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Evaluating productive performance: A new approach based on the product-mix problem consistent with Data Envelopment Analysis ☆

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ABSTRACT

We propose a new approach to estimate technical coefficients from a set of Decision Making Units (DMUs) under the assumption that their production plans are set by process engineers through Linear Programming (LP) techniques. The idea behind this approach is that most manufacturing and agricultural firms routinely resort to LP-based modeling in their decision making processes in order to plan output production and, therefore, this particularity should be taken into account when estimating their technical efficiency. A usual model of LP for these sectors is the so-called product-mix problem, which we relate to a standard Data Envelopment Analysis (DEA) model in terms of the Directional Distance Function. In this paper, we finally show how to estimate the technical coefficients of a sample of Andalusian farms in Spain and how this information can be seen as a complement to the usual by-products associated with estimating technical efficiency by DEA.

1. Introduction

Efficiency evaluation in production of whatever type of private firm or public organization has been a relevant topic for managers and policy makers, as well as an area of interest from a practical and methodological point of view in both engineering and economics [13,18]. The main aim of such assessment is to analyze the efficiency of a set of observations: generally termed DMUs (Decision Making Units) that use several inputs to produce several outputs, by comparing their performance with respect to the boundary of a production possibility set, and using to that end a sample of other observations operating in a similar technological environment. In the case of producing only one output the interest lies with the notion of production function, which represents the maximum product obtainable from the input combination at the existing state of technical knowledge. The usual methods for measuring technical efficiency of production need explicitly or implicitly to determine the boundary of the underlying technology, which constitutes the reference benchmark. Its estimation allows calculating the corresponding technical inefficiency value for each DMU as the deviation of each activity or production plan to the set of optimal ones, represented by the frontier of the production possibility set or, if it is the case, by the industry production function.

Regarding the determination of the technology in practice, before Farrell's [17] seminal contribution, economists used to specify parametrically the corresponding production functions, e.g., a Cobb-Douglas function [12], relying on Ordinary Least Squares (OLS) regression analysis to estimate an 'average' production function, and assuming that disturbance terms had zero mean. This was a patently unsatisfactory estimation, as it did not follow the traditional (frontier) definition of production functions in microeconomics as the maximal feasible output for each input combination considered. Farrell [17] was the first in showing, for a single output and multiple inputs, how to estimate an isoquant enveloping all the observations. He based his significant contribution on the construction of a production possibility set that satisfied two usual axioms: convexity and monotonicity. In this way, the most conservative estimation of the production possibility set may be obtained through the determination of the minimal set that envelops the observations and, at the same time, meets the two aforementioned axioms [16, p. 255]. Farrell's principle of conservation, now known as 'minimal extrapolation', leads to the estimation of a piece-wise linear isoquant in the input space. For the application of his method to a dataset from the US agricultural sector, Farrell resorted to finding out all the facets of the

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piece-wise linear frontier by applying more or less sophisticated combinatorial methods. The work of Farrell represented an enormous advance in the measurement of efficiency of production by showing how to decompose cost efficiency into technical and allocative counterparts. His contribution constitutes the first implementation of Debreu's coefficient of resource utilization [14] and Shephard's input distance function [35].

Although Farrell also dealt with the possibility of approximating a Cobb–Douglas production function from the previously estimated piece-wise linear isoquant, as a way of summarizing in a few parameters the main features of the underlying technology, Farrell's approach can be categorized in the current area of non-parametric techniques (see Diagrams 4, 5, 7 and 8 in [17]), since it is not necessary to identify *a priori* the specific mathematical formulation of the industry production function to be estimated. This line of research, initiated by Farrell, was later taken up by Charnes et al. [9] and Banker et al. [7], resulting in the development of the Data Envelopment Analysis (DEA) approach, in which the determination of the frontier is only restricted via its axiomatic foundation, mainly convexity and minimal extrapolation. Another paper working in this same line, is that by Afriat [2], showing how to determine a production function with the property *P* (e.g., non-decreasing concavity) that represents the set of observations to be as nearly efficient as possible. Under the production of only one output, the estimated production functions suggested by Afriat coincide with those associated with what later was known as DEA. A more natural sequel than the DEA literature of the previous work done by econometricians, even before Farrell's contribution, would be Aigner and Chu [1], who show how to apply a technique based on mathematical programming to yield an envelope 'parametric' Cobb–Douglas production function by controlling the sign of the disturbance terms and, therefore, allowing to make the traditional interpretation of the production function in microeconomics.

All the aforementioned methods for estimating the underlying technology from a data sample can be applied to any set of profit and not-for-profit organizations, from a farm to a university, without considering the specific methodologies and techniques that may have been used by managers to determine resource allocation and output mixes in their production processes. Indeed, these methods only depend on observing the inputs and outputs quantities and prices of each sample unit, as these were originally designed to overcome the lack of information (and the cost of collecting it) on what happens within each organization; i.e., they consider production processes as 'black boxes' where the intermediate transformation of inputs into outputs is not specified. However, some attempts to develop new methodologies for opening the black box have been recently published, mainly in the non-parametric field. Examples of these contributions are, on the one hand, those based on network DEA, which considers each unit as composed of distinct processes or stages, each one with its own inputs and outputs and with intermediate flows among the stages (Prieto and Zofio [34], Tone and Tsutsui [37] and Kao [23], to name but a few) and, on the other hand, the recent approach by Cherchye et al. [11], which explicitly includes information about the allocation of the output-specific inputs to their outputs; information that is collected from the activity-based costing (ABC) system of the firms, when it is available.

Ultimately, the way the firm internally plans the output mix has been usually neglected in the mainstream literature on production theory, being an issue that merits further research. This paper is devoted to making progress in this direction by showing the link between the product-mix problem (PMP) and DEA. While the former allows the characterization of the production technology *within* the firm at the individual level, the latter concerns the estimation of the boundary of the reference technology by comparing production plans *between* firms. In doing so, we propose a new approach based on the PMP that will take into account resources and technical coefficients. A recent survey showed that most Fortune 500 companies regularly use linear programming (LP) in their decision making [30]. A typical application of LP is the formulation and resolution of the product-mix problem. Process engineers worldwide have been trained to model such problems through the application of techniques of Operations Research.

A simplified example of a real-life situation of this type, represented by a firm producing leatherwork, taken from Winston [39], is adopted in the following section to illustrate the starting point of the new approach. Given output prices, input and output quantities, and the technical coefficients relating them, the product-mix problem is used to maximize revenue choosing the amount of output quantities to produce, subject to the inputs constraints. The information used for building these constraints are, on the one hand, resource endowments and, on the other, the technical coefficients, which represent the amount of each resource that are consumed in the production of one unit of each output product. In this way, the technology that is utilized for determining the firm's optimal output mix is given by the constraints of the PMP model. We show that it is possible to use the PMP to jointly determine a common technology[1] for a set of observations and estimate the corresponding technical coefficients.

Specifically, at a setting of manufacturing or agricultural production, where LP is a usual tool for determining the optimal output mix, and assuming that we observe a sample of DMUs operating in a similar technological environment, it is our aim to estimate a common set of technical coefficients for all the units associated with the underlying technology. To that end, we invoke Farrell's principle of conservation to determine the technical coefficients that yield the minimal set associated with the 'LP' resource constraints that, at the same time, envelops all the observations. In this way, a polyhedral production possibility set is estimated based on exactly m constraints, denoting by m the number of firm's resources (inputs). The geometrical shape of the estimated technology justifies comparing the new approach to DEA, since by this last technique a polyhedral production possibility set is also estimated based on an *a priori* undetermined number of hyperplanes (greater than the number of inputs with high probability). The distinctive feature of the approach that we introduce is that it contributes to opening the aforementioned 'black box' of efficiency measurement in two manners. First, incorporating into the analysis the usual way in which process engineers internally model the PMP. Second, estimating key information of the firms; in particular how inputs and outputs are linked to each other in the production process.[2]

Therefore, the proposed methodology represents a new approach to assess the production performance of firms that complements the results obtained with the standard DEA approach, as both are closely related. The new approach can be applied by process engineers in many industrial sectors, offering an alternative way to assess productive performance that is grounded on more familiar techniques, but without losing sight of the existing DEA methods. To sum up, we extend the notion of technical and economic efficiency to a wider audience who is

[1] It is commonly adopted in the literature, either implicitly or explicitly, that all firms share the same production possibility set and differ only with respect to their degree of inefficiency (see, e.g., [38,19,31]).

[2] It is also worth mentioning that linear production technologies and the behavior of producers have long been linked in the literature. Owen [33] used cooperative game theory to study a general class of linear production games in which multiple producers using the same technology decide whether to pool their available resources to produce some goods. Many researchers have extended this approach in different directions (see, e.g., Timmer et al. [36] and more recently Lozano [25], which mixes cooperative game theory and DEA).

routinely applying optimizing techniques based on the PMP and similar programs, and can incorporate the proposed methodology to their analytical toolbox.

The core of our investigation, contained in Section 2, is concerned with the introduction of the new methodology for estimating technologies in scenarios where it is usual to resort to LP in order to model the PMP. In Section 3 we relate the new approach to DEA, since this last technique also generates piece-wise linear efficient frontiers. In Section 4 an illustration of the proposed approach is undertaken and discussed. Section 5 summarizes and concludes.

2. The product-mix problem, linear programming and productive performance

In this section we introduce an innovative technique for estimating a common set of technical coefficients associated to the boundary of the underlying production possibility set of an industry from a sample of data. This new approach is created with the intention of adapting the way the frontier of a production possibility set is estimated to the manner in which the product mix is planned by a firm. In particular, the technique can be sensibly applied to the manufacturing and agricultural sectors, where mathematical programming is routinely used by engineers to determine the optimal output mix (see, for example, [22]).

We now introduce the necessary notation. Let R^k_+ be the non-negative Euclidean k-orthant, for $z, z' \in R^k_+$ we denote $z \leq z' \Leftrightarrow z_p \leq z'_p,\ p \in \{1,\ldots,k\}$ and $z < z' \Leftrightarrow z_p < z'_p,\ p \in \{1,\ldots,k\}$. We also denote 1_k and 0_k the column vectors belonging to R^k_+ with all components equal to one and all components equal to zero, respectively. Additionally, denote by n the number of observed DMUs (producers). We assume, as it is customary, that all of them operate in the same technological environment, are price takers, and their goal is to maximize profit from the sale of products. Each DMU$_j$, $j=1,\ldots,n$, is endowed with m inputs, denoted by $x^j = \left(x^j_1,\ldots,x^j_m\right)^T > 0_m$, which are used for producing s outputs, denoted by $y^j = \left(y^j_1,\ldots,y^j_s\right)^T \geq 0_s$. As Timmer et al. [36], we assume that markets clear so all outputs are sold at equilibrium prices. Regarding the technical coefficients, each DMU requires $a_{ir} \geq 0$ units of resource i to produce one unit of output r. These coefficients are independent on subscript j, since we assume that the technology is the same for all the observations. As for the prices, the marketing department of DMU j forecasts selling prices, so after production each unit of output r will contribute $\hat{p}^j_r$ (e.g., euros) to profit. We assume that each DMU$_j$ has a private forecast about the market prices for each product. In practice, all these estimations are usually obtained by DMUs using whatever market information and statistical tool at their disposal. Note also that our setting can accommodate the possibility of imperfect competition, incorporating the subscript j for prices. In this context, the profit maximization problem that the engineer of DMU$_0$ solves is

$$\begin{aligned} \max_{y}\quad & R_0 = \hat{p}^{0T} y \\ \text{s.t.}\quad & \\ & Ay \leq x^0, \\ & y \geq 0_s, \end{aligned} \tag{1}$$

where A denotes the matrix $\begin{pmatrix} a_{11} & \ldots & a_{1s} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{ms} \end{pmatrix}$ and $\hat{p}^{0T} = \left(\hat{p}^0_1,\ldots,\hat{p}^0_s\right)$. Model (1) is known as the Product-Mix Problem.

Model (1) is used to maximize net revenue R_0 by choosing the amount of outputs subject to the input constraints. The pieces of information used for building these constraints are the resource endowments and the technical coefficients, which represent the amount of each resource that is consumed in the production of one unit of each output. Winston [39] exemplifies model (1) with a simple process where a firm, Leather Limited, manufactures two kinds of belts per week: y_1=number of deluxe belts produced, and y_2=number of regular belts produced. Each type requires 1 square yard of leather, $a_{1r}=1$, $r=1,2$. A regular belt requires 1 hour of skilled labor, $a_{21}=1$ and a deluxe belt requires 2 hours: $a_{22}=2$. Each week, $x_1=40$ square yards of leather and $x_2=60$ hours of skilled labor are available. The marketing department of the firm estimates that at the equilibrium market prices each deluxe belt contributes $\hat{p}_1=4$ monetary units to profit and each regular belt $\hat{p}_2=3$.

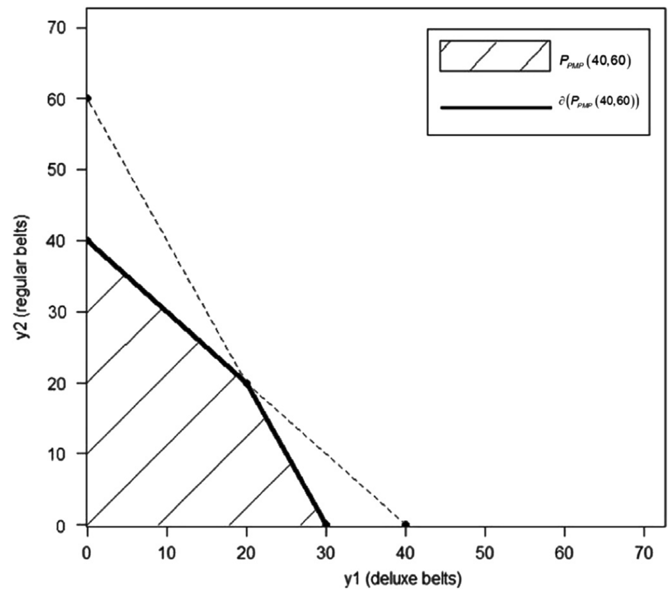

Fig. 1. The PMP output production set for the example of Leather Limited.

Given the vector of endowments $x > 0_m$ and assuming that the technical coefficients are known, the set of feasible (producible) outputs associated with the PMP, called PMP output production set, is defined as follows:

$$P_{PMP}(x) = \left\{y \in R^s_+ : Ay \leq x\right\}. \tag{2}$$

Fig. 1 illustrates the PMP output production set corresponding to the example of Leather Limited. In this case, $P_{PMP}(40,60) = \left\{(y_1,y_2) \in R^2_+ : y_1+y_2 \leq 40,\quad 2y_1+y_2 \leq 60\right\}$. By construction, $P_{PMP}(40,60)$ is a polyhedral set (the lined region in Fig. 1).

$P_{PMP}(x)$ is convex and, additionally, $0_s \in P_{PMP}(x)$ since $x > 0_m$, i.e., inaction is possible. Moreover, we assume that the PMP output production sets are bounded[3] and closed. In other words, $P_{PMP}(x)$ must be a compact set. This means that sets as, for example $P_{PMP}(x) = \left\{y \in R^2_+ : y_1 \leq 4\right\}$, do not represent a suitable PMP output production set in our context. Compactness is a usual axiom in multi-output production theory (see [15]).

[3] By construction, $P_{PMP}(x)$ in (2) is closed. Additionally, boundedness may be assured from conditions on the technical coefficients. For example, a sufficient condition would be the existence of $a_{ir} > 0$ for some i, $i=1,\ldots,m$, for each r, $r=1,\ldots,s$.

2.1. How to estimate the technical coefficients: The Hölder distance function D_{ℓ_∞}

In order to determine the technical coefficients in the context of the product-mix problem we need to isolate a reference subset of $P_{PMP}(x)$. We are referring to the boundary of the PMP output production set, which we denote and define as follows:

$$\partial(P_{PMP}(x)) = \{y \in P_{PMP}(x) : y < \tilde{y} \Rightarrow \tilde{y} \notin P_{PMP}(x)\}. \quad (3)$$

In Fig. 1 the subset corresponding to the PMP boundary consists of the two facets of the polyhedron associated with the PMP output production set – depicted by bold lines. Therefore, while the lined region corresponds to the PMP output production set, the PMP boundary is the frontier of that region. From the constraints $y_1 + y_2 \le 40$ (leather constraint) and $2y_1 + y_2 \le 60$ (labor constraint), it is possible to derive $y_1 + y_2 = 40$ and $2y_1 + y_2 = 60$; i.e., the supporting hyperplanes for the two aforementioned facets.

In the last example the technical coefficients are known. However, in practice, this is not the usual scenario. So, the main objective of this paper is to determine the unknown technical coefficients from a sample of observations. In the new approach, we suggest estimating the values of the technical coefficients a_{ir}, $i = 1, \dots, m$, $r = 1, \dots, s$, by minimizing the sum of the ℓ_∞-distances[4] from the observations, y^j, to the corresponding PMP boundary for each DMU, $j = 1, \dots, n$; i.e.,

$$\begin{aligned} &\underset{a_{ir}, z_r^j \ge 0}{\text{Min}} \quad \sum_{j=1}^{n} \epsilon_j, \text{ where} \\ &\epsilon_j := D_{\ell_\infty}\left(y^j, \partial\left(P_{PMP}\left(x^j\right)\right)\right) \\ &\quad = \min_{z^j}\left\{\max_{r=1,\dots,s}\left\{\left|z_r^j - y_r^j\right|\right\} : z^j \in \partial\left(P_{PMP}\left(x^j\right)\right)\right\}. \end{aligned} \quad (4)$$

Let $(\hat{A}, \hat{Z})$ be an optimal solution of (4) with $\hat{A} = \begin{pmatrix} \hat{a}_{11} & \dots & \hat{a}_{1s} \\ \vdots & \ddots & \vdots \\ \hat{a}_{m1} & \dots & \hat{a}_{ms} \end{pmatrix}$ and $\hat{Z} = \begin{pmatrix} \hat{z}_1^1 & \dots & \hat{z}_1^n \\ \vdots & \ddots & \vdots \\ \hat{z}_s^1 & \dots & \hat{z}_s^n \end{pmatrix}$, then we suggest to use $\hat{a}_{ir}$, $i = 1, \dots, m$, $r = 1, \dots, s$, as estimations of the technical coefficients a_{ir}, $i = 1, \dots, m$, $r = 1, \dots, s$. Accordingly, we define the estimation of the PMP output production set as $\hat{P}_{PMP}(x) = \left\{y \in R_+^s : \hat{A}y \le x\right\}$. In this way, the optimal value of ε_j is $\hat{\varepsilon}_j = D_{\ell_\infty}\left(y^j, \partial\left(\hat{P}_{PMP}(x^j)\right)\right)$.

Why do we select the ℓ_∞ norm to calculate the distance from y^j to the PMP boundary in our setting? In the DEA literature we find many possible measures representing the distance from a DMU to the efficient frontier in the output space: radial measures, the Russell output measure, output-oriented weighted additive models, and the output directional distance function, among others. Obviously, the ℓ_∞-distance is *a priori* as good as any. Nevertheless, this measure satisfies an interesting property given the characteristics of our problem. In particular, it is a mathematical distance in contrast to most approaches and, therefore, the estimated ℓ_∞-distances will indicate the smallest required modifications in outputs to reach the PMP boundary for each DMU.

Next, we illustrate model (4) by means of a simple numerical example. In Table 1 there are nine observations, labeled from A to I, which consume two inputs to produce two outputs. There are two different levels of inputs. While DMUs A, B, C D and E consume 5 units of input 1 and 11 units of input 2, DMUs F, G, H and I consume 7 units of input 1 and 15 units of input 2. In this way, it is possible to estimate the boundaries of two PMP output production sets. In particular, we draw two feasible solutions of

Table 1
Simple numerical example.

DMU	x_1	x_2	y_1	y_2
A	5	11	2	4
B	5	11	4	3
C	5	11	5	1
D	5	11	1	3.5
E	5	11	4	2
F	7	15	3	3
G	7	15	6	3
H	7	15	2	6
I	7	15	5.5	3

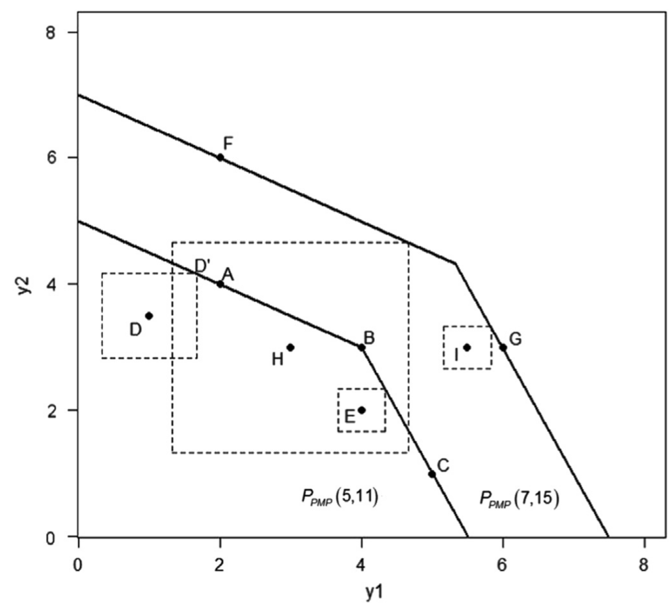

Fig. 2. Illustration of model (4), first feasible solution.

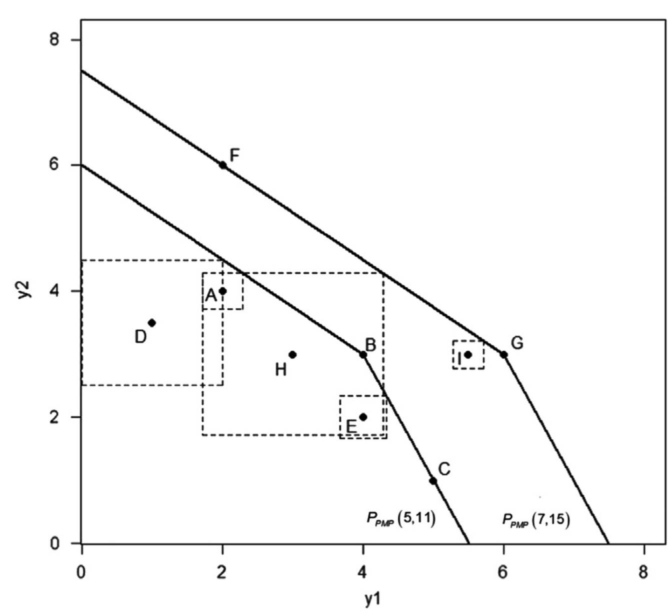

Fig. 3. Illustration of model (4), second feasible solution.

[4] The ℓ_∞ norm is not units invariant. Nevertheless, in order to impose this property, Briec [8, p. 125] introduced the weighted Hölder distance functions. In our case, this means that it is possible to resort in model (4) to a weighted ℓ_∞ norm with weights related to the data and satisfying units invariance.

model (4) for this example. On the one hand, $a_{11}=\frac{1}{2}$, $a_{12}=1$, $a_{21}=2$ and $a_{22}=1$, represented in Fig. 2. On the other, $a_{11}=\frac{3}{4}$, $a_{12}=1$, $a_{21}=2$ and $a_{22}=1$, in Fig. 3.

In both figures, we graphically represent the ball of radius d with respect of the norm ℓ_∞ for all DMUs. These distances are calculated using as a reference the boundary of $P_{PMP}(5,11)=\left\{(y_1,y_2)\in R^2_+ : a_{11}y_1+a_{12}\le 5,\quad a_{21}y_1+a_{22}y_2\le 11\right\}$ for units A, B, C, D and E, and with respect to $P_{PMP}(7,15)=\left\{(y_1,y_2)\in R^2_+ : a_{11}y_1+a_{12}\le 7,\quad a_{21}y_1+a_{22}y_2\le 15\right\}$ for units F, G, H and I. Using as optimizing criterion the minimization of the sum of the distances in model (4), the first feasible solution, with an objective function value equal to 3, is better than the second feasible solution, which presents an associated objective function value equal to 3.12. Overall, model (4) seeks the value of the technical coefficients that minimizes the sum of all the distances.

As a by-product of the solution, we determine for each observation a projection located on the boundary of the PMP output production set associated with the product-mix problem. These projections may be defined from the optimal solution of (4). Let $\left(\hat{A},\hat{Z}\right)$ be an optimal solution of model (4). Then, the corresponding PMP projection plan for DMU$_j$, $j=1,\dots,n$, is $\hat{z}^j$. In Fig. 2, point D′ corresponds to the PMP projection plan of unit D in the sample.[5]

As the next proposition establishes, the PMP projection plan generated by the ℓ_p norm dominates, in the sense of Pareto, the observed output vector that is being evaluated.

Proposition 1. *Let $\left(\hat{A},\hat{Z}\right)$ be an optimal solution to the minimization problem in (4), then $\hat{z}^j\ge y^j$, $j=1,\dots,n$.*

Proof. The result may be proved by following the same steps as Briec [8, p. 115] for proving Lemma 1(2).□

On the other hand, by (4), we are calculating the least distance measure from the assessed points to the boundary of the PMP output production set. This is related to a stream of the literature devoted to the application of the Principle of Least Action (see, for example, [3–5]). PMP projection plans in this way are the closest points on the boundary of the PMP output production set. In this way, the coordinates of the projected output vector are as similar as possible to the observed ones.

Nevertheless, there are a lot of possibilities for selecting a least distance measure (see [8]). The family of Hölder norms is wide. One member of this family is the ℓ_∞ distance that we use. Elaborating further on the justification of this particular choice, among all Hölder norms there are two that are known as polyhedral metrics: ℓ_1 and ℓ_∞. In both cases, the corresponding balls of radius d are polyhedral convex sets. This is an advantage from a computational point of view if we compare ℓ_1 and ℓ_∞ metrics with respect to the other Hölder norms (Euclidean, etc.) because of the existence of corner points that facilitates its implementation by means of mathematical programming. In particular, as we will show later, using ℓ_∞ allows us to solve (4) by applying Mixed Integer Linear Programming (MILP).

We now compare the new approach with a standard Data Envelopment Analysis measure. In this respect, the output directional distance function[6] [10] is defined in DEA as $\vec{D}\left(y^0;g^0;P_{DEA}(x^0)\right)=\max_{\beta}\left\{\beta : y^0+\beta g^0\in P_{DEA}\left(x^0\right)\right\}$, where $g^0=\left(g^0_1,\dots,g^0_s\right)^T\in R^s_+$ with $g^0\ne 0_s$ is the so-called reference direction for the evaluated point y^0 and $P_{DEA}\left(x^0\right)$ denotes the set of producible outputs under the usual assumptions of DEA. Specifically, $P_{DEA}(x)=\left\{y\in R^s_+ : (x,y)\in T_{DEA}\right\}$, where $T_{DEA}=\left\{(x,y)\in R^{m+s}_+ : X\lambda\le\ x, Y\lambda\ge y, \lambda\ge 0_n\right\}$ with $X=\left(x^1,\dots,x^n\right)\in R^{m\times n}$ and $Y=\ \left(y^1,\dots,y^n\right)\in R^{s\times n}$. In this way, the output directional distance function can be implemented in DEA by way of the following linear program:

$$\begin{aligned} &\max_{\beta,\lambda}\quad \beta\\ &\text{s.t.}\\ &\qquad X\lambda\le x^0,\\ &\qquad Y\lambda\ge y^0+\beta g^0,\\ &\qquad \lambda\ge 0_n,\\ &\qquad \beta\ge 0. \end{aligned} \tag{5}$$

For a recent revision of the main properties of the directional distance function, see Aparicio et al. [6].

The new approach, based on the ℓ_∞ metric, is related to the directional distance function since it coincides with the directional output distance function for $g^0=1_s$, as the next proposition establishes.

Proposition 2. *Let $\left(\hat{A},\hat{Z}\right)$ be an optimal solution to the minimization problem in (4). The following relationship holds:*

$$\hat{\varepsilon}_j=\vec{D}\left(y^j;1_s;\hat{P}_{PMP}\left(x^j\right)\right)=\max_{\beta}\left\{\beta : y^j+\beta 1_s\in\hat{P}_{PMP}\left(x^j\right)\right\},\ \forall j=1,\dots,n. \tag{6}$$

Proof. See a similar proof in Briec [8], Lemma 2(1).□

Proposition 2 implies that the new approach can be compared with the Directional Distance Function in DEA in a natural way. That is what we do in the empirical application.

2.2. Determining D_{ℓ_∞} in the product-mix problem

We now characterize the boundary of the estimated PMP output production set, $\partial\left(\hat{P}_{PMP}(x^j)\right)$, in terms of the next proposition. An output vector $\tilde{y}$ is located on $\partial\left(\hat{P}_{PMP}(x^j)\right)$ if and only if it is a feasible output vector given x^j and there also exists at least a resource constraint that supports $\tilde{y}$.

Proposition 3. *Let $\left(\hat{A},\hat{Z}\right)$ be an optimal solution to the minimization problem in (4). Let also $\sum_{r=1}^{s}\hat{a}_{ir}>0$ for $i=1,\dots,m$. Then, $\tilde{y}\in\partial\left(\hat{P}_{PMP}(x^j)\right)$ if and only if $\tilde{y}\in\hat{P}_{PMP}(x^j)$ and $\exists i'=1,\dots,m$ such that $\sum_{r=1}^{s}\hat{a}_{i'r}\tilde{y}_r=x^j_{i'}$.*

Proof. See Appendix A.

Given the possibility of rewriting the boundary of the PMP output production set through the hyperplanes associated with the constraints of (2), one may calculate each $\hat{\varepsilon}_j$, with $j=1,\dots,n$, by resorting to Ascoli's formula, as the following proposition states.

[5] While in DEA the notions of peers and targets are clear and well-known, this does not happen in the case of the new approach. The PMP projection plans might not match a convex or conical combination of observations located on the frontier.

[6] Luenberger [26] introduced the concept of benefit function as a representation of the amount that an individual is willing to trade, in terms of a specific reference commodity bundle g, for the opportunity to move from a consumption bundle to a utility threshold. Luenberger also defined a so-called shortage function, which basically measures the distance in the direction of a vector g of a production plan from the boundary of the production possibility set. In other words, the shortage function measures the amount by which a specific plan is short of reaching the frontier of the technology. Chambers et al. [10] redefined the benefit function and the shortage function as efficiency measures, introducing to this end the so-called directional distance function.

Proposition 4. *Let* $(\hat{A},\hat{Z})$ *be an optimal solution to the minimization problem in* (4) *with* $\sum_{r=1}^{s}\hat{a}_{ir}>0$ *for* $i=1,\dots,m$. *Then* $\hat{\varepsilon}_j=\min_{i=1,\dots,m}\left\{\frac{x_i^j-\sum_{r=1}^{s}\hat{a}_{ir}y_r^j}{\sum_{r=1}^{s}\hat{a}_{ir}}\right\}$.

Proof. It is a direct consequence of Ascoli's formula (see, for example [29]).□

As we suggested, a way of estimating the values of the technical coefficients a_{ir}, $i=1,\dots,m$, $r=1,\dots,s$, is by minimizing the sum of the ℓ_∞-distances from the observations, y^j, to the corresponding PMP boundary, i.e., Min $\sum_{j=1}^{n}\epsilon_j$. By Proposition 4, the problem to be solved in order to estimate a_{ir}, $i=1,\dots,m$, $r=1,\dots,s$, from a dataset of n DMUs is as follows[7]:

$$\begin{aligned}
&\min_{a_{ir}} \sum_{j=1}^{n}\min_{i=1,\dots,m}\left\{\frac{x_i^j-\sum_{r=1}^{s}a_{ir}y_r^j}{\sum_{r=1}^{s}a_{ir}}\right\}\\
&\text{s.t.}\\
&Ay^j\le x^j, \quad j=1,\dots,n \quad (7.1)\\
&A1_s>0_m, \quad (7.2)\\
&a_{ir}\ge 0, \quad i=1,\dots,m,\ r=1,\dots,s \quad (7.3)
\end{aligned}\tag{7}$$

In (7), constraint (7.2) has been added to assure that each resource has a technological relationship with at least one output of the production process. Otherwise, a resource of this type could be deleted from the approach. Additionally, model (7) is not a standard "linear" program, making its implementation difficult in practice. Fortunately, as the next proposition establishes, model (7) may be solved through a Mixed Integer Linear Programming model.

Proposition 5. *Let* $\{\alpha^*,B^*,\delta^*,h^{j*},d^{j*}\}$ *be an optimal solution of* (8) *with* $\alpha^*>0_m$. *Then the optimal value of* (8) *coincides with the optimal value of* (7) *and* $\hat{a}_{ir}=b_{ir}^*/\alpha_i^*$, $\forall i,r$, *is an optimal solution to* (7).

$$\begin{aligned}
&\min_{\alpha,b,\delta,h^j,d^j} 1_n^T\delta\\
&\text{s.t.}\\
&By^j+\delta+h^j=\alpha^Tx^j, \quad j=1,\dots,n \quad (8.1)\\
&B1_s=1_m, \quad (8.2)\\
&h^j\le Md^j, \quad j=1,\dots,n \quad (8.3)\\
&1_m^Td^j\le m-1, \quad j=1,\dots,n \quad (8.4)\\
&\alpha\ge 0_m, \quad (8.5)\\
&b_{ir}\ge 0, \quad i=1,\dots,m, r=1,\dots,s \quad (8.6)\\
&\delta\ge 0_n, \quad (8.7)\\
&h^j\ge 0_m, \quad j=1,\dots,n \quad (8.8)\\
&d_i^j\in\{0,1\}, \quad i=1,\dots,m, j=1,\dots,n \quad (8.9)
\end{aligned}\tag{8}$$

where $\alpha=(\alpha_1,\dots,\alpha_m)^T$, $B=\begin{pmatrix} b_{11} & \dots & b_{1s}\\ \vdots & \ddots & \vdots\\ b_{m1} & \dots & b_{ms}\end{pmatrix}$, $\delta=(\delta_1,\dots,\delta_n)^T$, $h^j=\left(h_1^j,\dots,h_m^j\right)^T$, $d^j=\left(d_1^j,\dots,d_m^j\right)^T$ and $M\in R_+$ is a large positive number.

[7] The constraints ((7.1)–(7.3)) and, in particular, constraint (7.2), yield an associated set for the decision variables in (7) that is, in general, neither open nor closed.

Proof. See Appendix A.

The hypothesis that $\alpha^*>0_m$ in Proposition 5 is not really demanding. Indeed, if $y^j>0_s$ for all $j=1,\dots,n$, we have that by (8.1), (8.2) and the non-negativity of δ and h^j, α_i has to be strictly greater than zero for all $i=1,\dots,m$.

Model (8) yields directly the estimations of the technical coefficients a_{ir} and the optimal value of model (7). In addition, the optimal solution of model (8) generates the value of each $\hat{\varepsilon}_j$ through the variables δ_j^*: $\hat{\varepsilon}_j=\min_{i=1,\dots,m}\left\{\frac{x_i^j-\sum_{r=1}^{s}\hat{a}_{ir}y_r^j}{\sum_{r=1}^{s}\hat{a}_{ir}}\right\}=\delta_j^*+\min_{i=1,\dots,m}\left\{h_i^{j*}\right\}=\delta_j^*$ since, by (8.4) and (8.3), we know that $\exists i'=1,\dots,m$ such that $h_{i'}^{j*}=0$ and, therefore, $\min_{i=1,\dots,m}\left\{h_i^{j*}\right\}=0$.

In the next section we compare the output production set generated from the PMP and that associated with the standard Data Envelopment Analysis.

3. A comparison with respect to Data Envelopment Analysis

In a context of engineering production processes such as manufacturing or agriculture, where the production-mix problem is frequently used to plan optimal outputs through LP, we introduced in the previous section an approach to estimate a common set of technical coefficients for all the units characterized by the same underlying technology. To that end, we invoked Farrell's principle of conservation to determine the minimal production set associated with the linear resource constraints that envelops all the observations. As implication, a polyhedral output production set is estimated based on exactly m hyperplanes, being m the number of firm's different inputs. In this way, the geometrical shape of the estimated technology justifies the comparison between DEA, which also generates polyhedral production possibility sets, and the new approach. Accordingly, in this section we develop the existing relationship between these two techniques, which complements the previous equivalence between Hölder's D_{ℓ_∞} distance and the directional distance function in the approach based on the PMP.

In the following we particularly show that the output production set generated by DEA is a subset of the estimated PMP output production set $\hat{P}_{PMP}(x)$. Given the input vector x, as we aforementioned, $P_{DEA}(x)$ denotes the set of producible outputs under the assumptions of DEA. Specifically, $P_{DEA}(x)=\{y\in R_+^s:(x,y)\in T_{DEA}\}$, where T_{DEA} is estimated from a sample of observations assuming the postulates of convexity, inefficiency, ray unboundedness and minimal extrapolation (see [7, p. 1081]). Proposition 6 states the relationship between $P_{DEA}(x)$ and $\hat{P}_{PMP}(x)$.

Proposition 6. $P_{DEA}(x)\subseteq\hat{P}_{PMP}(x)$.

Proof. See Appendix A.

From Proposition 6, we know that $\hat{P}_{PMP}(x)$ is an outer approximation of $P_{DEA}(x)$ in the sense of Färe and Li [16] or, alternatively, $P_{DEA}(x)$ may be seen as an inner approximation of $\hat{P}_{PMP}(x)$. Moreover, by Proposition 6, it is likely in empirical applications that some technical efficient DMUs under DEA is no longer on the boundary under the new methodology. This fact will be illustrated in Section 4.

The following corollary is a direct implication of Proposition 6, using as reference direction that corresponding to the Tshebishev metric (see [8]).

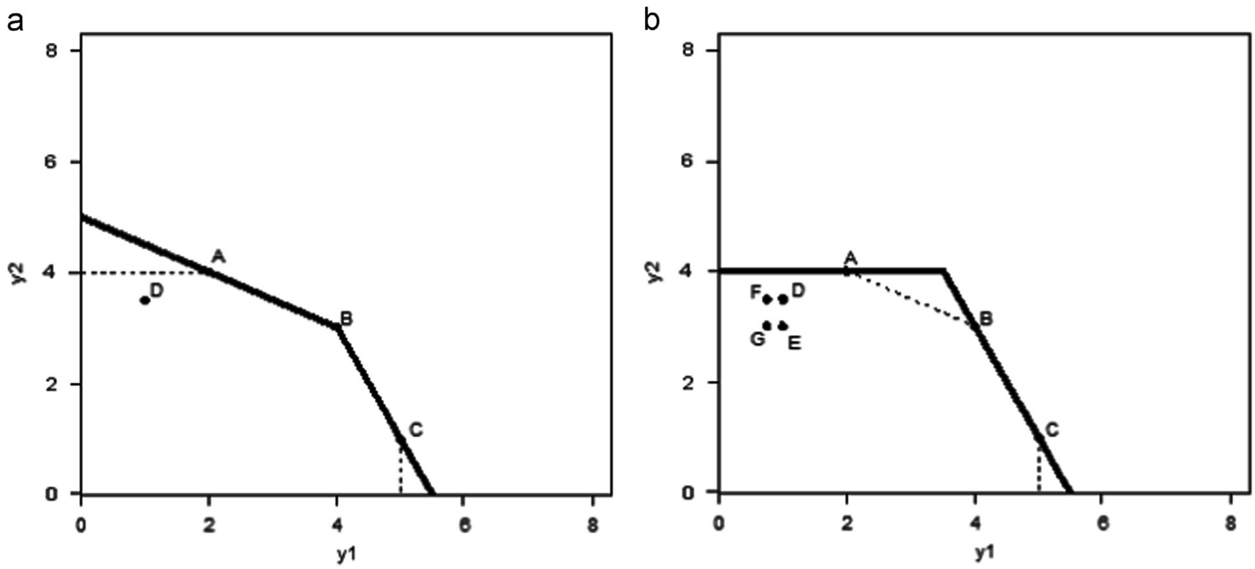

Fig. 4. DEA vs. the new approach.

Corollary 1. *For all* $j=1,\ldots,n$, *the following inequality holds*: $\vec{D}\left(y^j;1_s;P_{DEA}\left(x^j\right)\right) \le \vec{D}\left(y^j;1_s;\hat{P}_{PMP}\left(x^j\right)\right)$.

Corollary 1 states that the DEA technical inefficiency is never greater than the distance to the PMP boundary calculated with the new approach. Additionally, whereas DEA spans the production frontier on a "few" influential DMUs, those associated with Pareto-Koopmans efficient points, the introduced methodology uses the information of all observations in the sample for calculating the boundary of the PMP output production set. This is obviously consequence of the way in which the frontier is determined under the new approach: minimizing the sum of the technical inefficiencies, i.e. Min $\sum_{j=1}^{n} \epsilon_j$.

We illustrate these ideas in Fig. 4(a). Let us suppose that we have observed the data corresponding to four firms: A, B, C and D, which use two inputs to produce two outputs. Seeking simplicity, let us assume that all firms consume the same quantities of resources (inputs). In this scenario, the efficient frontier estimated by DEA would consist of two bold segments (AB and BC) and the dashed lines. In contrast, the new approach with the parameter $m=2$ (the number of inputs), yields the bold solid lines as boundary of the PMP output production set. Now, if we assume that three new firms are observed (E, F and G), improving the information of the sample (larger samples are always preferred in statistics), we obtain the same efficient frontier for DEA than with the initial four DMUs. This is because the three new firms are dominated in the sense of Pareto by a combination of A, B and C, constituting the Pareto-Koopmans DMUs in this example. In contrast, all the observations (even the technically inefficient E, F and G) reshape the frontier estimated with the new approach (see Fig. 4(b), where the bold solid lines change). This feature is shared with other modern techniques as, for example, the C^2NLS approach by Kuosmanen and Johnson [24] in comparison with DEA.

4. Empirical illustration

This section includes an empirical illustration of the use of the methodology proposed in this paper. In particular, we are interested in determining technical coefficients as a complementary information to that provided usually by DEA. Additionally, we are interested in showing that the output production sets determined by both methodologies are different in practice.

Table 2
Descriptive statistics (inputs and outputs).

	SAWU	FAWU	Citrus (ha)	T&S fruit trees (ha)
Mean	1429.4	909.4	612.4	374.0
Median	497.5	700.0	196.5	200.0
Min	8.0	125.0	1.0	6.0
Max	12,600.0	2750.0	5000.0	5200.0
Stand. dev.	2724.8	698.1	947.4	650.2

Kaiser and Messer [22] point out that agriculture is one of the main economic sectors using LP modeling and, particularly, the product-mix problem. Many land-grant universities, through their cooperative extension programs, offer numerous types of LP models to assist farmers in their decision-making process. Specifically, we estimate technical inefficiencies for a set of Spanish farms. Following Kaiser and Messer's [21, p. 137] agricultural example, and the specific set of inputs and outputs that they consider, we select data on the fixed number of hectares of land that each farm needs to best allocate to two types of trees (the outputs): citrus fruit, and temperate and subtropical (t&s) fruit trees. The inputs in this case are hired labor and family labor, both measured in full time equivalents. The data was taken from the Agricultural Census and the Survey on Agricultural Production Methods (Spanish National Statistics Institute [20]).

Farms cultivating jointly and solely citrus and t&s fruit trees have been considered. In addition, all of them are farms with irrigation systems, in an attempt to increase technological homogeneity. Farms are located in the southern Spanish region of Andalusia, where it is possible to find the highest number of farms that share both kinds of fruit trees. Andalusia, with more than 104,500 ha, accounts for more than 20% of the Spanish irrigated area devoted to citrus and t&s fruit trees, citrus accounting for 75,249 ha (72%) while t&s fruit trees cover 29,314 ha (28%). It should finally be noted that according to the Spanish Agricultural Census [20], there were 86 fruit trees farms in Andalusia that met the above-mentioned conditions, constituting the studied production units. In the last years this region has experienced a remarkable growth in productivity as a result of technological progress [27].

Table 3
Estimated distances to the frontier with the new methodology and DEA.

	PMP model (8)	DEA model (5)		PMP model (8)	DEA model (5)
Farm 1	475.6	61.6	Farm 44	2389.7	300.8
Farm 2	2411.8	321.0	Farm 45	254.8	35.4
Farm 3	7117.3	932.8	Farm 46	269.4	38.0
Farm 4	6727.9	774.3	Farm 47	517.5	67.4
Farm 5	7887.4	966.0	Farm 48	848.1	102.5
Farm 6	533.0	63.7	Farm 49	29.4	3.1
Farm 7	5689.6	748.5	Farm 50	0.0	0.0
Farm 8	5626.8	617.6	Farm 51	1790.4	223.6
Farm 9	7531.9	929.0	Farm 52	5667.7	701.2
Farm 10	4192.2	567.9	Farm 53	262.5	35.2
Farm 11	541.0	34.9	Farm 54	580.1	79.0
Farm 12	1213.1	0.0	Farm 55	361.8	0.0
Farm 13	7556.2	901.9	Farm 56	1365.7	181.3
Farm 14	7441.1	924.3	Farm 57	673.5	42.6
Farm 15	0.0	0.0	Farm 58	694.6	89.5
Farm 16	42.2	6.0	Farm 59	526.1	67.8
Farm 17	1506.9	191.4	Farm 60	600.8	84.8
Farm 18	2310.2	283.9	Farm 61	493.4	49.8
Farm 19	6812.0	777.7	Farm 62	12,423.8	1518.9
Farm 20	11,772.5	1279.5	Farm 63	1608.1	35.6
Farm 21	3036.9	368.0	Farm 64	509.0	64.4
Farm 22	723.9	93.3	Farm 65	8525.3	1147.0
Farm 23	1446.1	167.4	Farm 66	515.5	66.5
Farm 24	3044.3	400.4	Farm 67	565.1	79.8
Farm 25	811.9	112.1	Farm 68	706.7	95.4
Farm 26	5736.3	757.2	Farm 69	9.3	1.3
Farm 27	8220.1	996.2	Farm 70	0.0	0.0
Farm 28	1.4	0.2	Farm 71	13,151.2	1344.3
Farm 29	909.6	110.7	Farm 72	6349.5	662.8
Farm 30	933.6	114.6	Farm 73	7292.5	939.8
Farm 31	20.9	2.9	Farm 74	6448.4	788.9
Farm 32	819.8	101.5	Farm 75	484.1	0.0
Farm 33	1518.4	181.4	Farm 76	667.3	5.9
Farm 34	751.0	95.7	Farm 77	2773.1	363.0
Farm 35	775.4	95.1	Farm 78	11,693.1	1362.5
Farm 36	2836.3	381.2	Farm 79	12,422.3	1535.7
Farm 37	749.7	96.3	Farm 80	1827.3	221.9
Farm 38	3634.8	491.1	Farm 81	2267.4	297.0
Farm 39	1642.5	196.4	Farm 82	2499.3	282.8
Farm 40	1405.0	172.6	Farm 83	5788.8	685.7
Farm 41	817.6	99.7	Farm 84	1060.2	114.6
Farm 42	230.7	15.2	Farm 85	3487.0	469.6
Farm 43	5064.3	655.5	Farm 86	2243.3	264.7

Descriptive data for the selected farms are shown in Table 2. Inputs are: (1) Salaried Annual Working Units (SAWU). Data for this variable have been compiled considering the number of agricultural working days during the campaign comprised between 1 October 2008 and 30 September 2009. Usual working day refers either to at least one full day or full time equivalent ("eight-hour day", i.e., the sum of several part time days until an 8 h day is reached); and (2) Family Annual Working Units (FAWU). The same that in the former case but considering family AWU instead of salaried one. Outputs are: (1) Citrus fruit area (hectares) and (2) Temperate and subtropical (t&s) fruit trees area (ha)

Using model (8), we obtain the following estimated technical coefficients: $\hat{a}_{11}=0.159$, $\hat{a}_{12}=0.005$, $\hat{a}_{21}=0.051$ and $\hat{a}_{22}=0.097$. These values lead to the following constraints:

$$\begin{aligned} &0.159y_1+0.005y_2 \le x_1 \quad [\text{SAWU}] \quad (9.1)\\ &0.051y_1+0.097y_2 \le x_2 \quad [\text{FAWU}] \quad (9.2) \end{aligned} \tag{9}$$

These coefficients show, on one hand, that each hectare devoted to citrus employs 0.159 SAWU and 0.051 FAWU, while each t&s fruit trees hectare needs 0.005 SAWU and 0.097 FAWU. These results are coherent with the agronomic reality; especially in Andalusia, where the citrus crops has shown a strong growth over the past few years, with the average farm size increasing and experiencing a significant improvement in management practices. This means a considerable weight of the salaried labor (pruning, picking, fertilization, etc.), while t&s fruit trees are cultivated in more traditional family farms, with a relatively minor dimension, and where the externalization of agricultural works is lower [21,28].

Table 4
Hyperplanes defining the standard DEA technology.

DEA hyperplane	p1	p2	c1	c2	Ratio
1	11.083	1.534	1.000	71.102	0.138
2	13.201	1.000	40.679	8.559	0.076
3	36.751	1.000	165.408	1.886	0.027
4	8.169	7.663	1.000	79.354	0.938
5	0.149	0.000	0.050	0.801	–
6	0.135	0.000	0.010	0.856	–
7	0.000	0.007	0.993	0.000	–
8	0.000	0.089	0.000	0.911	–
9	0.138	0.000	0.862	0.000	–
10	0.218	0.000	0.626	0.156	–
11	0.101	0.000	0.000	0.899	–

These coefficients constitute a useful management tool for farmers, when quantifying for each kind of fruit trees the optimal labor input, while this allocation provides guidelines for public authorities in setting subsidies or adequate unemployment benefits policies for farmers and farm workers.

Table 3 reports the optimal values of model (5), with $g^0 = 1_s$, and model (8), which represent the distances from the analyzed 86 DMUs to the corresponding boundary. As seen from this table, under DEA there are six efficient DMUs, while under the new approach half of them do not belong to the PMP boundary. Besides the mean distance is 2967 for the product-mix problem against a mean value of 355.1 in DEA. Indeed, we find statistically significant differences between the two methods running a Wilcoxon signed rank test on two paired samples (p-value$=2.55 \cdot 10^{-15}$).

These differences are due, in part, to the different number of facets estimated with each method. While in DEA this number is not a priori restricted, under the new approach this quantity coincides with the number of inputs. In this way, in order to compute the number of facets that determine the standard DEA production possibility set, we resort to Qhull (see [32]). Accordingly, we find four hyperplanes associated with strongly efficient facets (all coefficients are strictly positive) and seven hyperplanes related to weakly efficient facets (some coefficient is zero). The general mathematical expression for any of these hyperplanes is $p_1y_1+p_2y_2-c_1x_1-c_2x_2=0$. Table 4 shows the corresponding coefficients. Additionally, we have reported in this same table the ratio between the output coefficients (see the last column) for the strongly efficient facets, in order to compare the DEA hyperplanes with those determined by the PMP approach. In this way, for (9.1) we have that this ratio equals 0.031 ($=0.005/0.159$), while for (9.2) we have a value of 1.902 ($=0.097/0.051$). Consequently, we observe that the first PMP facet has a similar 'slope' than the third DEA hyperplane, while we cannot conclude the same for the second PMP facet.

To understand in more detail what the gaps between the two frontiers mean, and inspired by Fig. 4(b) where a group of units is attracting down one of the PMP facets of the example, we carried out a K-Means Cluster Analysis. Accordingly, we determined seven clusters of farms (see Table 5). Additionally, we calculated the PMP facet where each unit is projected by solving model (8) (see the last column in Table 5). Note, however, that the determination of the facet where each unit is projected through model (8) is independent of the clusters computed. Each farm is associated with a PMP facet regardless of the cluster to which it belongs and the

Table 5
Clusters and PMP projection facets.

Farm	Cluster	PMP projection facet	Farm	Cluster	PMP projection facet
21	1	2	64	3	1
34	1	2	67	3	1
37	1	2	81	3	1
44	1	2	82	3	1
52	1	2	84	3	1
58	1	2	86	3	1
59	1	2	15	4	2
66	1	2	2	5	1
80	1	2	6	5	1
8	2	1	7	5	1
12	2	2	10	5	1
55	2	1	11	5	1
57	2	1	23	5	1
63	2	1	24	5	1
75	2	2	26	5	1
76	2	1	36	5	1
1	3	1	38	5	1
16	3	1	43	5	1
17	3	1	60	5	1
18	3	1	68	5	1
25	3	1	69	5	1
28	3	1	77	5	1
29	3	2	85	5	1
30	3	2	20	6	2
31	3	1	22	6	2
32	3	2	70	6	2
33	3	2	71	6	2
35	3	2	72	6	2
39	3	2	74	6	2
40	3	2	83	6	2
41	3	1	3	7	1
42	3	1	4	7	1
45	3	1	5	7	2
46	3	1	9	7	1
47	3	1	13	7	2
48	3	2	14	7	1
49	3	1	19	7	1
50	3	1	27	7	2
51	3	1	62	7	2
53	3	1	65	7	1
54	3	1	73	7	1
56	3	1	78	7	1
61	3	1	79	7	2

Table 6
Descriptive statistics of the results.

	PMP model (8)	DEA model (5)
Mean	2967.0	355.1
Median	1425.5	170.0
Min	0.0	0.0
Max	13,151.2	1535.7
Stand. dev.	3403.9	408.4

number of groups determined by the cluster analysis technique. From the results, we conclude that 54 out of 86 farms are projected onto the first PMP facet (9.1), while 32 units have the second PMP facet (9.2) as reference. Regarding the clusters, on the one hand, there is a group of units that exert an effect of attraction exclusively on the first PMP facet (cluster 5, consisting of 16 farms). On the other hand, two clusters pull down the second PMP facet in an exclusive way: cluster 1 (with 9 units) and cluster 6 (with 7 units). However, we observe a mix of facets for the remaining clusters (except for the case of cluster 4, which only consists of one farm).

Finally, descriptive statistics for the results of Table 3 are shown in Table 6. Even though results are different both statistically and in one order of magnitude, they are nevertheless correlated as Pearson's coefficient is 0.993, and Spearman's rank-order coefficient is 0.967.

5. Conclusions

We introduced a new methodology based on the product-mix problem to assess the production performance of firms, which is related to the well-known Data Envelopment Analysis approach, thereby complementing its results with information about the value of technical coefficients. As in DEA the new approach assumes a common technology, offering novel analytical possibilities to researchers interested in this field by introducing the ability of performing efficiency evaluation. An outstanding feature of the new approach is, therefore, the possibility of estimating the technical coefficients in the evaluation process, thereby determining the underlying relationship between each pair of inputs and outputs.

The new approach can be readily implemented as it departs from the product-mix problem used by process engineers in many industrial and primary sectors, particularly manufacturing and agriculture. The derived piece-wise linear output production set, based on the Tshebishev metric, is compared with the Directional Distance Function in the context of Data Envelopment Analysis. As a result of this comparison we prove that the new approach yields a production possibility set representing an outer approximation of that corresponding to DEA (vice versa, DEA is an inner approximation of the former), which extends to their efficient subsets.

In this respect, we have found statistically significant differences between the two methodologies in an empirical application using data from Andalusian farms in Spain, reflecting the difference that exists between the geometry of the estimated boundary corresponding to each technique. Research exploring the relationship between the new approach and several DEA models, thereby considering more complex applications and larger databases, represents a good avenue for further follow-up research.

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Appendix A

Proof of Proposition 3. Without loss of generality, we may assume that $\tilde{y} \neq 0_s$.

(1) If $\tilde{y} \in \partial(P_{PMP}(x^j))$ then $\tilde{y} \in P_{PMP}(x^j)$ since $\partial(P_{PMP}(x^j)) \subset P_{PMP}(x^j)$. Let us assume that $\hat{A}\tilde{y} < x^j$. Under this assumption, we will use $\overrightarrow{D}(\tilde{y}; 1_s; P_{PMP}(x^j)) = \max\{\beta : \tilde{y} + \beta 1_s \in P_{PMP}(x^j)\}$, which coincides with the definition of the directional output distance function considering $g = 1_s$. By (2), this optimization program is equivalent to $\max\left\{\beta : \hat{A}(\tilde{y} + \beta 1_s) \le x^j\right\}$. From the set of constraints, we obtain that $\beta \le \min\limits_{i=1,\ldots,m}\left\{\dfrac{x_i^j - \sum_{r=1}^{s} \hat{a}_{ir}\tilde{y}_r}{\sum_{r=1}^{s} \hat{a}_{ir}}\right\}$. In this way, β^*, the optimal solution of the optimization program,

is $\beta^* = \min_{i=1,\dots,m} \left\{ \frac{x_i^j - \sum_{r=1}^{s} \hat{a}_{ir}\tilde{y}_r}{\sum_{r=1}^{s} \hat{a}_{ir}} \right\}$. Therefore, $\overrightarrow{D}(\tilde{y}; 1_s; P_{PMP}(x^j)) = \beta^* \neq 0$, implying that $\tilde{y} \notin \partial(P_{PMP}(x^j))$ (see [10]). This is a contradiction. Consequently, $\exists i' = 1, \dots, m$ such that $\sum_{r=1}^{s} \hat{a}_{i'r}\tilde{y}_r = x_{i'}^j$.

(2) Let $\tilde{y} \in P_{PMP}(x^j)$ such that $\exists i' = 1, \dots, m$ with $\sum_{r=1}^{s} \hat{a}_{i'r}\tilde{y}_r = x_{i'}^j$. We prove that $\tilde{y} \in \partial(P_{PMP}(x^j))$. Let us assume that $\exists \tilde{\beta} > 0$ such that $\tilde{y} + \tilde{\beta} 1_s \in P_{PMP}(x^j)$, which is equivalent to say that $\overrightarrow{D}(\tilde{y}; 1_s; P_{PMP}(x^j)) > 0$, i.e. $\tilde{y} \notin \partial(P_{PMP}(x^j))$ (see [10]). Then, by (2), $A\left(\tilde{y} + \tilde{\beta} 1_s\right) \leq x^j$. In particular, for i' we have that $\sum_{r=1}^{s} \hat{a}_{i'r}\tilde{y}_r < \sum_{r=1}^{s} \hat{a}_{i'r}\left(\tilde{y}_r + \tilde{\beta}\right) \leq x_{i'}^j$, which is a contradiction. Note that $\sum_{r=1}^{s} \hat{a}_{i'r}\tilde{y}_r \neq 0$ since $\sum_{r=1}^{s} \hat{a}_{i'r}\tilde{y}_r = x_{i'j}$ and we assume that $x^j > 0_m$. Therefore, $\tilde{y} \in \partial(P_{PMP}(x^j))$.□

Proof of Proposition 5. First, let us check whether $\hat{a}_{ir} = b_{ir}^*/\alpha_i^*$ is a feasible solution of model (7). For each $i = 1, \dots, m$ and $j = 1, \dots, n$, we have that $\sum_{r=1}^{s} \hat{a}_{ir} y_r^j = \sum_{r=1}^{s} (b_{ir}^*/\alpha_i^*) y_r^j = \frac{1}{\alpha_i^*} \sum_{r=1}^{s} b_{ir}^* y_r^j \leq x_i^j - \frac{\delta_j^*}{\alpha_i^*} - \frac{h_i^{j*}}{\alpha_i^*} \leq x_i^j$ as a consequence of (8.1) and the non-negativity of δ and h^j. Additionally, $\sum_{r=1}^{s} \hat{a}_{ir} = \sum_{r=1}^{s} (b_{ir}^*/\alpha_i^*) = \frac{\sum_{r=1}^{s} b_{ir}^*}{\alpha_i^*} = \frac{1}{\alpha_i^*} > 0$ by (8.2) and $\hat{a}_{ir} = b_{ir}^*/\alpha_i^* \geq 0$ by (8.6) and $\alpha_i^* > 0$. Regarding the corresponding objective value in (7), we have that $\sum_{j=1}^{n} \min_{i=1,\dots,m} \left\{ \frac{x_i^j - \sum_{r=1}^{s} \hat{a}_{ir} y_r^j}{\sum_{r=1}^{s} \hat{a}_{ir}} \right\} = \sum_{j=1}^{n} \min_{i=1,\dots,m} \left\{ \alpha_i^* x_i^j - \sum_{r=1}^{s} b_{ir}^* y_r^j \right\} = \sum_{j=1}^{n} \delta_j^* + \sum_{j=1}^{n} \min_{i=1,\dots,m} \left\{ h_i^{j*} \right\}$ by (8.1). Now, by (8.4) and (8.3), we know that for each j, $j = 1, \dots, n$, $\exists i' = 1, \dots, m$ such that $h_{i'}^{j*} = 0$. Therefore, $\sum_{j=1}^{n} \min_{i=1,\dots,m} \left\{ h_i^{j*} \right\} = 0$ and $\sum_{j=1}^{n} \min_{i=1,\dots,m} \left\{ \frac{x_i^j - \sum_{r=1}^{s} \hat{a}_{ir} y_r^j}{\sum_{r=1}^{s} \hat{a}_{ir}} \right\} = \sum_{j=1}^{n} \delta_j^*$, which coincides with the optimal value of (8). Let us now suppose that there exists $\overline{a}_{ir}$, $i = 1, \dots, m$, $r = 1, \dots, s$, as feasible solution to (7), with $\sum_{j=1}^{n} \min_{i=1,\dots,m} \left\{ \frac{x_i^j - \sum_{r=1}^{s} \overline{a}_{ir} y_r^j}{\sum_{r=1}^{s} \overline{a}_{ir}} \right\} < \sum_{j=1}^{n} \delta_j^*$. From $\overline{a}_{ir}$, $i = 1, \dots, m$, $r = 1, \dots, s$, we define $\overline{\alpha}_i := \left(\sum_{r=1}^{s} \overline{a}_{ir} \right)^{-1}$, $i = 1, \dots, m$, and $\overline{b}_{ir} := \overline{a}_{ir}\overline{\alpha}_i$, $i = 1, \dots, m$, $r = 1, \dots, s$, $\overline{\delta}_j := \min_{i=1,\dots,m} \left\{ \overline{\alpha}_i \left(x_i^j - \sum_{r=1}^{s} \overline{a}_{ir} y_r^j \right) \right\}$, $j = 1, \dots, n$. Let $M(j)$ be the set of input indices where the minimum $\min_{i=1,\dots,m} \left\{ \overline{\alpha}_i \left(x_i^j - \sum_{r=1}^{s} \overline{a}_{ir} y_r^j \right) \right\}$ is achieved. In this way, we define the set of binary decision variables $\overline{d}_{ij}$, $i = 1, \dots, m$, $j = 1, \dots, n$, as

$$\overline{d}_{ij} = \begin{cases} 0 & \text{if } i \in M(j) \\ 1 & \text{otherwise} \end{cases}$$

Additionally, we define $\overline{h}_i^j$, $i = 1, \dots, m$, $j = 1, \dots, n$, as follows:

$$\overline{h}_i^j = \begin{cases} 0 & \text{if } i \in M(j) \\ \overline{\alpha}_i x_i^j - \sum_{r=1}^{s} \overline{b}_{ir} y_r^j - \overline{\delta}_j & \text{otherwise} \end{cases}$$

Next, we show that $\left\{ \overline{\alpha}, \overline{B}, \overline{\delta}, \overline{h}^j, \overline{d}^j \right\}$ is a feasible solution of model (8). Regarding (8.1), we have that if $i \in M(j)$, then $\sum_{r=1}^{s} \overline{b}_{ir} y_r^j + \overline{\delta}_j + \overline{h}_i^j = \overline{\alpha}_i \left(\sum_{r=1}^{s} \overline{a}_{ir} y_r^j + \min_{i=1,\dots,m} \left\{ x_i^j - \sum_{r=1}^{s} \overline{a}_{ir} y_r^j \right\} \right) = \overline{\alpha}_i x_i^j$ since $\min_{l=1,\dots,m} \left\{ \overline{\alpha}_l \left(x_l^j - \sum_{r=1}^{s} \overline{a}_{lr} y_r^j \right) \right\} = \overline{\alpha}_i \left(x_i^j - \sum_{r=1}^{s} \overline{a}_{ir} y_r^j \right)$ by the definition of $M(j)$, and if $i \notin M(j)$, then $\sum_{r=1}^{s} \overline{b}_{ir} y_r^j + \overline{\delta}_j + \overline{h}_i^j = \overline{\alpha}_i x_i^j$ by the definition of $\overline{h}_i^j$. Compliance of (8.2) is evident by the definition of $\overline{b}_{ir}$, $i = 1, \dots, m$, $r = 1, \dots, s$. As for (8.3), by the definitions of $\overline{h}_i^j$ and $\overline{d}_i^j$, this constraint is satisfied as long as M was a positive number sufficiently large. Constraint (8.4) is satisfied because $M(j) \neq \varnothing$ for all $j = 1, \dots, n$. Finally, ((8.5)–(8.9)) are evident. Consequently, $\left\{ \overline{\alpha}, \overline{B}, \overline{\delta}, \overline{h}^j, \overline{d}^j \right\}$ is a feasible solution to (8) with an objective function equaling $1_n^T \overline{\delta} = \sum_{j=1}^{n} \min_{i=1,\dots,m} \left\{ \frac{x_i^j - \sum_{r=1}^{s} \overline{a}_{ir} y_r^j}{\sum_{r=1}^{s} \overline{a}_{ir}} \right\}$, which is strictly less than $1_n^T \delta^*$. This last result contradicts the fact that $\left\{ \alpha^*, B^*, \delta^*, h^{j*}, d^{j*} \right\}$ is an optimal solution of (8). Hence, $\hat{a}_{ir} = b_{ir}^*/\alpha_i^*$, $i = 1, \dots, m$, $r = 1, \dots, s$, is an optimal solution of (7). □

Proof of Proposition 6. By minimal extrapolation [7], T_{DEA} is the intersection set of all $\hat{T}$ satisfying convexity, inefficiency and ray unboundedness, and subject to the condition that each observation belongs to $\hat{T}$. On the other hand, $\hat{P}_{PMP}(x)$ may be rewritten as $\hat{P}_{PMP}(x) = \left\{ y \in R_+^s : (x, y) \in \hat{T}_{PMP} \right\}$, where $\hat{T}_{PMP} = \left\{ (x, y) \in R_+^{m+s} : \hat{A}y \leq x \right\}$. Now, we prove that the set $\hat{T}_{PMP}$ satisfies convexity, inefficiency and ray unboundedness and, additionally, any observation belongs to $\hat{T}_{PMP}$. First, $\hat{T}_{PMP}$ is convex because it is the intersection of m semi-spaces. Second, let (x, y) belong to $\hat{T}_{PMP}$ and let $\overline{x} \geq x$ and $\overline{y} \leq y$. Then, $\hat{A}\overline{y} \leq \hat{A}y \leq x \leq \overline{x}$, which implies that $(\overline{x}, \overline{y}) \in \hat{T}_{PMP}$. Consequently, $\hat{T}_{PMP}$ meets the axiom of inefficiency. Third, for the same (x, y), we have that $\hat{A}(\theta y) \leq \theta x$, is equivalent to $\hat{A}y \leq x$, for all $\theta > 0$, which means that $(\theta x, \theta y) \in \hat{T}_{PMP}$. Therefore, $\hat{T}_{PMP}$ satisfies ray unboundedness. Finally, by (7.1) we have that $\hat{A}y^j \leq x^j$, $j = 1, \dots, n$, which by the definition of set $\hat{T}_{PMP}$ means that $(x^j, y^j) \in \hat{T}_{PMP}$ for all $j = 1, \dots, n$. As a consequence of all these steps, we have that $T_{DEA} \subseteq \hat{T}_{PMP}$. Finally, this implies that $P_{DEA}(x) \subseteq \hat{P}_{PMP}(x)$.□

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