide effect on accounting, a logical question would be: 'Do we really need this constraint?'. This question strikes at the heart of our system of financial record keeping and is unlikely to find a definitive answer in any single paper. It requires the careful consideration of pro's and con's of alternative choices. A first step is to answer the simpler question: 'What design choices lead to a bookkeeping system like the current one?'. To keep notation to a minimum, this section will use Lemma 2 and work with real numbers rather than with α-elements. Define for every element in $A \in \mathcal{A}^n$ a corresponding element in $\bar{A} \in \mathcal{R}^n$ via the transformation $\{[d_1//c_1], [d_2//c_2], ...[d_n//c_n]\} \mapsto \{d_1 - c_1, d_2 - c_2, ..., d_n - c_n\}$. Denote by $\bar{a}_i$ the i-th item in the list $\bar{A}$.

The choices leading to our system of bookkeeping form a set of basic axioms. If we want a recording system with different properties, we will have to change at least one of these axioms. This section proposes one such set of axioms:

1. There exists some set of events Ω with elements ω that we want to keep a record of and some entity to do the recording for.
2. Value is defined in the form of real, signed measures $q_i(\cdot)$ defined over Ω.
3. The system should have n elements to record value in, with $1 \leq n < \infty$.
4. All values should be defined in the same monetary unit: $q_i = q_j \rightarrow U(q_i) \simeq U(q_j),\ \forall q_i, q_j$, where $U(\cdot)$ measures some utility of value.
5. A history without economic events, ω_0, is valued at 0: $q_i(\omega_0) = 0\ \forall\ q_i$.
6. In the set of accounts $\bar{A}$, there exist a closing element $\bar{a}_n$ that absorbs the residual value in the organization: $-\bar{a_n} = \sum_i^{n-1} \bar{A}_i$

The first 4 axioms are equivalent to axioms found in earlier work, in particular that of Mattessich (1964) and Ijiri (1975). The first axiom captures the entity assumption in

accounting, plus the definition of the events that are accounted for (event set Ω). Events (transactions, or complete histories of transactions) are denoted as ω and the set of all possible events as Ω. All elements in the bookkeeping system record value, where value is defined via real, signed measures over the set of possible events. Different elements are indexed through subscript i as before. These measures attach an interpretable, numeric value to any possible outcome (event or history of events ω) in the set of all possible events, $q_i : \Omega \rightarrow \mathcal{R}\ \forall \omega \in \Omega$. This number is allowed to be positive, negative, or zero and denotes the value (or change in value) associated with the outcome. If $q_1(\cdot)$ is a measure for the cash at hand account, it should be equal to the net inflow of cash to our cash account for every possible outcome. Similarly, if $q_2(\cdot)$ measures the value of the inventory of raw material, it is equal to the monetary value of the net inflow into this inventory, and this is negative for all outflows from the raw materials inventory and positive for the inflows into the raw materials inventory.
The number, n, of different values in a bookkeeping system can vary from organization to organization, but has to be finite. To keep the elements and their measures apart, we use the index i to identify them, this leads to the third requirement.
The fourth axiom is the monetary-unit assumption in accounting. When everything is measured in money, we should be indifferent between an extra unit of value – an extra dollar– in one element or in the other. It also means that operations (journal entries) that move value from one element in the system to the next, simply reduce the value in one element (credit it) and increase the value in another element (debit it) by the same amount. Similarly, we can sum the value over accounts and compare value between them directly, since the numbers in each element are expressed in the same monetary units. This monetary-unit assumption is a strong requirement, but it greatly simplifies comparisons within and across organizations. For instance, the inventories in Figure 2 are elements 2 and 3. Because of axioms 2 and 4, the total value of inventories after a history of transactions ω is equal to $q_{inv} = \sum_{i=2}^{3} q_i(\omega)$. The monetary-unit assumption allows this comparison regardless of whether the inventories are farm products or medical supplies. We can compare the inventory values both within organizations (compare q_2 to q_3) as across organizations by comparing the monetary values of the inventories.

Under these first 4 requirements we logically have some collection of n elements each of which records a value. Since we require value to be defined as a measure, we have additivity within the value definitions of each individual element automatically. Furthermore, because elements are denoted in the same monetary units, adding values of different elements together can be done numerically and a value of 10 in one element is equivalent to a value of 10 in another element.

Requiring value to be a measure, implies that the empty event has zero value. This is a desirable property to have, no event implies we do not have any value or change in value to record. Before the organization starts we want the books to be empty. Similarly, after the organization starts – and thus after it starts to keep track of the books – if nothing happens, then the books should remain the same. We have to define our events in such a way that the event 'no transactions' is an empty set – which would make Ω the universe of *economic* events–, or we have to define the measures in such a way that an event with no economic relevance leads to $q_i = 0$. Axiom 5 specifies that one or the other has to occur for all accounts.

To complete the bookkeeping system as we currently use it, the balance constraint must be implied in the system. One reason for having balance can be found in the definition of owner's equity. If we single out the n-th element, we can write the balance constraint as $-\bar{\alpha_n} = \sum_i^{n-1} \bar{A}_i$ as is done in axiom 6. This is exactly how owner's equity ($\bar{\alpha_n}$) is defined in our accounting system (axiom 6): whatever remains after subtracting the value of liabilities from the value of all assets.

Requiring the presence of a residual account matches the development of double-entry bookkeeping in Venice. At the time of Pacioli, several different bookkeeping systems were in use. In Europe the most notable bookkeeping systems were the double-entry system used by the merchants of Venice and the factor bookkeeping used in the Hanseatic tradition (Yamey 1967; ten Have 1973; Funnell and Robertson 2011). The Venetians used the books to check on establishments that were run with money they invested at a distance, very similar to foreign subsidiaries in current time. This makes the existence of an "owner's equity" account very relevant. A residual account makes it possible to show how much money is owed to the Venetian merchant-bankers at any time, as well as how much profit was made over a period of time. In the Hanseatic tradition, individual traders in the trade network acted as trustees of the other traders for specific consignments of goods. When the goods were traded and the money was returned, the books were closed on this shipment. So the most important thing is that all goods are accounted for and no clear equivalent to owner's equity needed to exist.[8]

4.1 OCI and stock-awards revisited

Axiom 6 also provides a different interpretation of the balance constraint. Rather than casting the system as one where we keep track of n different values that are somehow related, it shows an unconstrained system of $n - 1$ values (all assets and all liabilities and changes therein) and a restricted, residual account. In this interpretation any imbalance between these $n - 1$ accounts is passively absorbed by the n-th, residual account.

[8] Historians of accounting argue from time-to-time about the exact starting point of modern "scientific" double-entry bookkeeping. Some place it when debits and credits are separated, some when they are balanced in such a way that a periodic profit number could be calculated, others only when a separate (owner's) capital account is maintained (ten Have 1973; Funnell and Robertson 2011). The owner's equity account is clearly an important one in the development of bookkeeping.

However, as we saw with the discussion of OCI and stock-based compensation, if we try to separately measure the value in the n-th account, equity, this creates problems. We only have $n-1$ degrees of freedom to define the value in n accounts, so that we cannot define n measures independently. If the resulting measures for earnings and equity (performance and value of the organization) do not have the desired result (are not congruent with the goals of measurement), we have to conclude that we want to use different value measures for performance than for valuation. This can only be achieved by adding accounts and associated value measures so that the residual values correspond to our desired definitions.

Adding extra accounts gives extra degrees of freedom, but also comes with extra valuation choices. As the discussion around extra 'awarded stocks' account shows, changing the value definitions on one account requires an opposing change elsewhere. The difference will either be short term costs and benefits, or have to remain in the books for a longer period of time. Depending on the situation, several solutions are possible. In a standard costing system, we see variances occur because two different measures of value for the goods are used, purchasing price and standard price. The variance accounts are the buffers used to balance the books. Variances are booked as a cost or benefit in the running year. Another solution would be to keep the buffers on the books. Corporations can use an account like Accumulated OCI – caused by two different measures of value for income, average of the year and closing date exchange rates – this means they make the opposite choice and keep accumulating the differences in their books. It is unclear if either of these solutions is strictly better than the other. Most likely, the best solution depends on the situation and the questions at hand. To find the best solution in a specific measurement problem, one could compare the information generated by the spectrum of choices and see what is more useful to decision makers.

4.2 More general axioms

So far we focused on the common assumption of monetary units, but this restriction is too strong. A similar system would emerge if we expressed everything in other units, like 'one-kilogram-bushels-of-wheat equivalents'. However, monetary units are usually the most natural choice. When we relax the monetary-unit axiom, a more general version of the balance axiom is required as well:

4∗ The units of all q_i should be linearly related such that there are ratios, λ_j, such that 1 unit of q_1 is equivalent in value to λ_j units of q_j:
$\exists \lambda_j \ s.t. \ U(q_i) \simeq U(\lambda_j q_j), \ \forall q_i, q_j$, where $U(\cdot)$ measures some utility of value.

6∗ In the set $\bar{B}$, there exist an element $\bar{a_n}$ that absorbs the residual value:
$\exists \lambda_1, ..., \lambda_n, \ \lambda_i \neq 0 \ \forall i, \ s.t. - \lambda_n \bar{a_n} = \sum_{i=1}^{n-1} \lambda_i \bar{a_i}$.

The more general unit requirement in Axiom 4$*$ implies that for any two elements i, j in our system, we can find a scalar λ_j such that we find 1 unit of q_i is equivalent in value to λ_j units of q_j (we can set $\lambda = 1$ if $j = i$). If we are discussing the materials and completed goods inventories in Figure 2, it is straightforward to see that λ_j would be the price-ratio between the two inventory elements. If the price of our final product in inventory $i = 3$ is twice that of the kilograms of raw material in inventory $j = 2$, we would need to multiply the number of items in the final goods inventory by $\lambda_3 = 2$ to express the value of the final goods inventory in terms of kilograms of raw material. Note that axiom 4 requires $\lambda_i = 1 \; \forall \; i$ and then axiom 4$*$ holds automatically.[9]

With the more general unit requirement, axiom 6 requires some weighted sum of the elements to be balanced, where the weights are proportional to λ. Note that the λ weights are the same for all possible outcomes, they are fixed prices that allow us to denote everything in a common unit. The condition in axiom 6$*$ would hold vacuously if λ was zero, so none of the prices can be zero. Under these alternative assumptions we can show that there are many, equivalently powerful, specifications of our record keeping system.

Every possible bookkeeping system can be associated with the space of possible values it can record. The larger the space, the more states of the organization the system can record. Every point in the space associated with the bookkeeping system defines a possible recorded state of the entity – some combination of assets, liabilities, costs and revenues – that can be recorded in this system. We can say two of these systems are equivalently powerful if they can record the same set states of the entity and thus provide the same information about the entity. For the $\mathcal{A}$-group for instance, Lemma 2 shows this space is $\mathcal{R}^n$. The elements of a balanced system, $\bar{\mathcal{B}}$, form a (linear) subspace of $\mathcal{R}^n$. This means that the space $\bar{\mathcal{B}}$ is some (smaller) part of $\mathcal{R}^n$ with many of the same properties as $\mathcal{R}^n$. Because the $\mathcal{B}$-group consists only of those elements of $\mathcal{A}$ that are balanced, we know the corresponding space is smaller. This is formalized in the next lemma:

Lemma 4. *$\mathcal{B}^n$ is a strict subspace of $\mathcal{A}^n$.*
$\bar{\mathcal{B}}^n$, corresponding to $\mathcal{B}^n$ via the bijection of Lemma 2, is a strict subspace of $\mathcal{R}^n$.

Proof. See appendix. □

Under the more general axiom 6$*$, every vector λ can be used as a price list, and all of these lists have a corresponding record keeping system. In this system λ_i can be used

[9]The difference between axiom 4 and 4$*$ is subtle. Since q_i is a signed measure, we know that for any real scalar λ_i, $\tilde{q}_i = \lambda_i q_i$ is also a signed measure. If there is some underlying physical entity we can count and price, then we can set the count as q_i and the price as λ_i. In this case axiom 4 and 4$*$ would be equivalent since we would define value as $\tilde{q}_i = \lambda_i q_i$. This makes interpretation a lot easier, but is not generally true. For instance, what would you have to count, or price, to get the value of owner's equity or intangibles?

to relate the units of q_i to some common unit. Superscripts l will index different lists λ^l, while subscripts refer to items in the list as before. We can proof Lemma 4 for every possible list $\lambda^l = \{\lambda^l_1, ..., \lambda^l_n\}$ as long none of the items in the list are equal to 0. For every λ^l we can find a subspace, denoted by W^l_λ that describes all possible recorded states of an entity in the bookkeeping system, furthermore, all of these subspaces are equivalent.

Proposition 2. *There exists a bijection between every two spaces $W_{\lambda^1}, W_{\lambda^2}$ as long as* none *of the prices in λ are zero: $\lambda^1_i, \lambda^2_i \neq 0$.*

Proof. See appendix. □

The one-to-one relationships between all of these bookkeeping systems show that they can all record the same states of the entity. If the system corresponding to λ^1 can record a particular state (or set of values) of the entity, we can guarantee that the system corresponding to λ^2 can record it too. The monetary-unit assumption (axiom 4) selects a specific λ by requiring that all translations are done at a price-ratio of 1. The prices λ^l do not determine the strength of the system, they simply determine how we translate units from one account into units in the other accounts in the bookkeeping system. This selects a unique bookkeeping system from this class of equivalent systems.

The equivalence between the systems has a direct, physical reason for items with a clear quantity. For inventories, for example, we know that the value of the inventory i divided by the price of the units in system 1 (λ^1_i) gives the number of units in the inventory. So if we know the price and the value of the inventory, we can find the physical count in any record keeping system. To translate this physical count to the value in system 2, we multiply it by the price in system 2 (λ^2_i). The same can be done for any other quantity of property or liabilities, while the measures q_i can be seen as an underlying quantity. Proposition 2 indicates that the prices we attach to real quantities of properties determine the below-the-line value we get from our recording system, but the list of prices does not influence the states of the world the system can record. The fact that the price lists, and thus accounting valuation, has little impact on the states that the system can record can be made more strongly. Ellerman (1982, 1986) develop a multi-dimensional, or vector accounting system that extends double-entry bookkeeping to bookkeeping for property and liabilities without monetary values (i.e., bookkeeping without price-lists). This recording system relies on the stock-flow equation in the same way that standard double-entry bookkeeping relies on the balance constraint.

The equivalence between the record keeping systems can be used to generalize bookkeeping beyond 'merely' tracking monetary performance and value. If one has a real signed measure of some output of the entity and an associated unit price, it could be integrated directly in the double entry system. A clear example would be the (net) CO_2 emissions of the organization. This output is measured in metric tons, which satisfies the requirement of a real signed measure, denote it by q_{n+1}. If we define a price for it through the marginal social cost of emissions and denote that as λ_{n+1}, we have an additional flow. To complete the group, and thus keep a balance, we need a counter account. All stocks on the balance sheet are equal to the total net inflow into the account over the history of the organization. A similar total emissions account starting from zero at a given point would allow us to keep the balance in the record keeping system. In this record keeping a 'CO_2-corrected' performance measure is readily available: $Perf_{CO2} = -Val(A_E) - \lambda_{n+1} q_{n+1} = Val(A_E \bigcup \alpha_{n+1})$, shows the profit net of social costs of CO_2-emissions as a list definition like net income. This could bring a clear incentive to increase investment in abatement or compensation of CO_2 emissions. All results of such investment would be immediately visible in this extended record keeping system, in similar fashion to the visibility of investment results on realized profit.

Under these 6 requirements – axioms 1-3, 4*, 5 and 6* – we get a system equivalent to double-entry bookkeeping. There is something to do bookkeeping for (axiom 1), we get a system that has a list (axiom 3) that is balanced after all possible outcomes (axiom 6*). The list (or vector) of values is closed under addition because we have well defined measures (axiom 2 and 4*) of value, it has an identity element (axiom 5), the inverse of a signed measure is the same measure with the opposing sign (axiom 2), such that we also have an inverse element in all cases, and thus the necessary and sufficient requirements of a group are met. Since all $\hat{q}_i$ map to the real line, Lemma 2 together with Proposition 2 show that this is equivalent to our double-entry bookkeeping system based on the $\mathcal{B}$-group. This system is not unique, and Proposition 2 shows that there is a family of equivalently strong systems under axiom $4*$. The monetary-unit assumption in axiom 4 simply selects a unique one from this class. However, Lemma 2 shows that even this system has several representations. We present no indication that this set of axioms are the only set of axioms that lead to our current system, others might exist. However, if one wants to have the 6 properties captured in the axioms, then a system equivalent to the $\mathcal{B}$-group will result.

5 Different mathematical descriptions

The bookkeeping system can be modeled mathematically in many different ways. The group-formalism used in the first part of this paper is intentionally chosen to describe the bookkeeping in a very 'tight' form. The group has easily recognizable underlying elements and the group law corresponds to updating the books by processing a journal entry, a well-known process in accounting. Different, complementary mathematical descriptions of this system exist.

In a different form, the group-set $\mathcal{B}^n$ has appeared in the accounting literature before.

Mattessich's (1964) accounting matrices are formed by putting the list of accounts, i.e. the list corresponding to B, on both the horizontal and vertical dimensions of an $n \times n$ matrix. A transaction is posted to the matrix by adding some value to a cell. The row it is placed in determines the debited account, the column the credited account. The value of account i is then found by comparing the total of the i-th row to the total of the i-th column. Each time a value is posted to a cell in this matrix, it automatically adds the same amount to the debit total (via the row totals) and the credit totals (via the column totals), thus the accounting system is balanced by construction. The combinations of the row and sum totals in this matrix are equivalent to the lists in $\mathcal{B}^n$. This type of accounting matrix is also used in the vector accounting of Ellerman (1982, 1986).

Several papers study accounting through a different type of matrix combined with matrix algebra (Ijiri 1967, 1975; Butterworth 1972; Ijiri 1993; Arya et al. 2000, e.g.). In this matrix formalism, the process of updating the books forms the main focus point. In an very interesting application, Li et al. (2019) show how the resulting matrices can be studied using graph theory to create an internal control mechanism for inter-firm transactions. There are clear links between this matrix formalism and the group formalism in this paper. In this matrix notation, the changes in values of the accounts ($\lambda_i q_i$ in the definitions of this paper), are stacked in a m dimensional vector y. All transactions in a period are described in an $n \times m$ matrix A that contains only values $\{1, 0, -1\}$. The cells in each row of A add up to 0 (which captures the balance constraint). The cells with a 1 in this matrix indicate debit values, while the cells with a -1 indicate a credit value. This matrix is multiplied by vector y that captures the values of the transactions to yield the resulting books captured in vector x. This allows one to denote all the journal entries in a given period:

$$x = A \cdot y \tag{1}$$

The matrix A has a nullspace that is defined by the amount of aggregation that occurs in the bookkeeping process. If the resulting vector x would allow you to identify the net value of all transaction between any two accounts, the nullspace of A would be the zero-vector. However, typically there are many values of y that satisfy this equation. In the inventory example, increasing both the purchases and use of material by 1 leaves the change of inventory-value the same. In this paper we look at the difference between debit and credit values in any account. Hence, any change on the right-hand side of Equation 1 that yields the same difference between debit and credit (the nullspace of A) is included in the equivalence definition of Section 2.

Barra et al. (2010) and Gentili and Giacomello (2017) study the updating process under the balance constraint in two closely related papers. Barra et al. (2010) models the accounting system with balance constraint through equational zero-vectors like in axiom 6. They link this description to the databases used in bookkeeping, showing how the zero-vectors can be used for the detection of fraud and errors. Altough the modeling approach is different, this is very much in the same spirit as Li et al. (2019). Gentili and Giacomello (2017) study the accounting systems through zero-terms, that are directly related to the zero-vector equations. In this model, accounting data is updated through a finite differences system in which the development of accounts over time becomes more explicitly. In both cases the properties found are determined by the balance constraint, rewritten as a zero-term equation of the form:

$$assets - equity - liabilities = 0$$

Another approach is to describe bookkeeping as a computational problem, for instance in terms of Turing machines (Rambaud and Pérez 2005). Turing machines are closely related to lambda-algebra's, like the one formed by the group-set $\mathcal{B}$. This can be used to model the operations needed in a computer program to do the bookkeeping. The links between the algebraic structure of the bookkeeping, like the Pacioli group, and the computational problem, modeled via automatons, are studied extensively in Rambaud (2010).

6 discussion

The balance constraint in bookkeeping gets a prominent role in earlier fundamental work like Mattessich (1964) and Ijiri (1975). Both these authors describe the balance constraint as part of accounting duality. They discuss its origins as either a consequence of value definitions (causal duality) or as a consequence of descriptive reality (claims on assets and the assets should be worth the same). There is, however, another reason for duality in bookkeeping. We keep track of all the stocks and all the flows of value in the company. Given that we do not allow for value to spontaneously appear or disappear, bookkeeping for the values in the entity is a closed system. Whatever value is created or lost has to be absorbed by or taken from the residual account, a role taken by owner's equity.

The balance constraint creates a fundamental link value definitions in earnings statements and balance-sheets. The earnings statement and balance-sheet pertain to two different dimensions of the entity, performance and valuation. We want to record and report both of these dimensions as accurately as possible, but they cannot be separated in our bookkeeping system. This makes the trade-off between the reliability and relevance of the generated financial statements unavoidable. If we do not use double-entry bookkeeping, we are free to define balance sheet value and earnings value in any way we like, but we loose the $debit = credit$ link that is so instrumental in verifying the reliability of the financial statements. We also loose the definition of owner's equity. This interdependence is not due to a lack of imagination or ingenuity of accountants and standard setters, this trade-off is baked into the bookkeeping process via the design choices (axiom 6). It is a direct consequence of the requirements of having a verifiable and thus reliable bookkeeping system that keeps track of owner's equity.

Double-entry bookkeeping is not so restrictive that we cannot improve value definitions. This is clear from the discussion of OCI and stock-based compensation in Section 3. If we want to accurately measure performance of the company via an income measure, we can separate out incidental or transitory items by adding buffer accounts like OCI. However, the change in equity always has include all changes. The balance constraint does not allow us to treat the costs of stock-based compensation as a normal cost that reduces equity. Here too, if we want to report it as a cost, the balance constraint would force us to to keep extra buffer accounts. However, as with OCI, the balance constraint means that the introduction of such extra accounts change more value definitions then just the newly introduced accounts.

The value recorded in these buffer accounts shows the restrictiveness of the balance constraint for a particular company and measurement problem. If change in equity is highly volatile for non-performance related reasons, this would reduce the relevance of change in equity as a performance measure. The swings in equity that are not related to performance, are just noise in performance measurement. By changing the definitions of value so that the noise is recorded under OCI, the resulting earnings number becomes more relevant for performance measurement. The statistical properties of the new account can be used to analyze its function for the record-keeper. Statistical noise, by definition, has a zero average. If the buffer account is used specifically to reduce noise, it should have an average value near 0 over time (ignoring the effect of inflation). The variance in OCI thus shows how constraining the balance constraint was in recording performance. If we see that OCI (or any other buffer account) is systematically different from 0, this could indicate that the new earnings number is a biased performance measure. This bias makes earnings (or the relevant measure) less relevant, but due to the balance constraint it is detectable in principle.

The interdependence between value definitions also implies that the balance-sheet approach of measuring value that the IASB seems to adhere to and an earnings-based approach as is proposed by some authors Benston et al. (2007); Dichev (2008, e.g.) are not independent. Any change in the definition of a cost or benefit that changes recorded earnings has to be matched by an opposing change of valuation on the balance sheet (and vice versa). The difference in focus is mostly a rhetorical or didactic one. This distinction is clearly important to clarify the conceptual framework to all users, but both approaches have to take into account that the measurement of flows and the measurement of stocks are related. Furthermore, any difference that occurs because of the different focus of both approaches can be resolved through buffer accounts.

If a new record keeping system is set up, fundamental choices have to be made about its properties. These choices can be studied efficiently using mathematical formalisms like the one in this paper. The double-entry accounting system has survived the test of time, which is an indication it has desirable properties to its users. To show the design choices made, a backwards-engineering exercise shows a set of axioms that lead to our current system. A system that adheres to these 6 axioms will be mathematically equivalent, regardless of the exact techniques used to record or store the data. A commonly heard proposal for blockchain's role in accounting is a triple-entry accounting system. The third entry refers to a record indicating the other entity involved in the transaction on top of a double-entry system. With the extra entry, each transaction record forms a signed receipt that can be audited by comparing it to the transaction record at the counterpart. By putting the third entries of both entities on a public ledger, fraud can be reduced as any interested party can verify that both parties recorded the same transaction for the same value, see for instance Dai and Vasarhelyi (2017) and references therein. If both sides of the transaction are correctly recorded on the public ledger, the incoming monetary values should equal the outgoing monetary values on this ledger. This public ledger thus represents a balanced set of accounts and can be described, analyzed and studied through the $\mathcal{B}$-group just like the internal accounting system of an organization. Smart contracts that rely on the blockchain record can thus use the apparatus developed for traditional accounting systems, as well as the apparatus of the blockchain itself.

Identifying efficient ways of implementing required checks for data consistency is an important step to making a record keeping systems viable. Proposition 1 shows that several options for a consistency test on balance of the record exit. Via Lemma 2 we can reduce the amount of numbers to store, process and transmit, from 2 per element to 1. This creates a single-entry system mathematically, but it has the data-integrity protection of double-entry bookkeeping (Christiaanse and Hulstijn 2013). This system corresponds to an equational zero-vector database as studied in for instance Barra et al. (2010) or the zero-terms of Gentili and Giacomello (2017). The mathematical abstraction also makes it much easier to think about extensions to the current record keeping system. A simple example shows, that we can use double-entry record keeping to evaluate CO_2-corrected performance of organizations. If we can find real measures and real prices for the relevant outputs, it is technically feasible to expand our bookkeeping and generate a single performance measure of the organization capturing the triple bottom-line of people, planet and profit. This would allow much clearer incentives to be placed on the societal impact of organizations, since all the incentive systems developed for profit-maximization could be directly adapted.

The axioms in this paper do not define what kind of events we want to record, or against what value we record them (except for empty events, and the equity account). These valuation questions are left as questions on the shape of the q_i. This paper focuses on the bookkeeping aspects, so it only requires that these measures exist. The form and value of these measures q_i are the core of a mathematical description of the valuation and classification questions discussed accounting papers and standards (see e.g. Ijiri 1975; Mattessich 1964; Mock 1976)). Further definitions of these measures and how we can use their properties to better define accounting standards could be an interesting topic for future research.

The group formalism used to study the properties of the bookkeeping system, gives a clear algebra to derive properties of our bookkeeping system. Mathematical structures like these allow us to think about accounting and bookkeeping in a more abstract manner. This can show commonalities between problems that do not, on the surface, look that similar. One example of such a relation is the common source of accounting issues in OCI and stock awards. Mathematical structures allow us to think about such problems in a genera, but precise way. This analysis shows the trade-offs that cannot be avoided, as well as potential choices we have in creating a system that would create different trade-offs.

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A Appendix

A.1 Differences and the equivalence relation

In the main text we immediately subtract the credit entry from the debit entry in an element to get to the differences between these entries. Strictly speaking, however, this minus operation needs a proper definition before we can use it to define the equivalence relation. Therefore, a cleaner definition of the equivalence relation and the difference operation is in this appendix, in reversed order.

Equivalence is defined via the cross-product, two elements are equivalent iff their cross-addition yields the same debit value as credit value. First take two elements: $\alpha_1 = [d_1//c_1], \alpha_2 = [d_2//c_2]$ Then we can take one of these elements and reverse the order of the entries and add it to the other:

$$\alpha_1 + \alpha_2^{-1} = [d_1//c_1] +_{dc} [c_2//d_2] = [d_1 + c_2//c_1 + d_2] \tag{A.1}$$

Then we define these elements to be equivalent iff the resulting sum has the same debit as credit value:

$$\alpha_1 \simeq \alpha_2 \Leftrightarrow d_1 + c_2 = c_1 + d_2 \tag{A.2}$$

However, if we rearrange the terms on the right-side of this definition we get:

$$d_1 + c_2 = c_1 + d_2 \rightarrow d_1 - c_1 = d_2 - c_2$$

such that we see that this implies the difference between the debit and credit in both elements is the same. Here we can avoid the problem with the undefined minus sign, as the debit and credit entries (but not the α-elements) are defined as real numbers, such that the subtraction operation is the standard difference operation. The difference operation on the α-elements can then be defined via the inverse element as is done formally in subsection B.3.

B $\mathcal{A}$ groups, Lemma 1

This section shows that the $\mathcal{A}$-accounts meet the definition of an Abelian (symmetric) group under the operation of addition of the debit and credit values as is stated in Lemma 1. The underlying set, $\mathcal{A}$, consists of the set of pairs of non-negative real numbers. The group law is addition of the credit and debit values of the two accounts. All under the equivalence relation defined in subsection A.1. To proof the group-structure, we need to proof closure, symmetry, associativity, and the existence of a unique identity element, as well as an inverse element for each element in the set. By also showing symmetry of the binary operation, we show that the group is Abelian.

B.1 Closure and symmetry

If we have 2 accounts $\alpha_1 = [a\ //\ b]$ and $\alpha_2 = [c\ //\ d]$, the outcome of operation of addition is always the same:

$$\alpha_2 +_{dc} \alpha_1 \simeq \alpha_1 +_{dc} \alpha_2 \simeq [a + c\ //\ b + d] \simeq \alpha_3$$

since if $0 \geq a, b, c, d < \infty$ then $0 \geq a + c, b + d < \infty$ and thus α_3 is finite and the outcome is again a pair of non-negative numbers and thus $\alpha_3 \in \mathcal{A}$. So that the set is closed under this operation and the operation is symmetric.

B.2 Associativity

Associativity requires that $(\alpha_1 +_{dc} \alpha_2) +_{dc} \alpha_3 \simeq \alpha_1 +_{dc} (\alpha_2 +_{dc} \alpha_3) = [d_1 + d_2 + d_3\ //\ c_1 + c_2 + c_3]$. Given that we are dealing with addition, the order of addition does not impact the results, such that the operation is indeed associative.

B.3 Identity element

There is a unique identity element, $e = [0\ //\ 0]$. By definition of the addition operation, we have $\alpha +_{dc} [0\ //\ 0] \simeq \alpha$. Uniqueness is given by the fact that under the equivalence relationship we know that if

$$\alpha +_{dc} [x\ //\ x] \simeq \alpha +_{dc} [0\ //\ 0] \tag{A.3}$$

$$[x\ //\ x] \simeq [0\ //\ 0] \tag{A.4}$$

$$[x - x\ //\ 0] \simeq [0\ //\ 0] \tag{A.5}$$

Where the last equivalence holds since $x - x = 0$ by definition. Hence any element e that can be applied to another element a to yield a is equivalent to the identity element, $[0\ //\ 0]$.

B.4 Inverse element

An inverse of an element, maps the element to identity via the group law. Hence we need to guarantee that for each element in the set, $\alpha \in \mathcal{A}$, we can find an inverse element, $\alpha^{-1} \in \mathcal{A}$ such that we have $\alpha^{-1} +_{dc} \alpha = [0\ //\ 0]$. Via the equivalence relationship this element is easily identified for each pair $[x\ //\ y]$ as $[y\ //\ x]$. We use this to denote the operation $+_{dc}\ \alpha_i^{-1}$ as $-_{dc}\ \alpha_i$ which is equivalent to the definition of subtraction on the real numbers.

Since this means the set $\mathcal{A}$ with the operation addition of the debit and credit values, satisfies the necessary and sufficient conditions of having closure, associativity, a unique identity and an inverse element (under the difference equivalence relation), so that it forms a group.

Note that journalization is symmetric. If we have 2 accounts $\alpha_1 = [a\ //\ b]$ and $\alpha_2 = [c\ //\ d]$, it does not matter whether we say that α_1 is the original account and α_2 comes from the journal entry or the other way round, the outcome of the application is always the same:

$$\alpha_2 +_{dc} \alpha_1 \simeq \alpha_1 +_{dc} \alpha_2 \simeq [a + c\ //\ b + d] \simeq \alpha_3$$

and the outcome is again a pair of non-negative numbers, so the group is an Abelian group. In most cases in normal bookkeeping, however, the journal entry will only involve a couple of accounts that change value, so that we usually do not denote the accounts with zero-value.

C Isomorphism real line

To proof a group-isomorphism, we have to show a one-to-one correspondence between the elements of the real line and the α-elements, as well as the correspondence in operations.

Through the equivalence relationship, we can organize all α-elements based on the size of their debit or credit balance. If we make a line starting at 0 and increase the credit balances to the right and the debit balances to the left, we get a line of elements in either direction. We can put the real line orthogonal to this as is done in Figure 1, then for every possible value of an α-element we find exactly 1 value on the real line as is shown by the red line. The relationship between the sets is one-to-one and can be defined for every element in both sets. Furthermore, the group law corresponds to normal addition on the real line, which is quickly verified by looking at the corresponding operations:

$$[d_1//c_1] +_{dc} [d_2//c_2] \simeq [d_1 + d_2//c_1 + c_2] \tag{A.6}$$

$$(d_1 - c_1) + (d_2 - c_2) = (d_1 + d_2) - (c_1 + c_2) \tag{A.7}$$

Note that the first α-element in A.6 is mapped to the real number between the first set of brackets in A.7, etc. This one-on-one correspondence holds for all non-negative numbers, and hence for all α-elements. So that the operation $+_{dc}$ corresponds to normal addition on the real line, and that the equivalence relationship corresponds to equality of real numbers.

D The list of accounts, Lemma 3

To prove Lemma 3, we have to prove the same properties for the lists of accounts that form the entire accounting system.

Associativity is given by the fact that $\mathcal{B}^n$ elements consist of a list of $\mathcal{A}$-elements and each α-group is associative, such that the $\mathcal{B}^n$-elements consists of associative elements. Since we only apply elements with the same index to each other, associativity is given by the fact that each of the elements are associative.

Similarly the identity element can be obtained by combining n identity elements from the $\mathcal{A}$-group: $[0\ //\ 0], ..., [0\ //\ 0]$.

The inverse element to $\mathcal{B}^n = [\alpha_1, ..., \alpha_n]$ is $(\mathcal{B}^n)^{-1} = [\alpha_1^{-1}, ..., \alpha_n^{-1}]$, which is unique as all of the elements have a unique inverse.

This group is also an Abelian group, as we have to do the addition per account, and thus have the same result per account as we had under the $\mathcal{A}$-group.

Any finite number of addition applications of accounts to each other, or any finite number of journal entries applied to some set of accounts yields a set of transformed accounts $B' \in \mathcal{R}^n$ by the same logic. That is, we can conclude that the set of elements $B \in \mathcal{R}^n$ is closed under finite debit-credit addition. While for each element in $\mathcal{B}^n$ we know that $\sum_i^{dc} B \simeq [0\ //\ 0]$, so that $\sum_i B +_{dc} \sum_i B' \simeq [0\ //\ 0] +_{dc} [0\ //\ 0] \quad \forall B, B' \quad \in \quad \mathcal{B}^n$. Or in bookkeeping terms, as long as we process valid journal entries, we end up with a valid set of financial statements for any finite number of journal entries. Which proofs Lemma 3.

E Balanced differences

The difference between two balanced lists of accounts is balanced:
$\forall B, B' \in \mathcal{B}^n \quad (B -_{dc} B') = B +_{dc} (B')^{-1} \simeq B'' \ \in \ \mathcal{B}^n$.

Proof. By definition of the inverse elements, the credit-balance of B' is the debit-balance of $(B')^{-1}$. By associativity, the order of addition of the balances does not matter. So that we have $\sum_{i=1}^n (B -_{dc} B') = \sum_{i=1}^n B \sum_{i=1}^n (B')^{-1} = 0 - 0 = 0$, so the difference between to balanced lists is similarly balanced. □

F Corollary 2, links value definitions

Corollary 2 proposes a link between the value on the earnings statement and the value recorded in the balance sheet.

First recall the definitions. We make a list of accounts that are added or subtracted from Earnings, A_E, and its complement A_E^C, so that each account is on only one of these sub-list:

$$B = A_E \bigcup A_E^C, \quad A_E \bigcap A_E^C = \emptyset \quad \rightarrow Val(A_E) + Val(A_E^C) = 0$$

Furthermore, by definition, net income equals $-val(A_E)$. For a given history of events ω (i.e. whatever happened in the fiscal year), we record its consequences under the balance constraint. Let the resulting books for history ω denoted as $B(\omega), B'(\omega) \in \mathcal{B}^n$ respectively, where B, B' refer to two different sets of books where the differences are caused by differences in value definitions, i.e. different measures q, q'. These value definitions could, for instance, be managed vs un-managed earnings, or IFRS and local GAAP, or fair value and historic cost. We know that for any possible history, ω, we must have: $B - B' \in \mathcal{B}^n$ by Proposition 1. Hence any difference between two sets of balanced valuations, has to be balanced itself.

If reported earnings differ between B and B', i.e. $Val(A_E) \neq Val(A'_E)$, this difference has to be reflected in a balance sheet account. We will show this by contradiction. Assume that there are 2 changes in value definitions, one in account i and one in j, denoted as $\Delta Val(q_i), \Delta Val(q_j)$. Proposition 1 tells us that $\Delta Val(q_i) = -\Delta Val(q_j)$. Furthermore, assume that both of the associated elements are in the earnings list: $\alpha_i, \alpha_j \in A_E$. Then we know:

$$-Val(A'_E) = \sum_{A_E} d_i - \sum_{A_E} c_i + \Delta Val(q_i) + \Delta Val(q_j) = \sum_{A_E} d_i - \sum_{A_E} c_i = Val(A_E)$$

which violates the initial assumption that the recorded net income was different under the new definitions. So that, if we assume $Val(A_E) \neq Val(A_E)'$ and we assume that $B, B' \in \mathcal{B}^n$, then i and j cannot both be in the list of accounts in A_E. One account has to be in A_E^C. Since at the moment that we calculate earnings, all costs and revenues go to earnings, the remaining accounts have to be some inventory that is reported on the balance sheet. which completes the proof.

G Subspaces, Lemma 4 and Proposition 2

First the proof of Lemma 4. A sufficient condition for a subspace $\mathcal{W} \in \mathcal{R}^n$ requires that, for two elements $\bar{S}, \bar{S}' \in \mathcal{W}$, we can proof:

$$v\hat{S} + u\hat{S}' \in \mathcal{W}$$

(set $u, v = 1$ for the addition property; set $v = 0$, and $u \neq 0$ for the multiplication property). For any n-vector μ of real numbers, define the space of elements $\mathcal{W}$:

$$\mathcal{W} = \{x \in \mathcal{R}^n | \mu x = 0\}$$

In words, $\mathcal{W}$ consists of all vectors that have a zero inner product with μ. Assume that both $\mu\bar{S} = \mu\bar{S}' = 0$, so that these two vectors are part of $\mathcal{W}$. Then we can check that $(v\bar{S} + u\bar{S}')\mu = v\bar{S}\mu + u\bar{S}'\mu = v0 + u0 = 0 \; \forall u, v \in \mathcal{R}$ so that indeed the space $\mathcal{W}$ is a subspace of $\mathcal{R}^n$. This holds for any value of $\mu \in \mathcal{R}^n$ and every value of μ defines a subspace $\mathcal{W}_\mu$. Now set $\mu = 1$ and we find our space $\bar{\mathcal{B}}^n$. The isomorphism of Lemma 2 completes the result.

G.1 Proposition 2

Note that Lemma 4 holds for any value of $\mu \in \mathcal{R}^n$, so that by setting $\mu = \lambda$ we show that in fact every bookkeeping system – each subspace W_λ found through the corresponding vector λ – forms a linear subspace. All these linear subspaces are equivalent, as a one-to-one mapping exists between any two of them. This is formally shown in Proposition 2

To prove this, it is without loss of generality to show a bijection between W_{λ^l} and $\bar{\mathcal{B}}^n = W_{\mathbf{1}}$. First define $\kappa^l = \{\frac{1}{\lambda^l_i}, ..., \frac{1}{\lambda^l_n}\}$, this is a vector holding the multiplicative-inverse values of the elements in λ^l. This vector must exist, as we required that none of the elements in the vector λ are equal to zero. Now take some vector $\bar{b} \in \bar{\mathcal{B}}^n$, then we know that $\sum_{i=1}^n \bar{b}_j = 0$. We can define a transformation $\bar{b} \mapsto^{\kappa^l} \{\kappa^l \bar{b_1}, ..., \kappa^l \bar{b_n}\} = k^l(\bar{b})$. Then we can directly verify that

$$\lambda^l k^i(\bar{b}) = \sum_{i=1}^n \lambda^l_i \kappa^l_i \bar{b_i} = \sum_{i=1}^n 1\bar{b_i} = 0$$

This shows that $k^l(\bar{b}) \in W_{\lambda^l}$. Furthermore, every κ^l_i defines a linear mapping $\mathcal{R} \mapsto \mathcal{R}$ that is invertible, one-to-one and covers the entire space $\mathcal{R}$. Hence, it defines a bijection for every element in the vector $\bar{b}$. Which means κ^l defines a bijective mapping $\bar{\mathcal{B}}^n \mapsto^{\kappa^l} W_{\lambda^l}$, while the inverse of this mapping is determined in the same way through λ^l. As we have bijective mappings from any W_{λ^l} to and from $\bar{\mathcal{B}}^n$, to find a bijective mapping from W_{λ^l} to W_{λ^k}, simply combine the two relevant mappings to and from $\bar{\mathcal{B}}^n$:

$$W_{\lambda^l} \mapsto^{\kappa^l} \bar{\mathcal{B}}^n \mapsto^{\lambda^k} W_{\lambda^k}$$

Again, The isomorphism of Lemma 2 completes the result.

H Figures

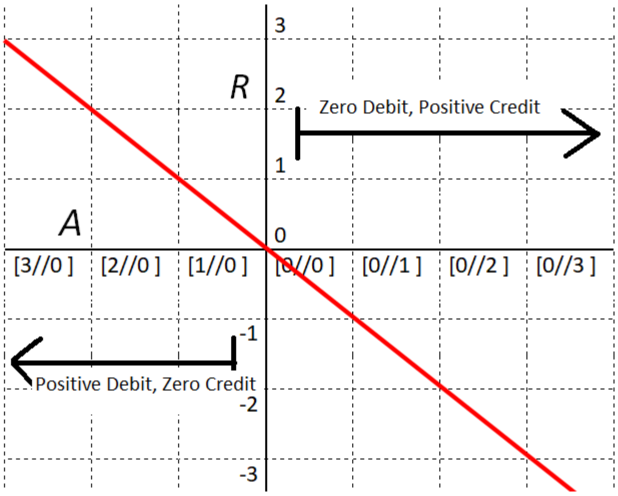

Figure 1

(a) Ledger

$$a_1 \ [\, x_1 \,//\, 0 \,]$$
$$a_2 \ [\, x_2 \,//\, 0 \,]$$
$$a_3 \ [\, x_3 \,//\, 0 \,]$$
$$a_4 \ [\, 0 \,//\, x_4 \,]$$
$$a_5 \ [\, 0 \,//\, x_5 \,]$$

(b) Journal entry

$$a_1 \ [\, 0 \,//\, y_1 \,]$$
$$a_2 \ [\, y_2 \,//\, 0 \,]$$
$$a_3 \ [\, 0 \,//\, 0 \,]$$
$$a_4 \ [\, 0 \,//\, 0 \,]$$
$$a_5 \ [\, 0 \,//\, 0 \,]$$

(c) Updated ledger

$$a_1 \ [\, x_1 \,//\, y_1 \,]$$
$$a_2 \ [\, x_2 + y_2 \,//\, 0 \,]$$
$$a_3 \ [\, x_3 \,//\, 0 \,]$$
$$a_4 \ [\, 0 \,//\, x_4 \,]$$
$$a_5 \ [\, 0 \,//\, x_5 \,]$$

Index, *i*	Account name
1	Cash
2	Materials Inventory
3	Completed goods Inventory
4	Owner's equity
5	Loan, 5%
...	

(d) Association α-elements and accounts

Figure 2

H.1 Figure captions

Figure 1 Bijection between set of α-elements and set of real numbers.

Figure 2 Processing of a journal entry in the notation of a $n = 5$ list of α-accounts.