# Comparison between LP bound of the Two-Index and the Three-Index Vehicle Flow Formulation for the Capacitated Vehicle Routing Problem

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**Abstract**

This paper looks into the Two-Index Vehicle Flow Formulation and the Three-Index Vehicle Flow Formulation for the Capacitated Vehicle Routing Problem. Aside from comparing the number of variables and constraints, we also compare the strength of the LP bound of each formulation. As a result, we found that the Two-Index Vehicle Flow Formulation produces stronger LP bound than the Three-Index Formulation. Furthermore, we also found that these formulation deals with slightly different problems. Thus, we adjusted the two formulations so that they solve the same problem, while also prove that the LP bound of the Two-Index Formulation is at least that of the Three-Index Formulation.

**Keywords:** Vehicle Routing Problem, Formulations, LP bounds.

## 1 Introduction

The Capacitated Vehicle Routing Problem (CVRP) is a very well-known integer programming problem that deals with the common delivery problem to find routes that optimizes the the costs vehicles inducing when travelling to deliver to a given set of customers and their demand for goods. This problem can be applied to many areas, such as delivery - pick-up services, goods transportation, fuel consumption optimization, reverse logistics of disposed food waste, etc. However, the CVRP is categorized as an NP-hard problem, meaning that there is a limitation to the size of the problem that can be solved with mathematical programming or combinatorial optimization.

Thus, oftentimes, instead of solving the CVRP itself, we look into the LP relaxation of the problem. This is due to a fact that the LP relaxation turns the CVRP into a Linear Programming problem that can be solved in polynomial time. Also, these LP bounds provide a lower bound to the objective of a minimization problem, as said in Wolsey (1998). Therefore, comparing LP bounds of different formulations give us the strongest lower bound to our problem. Furthermore, with a heuristic, we can obtain a feasible solution to the problem.

Furthermore, the CVRP has many different formulations to date, and each has different characteristics, such as number of constraints or variables, and, especially, a different LP bound. Hence, we would like to investigate which formulation provides us with the strongest LP bound while having minimum number of variables and constraints. In this paper, we are particularly interested in the LP bound of the Two-Index Vehicle Flow Formulation proposed by Laporte, Nobert, and Desrochers (1985) as well as the Three-

Index Vehicle Flow Formulation by Golden, Magnanti, and Nguyen (1977). The reason is that these two formulations are both single commodity, flow based formulation and are widely known formulations.

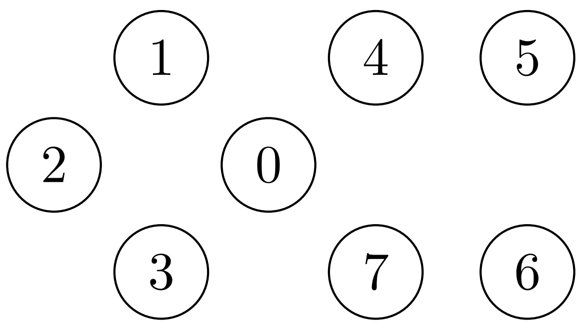
This paper is organized as follows. Section 2 briefly describes the CVRP and the formulations involved. Section 3 introduces our main theorem, together with its proofs. Section 4 includes the further discussion of this comparison and offers possible modifications of the formulations. And Section 5 offers the conclusion of this paper as well as its limitation. In this paper, we will refer to the Two-Index Vehicle Flow Formulation as the Two-Index Formulation and the Three-Index Vehicle Flow Formulation as the Three-Index Formulation.

# 2 Problem description and mathematical formulations

## 2.1 Description

Assume that we have $m$ delivery vehicles and are supposed to distribute goods to a list of $n$ customers given the amount that they demand and their locations. Each of our vehicles has a capacity of $Q$ goods, while customer i wants $q_i$ goods. Additionally, we can compute the cost of travelling from location i to location j and denote it as $d_{ij}$. Hence, we would like find a route for each vehicles such that the costs induced is minimal and we deliver enough goods to all customers while not violating the capacity of the vehicle.

**Example** Below is a map of customer's locations with the depot in the middle, denoted with 0:




Each customer has a demand for 1 goods. We have 2 vehicles, each with capacity of 100. The distance acts as the sole transportation cost and can be computed based on the customer's location, however, it has little to do with our paper, thus, we omit the distance for our example.

From the above description, we can identify 3 types of constraints, namely: Logical constraints, Vehicle constraints and Customer constraints. Logical constraints makes sure that a vehicle is moving rationally, in other words, starting from the depot, go to different locations, immediately leave that location after entering and eventually return to the depot, creating a route. This type of constraints also include the subtour elimination constraints, ensuring that every route leaving and returning to the depot, and variable range constraints, guaranteeing the variables taking correct values for interpretation. Vehicle constraints make sure that the number of vehicles used is less than or equal to the number of vehicles that we have. And Customer constraints assure that all customers are served and their demands are met with regards to the vehicle capacity, which we name the Demand constraints. Further interpretation of the constraints can be found in the literature of the formulations.

Furthermore, we denote the set of customers as $V_c$, the depot as customer 0, and the set of customers and depot $V$. Thus, $V_c \cup \{0\} = V$. In addition, we use $\delta(S)$ to denote set of edges $(i, j)$ such that either $i \in S$ and $j \notin S$ or the reverse.

## 2.2 Two-Index Vehicle Flow Formulation with LP relaxation

The following formulation is taken from Laporte et al. (1985).

The decision variable in this formulation is defined as $x_{ij}$, which represents the number of times edge (i,j) appears in the optimal solution, for all $i$ and $j \in S$.

$$\min \sum_{i,j \in V_c} d_{ij} x_{ij}$$

$$\text{subject to} \quad \sum_{j \in V_c} x_{0j} = 2m \tag{1}$$

$$\sum_{i<k} x_{ik} + \sum_{j>k} x_{kj} = 2 \quad (k \in V_c) \tag{2}$$

$$\sum_{(i,j) \in \delta(S)} x_{ij} \geq 2b(S) \quad (S \subset V_c) \tag{3}$$

$$0 \leq x_{ij} \leq 1 \quad (i, j \in V_c) \tag{4}$$

$$0 \leq x_{0j} \leq 2 \quad (j \in V_c) \tag{5}$$

In this paper, we use that $b(S) = \left\lceil \frac{\sum_{i \in S} q_i}{Q} \right\rceil$ as it is an often used value.

We can interpret the constraints as follows. Constraint (1) is the Vehicle constraint, assuring that exactly $m$ vehicles are used; (2) is a Logical constraint together with Customer constraint, dictating that exactly one vehicle go in and out of a node; (3) combines subtour elimination constraint with the Demand constraint, which makes sure that no subtour is formed and that the demand is met without violating the capacity of the vehicle; and the others are variables range constraints.

It is also worthy to note that this formulation has $\frac{n(n+1)}{2}$ variables and $2^n + n$ constraints with $n+1$ equalities and $2^n - 1$ inequalities, all excluding Constraint (4) and (5).

## 2.3 Three-Index Vehicle Flow Formulation with LP relaxation

While this formulation originates from Golden et al. (1977), this specific notation of the formulation is taken from Baldacci, Toth and Vigo (2009).

In this formulation, we have two variables, those are $x_{ij}^k$, representing whether vehicle k travels directly from customer i to j in the optimal solution for $i < j$, and $y_i^k$, representing whether customer i is served by vehicle k in the optimal solution.

$$\min \sum_{i,j \in V_c} d_{ij} \sum_{k=1}^{m} x_{ij}^k$$

$$\text{subject to} \quad \sum_{(i,j) \in \delta(h)} x_{ij}^k = 2y_h^k \quad (\forall h \in V_c, k = 1, ..., m) \tag{6}$$

$$\sum_{(i,j) \in \delta(S)} x_{ij}^k \geq 2y_h^k \quad (\forall S \subset V_c, h \in S, k = 1, ..., m) \tag{7}$$

$$\sum_{k=1}^{m} y_i^k = 1 \quad (\forall i \in V_c) \tag{8}$$

$$\sum_{k=1}^{m} y_0^k = m \tag{9}$$

$$\sum_{i \in V_c} q_i y_i^k \leq Q \quad (k = 1, ..., m) \tag{10}$$

$$0 \leq x_{ij}^k \leq 1 \quad (\forall (i,j) \in E \backslash \{(0,j) : j \in V_c\}, k = 1, ..., m) \tag{11}$$

$$0 \leq x_{0j}^k \leq 2 \quad (\forall (0,j)\,, j \in V_c, k = 1, ...m) \tag{12}$$

$$0 \leq y_i^k \leq 1 \quad (\forall i \in V, k = 1, ..., m) \tag{13}$$

The constraint of this formulation can be interpreted in the following way. Constraint (6) defines whether a customer is served; (7) is the subtour elimination constraint; (8) ensures that all customers are fully served; (9) dictates that m vehicles should be used; (10) requires that the capacity of vehicles cannot be violated; and the others are variable range constraints.

Additionally, we have that this formulation has $m \cdot \frac{(n+1)(n+2)}{2}$ variables and $n \cdot 2^{n-1} + (n+1)(m+1)$ constraints, excluding Constraint (11), (12) and (13).

# 3 Theorem

In this section, we propose our hypotheses and prove this hypotheses.

**Theorem** The Two-Index Vehicle Flow Formulation produces a stronger LP bound than the Three-Index Vehicle Flow Formulation.

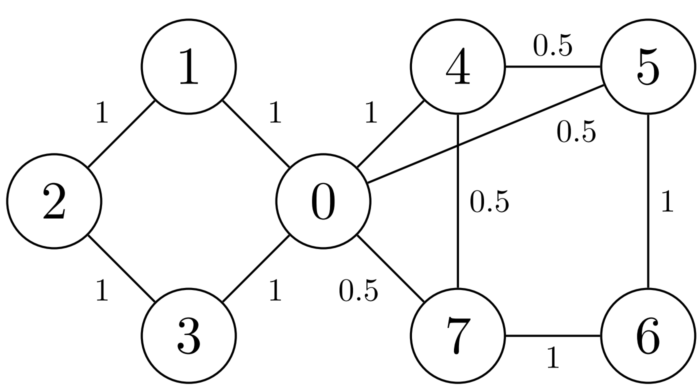
In order to prove this theorem, we will first prove that the feasible region of the Two-Index Formulation lies inside that of the Three-Index, then enhance our proof with a computed example. First, we define the variables that we will use in Section 3.1 and then prove that our variables satisfy every constraints of the Three-Index Vehicle Flow Formulation in Section 3.2. Then, the computed example can be found in Section 3.3. Section 3.4 gives the interpretation of this proof while we discuss further about the two problems in Section 3.5.

## 3.1 Defining Variables

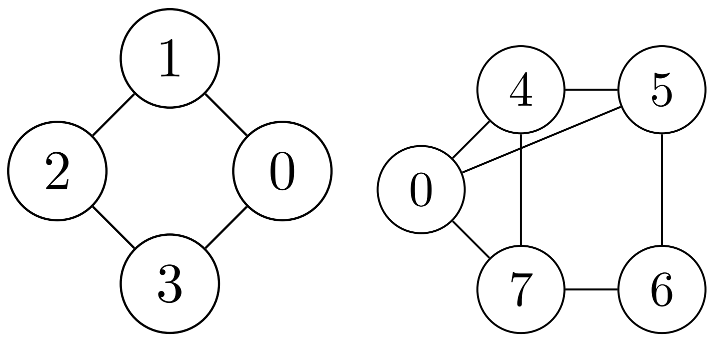
Every feasible solution of the Two-Index Vehicle Flow may be graphed as a map. Let X be an arbitrary feasible solution for the Two-Index Vehicle Flow Formulation. Thus, X can be expressed through a map.

We split this graph into different subgraphs connected at the depot (0). That is by removing (0), we group vertices that are connected to each other into a subgraph and then add back (0) for every subgraph formed.

**Example** Use the problem stated in the example above, assume we have a map with feasible solution:



Here, because (1), (2) and (3) are connected to each other and to (0). Since these 3 nodes are not connected to others nodes except for connected through (0), we split these 3 nodes and (0) into a subgraph. The same reasoning applied to (4), (5), (6) and (7).




We shall name these subgraph 1 and 2 respectively.

Now, for each subgraph S*, we do the following procedure.

Define $m^* = \left\lfloor \frac{\sum_{j \in S^* \setminus \{0\}} x_{0j}}{2} \right\rfloor$

The value $m^*$ can be interpreted as the number of vehicles assigned to this subgraph. This can happen because $m^*$ is a positive value and that the total $m^*$ assigned across subgraphs is less than or equal to m, in other words, the total number of vehicles we assigned is less or equal to the number of vehicles available.

**Lemma 3.2.** *$m^*$ is larger than or equal to 1*

This lemma will be proven at later stage of this proof, to be more specific, Section 3.2.4. Thus, for now, we shall take this result for granted.

**Lemma 3.3.** *Total number of vehicles we assigned is less than or equal to m.*

*Proof.* Let M be the number of vehicles that we assigned and $m_i^*$ be the number of vehicles that we assigned for subgraph $S_i$, i = 1, 2,...

$$\begin{aligned}
M &= m_1^* + m_2^* + ... \\
&= \left\lfloor \frac{\sum_{j \in S_1 \setminus (0)} x_{0j}}{2} \right\rfloor + \left\lfloor \frac{\sum_{j \in S_2 \setminus (0)} x_{0j}}{2} \right\rfloor + ... \\
&\leq \frac{\sum_{j \in S_1 \setminus (0)} x_{0j}}{2} + \frac{\sum_{j \in S_2 \setminus (0)} x_{0j}}{2} + ... \\
&= \frac{\sum_{j \in V_c} x_{0j}}{2} \\
&= m
\end{aligned}$$

□

For k = 1,2,...,m*, assign

$$x_{ij}^k = \begin{cases} \frac{x_{ij}}{m^*} & \text{if } (i,j) \in S^* \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

Note that these k are relative in a sense that it is from 1 to $m^*$ for the first subgraph but is from $m^*+1$ onward for the second subgraph and so on. Moreover, we are able to make such division because of Lemma 3.1, $m^* \geq 1$.

From (6), we have: for all $h \in S^* \backslash (0)$

$$\begin{aligned} y_h^k &= \frac{1}{2} \sum_{(i,j)\in\delta(h)} x_{ij}^k \\ &= \frac{\sum_{(i,j)\in\delta(h)} x_{ij}}{2m^*} \\ &= \frac{1}{m^*} & \text{for } h \neq 0 \end{aligned}$$

The last equation happens thanks to (2).

As there is no defining function for $y_0^k$, we define: for all k

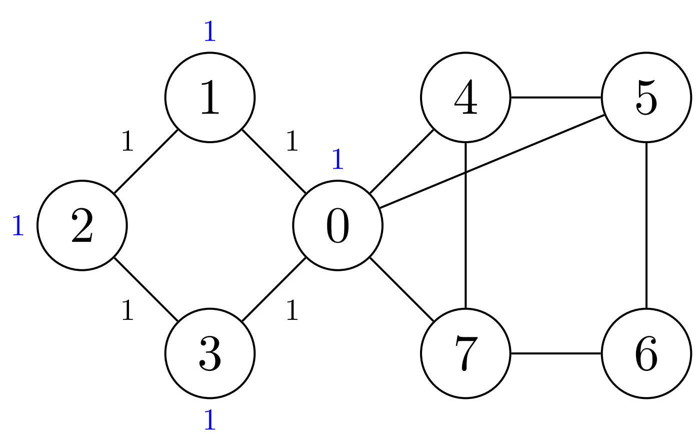
$$y_0^k = 1$$

Hence, we obtain:

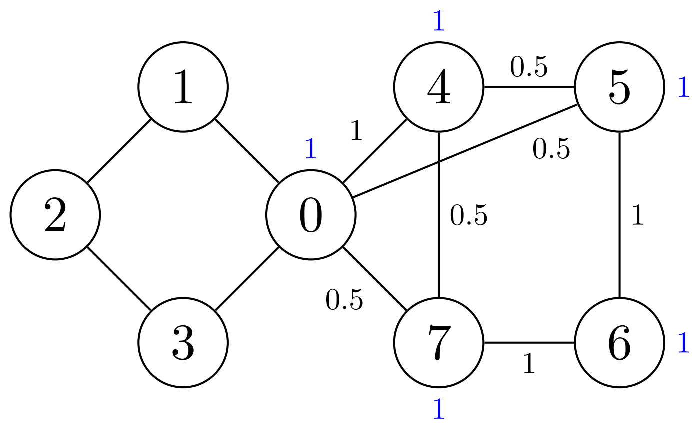
$$y_i^k = \begin{cases} \frac{1}{m^*} & \text{if } i \in S^* \backslash 0 \\ 1 & \text{i = 0} \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

**Example** Following Example above, we obtain the following.

For subgraph 1, $m^* = 1$, and through calculation, we obtain the following values for $x_{ij}^1$ and $y_i^1$. Black edge values denote $x_{ij}^k$, while blue values denote $y_i^k$.




For subgraph 2, $m^* = 1$, we also obtain the following for $x_{ij}^2$ and $y_i^2$.




Empty edges or node denote that $x_{ij}^k = 0$ or $y_i^k = 0$.

Thus, we have several properties of subgraph $S^*$ and $m^*$:

- Vertices in $S^*$ are only connected to other vertices in the subgraph except for the depot.
- $m^*$ vehicles only operates, or provides for customers, in $S^*$ and reverse.

## 3.2 Proof

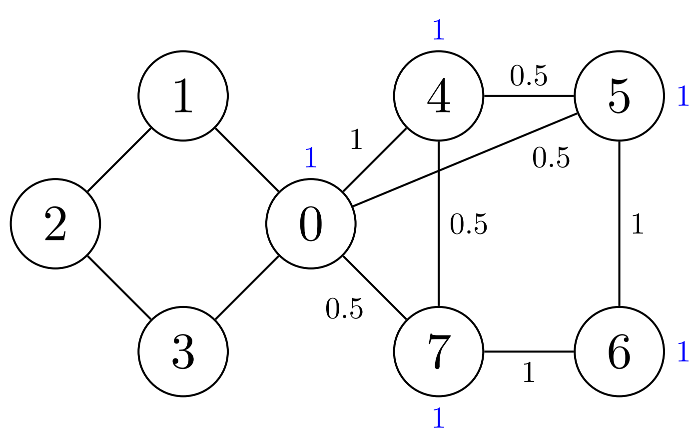
Now, we prove that the above assignment satisfies every constraints in the Three-Index Vehicle Flow Formulations.

### 3.2.1 Constraint (7)

*Proof.* We only assign non-negative values to $x_{ij}^k$ for $(i, j) \in S^*$ that k operates in. All values of $x_{ij}^k$ and $y_i^k$ are 0 if they are not in the subgraph. Thus, if we take S as a set that include any node not included in the subgraph, it can be reduced to a set $S'$ in the subgraph.

To illustrate this, we shall take a look at an example.

**Example** Take subgraph 2 from the example above, $m^* = 1$, we also obtain the following for $x_{ij}^2$ and $y_i^2$.



If we consider set S = 1, 4, 5.

Thus the left hand side of the inequality becomes.

$$\begin{aligned}\sum_{(i,j)\in\delta(S)} x_{ij}^k &= x_{01}^k + x_{04}^k + x_{05}^k + x_{12}^k + x_{47}^k + x_{56}^k \\ &= 0 + x_{04}^k + x_{05}^k + 0 + x_{47}^k + x_{56}^k \\ &= \sum_{(i,j)\in\delta(\{4,5\})} x_{ij}^k\end{aligned}$$

And the set of right hand side becomes.

$$\begin{aligned}\{2y_h^k, h \in S\} &= \{2y_1^k, 2y_4^k, 2y_5^k\} \\ &= \{0, 2y_4^k, 2y_5^k\}\end{aligned}$$

As all our variables takes values larger or equal to 0 (proven in Section 3.2.5, 3.2.6 and 3.2.7). The constraint when $y_h^k = 0$ holds without any proof needed. Thus, this set becomes $\{2\mathrm{y}_h^k, h \in \{4,5\}\}$.

Hence, S could be reduced to $S^{'} = \{4,5\}$.

Therefore, we only need to consider $S \subset S^*\backslash(0)$. For every h in S

$$\begin{aligned}\sum_{(i,j)\in\delta(S)} x_{ij}^k &= \frac{1}{m^*}\sum_{(i,j)\in\delta(S)} x_{ij} && \text{(Equation (14))} \\ &\geq \frac{1}{m^*}2b(S) && \text{(Constraint (3))} \\ &= 2y_h^k b(S) \qquad \text{for } h \neq 0 && \text{(Equation (15))}\end{aligned}$$

Use that $b(S) = \left\lceil \frac{\sum_{i\in S} q_i}{Q} \right\rceil$ and that $\frac{\sum_{i\in S} q_i}{Q} > 0$

$$\begin{aligned}&\Rightarrow b(S) \geq 1 \\ &\Rightarrow 2y_h^k b(S) \geq 2y_h^k \\ &\Rightarrow \sum_{(i,j)\in\delta(S)} x_{ij}^k \geq 2y_h^k\end{aligned}$$

□

### 3.2.2 Constraint (8)

*Proof.* For every $i \neq 0$, we have:

$$\begin{aligned}\sum_{k=1}^{m} y_i^k &= \sum_{k=1}^{m^*} \frac{1}{m^*} && \text{(Equation (15))} \\ &= \frac{m^*}{m^*} \\ &= 1\end{aligned}$$

□

### 3.2.3 Constraint (9)

*Proof.*

$$\begin{aligned}\sum_{k=1}^{m} y_0^k &= \sum_{k=1}^{m} 1 && \text{(Equation (15))}\\ &= m\end{aligned}$$

□

### 3.2.4 Constraint (10)

*Proof.* Consider Inequality (3) with S as S*\{0}, we have:

$$\begin{aligned}
& \sum_{(i,j)\in\delta(S)} x_{ij} \geq 2\left\lceil \frac{\sum_{i\in S} q_i}{Q} \right\rceil && \text{(Constraint (3))}\\
\Rightarrow\quad & \sum_{j\in\delta(S^*\setminus\{0\})} x_{0j} \geq 2\left\lceil \frac{\sum_{i\in S^*\setminus\{0\}} q_i}{Q} \right\rceil\\
\Leftrightarrow\quad & \frac{\sum_{j\in\delta(S^*\setminus\{0\})} x_{0j}}{2} \geq \left\lceil \frac{\sum_{i\in S^*\setminus\{0\}} q_i}{Q} \right\rceil\\
\Rightarrow\quad & \left\lfloor \frac{\sum_{j\in\delta(S^*\setminus\{0\})} x_{0j}}{2} \right\rfloor \geq \left\lceil \frac{\sum_{i\in S^*\setminus\{0\}} q_i}{Q} \right\rceil\\
\Leftrightarrow\quad & m^* \geq \left\lceil \frac{\sum_{i\in S^*\setminus\{0\}} q_i}{Q} \right\rceil && \text{(Definition of } m^*)\\
\Rightarrow\quad & m^* \geq \frac{\sum_{i\in S^*\setminus\{0\}} q_i}{Q}\\
\Leftrightarrow\quad & \frac{1}{m^*} \sum_{i\in S^*\setminus\{0\}} q_i \leq Q\\
\Leftrightarrow\quad & \sum_{i\in S^*\setminus\{0\}} q_i y_i^k \leq Q && \text{(Equation (15))}
\end{aligned}$$

□

As a by-product of this proof, we notice that

$$\begin{aligned} m^* &\geq \left\lceil \frac{\sum_{i\in S^*\setminus\{0\}} q_i}{Q} \right\rceil\\ &\geq 1\end{aligned}$$

Hence, proven Lemma 3.1.

### 3.2.5 Constraint (11)

*Proof.* We have from Equation (14):

$$x_{ij}^k = \begin{cases} \frac{x_{ij}}{m^*} & \text{if } (i,j) \in S^* \\ 0 & \text{otherwise} \end{cases}$$

For $(i, j) \in S^*$:

$$\begin{aligned} x_{ij}^k &= \frac{x_{ij}}{m^*} \\ &\leq \frac{1}{m^*} \\ &\leq 1 \end{aligned}$$

The second and third sign is thanks to the fact that m* is larger than or equal to 1 (Lemma 3.1). Furthermore, because m* is positive (Lemma 3.1) and $x_{ij}$ is also non-negative, $x_{ij}^k \geq 0$. For other cases, $x_{ij}^k$ is 0, hence, also falls in the given range. □

#### 3.2.6 Constraint (12)

*Proof.* We have from Equation (14):

$$x_{ij}^k = \begin{cases} \frac{x_{0j}}{m^*} & \text{if } (0, j) \in S^* \\ 0 & \text{otherwise} \end{cases}$$

For $(0, j) \in S^*$:

$$\begin{aligned} x_{0j}^k &= \frac{x_{0j}}{m^*} \\ &\leq \frac{2}{m^*} \\ &\leq 2 \end{aligned}$$

The second and third sign is thanks to the fact that m* is larger than or equal to 1 (Lemma 3.1). Furthermore, because m* is positive (Lemma 3.1) and $x_{0j}$ is also non-negative, $x_{0j}^k \geq 0$. For other cases, $x_{0j}^k$ is 0, hence, also falls in the given range. □

#### 3.2.7 Constraint (13)

*Proof.* We have from Equation (15):

$$y_h^k = \begin{cases} \frac{1}{m^*} & \text{for } h \neq 0 \\ 1 & \text{for } h = 0 \end{cases}$$

For $h \neq 0$:

$$y_h^k = \frac{1}{m^*}$$

Since m* is larger than 1, $0 \leq y_h^k \leq 1$
For $h = 0$:

$$y_0^k = 1$$

Here, $y_0^k$ falls in our constraint region. □

#### 3.2.8 Conclusion

Considering the objective functions of two formulations:

$$
\begin{aligned}
\sum_{i,j\in V_c} d_{ij} \sum_{k=1}^{m} x_{ij}^k &= \sum_{i,j\in V_c} d_{ij} \sum_{S^*\in S} \sum_{k=1}^{m^*} x_{ij}^k \\
&= \sum_{i,j\in V_c} d_{ij} \sum_{S^*\in S,(i,j)\in S^*} \sum_{k=1}^{m^*} \frac{1}{m^*} x_{ij} \qquad \text{(Equation (14))} \\
&= \sum_{i,j\in V_c} d_{ij} \sum_{S^*\in S,(i,j)\in S^*} x_{ij} \\
&= \sum_{i,j\in V_c} d_{ij} x_{ij}
\end{aligned}
$$

Hence, the value of the objective function of the solution for the Two-Index Vehicle Flow Formulation is equal to that of the newly defined solution to the Three-Index Vehicle Flow Formulation. It follows that every objective value of the Two-Index Vehicle Flow Formulation is also an objective value of the Three-Index Vehicle Flow Formulation.

## 3.3 Computation Result of Example

To illustrate this, we solved the above example into Python using the DOCPLEX package with the following coordinates of each node.

Table 1: Node coordinates of Example

| **Node** | **0** | **1** | **2** | **3** | **4** | **5** | **6** | **7** |
|---|---|---|---|---|---|---|---|---|
| **Coordinate** | [0, 0] | [-1, 1] | [-2, 0] | [-1, -1] | [1, 1] | [2, 1] | [2, -1] | [1, -1] |

Thus, we receive the following results for the LP bound of each formulation.

Table 2: Computation result for Example

| **Formulation** | **LP bound** |
|---|---|
| **2-index** | 12.485 |
| **3-index** | 11.657 |

This result show that the LP bound of the Three-Index Formulation is smaller, or weaker, than that of the Two-Index Formulation. Thus, enhance our hypothesis that the Two-Index Formulation provides a stronger LP bound than the Three-Index Formulation. Additionally, this result acts as a counter-example for whether the LP bound of the two formulations are equal instead of an existing superiority.

In conclusion, Two-Index Vehicle Flow Formulation produces stronger LP bound that that of the Three-Index Vehicle Flow Formulation.

## 3.4 Interpretation

The assignment of flow from the Two-Index Formulation to the Three-Index Formulation can be interpreted as follow. For single cycle case, there would only be one vehicle going through the route and deliver goods to all customers. However, for cases of multiple cycles, there will be as many as $m^*$ vehicles repeating this same route over and over to deliver $\frac{1}{m^*}$ of the total demand goods until the customers in that route receive everything demanded. There could possibly be vehicles left unused, however, as they only remain in the depot because of Lemma 3.2.

### 3.5 Discussion

While the two formulations should describe similar problem, there is an existing differences. The Two-Index Formulation indicated that the number of vehicle used have to be exactly m through equation (1), by making sure that there are $2m$ flows from the depot. Meanwhile, the Three-Index Formulation allows vehicles to stay at the depot for optimization as they only require the total of $y_h^k$ with regard to k equals to m at the depot. Thus, we lose vehicles in the three-index formulation as compared to the two-index. Hence, these formulation deal with slightly different problems. To modify these formulations so that they describe the same problem, we may add or change constraints of these formulation. This is done in section 4, while also proving that it holds for the new case that the LP bound of the Two-index is at least that of the Three-index.

# 4 Formulation adjustments

Due to their differences, we proposed two new formulations that mends this difference. As the Two-Index Formulation constraint that every vehicle must leave the depot while the Three-Index Formulation allows vehicles to stay at the depot. We will first partially relax Constraint (1) of the Two-Index Formulation so that it allows some vehicles to remain unused. Thus, compare its LP bound with that of the Three-Index Formulation. Then, we add a constraint to the Three-Index Formulation to ensure all vehicles are used and compare its LP bound to the unadjusted Two-Index Formulation.

With that said, we continue to prove the following theorem through similar methods as Section 3, excluding the use of an example.

**Theorem** With the modification of the formulations, the LP bound of the Two-index formulation is be at least that of the Three-index formulation.

### 4.1 Two-Index Vehicle Flow Formulation with LP relaxation with flexible number of vehicles

$$\min \sum_{i,j \in V_c} d_{ij} x_{ij}$$

$$\sum_{j \in V_c} x_{0j} \leq 2m \tag{16}$$

$$\sum_{j \in V_c} x_{0j} \geq 2 \tag{17}$$

(2), (3), (4) and (5)

#### 4.1.1 Interpretation

Here, we replace the original constraint (1) with constraint (16) and (17) while keeping all others.

In the unadjusted formulation, Constraint (1) can be interpreted as constraint for the number of vehicles, or to keep the number of vehicles used at exactly m. Therefore, by replacing it with Constraint (16) and (17), we allow the solution to take as many vehicles as it wants, as long as it is lower than m and bigger than 1.

#### 4.1.2 Proof

We shall use the same transformation as in Section 3 and take into consideration that the first constraint of the formulation is different. Because only the first constraint is different from before, we may filter out the proof that does not make use of this constraint and take them for granted. By doing that, we notice that none of the proofs involves Constraint (1). Hence, the same conclusion still holds in this case.

However, we should note that Lemma 3.3 now has to be re-proven, which could be done by replacing the last equality with less than or equal sign.

## 4.2 Adjusted Formulation of Three-Index Vehicle Flow Formulation with LP relaxation

$$\min \sum_{i,j \in V_c} d_{ij} \sum_{k=1}^{m} x_{ij}^k$$

$$\text{subject to} \quad \sum_{i \in V_c} y_i^k \geq 1 \quad (k = 1, ..., m) \tag{18}$$

$$(6), (7), (8), (9), (10), (11), (12) \text{ and } (13)$$

### 4.2.1 Interpretation

In this formulation, we added constraint (18) to ensure every vehicle go through at least 1 other node, as we consider that constraint (9) is not sufficient to assure the number of vehicles leaving the depot is exactly m.

### 4.2.2 Proof

For this section, while keeping all other definitions, we define m* differently:

At first, we shall define m* as: $m^* = \left\lceil \frac{\sum_{j \in S^* \setminus (0)} x_{0j}}{2} \right\rceil$.

Thus, we obtain the following properties of $m^*$.

**Lemma 4.2.** *If we round up instead of down for m*, total number of vehicles we assigned is greater than or equal to m.*

*Proof.* Similar to Lemma 3.3, let M be the number of vehicles that we assigned and $m_i^*$ be the number of vehicles that we assigned for subgraph $S_i$, i = 1, 2,...

$$\begin{aligned} M &= m_1^* + m_2^* + ... \\ &= \left\lceil \frac{\sum_{j \in S_1 \setminus (0)} x_{0j}}{2} \right\rceil + \left\lceil \frac{\sum_{j \in S_2 \setminus (0)} x_{0j}}{2} \right\rceil + ... \\ &\geq \frac{\sum_{j \in S_1 \setminus (0)} x_{0j}}{2} + \frac{\sum_{j \in S_2 \setminus (0)} x_{0j}}{2} + ... \\ &= \frac{\sum_{j \in V_c} x_{0j}}{2} \\ &= m \end{aligned}$$

□

**Lemma 4.3.** *The number of vehicles $m^*$ assigned to a subgraph $S^*$ is less than or equal to than the number of nodes in that subgraph excluding the depot.*

*Proof.* To prove this lemma, we consider that any value $m^*$ is less than or equal to the maximum value possible value of $m^*$ with regards to the values of $x_{ij}$ for $i, j \in S^*$.

$$\begin{aligned}
m^* &\leq \max_{x_{0j},\, j \in S^*} m^* \\
&= \max_{x_{0j} for j \in S^*} \frac{\sum_{j \in S^* \backslash (0)} x_{0j}}{2} && (x_{0j} \leq 2) \\
&= \frac{\sum_{j \in S^* \backslash (0)} 2}{2} \\
&= \frac{2|S^* \backslash \{0\}|}{2} \\
&= |S^* \backslash \{0\}|
\end{aligned}$$

As the result of the max function is an integer, ceiling and floor functions were ignored. □

According to Lemma 4.1, the total number of vehicle assigned is larger than or equal to m. Hence, we fix this problem using a simple technique: take away 1 vehicle from some m* that is larger than 1 and its $\frac{\sum_{j \in S^* \backslash \{0\}} x_{0j}}{2}$ is not an integer at random until our total is exactly equal to m. Thus, m* is either $\left\lceil \frac{\sum_{j \in S^* \backslash \{0\}} x_{0j}}{2} \right\rceil$ or $\left\lfloor \frac{\sum_{j \in S^* \backslash \{0\}} x_{0j}}{2} \right\rfloor$ at random.

We shall prove the validity of this technique.

*Proof.* Let $m_i = \frac{\sum_{j \in S_i \backslash \{0\}} x_{0j}}{2}$ for i = 1, 2, ...

M and $m_i^*$, i = 1, 2,..., be defined exactly like M in the proof of Lemma 4.1.

$$\begin{aligned}
M &= m_1^* + m_2^* + ... \\
&= \lceil m_1 \rceil + \lceil m_2 \rceil + ...
\end{aligned}$$

For this trick to be valid, we need to confirm that it is possible to subtract 1 until we reach m. Thus, we have to prove that the number of $m_i$ with fractional part is larger or equal to $M - m$.

Now consider M' and $m_i^{'}$, i = 1, 2,..., be defined exactly like M in Lemma 3.3.

$$\begin{aligned}
M^{'} &= m_1^{'} + m_2^{'} + ... \\
&= \lfloor m_1 \rfloor + \lfloor m_2 \rfloor + ...
\end{aligned}$$

From Lemma 3.3 and 4.1, we have:

$$\begin{aligned}
&M^{'} \leq m \leq M \\
\Leftrightarrow \quad &M - M^{'} \geq M - m \geq 0
\end{aligned}$$

Notice that the difference of M and M' is only whether it is ceiling for floor function. Therefore, in $M - M^{'}$, those with $m \in \mathbb{N}$ are cancelled out, and left only the m with decimal part. Furthermore, the difference between ceiling and floor of decimal numbers is exactly 1. Thus, $M - M^{'}$ gives the number of $m_i$ with decimal parts and is the value that we are looking for. Hence, the number of $m_i$ with fractional part is larger or equal to $M - m$. □

Now, we do the same procedure as we did in Section 4.1 and find sub-proofs that used the definition of m* and prove the additional constraint (18). Thus, only constraint (10) needs to be reworked.

**For constraint (10):**

*Proof.* For $m^* = \left\lfloor \frac{\sum_{j \in S^* \setminus \{0\}} x_{0j}}{2} \right\rfloor$, we accept the proof from Section 3.2.4 as valid for this part as well.

For $m^* = \left\lceil \frac{\sum_{j \in S^* \setminus \{0\}} x_{0j}}{2} \right\rceil$, we have:

$$\begin{aligned}
& m^* \geq \left\lfloor \frac{\sum_{j \in S^* \setminus \{0\}} x_{0j}}{2} \right\rfloor \geq \left\lceil \frac{\sum_{i \in S^* \setminus \{0\}} q_i}{Q} \right\rceil && \text{(Inequality from line 4 of (3.2.4))} \\
\Leftrightarrow \quad & m^* \geq \frac{\sum_{i \in S^* \setminus \{0\}} q_i}{Q} \\
\Leftrightarrow \quad & \frac{1}{m^*} \sum_{i \in S^* \setminus \{0\}} q_i \leq Q \\
\Leftrightarrow \quad & \sum_{i \in S^* \setminus \{0\}} q_i y_i^k \leq Q
\end{aligned}$$

□

**For constraint (18):**

*Proof.*

$$\begin{aligned}
\sum_{i \in V_c} y_i^k &= \sum_{i \in S^* \setminus \{0\}} y_i^k \\
&= \sum_{i \in S^* \setminus \{0\}} \frac{1}{m^*} \\
&= |S^* \setminus \{0\}| \frac{1}{m^*} \\
&\geq 1 && \text{(Lemma 4.2)}
\end{aligned}$$

□

Therefore, we may conclude that our hypothesis holds true.

# 5 Conclusion

In this paper, we have compared the LP bounds of Two-Index Vehicle Flow Formulation and Three-Index Vehicle Flow Formulation of the CVRP. From thorough investigation and proof, we conclude that the Two-Index Vehicle Flow Formulation produces a stronger LP bound than the Three-Index Vehicle Flow Formulation. Furthermore, we discussed that these two formulations may formulate different formulations from each other due to their Vehicle constraints. Thus, we then try to modify their Vehicle constraints such that they describe a similar problem. Even with these changes, we are still able to show that the LP bound of the Two-Index Formulation is at least that of the Three-Index Formulation.

Nonetheless, this paper still lacks the simulated results and benchmark of each formulations to support this proof. Thus, we suggest adding computation results to complete this research in the future.

Finally, this paper can be used as future reference when choosing a CVRP formulation. Our modified formulation may also be used according to their newly formed problems.

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