LP TESTS FOR MV EFFICIENCY

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LP tests for MV efficiency

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ABSTRACT

We derive empirical tests for the mean-variance efficiency of a given portfolio. The tests can be computed using straightforward linear programming, and they give substantial flexibility in modeling the investment possibilities. Using this test, we can reject the hypothesis that the S&P 500 index is mean-variance efficient relative to the 25 Fama and French (1993) equity portfolios.

Mean-variance analysis (MVA; Tobin (1958), Markovitz (1952, 1959)), is the dominant framework for analyzing investment behavior. It is useful both for positive analysis (where the objective is to analyze the decision rules actually used by decision-makers) as well as in normative analysis (where the objective is to support practical decision making). Applying MVA criteria to empirical data typically requires solving large-scale quadratic programming (QP) models. QP problems are solvable in polynomial time using interior-point methods (see e.g. Nesterov and Nemirovskii (1994)). However, for large problem, involving hundreds or thousands of assets and/or investment restrictions, the computational burden can be prohibitive. To reduce the computational burden, various alternatives to MVA use alternative, linear measures of risk. For example, Yamakazi and Konno (1991) present a model based on Mean Absolute Deviation (MAD), and Young (1998) presents a model based on the minimum return (or maximum loss). Both models require straightforward Linear Programming (LP), and hence are computationally more attractive than MVA. This paper extends this literature by presenting LP tests based on the Hanoch and Levy (HL; 1970) definition of mean-variance (MV) efficiency.

The standard definition of MV efficiency classifies a portfolio as efficient if and only if no other portfolios exist with a higher mean and a lower variance. This definition is consistent with the expected utility theory (EUT) if the probability distribution of return has a normal shape or, more generally, an elliptical shape (see e.g. Bigelow (1993) for a complete characterization of the necessary and sufficient conditions). By contrast, our tests are derived from an alternative MV efficiency definition by Hanoch and Levy (HL; 1970): a portfolio is efficient if and only if there exists an increasing and concave, quadratic utility function that rationalizes the portfolio. The HL definition is more powerful than the standard definition; the HL efficient set is subset of the standard MV efficient set (see e.g. HL, Levy and Hanoch (1970), and the Corollary to Theorem 2 in Section III). Further, the HL efficient set is a proper subset of the efficient set of second-order stochastic dominance (SSD; see e.g. Hadar and Russel (1969) and Hanoch and Levy (1969)), which considers the general class of increasing and concave utility functions. HL and Meyer (1979) present necessary and sufficient conditions for testing if a given portfolio is MV efficient in the HL definition. As is true for the standard MV

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criterion, applying these criteria to empirical data requires QP and hence may involve substantial computational burden. However, we will demonstrate in this paper that it is possible to derive necessary and sufficient conditions that require LP only. Specifically, we derive empirical tests for the MV efficiency of a given portfolio. The tests substantially reduce the computational burden associated with the QP approach, and in addition they offer substantial flexibility in modeling the investment possibilities (e.g. including short selling restrictions or transaction costs).

The remainder of this paper is organized as follows. Section I recaptures the HL definition of MV efficiency. Section II gives our LP test of MV efficiency. Section III provides an equivalent dual formulation and it formalizes the relationship between the HL efficient set and the standard efficient set. Section IV illustrates our approach by means of an empirical application for US stock market data. Finally, Section V gives conclusions and suggests directions for future research.

I. MEAN-VARIANCE EFFICIENCY

Consider an investment universe consisting of N assets, associated with returns $\boldsymbol{x} \in \Re^N$.[1] Throughout the text, we will use the index set $\mathrm{I} \equiv \{1, \ldots, N\}$ to denote the different assets. In addition, we will treat the returns as serially independent and identically distributed random variables with a continuous joint cumulative distribution function (CDF) $G : \langle -\infty, d]^N \rightarrow [0,1]$, with $d \in \langle 0, \infty \rangle$ for an upper bound to the return distribution.[2] Investors may diversify between the assets, and we will use $\boldsymbol{l} \in \Re^N$ for a vector of portfolio weights. For simplicity, we will consider the case where short selling is not allowed, and the portfolio weights belong to the portfolio possibilities set $\Lambda \equiv \{\boldsymbol{l} \in \Re^N_+ : \boldsymbol{l}^T \boldsymbol{e} = 1\}$. However, it is possible to generalize the analysis towards cases where short selling is allowed and cases where additional restrictions are imposed on the portfolio weights (see the Conclusions).

We consider the problem of establishing whether a particular portfolio, say $\boldsymbol{t} \in \Lambda$, is optimal, i.e. whether it maximizes the expected value of the investor's utility function $u : \langle -\infty, d] \rightarrow \Re$, $u \in U$, with U for the class of twice continuously differentiable, von Neuman-Morgenstern utility functions. The portfolio $\boldsymbol{t}$ is optimal if and only if:

$$(1) \qquad \int u(\boldsymbol{x}\boldsymbol{t})\partial G(\boldsymbol{x}) = \max_{\boldsymbol{l} \in \Lambda} \int u(\boldsymbol{x}\boldsymbol{l})\partial G(\boldsymbol{x}).$$

In practical applications, full information about the utility function typically is not available, and this condition cannot be verified directly. This provides the rationale for using efficiency criteria that rely on a set of general assumptions rather than a full

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[1] Throughout the text, we will use $\Re^m$ for an m-dimensional Euclidean space, and $\Re^m_+$ denotes the positive orthant. Further, to distinguish between vectors and scalars, we use a bold font for vectors and a regular font for scalars.

[2] A natural candidate for d is the sample maximum, i.e. in terms of the notation developed below: $d = \max_{i \in \mathrm{I}, t \in \Theta} \boldsymbol{x}_{it}$.

specification of the utility function. The HL definition of MV efficiency restricts attention to the class of monotone and concave quadratic utility functions:

(2) $$Q \equiv \{u \in U : u(x) = a + bx + gx^2 \quad g \leq 0, b + 2gd \geq 1\}$$

Note that the normalizing constraint $b + 2gd \geq 1$ restricts the functions in Q to be strictly increasing over the entire return interval. The unity value can be replaced by an arbitrary strictly positive value, as $u \in Q$ implies $ku \in Q$ for all $k > 0$.

Apart from the utility function, also the CDF generally is not known in practical applications. Rather, information typically is limited to a discrete set of time series observations, say $(\boldsymbol{x}_1 \text{ L } \boldsymbol{x}_T)^T$ with $\boldsymbol{x}_t \equiv (\boldsymbol{x}_{1t} \text{ L } \boldsymbol{x}_{Nt}) \in \Re^N$, which can be treated as independent random samples from the CDF. Throughout the text, we will use the index set $\Theta \equiv \{1, \text{L}, T\}$ to denote different points in time. Using the observations, we can construct the empirical distribution function (EDF):

(3) $$F(\boldsymbol{x}) \equiv \text{card}\{t \in \Theta : \boldsymbol{x}_t \leq \boldsymbol{x}\} / T .$$

In this paper, we analyze SD for the EDF rather than for the CDF, so as to focus on the computational problems encountered in practical applications. Still, in the application section, we do use bootstrap techniques to assess the sensitivity of our results with respect to sampling variation.

Using the above notation, MV efficiency can be defined as follows:

DEFINITION 1 *Portfolio $t \in \Lambda$ is MV efficient if and only if it is optimal relative to some monotone and concave quadratic utility functions $u \in Q$, i.e.*

(4) $$\min_{u \in Q}\left\{\max_{l \in \Lambda}\left\{\int u(\boldsymbol{x}l)\partial F(\boldsymbol{x}) - \int u(\boldsymbol{x}t)\partial F(\boldsymbol{x})\right\}\right\} =$$
$$\min_{u \in Q}\left\{\max_{l \in \Lambda}\left\{\sum_{t \in \Theta}\left(u(\boldsymbol{x}_t l) - u(\boldsymbol{x}_t t)\right)/T\right\}\right\} = 0 .$$

This minimax formulation is similar to Post's (2001a) formulation of SSD. Specifically, Definition 1 replaces the set of monotone and concave utility functions (used by Post) with the set of monotone and concave, quadratic utility functions. The minimax formulation is also reminiscent of several measures of productive efficiency, including the well-known Debreu (1951)-Farrell (1957) measures.

We stress that Definition 1 adheres to the HL definition of MV efficiency rather than the standard definition (a portfolio is efficient if and only if no other portfolio achieves a higher mean and a lower variance). The HL definition is more powerful than the standard definition; the HL efficient set is subset of the standard MV efficient set (see also the Corollary to Theorem 2 in Section III). For example, the standard definition typically classifies a riskless fund as efficient (no portfolio of risky assets achieves a zero variance). By contrast, the HL definition typically classifies a riskless fund as inefficient, reflecting Arrow's theorem - 'A risk averter takes no part of an unfavorable or barely fair game; on the other hand, *he always takes some part of a favorable gamble*' (Arrow, 1970, p. 100, italics as in the original text).

II. LINEAR PROGRAMMING FORMULATION

The Post (2001a) SSD tests check whether there exist monotone and concave utility functions that rationalize the evaluated portfolio. In this spirit, we may ask if we can construct monotone and concave, quadratic utility functions $u \in Q$ that rationalize the evaluated portfolio $t \in \Lambda$:

THEOREM 1 *We may test MV efficiency of portfolio* $\boldsymbol{t} \in \Lambda$ *using the MV test statistic*

$$(5) \qquad x(\boldsymbol{t}) \equiv \min_{q,(b,g)\in\Omega} \left\{ q : \sum_{t\in\Theta} (b + 2g\boldsymbol{x}_t t)(\boldsymbol{x}_t t - \boldsymbol{x}_{it})/T + q \geq 0 \quad \forall i \in \mathrm{I} \right\},$$

with $\Omega \equiv \{(b,g) \in \Re \times \Re_- : b + 2gd \geq 1\}$. *Specifically, portfolio* $t \in \Lambda$ *is MV efficient if and only if* $x(t) = 0$.

PROOF The necessary condition follows from the well-known Kuhn-Tucker conditions for selecting an optimal portfolio if short selling is not allowed. Specifically, t is an optimal portfolio i.e. $t = \arg\max_{l\in\Lambda} \sum_{t\in\Theta} u(\boldsymbol{x}_t l)/T$ for $u \in Q$ only if all portfolios $l \in \Lambda$ are enveloped by the tangent hyperplane defined by the gradient vector $u'(t) \equiv (u'(\boldsymbol{x}_1 t) \,\mathrm{L}\, u'(\boldsymbol{x}_T t))$, i.e.

$$(6) \qquad \sum_{t\in\Theta} u'(\boldsymbol{x}_t t)(\boldsymbol{x}_t t - \boldsymbol{x}_t l)/T \geq 0 \quad \forall\, l \in \Lambda .$$

By construction, $(u'(t), u''(t))$ is a feasible solution to the primal problem, i.e. $(u'(t), u''(t)) \in \Omega$. The inequality (6) implies that this solution is associated with a solution value of zero. Hence, we find the necessary condition; t is MV efficient only if $x(t) = 0$.

To establish the sufficient condition, use $(b^*, g^*) \in \Omega$ for the optimal solution and use $p(x) \equiv b^* x + g^* x^2$. If $x(t) = 0$, then

$$(7) \qquad \sum_{t\in\Theta} p'(\boldsymbol{x}_t t)\boldsymbol{x}_t t = \max_{l\in\Lambda} \sum_{t\in\Theta} p'(\boldsymbol{x}_t t)\boldsymbol{x}_t l \; .$$

Since p is concave, we have $\sum_{t\in\Theta} p(\boldsymbol{x}_t l) \leq \sum_{t\in\Theta} p'(\boldsymbol{x}_t t)\boldsymbol{x}_t l$ for all $l \in \Lambda$. Combining this with $\sum_{t\in\Theta} p(\boldsymbol{x}_t t) = \sum_{t\in\Theta} p'(\boldsymbol{x}_t t)\boldsymbol{x}_t t$ and equality (10), we find that t is optimal relative to $p(x)$ i.e.

$$(8) \qquad \sum_{t\in\Theta} p(\boldsymbol{x}_t t) = \max_{l\in\Lambda} \sum_{t\in\Theta} p(\boldsymbol{x}_t l) \; .$$

Therefore, we find the sufficient condition; portfolio $t \in \Lambda$ is MV efficient if $x(t) = 0$. *Q.E.D.*

The test statistic $x(t)$ basically asks if we can find support lines for a monotone and concave quadratic utility function $u \in Q$ that rationalizes the evaluated portfolio. If the evaluated portfolio is efficient, then such support lines must exist, and if such support lines exist then the portfolio must be efficient (see the proof above). The necessary and sufficient condition can separate efficient portfolios from inefficient ones. However, we stress that the test statistic does not represent a meaningful performance measure that can be used for ranking portfolios based on the 'degree of efficiency'. For selecting the optimal portfolio from the efficient set, and for measuring the deviation from optimum, we typically need more information on investor preferences than is assumed in MVA.

The test statistic $x(t)$ involves a linear objective function and linear constraints, and it can be computed using straightforward LP. The following is a full LP formulation for $x(t)$:

$$
\begin{aligned}
(\mathrm{P}) \quad & \min_{b,g,q} q \\
& \text{s.t.} \sum_{t\in\Theta}(b + 2g\boldsymbol{x}_t t)(\boldsymbol{x}_t t - \boldsymbol{x}_{it})/T + q \geq 0 \quad i = 1, \mathrm{L}, N \quad (l_i) \\
& b + 2gd \geq 1 \quad (r) \\
& -g \geq 0 \\
& b \quad \text{free} \\
& q \quad \text{free}
\end{aligned}
$$

The shadow prices to the restrictions are given within brackets. This information is useful for interpreting the dual formulation (see Section III). The problem involves only 3 variables and N+1 constraints. Further, the model always has a feasible solution, as e.g. $b = 1$, $g = 0$ and $q = \max_{i\in \mathrm{I}} \sum_{t\in\Theta}(\boldsymbol{x}_{it} - \boldsymbol{x}_t t)/T$ necessarily satisfies all constraints. (This solution effectively represents risk neutral investors; risk neutral investors have linear utility functions and compare portfolios solely in terms of the expected return.) For small data sets up to hundreds of observations and/or assets, the problem can be solved with minimal computational burden, even with desktop PCs and standard solver software (like LP solvers included in spreadsheets). Still, the computational complexity, as measured by the required number of arithmetic operations, and hence the run time and memory space requirement, increases progressively with the number of variables and restrictions. Therefore, specialized LP solver software is recommended for large-scale problems involving thousands of assets (and/or investment restrictions).[3]

In addition to the value of the test statistic, the model gives information on the shape of the optimal quadratic utility function. If the evaluated portfolio is MV efficient,

[3] For an elaborate introduction in LP, we refer to Chvatal (1983). In practice, very large LPs can be solved efficiently by both the simplex method and interior-point methods. An elaborate guide to LP solver software can be found at the homepage of the Institute for Operations Research and Management Science (INFORMS); http://www.informs.org/.

then the empirical utility function gives an example of the type of utility functions that rationalize the portfolio. Still, we stress the empirical utility function is used as an instrument for testing MV efficiency, rather than as an estimate for the utility function of the investor that holds the evaluated portfolio. One complication is that there typically exist multiple optimal solutions if the evaluated portfolio is MV efficient. In addition, the interpretation of the optimal utility function is not clear if the evaluated portfolio is MV inefficient.

III. DUAL FORMULATION

Linear duality theory can derive an interesting alternative formulation for the MV test statistic. That formulation phrases in terms of the mean difference $m(l,t) \equiv \sum_{t\in\Theta}(x_{it}l - x_t t)/T$ and the co-movement measure $s(l,t) \equiv \sum_{t\in\Theta}(d - x_t t)(x_{it}l - x_t t)/T$.

THEOREM 2 *We may test MV efficiency of portfolio* $t \in \Lambda$ *using the dual test statistic*

(9) $$y(t) \equiv \max_{l\in\Lambda}\{m(l,t) : s(l,t) \geq 0\}.$$

Specifically, portfolio $t \in \Lambda$ *is MV efficient if and only if* $y(t) = 0$.

PROOF The test statistic $y(t)$ is obtained directly from the LP dual of (P):

$$(\mathrm{D}) \quad \max_{l,r} r$$

$$\text{s.t.} \sum_{t=1}^{T}(\boldsymbol{x}_t t - \boldsymbol{x}_t l)/T + r = 0 \quad (b)$$

$$-\sum_{t=1}^{T}(\boldsymbol{x}_t t)(\boldsymbol{x}_t t - \boldsymbol{x}_t l)/T - rd \leq 0 \quad (g)$$

$$\sum_{i=1}^{N} l_i = 1 \quad (q)$$

$$l_i \geq 0 \quad i = 1, \mathrm{L}, N$$

$$r \geq 0$$

It follows directly from the first equality that the optimal solution occurs if $r = m(l,t)$. Substituting the optimal solution in the first inequality yields $s(l,t) \geq 0$. Hence, the solution to (D) equals $y(t)$. Since (P) always has a feasible solution (see Section II), $y(t) = x(t)$. Hence, Theorem 1 implies that portfolio $t \in \Lambda$ is MV efficient if and only if $y(t)=0$. *Q.E.D.*

In addition to the value of the MV test statistic, the dual identifies a solution portfolio. Specifically, the optimal values of the dual variables l_i, $i \in \mathrm{I}$, represent the shadow prices for the constraints $\sum_{t \in \Theta} (b + 2g\mathbf{x}_t t)(\mathbf{x}_t t - \mathbf{x}_{it}) / T + q \geq 0$, $i \in \mathrm{I}$. These constraints are binding for the assets that constitute the portfolio that maximizes the value of the test statistic, and the optimal l gives the composition of this portfolio. Since the solution portfolio is found by maximizing the expected value of a quadratic utility functions $u \in Q$, the solution portfolio necessarily is efficient. Still, the solution portfolio is only one element of the MV efficient set, and that there is no prior reason to prefer this portfolio to other efficient portfolios. For selecting the optimal portfolio, one typically needs more information than is assumed in MVA. Hence, as is true for the empirical utility function discussed in section II, the solution portfolio is used as an instrument for testing if the evaluated portfolio is efficient, not as the optimal portfolio.

As discussed in Section I, the HL efficient set is a subset of the standard MV efficient set. The dual formulation can confirm this relationship:

COROLLARY If portfolio $l \in \Lambda$ has a higher mean than $t \in \Lambda$, i.e. $m(l,t) \geq 0$, and a lower variance $s^2(l) \equiv \sum_{t \in \Theta} (x_{it} l)^2 / T - \left(\sum_{t \in \Theta} (x_{it} l) / T \right)^2$, then l HL dominates t, i.e. $m(l,t) \geq 0$ and $s(l,t) \geq 0$.

PROOF If $m(l,t) \geq 0$ and $s^2(l) \leq s^2(t)$, then

(10) $\sum_{t \in \Theta} (x_{it} t)^2 / T \geq \sum_{t \in \Theta} (x_{it} l)^2 / T$.

Further, we have by definition:

(11) $\sum_{t \in \Theta} (x_{it} l)^2 / T + \sum_{t \in \Theta} (x_{it} t)^2 / T \geq 2 \sum_{t \in \Theta} (x_{it} l)(x_{it} t) / T$.

Combining (10) and (11) gives $\sum_{t \in \Theta} (x_t t)(x_{it} l - x_t t) / T \leq 0$, and hence $s(l,t) \geq m(l,t)$. Therefore, $m(l,t) \geq 0$ and $s^2(l) \leq s^2(t)$ (standard dominance) implies $m(l,t) \geq 0$ and $s(l,t) \geq 0$ (HL dominance). *Q.E.D.*

IV. EMPIRICAL APPLICATION

To illustrate our approach to MVA, we continue the empirical application used in Post (2001a). The application evaluates whether the Standard and Poors 500 (S&P 500) index is MV efficient relative to all possible portfolios of the 25 Fama and French (1993) benchmark portfolios. The benchmark portfolios are the intersections of 5 portfolios formed on size (market equity) and 5 portfolios formed on the ratio of book equity to market equity (BE/ME). The benchmark portfolios are constructed from all NYSE, AMEX, and NASDAQ stocks. By contrast, the S&P 500 index is based on a much smaller set of stocks (until 1979, the index was even limited to NYSE stocks only). The application uses data on monthly returns (month-end to month-end) from July 1926 to December 2000 (894 observations) obtained from the data library on the homepage of Kenneth French (http://web.mit.edu/kfrench/www/). Table 1 gives some descriptive statistics for these data.

Table 1: Descriptive statistics of the monthly returns (month-end to month-end) from July 1926 to December 2000 for the S&P 500 index and the 25 Fama and French benchmark portfolios.

Portfolio				Mean	Standard Deviation	Skewness	Kurtosis
S&P 500				0.0063	0.0463	0.7735	19.1381
Fama and French benchmark portfolios	No.	BE/ME	Size				
	1	Low	Small	0.0077	0.1266	2.8118	27.9185
	2	2	Small	0.0103	0.1091	3.9491	48.6091
	3	3	Small	0.0133	0.0951	2.0006	16.8556
	4	4	Small	0.0153	0.0887	2.7616	29.3945
	5	High	Small	0.0168	0.0983	3.2252	30.4773
	6	Low	2	0.0086	0.0805	0.4236	5.1488
	7	2	2	0.0127	0.0789	1.8290	19.7922
	8	3	2	0.0136	0.0754	2.3155	24.3154
	9	4	2	0.0140	0.0768	1.8028	18.6467
	10	High	2	0.0151	0.0877	1.6950	15.9496
	11	Low	3	0.0100	0.0770	1.0103	9.9357
	12	2	3	0.0122	0.0673	0.3120	7.0884
	13	3	3	0.0129	0.0685	0.9969	13.0653
	14	4	3	0.0133	0.0691	1.2614	13.8547
	15	High	3	0.0142	0.0870	1.9309	19.2382
	16	Low	4	0.0103	0.0628	-0.1606	3.7688
	17	2	4	0.0108	0.0639	1.0598	13.6494
	18	3	4	0.0121	0.0641	1.0731	14.9477
	19	4	4	0.0131	0.0715	1.9706	21.5985
	20	High	4	0.0144	0.0927	2.1360	21.9067
	21	Low	Big	0.0099	0.0557	-0.0270	5.5123
	22	2	Big	0.0095	0.0536	-0.0703	5.2748
	23	3	Big	0.0102	0.0581	0.7875	13.6679
	24	4	Big	0.0110	0.0702	1.8283	21.3741
	25	High	Big	0.0009	0.1441	-3.9635	32.3209

As in Post (2001a), a number of disclaimers apply. First, we use the basic MV tests developed in Section II and III. The tests assume that it is possible to invest in all convex combinations of the individual assets, and they do not analyze the effects of short selling or investment restrictions.[4] Second, we abstract from transaction costs. This puts the S&P 500 index in an unfavorable perspective relative to the Fama and French portfolios, which generally include more stocks and relatively more small caps, and hence involve higher transaction costs. Finally, we do not analyze the sensitivity of our results to the return horizon and the sample period (e.g. the inclusion of NASDAQ and AMEX stocks after 1979 may have changed the efficiency of the S&P 500 index).

Post (2001a) found that the S&P 500 index is SSD inefficient. The monotone and concave, quadratic utility functions $u \in Q$ considered in MVA are special cases of the monotone and concave functions considered in SSD. Hence, we expect to find that the S&P 500 index is also MV inefficient. The results confirm this expectation; we find $x(t) = 0.0105$. The empirical utility function (as obtained from the primal solution) is simply $u(x) = x$. This suggests that the S&P 500 index is relatively favorable for

[4] As discussed in the Conclusions, the tests can be extended in a straightforward manner towards a general polyhedral possibilities set that can account for short selling and investment restrictions.

investors that are risk neutral. Still, even these investors would not select the S&P 500 index as the optimal portfolio. The 'optimal portfolio' (as obtained from the primal solution) consist of benchmark portfolio 5, i.e. the portfolio with the maximum expected return (for our sample period: 0.0168).

These results are based on the EDF and they are likely to be affected by sampling error in a non-trivial way. To approximate the sampling distribution of our results, we use the bootstrap method. Bootstrapping, first introduced by Efron (1979) and Efron and Gong (1983), is based on the idea of repeatedly simulating the CDF, usually through resampling, and applying the original estimator to each simulated sample or pseudo-sample so that the resulting estimators mimic the sampling distribution of the original estimator. Key to the success of the bootstrap is the selection of an appropriate approximation for the CDF. If the approximation is statistically consistent, then the bootstrap distribution gives a statistically consistent estimator for the original sampling distribution. In the context of our tests, the EDF is an appropriate approximation for the CDF; under the assumption that the return distribution is serially IID (see Section I), the EDF is a consistent estimator of the true CDF. This suggests bootstrapping samples would be simply obtained by randomly sampling with replacement from the EDF along the lines of the 'correlation model' proposed by Freedman (1981) in a regression framework. Figure 1 gives the bootstrap distribution resulting from this approach, resulting from 1000 random pseudo-samples. The S&P 500 index is classified as MV inefficient in all the pseudo-samples. This finding suggests that the index is MV inefficient to a statistically significant degree.

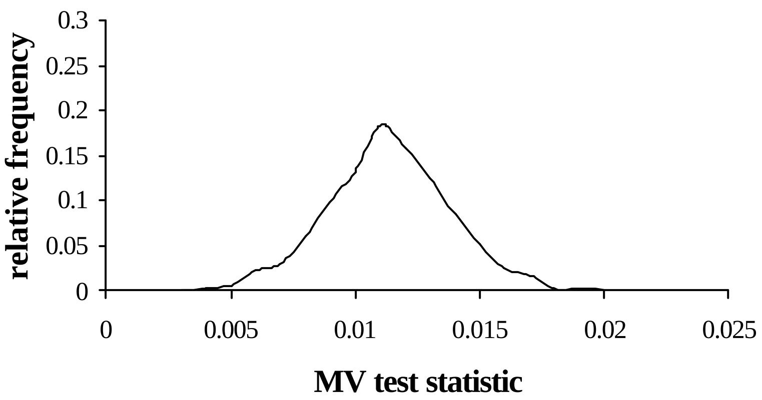

Fig. 1: Bootstrap distribution for the MV test statistic, based on 1000 replications. The S&P 500 index is classified as SSD inefficient in all 1000 pseudo-samples, which suggests that the index is MV inefficient to a statistically significant degree.

V. CONCLUDING REMARKS AND SUGGESTIONS

We have derived necessary and sufficient empirical tests for MV efficiency that can be computed using straightforward LP. Our analysis allows for the following straightforward generalizations:

1. Our analysis is based on the optimality conditions for optimizing a concave utility function over a convex portfolio possibilities set (see the proof to Theorem 1).

These conditions apply for any non-empty, closed and convex portfolio set. We may therefore generalize our analysis by simply replacing ? by a more general polyhedron in the dual formulation (D) developed in Section III. The generalized primal formulation can then be obtained by applying linear duality theory to the generalized dual.

2. The HL definition of MV efficiency focuses on quadratic utility functions. The analysis can be extended in a straightforward manner to higher-order polynomials, while preserving the LP structure (the monotonicity and concavity restrictions remain linear). For example, the cubic utility function discussed in Hanoch and Levy (1970) can account for the first three moments of the return distribution (mean, variance and skewness) rather than the first two moments. In fact, increasing the order of the polynomial induces a gradual transition from the MV criterion (considered in this paper) to the SSD criterion (considered in Post (2001a), which represents the limiting case.

We see the following routes for further research:

1. Our LP tests can be very useful for portfolio evaluation i.e. for evaluating whether a given portfolio is efficient. For selecting the optimal portfolio, one typically needs more information on investor preferences than is assumed in MVA, and MVA has to be complemented with other research instruments, like interactive Multi-Criteria Decision-Making procedures. Still, MVA can be very useful as a pre-analysis screening device for reducing the number of choice alternatives. Further research could focus on obtaining a full characterization of the set of MV efficient portfolios.

2. We have focussed on the case where the production possibilities set is given. In many cases, it is also useful to analyze the impact of the introduction of new assets (e.g. introduced via IPOs) or the relaxation of investment restrictions for existing assets (e.g. liberalization in emerging markets). For this purpose, various tests for MV *spanning* and *intersection* have been developed (see e.g. Huberman and Kandel (1987) and De Roon et al. (2001)). Further research could extend our LP tests towards spanning and intersection, e.g. along the lines of Post's (2001b) tests for SSD spanning and intersection.

3. The empirical application used bootstrapping to analyze the sampling properties of our results. Further research may focus on an analytical characterization of the sampling distribution. In this respect, it easy to verify that the asymptotically least favorable distribution for Post’s (2001a, Theorem 5) SSD statistic also applies for the MV statistic. The least favorable distribution minimizes Type I error (wrongly classifying an efficient portfolio as inefficient). Further research may focus on asymptotic tests that minimize Type II error (wrongly classifying an inefficient portfolio as efficient).

4. We have thus far assumed that the return distribution is serially IID (see Section I), and our tests are unconditional by nature. Still, there is substantial evidence that the distribution of assets returns (e.g. risk premia and volatilities) varies through time. Further research could focus on developing conditional tests that relax the assumption that the observations are serially IID.

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