BASIS- AND TRIPARTITION IDENTIFICATION FOR QUADRATIC PROGRAMMING AND LINEAR COMPLEMENTARITY PROBLEMS

From an interior solution to an optimal basis and viceversa

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Abstract

Optimal solutions of interior point algorithms for linear and quadratic programming and linear complementarity problems provide maximal complementary solutions. Maximal complementary solutions can be characterized by optimal (tri)partitions. On the other hand, the solutions provided by simplex–based pivot algorithms are given in terms of complementary bases. A basis identification algorithm is an algorithm which generates a complementary basis, starting from any complementary solution. A tripartition identification algorithm is an algorithm which generates a maximal complementary solution (and its corresponding tripartition), starting from any complementary solution. In linear programming such algorithms were respectively proposed by Megiddo in 1991 and Balinski and Tucker in 1969. In this paper we will present identification algorithms for quadratic programming and linear complementarity problems with sufficient matrices. The presented algorithms are based on the principal pivot transform and the orthogonality property of basis tableaus.

Key words: Basis Recovery, Tripartitions, Principal Pivot Transforms, Balinski-Tucker Tableaus, Quadratic Programming, Linear Complementarity Problems, Interior Point Methods, Sufficient Matrices, Crossover, Criss-Cross Method.

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1 Introduction

Interior point algorithms provide maximal complementary solutions, which can be characterized by optimal partitions in linear programming (LP) and optimal tripartitions in quadratic programming (QP) and linear complementarity problems (LCPs). Pivot algorithms provide basic solutions; these basic solutions have a one-to-one relation to their characterization, the so-called bases. Thus, optimal solutions of interior point algorithms for LP and QP and LCPs can be characterized in terms of optimal (tri)partitions. Optimal solutions of pivot algorithms (such as the simplex method) are represented by optimal bases. In the case of uniqueness of the optimal solution both concepts coincide, however non-uniqueness is quite common in practical problems. In case of non-uniqueness, an optimal solution of an interior point method is a point in the relative interior of the optimal face [16, 21, 31, 33]. Recently the use of combined interior-point and pivot algorithms is being considered (see, e.g. [1]). Both from a theoretical and a practical point of view it is interesting to investigate how maximal complementary solutions and complementary basic solutions can be found from each other. Below some reasons will be mentioned why an optimal bases might be needed in some cases. In the sequel we will speak of an optimal basis identification (BI) algorithm and an optimal (tri)partition identification (TI) algorithm. A BI-algorithm is an algorithm which generates an optimal complementary basis, starting from a complementary solution (and its corresponding (tri)partition). A TI-algorithm generates an optimal (tri)partition and a maximal complementary solution, starting from an optimal basis.

It is well known that for LP a BI-algorithm exists. This algorithm is due to Megiddo [32] and is strongly polynomial (see also [14, 48]). For linear programming this algorithm is very efficient in practice (see, e.g. [5, 6]), but, to our best knowledge, until now, no such algorithm exists for QP and LCPs. In this paper we present a strongly polynomial BI-algorithm for QP and LCPs with sufficient matrices (to be defined later). This algorithm is a generalization of the algorithm of Megiddo. Although our algorithm is of both theoretical and practical interest, we will mainly be concerned with the theoretical aspects.

For LP there also exists a TI-algorithm. This algorithm is due to Balinski and Tucker [3]. Balinski and Tucker used this algorithm or procedure to proof the existence of strictly complementary solutions for LP by constructing so–called Balinski-Tucker tableaus. In this paper their approach is generalized to QP and LCPs with sufficient matrices to obtain a TI-algorithm. To our best knowledge such an algorithm did not exist yet for QP and LCPs.

In LP there are several reasons ([5, 14]) why it might be important to have an optimal basis. For each of these reasons we will point out a similar reason for QP and LCPs. We will only slightly mention the reasons for QP and LCPs since many results are not completely known or are still in progress.

- Basic solutions typically have a small number of non–zero coordinates, which might be advantageous in some practical applications. This can also be said of QP and LCPs. However, there are also situations in which a strictly complementary solution or the knowledge of the optimal partition is needed (see [15]).
- In commercial OR-packages sensitivity and postoptimal analysis are based on optimal bases. Although several papers have pointed out that this approach can lead to misleading and incorrect conclusions (see e.g. [15, 18, 36]), people are still used to it. However, we would like to stress that it is unnecessary to stick to sensitivity analyses using optimal bases (see e.g.[15, 21]). In QP and LCPs sensitivity analysis is hardly being performed at a commercial level at all. Berkelaar et al. [4] considered sensitivity analysis using tripartitions instead of bases for QP. As an example they consider the mean-variance portfolio models of Markowitz [30]. These models are often used by financial institutions.
- Simplex based solvers for both LP and QP are until date more efficient than interior point methods

in the case of reoptimization. This is particularly the case in mixed integer programming, based on cutting planes and branch and bound techniques. Techniques based on concepts in interior point theory are still in research, see for example [34].

- For numerical reasons it sometimes might be necessary to have an optimal basis solution instead of an optimal (tri)partition. Interior point methods can sometimes suffer from round–off errors with respect to complementarity.

The two basic tools that are used in both the BI-algorithm and the TI-algorithm are

1. Principal pivoting (see, e.g. [8, 9, 10, 11, 43, 44]).
2. Elimination based on the orthogonality property of pivot tableaus (see, e.g. [12, 17, 25, 39]).

The principal pivot transform was first introduced by Tucker [43, 44]. Cottle and Dantzig were the first to use this transform in their principal pivoting method (see [8, 9, 10, 11]) for linear complementarity problems with P-matrices and positive semidefinite matrices (see Section 2). The principal pivot transform is also a basic ingredient of the criss-cross methods presented by Terlaky et al. ([17, 23, 25, 39, 40]).

Unless stated explicitly otherwise we will stick to the following notational convention. Lowercase letters will denote scalars or vectors. Matrices will be denoted by capitals, and sets will be denoted by calligraphic letters. If $x \in \mathbb{R}^n$ is a vector, then we denote by capital X the diagonal matrix $X = \mathrm{diag}(x_i)$.

This paper is organized as follows. Section 2 contains a brief overview of the principal pivot transform and some invariance theorems. Section 3 deals with the concepts of tripartitions and bases and describes the orthogonality property of pivot tableaus. In Section 4 the BI-algorithm is presented. We will describe the Balinski-Tucker tableaus for LCPs and present the TI-algorithm in Section 5. Finally, in Section 6 we will give a short summary and mention directions for further research.

2 Principal pivot Transform and some Invariance Theorems

Consider the linear complementarity problem (LCP).

$$\begin{array}{c}\text{Find } x, s \in \mathbb{R}^n \text{ such that} \\ -Mx + s = q, \quad x \geq 0,\ s \geq 0,\ x^T s = 0,\end{array} \tag{1}$$

where $q \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n \times n}$. We will refer to this problem by the pair LCP(q, M). Let us first recall the definitions of some matrix classes from literature.

Definition 2.1 *A matrix $M \in \mathbb{R}^{n\times n}$ is said to be a P-*matrix *if all its principal minors are positive. The set of such matrices is denoted by $\mathcal{P}$. A matrix $M \in \mathbb{R}^{n\times n}$ is said to be a P_0-*matrix *if all its principal minors are nonnegative. The set of such matrices is denoted by $\mathcal{P}_0$.*

Definition 2.2 *A matrix $M \in \mathbb{R}^{n\times n}$ is said to be* positive semi-definite *if $x^T Mx \geq 0$ for all $x \in \mathbb{R}^n$. It is called* positive definite *if $x^T Mx > 0$ for all $x \in \mathbb{R}^n$, $x \neq 0$.*

It is clear by the following proposition, that if M is symmetric, the set $\mathcal{P}$ and the set of positive definite matrices are equivalent. Analogous, the set $\mathcal{P}_0$ and the set of positive semi–definite matrices are equivalent if M is symmetric.

Proposition 2.1 [38, Theorem 6E, pp. 341]
Let $M \in \mathbb{R}^{n\times n}$ be symmetric. The following statements are equivalent

1. *M positive (semi–) definite.*
2. *All principal minors of M are positive (nonnegative).*
3. *All eigenvalues of M are positive (nonnegative).*

It is well known [8, 11, 25] that a QP problem can be formulated as an LCP. This goes as follows. Consider the QP in symmetrical form, as found in [25, 35, 42].

$$\begin{array}{c} \min_{x,z} c^T x + \frac{1}{2} x^T C^T C x + \frac{1}{2} z^T z \\ \text{s.t. } Ax + Bz \geq b, \;\; x \geq 0 \end{array}$$

This primal QP problem and its dual form yield a pair LCP$(q^{(1)}, M^{(1)})$, where

$$M^{(1)} = \begin{bmatrix} -R & -A \\ A^T & -Q \end{bmatrix} \quad q^{(1)} = \begin{pmatrix} -b \\ c \end{pmatrix},$$

and $R = BB^T$, $Q = C^T C$. The QP problem in the standard formulation, often found in literature, is stated as follows

$$\text{(QP)} \quad \min_x \left\{ c^T x + \frac{1}{2} x^T Q x \; : \; Ax = b, \; x \geq 0 \right\},$$

where $c, x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, A an $m \times n$ matrix with full row rank and Q a symmetric positive semidefinite $n \times n$ matrix. The Wolfe–dual of (QP) is given by

$$\text{(QD)} \quad \max_{u,y,s} \left\{ b^T y - \frac{1}{2} u^T Q u \; : \; A^T y + s - Qu = c, \; s \geq 0 \right\}.$$

These problems yield a pair LCP$(q^{(2)}, M^{(2)})$, where

$$M^{(2)} = \begin{bmatrix} 0 & -A \\ A^T & -Q \end{bmatrix} \quad q^{(2)} = \begin{pmatrix} -b \\ c \end{pmatrix}.$$

According to the following definition both $M^{(1)}$ and $M^{(2)}$ are called bisymmetric.

Definition 2.3 *A square matrix $M \in \mathbb{R}^{n\times n}$ is said to be* bisymmetric *[43] ($\mathcal{BS}$) if there exists a permutation matrix $P \in \mathbb{R}^{n\times n}$ such that*

$$P^T M P = \begin{bmatrix} D & E \\ -E^T & F \end{bmatrix}, \tag{2}$$

where D and F are symmetric. This form will be referred to as the canonical bisymmetric form, *or, if it is clear from the context that a matrix is bisymmetric, simply as canonical form.*

Remark 2.1 *When both D and F are zero matrices, the matrix M is a* skew-symmetric *($\mathcal{SS}$) matrix. In general a matrix M is skew-symmetric if $M^T = -M$.*

Skew-symmetric matrices are always positive semidefinite. In the sequel we will consider positive semidefinite bisymmetric matrices only.

Definition 2.4 *A matrix $M \in \mathbb{R}^{n\times n}$ is called* row sufficient *($\mathcal{RS}$) if $XM^Tx \le 0$ implies $XM^Tx = 0$ for every vector x. It is called* column sufficient *($\mathcal{CS}$) if $XMx \le 0$ implies $XMx = 0$ for every vector x. If it is both row and column sufficient, M is called* sufficient *($\mathcal{S}$).*

Väliaho [45] proved that the class of sufficient matrices is equivalent to the class $\mathcal{P}_*$ of Kojima et al. [26]. Furthermore, the following inclusions are well known from literature.

$$\mathcal{SS} \subset \mathcal{BS} \subset PSD \subset \mathcal{S} = \mathcal{P}_* \subset \mathcal{CS} \subset \mathcal{P}_0,$$

$$\mathcal{S} \subset \mathcal{RS} \subset \mathcal{P}_0.$$

$$\mathcal{P} \subset \mathcal{CS},$$

For sufficient matrices the following sign–properties hold.

Proposition 2.2 *Let $M \in \mathbb{R}^{n\times n}$.*

- *Let $M \in \mathcal{RS}$ with $m_{ii} = 0$ for some i. If $m_{ij} \neq 0$ for some j, then $m_{ji} \neq 0$, and in this case $m_{ij}m_{ji} < 0$.*
- *Let $M \in \mathcal{CS}$ with $m_{ii} = 0$ for some i. If $m_{ji} \neq 0$ for some j, then $m_{ij} \neq 0$, and in this case $m_{ji}m_{ij} < 0$.*
- *Let $M \in \mathcal{S}$ with $m_{ii} = 0$ for some i. Then, $m_{ij} \neq 0$ for some j if and only if $m_{ji} \neq 0$, and $m_{ij}m_{ji} < 0$.*

Corollary 2.1 *Let for some i, $m_{ii} = 0$. Then, by Proposition 2.2*

- *Let $M \in \mathcal{RS}$. If $m_{ji} \ge 0$ for all j, then $m_{ij} \le 0$ for all j. If $m_{ji} \le 0$ for all j, then $m_{ij} \ge 0$ for all j.*
- *Let $M \in \mathcal{CS}$. If $m_{ij} \ge 0$ for all j, then $m_{ji} \le 0$ for all j. If $m_{ij} \le 0$ for all j, then $m_{ji} \ge 0$ for all j.*
- *Let $M \in \mathcal{S}$. $m_{ij} \ge 0$ for all j if and only if $m_{ji} \le 0$ for all j,*
  and $m_{ij} \le 0$ for all j if and only if $m_{ji} \ge 0$ for all j.

Before we will consider the principal pivot transform, let us describe submatrices and principal rearrangements first.

Definition 2.5 *Let a matrix $M \in \mathbb{R}^{m\times n}$ be given. For index sets $\mathcal{I} \subseteq \{1,\dots,m\}$ and $\mathcal{J} \subseteq \{1,\dots,n\}$ the matrix $M_{\mathcal{IJ}}$ is the* submatrix *of M whose entries lie in the rows of M indexed by $\mathcal{I}$ and the columns indexed by $\mathcal{J}$. If $m = n$ and $\mathcal{I} = \mathcal{J}$, the submatrix $M_{\mathcal{II}}$ is called a* principal submatrix *of M and the determinant of $M_{\mathcal{II}}$ is called a* principal minor *of M.*

Thus, a submatrix is simply a composition of some entries of a matrix in the same row- and columns–order as the matrix itself. A special submatrix of an arbitrary matrix is a so–called leading submatrix. Leading submatrices will be used in the principal pivot transform (see Section 4.3).

Definition 2.6 *Let $M \in \mathbb{R}^{n\times n}$ be given. For a given integer k : $(1 \leq k \leq n)$, the principal submatrix $M_{\mathcal{I}\mathcal{I}}$, where $\mathcal{I} = \{1, \ldots, k\}$ is called a* leading submatrix *of M. The corresponding determinant is called a* leading principal minor *of M.*

Closely related to the concept of leading submatrix is the principal rearrangement of a matrix.

Definition 2.7 *Let $M \in \mathbb{R}^{n\times n}$ and $\mathcal{I} \subseteq \{1, \ldots, n\}$ be given. Then, there exists a permutation matrix P such that the principal submatrix $M_{\mathcal{I}\mathcal{I}}$ appears as a leading submatrix in the matrix PAP^T. The matrix PAP^T is called a* principal rearrangement *of M with respect to $\mathcal{I}$.*

Proposition 2.3 [10] *Every principal rearrangement of a row or column sufficient matrix is row or column sufficient.*

Proposition 2.4 [10] *Every principal submatrix of a row or column sufficient matrix is row or column sufficient.*

The principal pivot transform

We will shortly describe the principal pivot transform, as found in [8, 9, 10, 43]. Consider a pair LCP(q, M). Let $\mathcal{I}$ be a subset of $\{1,\ldots,n\}$ and let $\mathcal{J}$ denote its complement. Suppose that the principal submatrix $M_{\mathcal{I}\mathcal{I}}$ is nonsingular. Furthermore, we suppose that $M_{\mathcal{I}\mathcal{I}}$ is a leading principal submatrix (this can be done by means of a principal rearrangement). We can now write $s = q + Mx$ in partitioned form

$$\begin{aligned} s_{\mathcal{I}} &= q_{\mathcal{I}} + M_{\mathcal{I}\mathcal{I}}x_{\mathcal{I}} + M_{\mathcal{I}\mathcal{J}}x_{\mathcal{J}} \\ s_{\mathcal{J}} &= q_{\mathcal{J}} + M_{\mathcal{J}\mathcal{I}}x_{\mathcal{I}} + M_{\mathcal{J}\mathcal{J}}x_{\mathcal{J}}. \end{aligned}$$

Using that $M_{\mathcal{I}\mathcal{I}}$ is nonsingular we may eliminate $x_{\mathcal{I}}$, and obtain a system of the form

$$\begin{aligned} x_{\mathcal{I}} &= q^*_{\mathcal{I}} + M^*_{\mathcal{I}\mathcal{I}}s_{\mathcal{I}} + M^*_{\mathcal{I}\mathcal{J}}x_{\mathcal{J}} \\ s_{\mathcal{J}} &= q^*_{\mathcal{J}} + M^*_{\mathcal{J}\mathcal{I}}s_{\mathcal{I}} + M^*_{\mathcal{J}\mathcal{J}}x_{\mathcal{J}}, \end{aligned}$$

where

$$M^* = \begin{bmatrix} M^*_{\mathcal{I}\mathcal{I}} & M^*_{\mathcal{I}\mathcal{J}} \\ M^*_{\mathcal{J}\mathcal{I}} & M^*_{\mathcal{J}\mathcal{J}} \end{bmatrix} = \begin{bmatrix} M^{-1}_{\mathcal{I}\mathcal{I}} & -M^{-1}_{\mathcal{I}\mathcal{I}}M_{\mathcal{I}\mathcal{J}} \\ M_{\mathcal{J}\mathcal{I}}M^{-1}_{\mathcal{I}\mathcal{I}} & M_{\mathcal{J}\mathcal{J}} - M_{\mathcal{J}\mathcal{I}}M^{-1}_{\mathcal{I}\mathcal{I}}M_{\mathcal{I}\mathcal{J}} \end{bmatrix}$$

and

$$q^* = \begin{bmatrix} q^*_{\mathcal{I}} \\ q^*_{\mathcal{J}} \end{bmatrix} = \begin{bmatrix} -M^{-1}_{\mathcal{I}\mathcal{I}}q_{\mathcal{I}} \\ q_{\mathcal{J}} - M_{\mathcal{J}\mathcal{I}}M^{-1}_{\mathcal{I}\mathcal{I}}q_{\mathcal{I}}. \end{bmatrix}$$

The resulting matrix M^* is called a principal pivot transform of M w.r.t. the index set $\mathcal{I}$. Similarly, the vector q^* is called a principal pivot transform of q. The above transformation defined by the matrix $M_{\mathcal{I}\mathcal{I}}$ is called a principal pivot transformation. This is in fact a block pivot on $M_{\mathcal{I}\mathcal{I}}$, which is called shortly also as principal pivoting. Following Keller [22] and Van de Panne and Whinston [35] a 1×1 principal pivot is called *diagonal pivot* and a 2×2 principal pivot is called *exchange pivot*.

The principal pivot transform as used in so–called complementary pivot algorithms consists of a basic transformation: a set of basic variables is replaced by its complementary set. Many algorithms to solve (LCP) are based on complementary pivot theory ([9, 29]). Lemke [29] describes three different schemes. Lemke's scheme 1, has become known as Lemke's pivot algorithm or Lemke's method (see also [28]). Most of these complementary pivot algorithms go through on so–called almost complementary solutions.

Definition 2.8 *Let $x, s \in \mathbb{R}^n$. We call (x, s) an* almost complementary solution *of the pair LCP(q, M) if there exists precisely one index $k \in \{1, \ldots, n\}$ such that*

$$x_i s_i = 0, \quad i = \{1, \ldots, n\} \backslash k, \ \textit{and} \quad x_k s_k \neq 0.$$

In Lemke's Scheme 1 an artificial variable x_0 is introduced in order to obtain feasible almost complementary solutions for the augmented problem. The block pivots in the principal pivot transform consists of a finite number of nonprincipal pivots each of which produces an almost complementary set of basic variables in the current transform. Since we only use diagonal pivots and exchange pivots in our BI-algorithm, almost complementarity occurs on exchange pivots only. The principal pivot transform, described in the preceding section, consists of a basic transformation: a set of basic vectors is replaced by its complementary set. Nonnegativity and complementarity conditions are preserved in Lemke's [28] and Wolfe's [47] methods. These algorithms stop if equality conditions are fulfilled. In Keller's [22], Cottle-Dantzig's [9] and Van de Panne-Whinston's [35] methods equality and complementarity conditions are satisfied while nonnegativity is partially preserved. In Klafszky-Terlaky's [17, 25, 39] criss-cross methods equality and complementarity conditions are satisfied while nonnegativity is not preserved.

The success of pivot algorithms such as the criss-cross method is related to invariance (see [8, 9, 10, 43]) of certain classes of matrices under principal pivoting. Such invariance theorems correspond to the four matrix classes mentioned above : P-matrices, P(S)D-matrices, bisymmetric matrices and sufficient matrices.

Theorem 2.1 [43] *The class of $\mathcal{P}$ is invariant under principal pivoting.*

Theorem 2.2 [8] *The classes of PD and PSD matrices are invariant under principal pivoting.*

Corollary 2.2 *The classes of negative definite and negative semidefinite matrices are invariant under principal pivoting.*

Theorem 2.3 [44] *The class $\mathcal{BS}$ is invariant under principal pivoting.*

Theorem 2.4 [10] *The classes $\mathcal{RS}, \mathcal{CS}$ and $\mathcal{S}$ are invariant under principal pivoting.*

Due to these invariance theorems and the fact that the class of sufficient matrices is the largest among the above mentioned classes we will develop our BI- and TI-algorithm for LCPs with sufficient matrices. The reader will easily verify that all the results hold also for the other special classes, including convex quadratic programming as well.

3 Optimal Tripartitions and Bases

3.1 Maximal Complementary Solutions and Tripartitions

The set of solutions in LCPs and QP can be characterized by *maximal complementary solutions* (cf. Güler and Ye [16]), and the corresponding *tripartition.* Define

$$\begin{aligned}
\mathcal{B} &:= \{i : x_i > 0 \text{ in at least one solution } x \text{ of (LCP)}\}\\
\mathcal{N} &:= \{i : s_i > 0 \text{ in at least one solution } s \text{ of (LCP)}\}\\
\mathcal{T} &:= (\mathcal{B} \cup \mathcal{N})^c = \{1, \ldots, n\} \backslash (\mathcal{B} \cup \mathcal{N}).
\end{aligned}$$

We denote this tripartition by $\pi := (\mathcal{B}, \mathcal{N}, \mathcal{T})$, and we will refer to it as the optimal tripartition. This is a straightforward generalization of the concept of optimal partitions in LP (see, e.g. [21]). A maximal complementary solution (x, s) is a solution for which

$$x_i > 0 \iff i \in \mathcal{B}; \qquad s_i > 0 \iff i \in \mathcal{N},$$

and it was shown to exist by Güler and Ye [16]. Moreover, they proved that interior point methods such as Carpenter et al. [7] and Vanderbei [46] generate such a solution (in the limit). In linear programming the set $\mathcal{T}$ is empty, which leads to the existence of *strictly complementary solutions.* This was first shown by Goldman and Tucker [13] (see e.g. Schrijver [37] and Jansen et al. [19] for an IPM based proof). The resulting partition $(\mathcal{B}, \mathcal{N})$ is called *the optimal partition.* In this paper we will be concerned with optimal tripartitions.

3.2 Orthogonality Property of Basic Tableaus

We will now describe the orthogonality properties (cf. [12, 17, 25, 39]) of basic tableaus, which will play a crucial role in our BI-algorithm. Let $\bar{T} \in \mathbb{R}^{m\times n}$ be an arbitrary matrix, let the columns of $\bar{T}$ be denoted by $\bar{t}_i$, $i = 1, \ldots, n$ and let B denote a basis chosen from the column vectors of $\bar{T}$. Furthermore we denote by J_B the set of basic indices, and similarly by J_N the set of nonbasic variables ($J = \{1, \ldots, n\} = J_B \cup J_N$). Then the tableau of $\bar{T}$ (in condensed form) w.r.t. B contains coefficients τ_{ik} of basic vector $\bar{t}_i$ in the basic representation of nonbasic vector $\bar{t}_k$, that is $\bar{t}_k = \sum_{i\in J_B} \tau_{ik}\bar{t}_i$ for all $k \in J$. Furthermore, we will introduce vectors $t^{(i)}$, $i \in J_B$ and $t_{(k)}$, $k \in J_N$ as follows

$$t^{(i)}_j = \begin{cases} \tau_{ij} & \text{if } j \in J_N,\\ 1 & \text{if } i = j,\\ 0 & \text{otherwise,}\end{cases} \qquad {t_{(k)}}_j = \begin{cases} \tau_{kj} & \text{if } j \in J_B,\\ -1 & \text{if } j = k,\\ 0 & \text{otherwise.}\end{cases}$$

Thus, $t^{(i)}$ is defined as a row of the canonical tableau corresponding to $\bar{T}$, and $t_{(k)}$ is defined as a column of the dual canonical tableau. In other words, $t^{(i)}$ lies in the rowspace of the canonical tableau of $\bar{T}$ and $t_{(k)}$ lies in the column space of the dual canonical tableau. The well–known orthogonality property can now be stated in the following lemma.

Lemma 3.1 [24] *Let B and $\tilde{B}$ denote two bases. Then,*

$${t^{(i)}}^T \tilde{t}_{(k)} = 0, \ \forall i \in J_B, k \in J_{\tilde{N}}.$$

Proof:

If $B = \tilde{B}$ the proof is obvious. The row space of the canonical tableau spanned by $\{t^{(i)} : i \in J_B\}$ is the orthogonal complement of the subspace spanned by $\{t_{(k)} : k \in \bar{J}_B\}$. The orthogonal complement of the row space of a matrix is its null space (see [38, p. 138]). Using two well–known results from linear algebra, that the row space and the null space of a matrix are invariant under pivoting (e.g. see [38, pp. 91,92]) the lemma can be concluded. □

The orthogonality of tableaus will be used here for the following (compact) tableau of LCPs. Let us record the data of LCP(q, M) in a *tableau*. The next table shows the standard initial tableau for LCPs in condensed form.

$$\begin{array}{c|c|ccc|}
 & 1 & x_1 & \cdots & x_n \\
\hline
s_1 & q_1 & m_{11} & \cdots & m_{1n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
s_n & q_n & m_{n1} & \cdots & m_{nn} \\
\hline
\end{array}$$

In LCP(q, M), $s_1, \ldots, s_n$ are the *basic variables* and $x_1, \ldots, x_n$ are the *nonbasic variables*. These terms have the synonyms *dependent variables* and *independent variables*, respectively. The columns associated with the basic variables are linearly independent. The matrix of columns with a maximal set of basic variables is called a *basis*.

4 Basis Identification (BI) Algorithm

4.1 Basis Identification

In this section the basis identification algorithm for LCPs with sufficient matrices is presented. From Section 2 we know that sufficient matrices include skew-symmetric matrices and bisymmetric matrices. Therefore, the presented algorithm is also applicable to LP and QP. Having a complementary (possibly maximal complementary) solution, the algorithm primarily consists of two phases. Given an arbitrary initial basis (we can for example choose initial slacks) we try to bring as much positive variables in the basis as possible. The first phase consists of diagonal and exchange pivots, the basic variables with value zero are replaced by the non–zero nonbasic variables. In the second phase the set $\mathcal{B}$ is reduced, using the orthogonality property described in the preceding section, and if necessary some principal pivots are performed. This continues until $\mathcal{B}$ is empty.

The complete basis identification algorithm is formally given as follows.

Algorithm 4.1 BASIS IDENTIFICATION ALGORITHM

Suppose that a complementary solution (x, s) of LCP(q, M) and its corresponding tripartition $\pi := (\mathcal{B}, \mathcal{N}, \mathcal{T})$ are available[5]. Let $\bar{M} = [-M, E]$ be the coefficient matrix of the LCP. Choose an arbitrary complementary basis. For simplicity let us assume that the initial basic variables are s_i, the nonbasic

[5]Note that it is not necessary to have a maximal complementary solution and an optimal tripartition.

$\mathcal{N}$	$\mathcal{T}$	$\mathcal{B}$		
$0 \ \cdots \ 0$	$0 \ \cdots \ 0$	$+ \ \cdots \ +$		
$M_{\mathcal{N}\mathcal{N}}$	$M_{\mathcal{N}\mathcal{T}}$	$M_{\mathcal{N}\mathcal{B}}$	$+ \ \vdots \ +$	$\mathcal{N}$
$M_{\mathcal{T}\mathcal{N}}$	$M_{\mathcal{T}\mathcal{T}}$	$M_{\mathcal{T}\mathcal{B}}$	$0 \ \vdots \ 0$	$\mathcal{T}$
$M_{\mathcal{B}\mathcal{N}}$	$M_{\mathcal{B}\mathcal{T}}$	$M_{\mathcal{B}\mathcal{B}}$	$0 \ \vdots \ 0$	$\mathcal{B}$

Table 1: Condensed starting tableau for the optimal basis identification algorithm.

variables are x_i. This can be done by using principal pivots and by renaming the variables. Let $j := 0$. We will denote the vector of basic variables by $\bar{s}$ and the vector of nonbasic variables by $\bar{x}$. Denote, $\bar{x}^0 = \bar{x}$ and $\bar{s}^0 = \bar{s}$, and $\mathcal{B}^0 = \mathcal{B}$, $\mathcal{T}^0 = \mathcal{T}$ and $\mathcal{N}^0 = \mathcal{N}$. Let $j = 1$. In Table 1 the starting tableau for the algorithm is presented.

Phase I: Increasing the number of positive variables in the basis

While $\mathcal{B}^j$ not empty **do**

1. Choose an arbitrary index k from $\mathcal{B}^j$, such that there exists an $m_{k\ell} \neq 0$, where $\ell \in \mathcal{T}^j \cup \mathcal{B}^j$. **If** no such k exists **then** goto Phase II.
2. **If** $m_{kk} < 0$ **do** a diagonal pivot, $\bar{s}^j_k$ leaves the basis and $\bar{x}^j_k$ enters the basis. Thus, $\mathcal{B}^j$ reduces by one index. Goto step 4.
3. **Else** we have $k \neq \ell$, such that $m_{\ell k} \neq 0$ (for sufficient matrices, $m_{kk} = 0$ and $m_{k\ell} \neq 0$). **Do** an exchange pivot on the pair (k, ℓ). In this case, $\bar{x}^j_k$ and $\bar{x}^j_\ell$ both enter the basis, while $\bar{s}^j_k$ and $\bar{s}^j_\ell$ both leave the basis. The set $\mathcal{B}^j$ reduces either by one or two indices.
4. $j \leftarrow j + 1$. Update $\mathcal{B}^j, \mathcal{N}^j$ and $\mathcal{T}^j$ and redefine x and s as the set of nonbasic and basic variables. **Goto** step 1.

At the end of this phase either $\mathcal{B}^j$ is empty or $M_{\mathcal{T}^j\mathcal{B}^j} = 0$ and $M_{\mathcal{B}^j\mathcal{B}^j} = 0$. If $\mathcal{B}^j$ is empty we are done.

Phase II: Reduction of $\mathcal{B}$.

While $\mathcal{B}^j$ not empty **do**

1. Choose an arbitrary index $k \in \mathcal{B}^j$, corresponding to a nonbasic variable $\bar{x}^j_k$. Let $t_{(k)}$ be as defined in the preceding section. Note that since $t_{(k)}$ is in the null space of $\bar{M}$, we have $\bar{M}t_{(k)} = 0$.
2. Eliminate a positive coordinate of either $x_{\mathcal{B}^j}$ or $s_{\mathcal{N}^j}$ using $t_{(k)}$. In the preceding phase, we have obtained $M_{\mathcal{T}^j\mathcal{B}^j} = 0$ and $M_{\mathcal{B}^j\mathcal{B}^j} = 0$. Therefore, when one element of $t_{(k)}$ is non–zero, the corresponding element in $(\bar{x}^j, \bar{s}^j)$ is positive. Eliminating positive coordinates of $x_{\mathcal{B}^j}$ can be done by

means of

$$(\bar{x}^j, \bar{s}^j) + \vartheta t_{(k)}.$$

Choosing an appropriate scalar $\vartheta \in \mathbb{R}$ (by a ratio-test), we can eliminate at least one positive variable, while preserving the nonnegativity. The variable(s) which is (are) eliminated will be called reducing variable(s), the operation itself will be referred to as reduction.

3. **If** $\mathcal{B}^j$ reduces by at least one index and $\mathcal{N}^j$ remains unchanged we update $\mathcal{B}^j, \mathcal{N}^j$ and $\mathcal{T}^j$, $j \leftarrow j+1$, redefine x and s, and **goto** step one.

4. Else the set $\mathcal{N}^j$ reduces by either one or more indices, then we call Phase I again.

5. $j \leftarrow j+1$, and **goto** step one.

Before proving that the algorithm is well defined, first some comments regarding the BI-algorithm. In Phase I, step 2, we process a diagonal pivot, and check for negativity of m_{kk}. From the sign–properties of sufficient matrices in Section 2, it is clear that m_{kk} cannot be positive for the classes considered. For exchange pivots (Phase I, step 3 and Phase II, step 5) we need $m_{kl} \neq 0$ and $m_{lk} \neq 0$, this is also provided by the sign–properties (see Section 2). Thus, the algorithm is correct and executable. We might ask for necessary and sufficient properties for the finiteness of the algorithm (see also [17]). The only properties used till now in this algorithm are that diagonal and exchange pivots are possible. From the sign–properties and the remarks above it is clear, that for sufficient matrices these pivotal operations are possible. Moreover, even for column sufficient matrices we can conclude that by the sign–properties diagonal and exchange pivots are possible (we need to change step 1 of Phase I in order to get this result). Note that after a diagonal or exchange pivot, the transformed matrix M is still in the same matrix class, due to the invariance theorems from Section 2. The algorithm reduces to a variant of Megiddo's procedure in the LP case. Then, no diagonal pivots are possible and, in fact, the algorithm is performed twice simultaneously, both on the primal and dual problem.

4.2 Finiteness & Strong Polynomiality of the BI-algorithm

In this section we will deal with the finiteness, the polynomiality and the flexibility of the presented BI-algorithm. These are the main results of this paper for the BI-algorithm. The BI-algorithm could have been presented more formally as is done by Megiddo. However, we have chosen for a representation of the BI-algorithm in terms of tableaus and pivots, both for simplicity and for clarity.

Theorem 4.1 *If M belongs to the class of sufficient matrices, then the BI-algorithm applied to the pair LCP(q, M) is finite and strongly polynomial. Moreover, the algorithm finds a complementary basis in at most $\mathcal{O}(n^3)$ operations.*

Proof:

Phase I consists of principal pivoting, and at most $|\mathcal{B}+\mathcal{T}|$ pivots are necessary to obtain that $M_{\mathcal{T}\mathcal{B}} = 0$ and $M_{\mathcal{B}\mathcal{B}} = 0$ or that $\mathcal{B}$ is empty. Note that at each diagonal and at each exchange pivot $\mathcal{B}$ reduces by at least one index. In Phase II, at each step at least one nonzero coordinate of $\mathcal{B}$ or $\mathcal{N}$ is reduced to zero. If set $\mathcal{N}$ reduces, Phase I is called and set $\mathcal{B}$ is reduced again, so at most $|\mathcal{B}+\mathcal{N}|$ pivots are necessary to obtain that $\mathcal{B}$ is empty. The total number of iterations is at most n, since the part of $\mathcal{B}$ reduced in Phase I doesn't need to be reduced in Phase II anymore. Thus, in the worst case, the algorithm needs at most $|\mathcal{B}+\mathcal{N}+\mathcal{T}| = n$ pivots. So, the algorithm is finite. Furthermore, in Phase I the solution remains the same, only pivoting is involved to rearrange the indices. In Phase II reduction takes place. The vector $t_{(k)}$ is chosen such that $\bar{M}t_{(k)} = 0$, and thus, although the solution changes, it remains feasible. That complementarity of the bases is preserved during the algorithm is a result of the principal pivot transform for diagonal and exchange pivots. The reduction step only reduces the number of positive coordinates in the solution, hence its complementarity is preserved. Since only pivoting and elimination are involved in the algorithm, strong polynomiality follows immediately. It is well–known that the complexity of one pivot is $\mathcal{O}(n^2)$. At most n pivots are needed and at most n reductions which cost at most $n \cdot \mathcal{O}(n)$ arithmetic operations. Therefore, the whole algorithm takes at most $\mathcal{O}(n^3)$ arithmetic operations, independently of the input length. □

Megiddo [32] used the BI-algorithm for LP to prove some complexity results for LP. Using the BI-algorithm presented here we can generalize these results to QP and LCPs with sufficient matrices. We sketch the proof only for the case of sufficient LCPs. Consider the next three problems.

Problem 1. Given M and q find a complementary basis or conclude that no such basis exists.

Problem 2. Given M and q such that a complementary solution exists, find a complementary basis.

Problem 3. Given M and q and $(x_I, s_{\bar{I}}) \geq 0$, where $I \subset \{1, \ldots, n\}$. Furthermore it is given that there exists a complementary solution $(\bar{x}, \bar{s})$ such that $\bar{x}_I = x_I$ and $\bar{s}_{\bar{I}} = s_{\bar{I}}$. Find a complementary basis.

Theorem 4.2 *If there exists a strongly polynomial time algorithm that finds an optimal basis, given an optimal solution for* either *the primal* or *the dual (QP), then there exists a strongly polynomial algorithm for the general QP problem.*

Theorem 4.3 *If there exists a strongly polynomial-time algorithm that solves LCP Problem 3, then there is a strongly polynomial algorithm for solving the sufficient LCP.*

Proof:

The proof essentially follows the same line as the proof in [32], with Megiddo's algorithm replaced by ours. We will sketch the proof here. We consider the three problems above. The proof follows from the following reductions. Assume that there exists a strongly polynomial algorithm for Problem 3 (this is in the theorem). Using feasibility and optimality arguments, Problem 2 can be reduced to Problem 3. Applying the BI-algorithm we can find a basis for Problem 2. This can be done in strongly polynomial time (cf. Theorem 4.1). We have obtained a strongly polynomial algorithm for Problem 2. Again using feasibility and optimality arguments, Problem 1 can be reduced to Problem 2 and we apply the BI-algorithm again. Now, we have obtained a strongly polynomial algorithm for Problem 1. For the complete proof for LP we refer to Megiddo [32]. □

The BI-algorithm is very flexible. One is free to choose the initial complementary basis and design numerically stable pivot rules for the sequence of the diagonal and the exchange pivots. The flexibility of the BI-algorithm could be exploited in efficient numerical implementations, the same way as Megiddo's algorithm is implemented for LP (see for example Bixby's implementation [5] of Megiddo's algorithm) by choosing an efficient starting basis. The BI-algorithm could also be embedded in combined interior point- pivot algorithms, see e.g. Andersen and Ye [1], or in adapted form can be used for generalized purification procedures [27] and crossover from IPM solvers to Simplex–based solvers.

The BI-algorithm has a broad range of applications. The only restriction for a class of matrices is invariance under principal pivot transforms and the sign–properties. Thus, the algorithm can for example be used on LP (skew symmetric matrices [19]), QP (bisymmetric matrices) and LCPs with sufficient matrices. Thus, the algorithm is also applicable to any LCP with matrices from the class $\mathcal{P}_*$ of Kojima et al. [26]. Even more, the algorithm is applicable to LCPs with column sufficient matrices[6], despite of the fact that no polynomial algorithm to solve such problems exists up till date. However, if we have any complementary solution, the algorithm can be used to find a complementary basis in strongly polynomial time.

5 Tripartition Identification (TI) Algorithm

5.1 Balinski-Tucker tableaus for LCPs

Balinski and Tucker [3] introduced a staircase-shaped optimal tableau for linear programming. This tableau is called a Balinski-Tucker tableau. Balinski and Tucker showed that it is possible to find a strictly complementary solution of an LP from such a Balinski-Tucker tableau very easily by performing some additions (summing up two vectors). Since the complexity of such operations is $\mathcal{O}(n)$, a strictly complementary solution of an LP can be found in strongly polynomial time from a Balinski-Tucker tableau. Recently, Zhang [48] showed that a Balinski-Tucker tableau can be found in polynomial-time, using a primal interior point algorithm. Moreover, he showed that if a pair of complementary solutions is known, it is possible to construct a Balinski-Tucker tableau in strongly polynomial time. In this section we will extend Balinski-Tucker tableaus to LCPs. From this Balinski-Tucker tableau for LCPs the optimal tripartition can easily be obtained, as we shall see in Section 5.2.

The following lemma is a generalization of a corollary of the main theorem in [3]. We prove it for LCPs with sufficient matrices by using the criss-cross method [17, 25].

Lemma 5.1 *Let $j, \bar{j}$ denote an arbitrary distinguished complementary pair. Then, one of the following three complementary tableaus can always be found in a finite number of pivot operations*

$$
\begin{array}{c|c|c|}
 & j & \\
\cline{2-3}
\bar{j} & - & \\
\cline{2-3}
 & \ominus & \\
 & \vdots & \\
 & \ominus & \\
\cline{2-3}
\end{array}
\qquad
\begin{array}{c|c|ccc|}
 & j & & & \\
\cline{2-5}
\bar{j} & 0 & \oplus & \cdots & \oplus \\
\cline{2-5}
 & \ominus & & & \\
 & \vdots & & & \\
 & \ominus & & & \\
\cline{2-5}
\end{array}
\qquad
\begin{array}{c|c|ccc|}
 & \bar{j} & & & \\
\cline{2-5}
j & 0 & \oplus & \cdots & \oplus \\
\cline{2-5}
 & \ominus & & & \\
 & \vdots & & & \\
 & \ominus & & & \\
\cline{2-5}
\end{array}
$$

where $\oplus$ and $\ominus$ denote nonnegative and nonpositive numbers respectively[7].

[6] It only takes a minor adaption to the algorithm; step 1 in Phase I should be adapted using the sign–properties for column sufficient matrices.

[7] Observe that the position of $\bar{j}$ and j in the first tableau can be interchanged by a diagonal pivot, resulting in an equivalent tableau for j and $\bar{j}$.

Proof:

The proof follows essentially the same line as the proof of the criss-cross method [39, 25, 17] for LP, QP and LCPs with sufficient matrices. We present the algorithm below and refer to the proof in [17, 24]. The criss-cross method for LCP with sufficient matrices is as follows

Algorithm 5.1 CRISS-CROSS METHOD

Initialization:

Define the initial starting basis by $s = q$ and let $x = 0$ be the nonbasic variables.
The initial tableau is given by $[-M, E, q]$.

Pivot rule:

1. *Leaving variable selection*:
   Let $k := \min\{i : s_i < 0 \text{ or } x_i < 0\}$. **If** there exists no such k, then **STOP**;
   (a feasible solution has been found).
   Else assume, without loss of generality, that $s_k < 0$.
2. *Entering variable selection*:
   2.1. *Diagonal pivot:*
      If $-m_{kk} < 0$, then perform a diagonal pivot and **goto** step 1.
      (s_k leaves the basis and x_k enters the basis).
      If $-m_{kk} = 0$, then select an exchange pivot.
      (the case $-m_{kk} > 0$ cannot occur).
   2.2. *Exchange pivot:*
      We know that $m_{kk} = 0$ in this case.
      Let $\ell := \min\{j : -m_{kj} < 0 \text{ or } -m_{jk} > 0\}$.
      If there exists no ℓ, **STOP**; LCP is infeasible.
      Else there exists an ℓ, with $m_{\ell k} m_{k\ell} < 0$
      (due to the sign properties for sufficient matrices),
      then perform an exchange pivot on (ℓ, k) and **Goto** step 1.
      (s_k and x_ℓ leave the basis, and x_k and s_ℓ enter the basis).
      NB. The case $m_{\ell k} m_{k\ell} \geq 0$ cannot occur.

Applying the criss-cross method to the LCP where the column of j plays the role of q, we get the first or second tableau when the criss-cross method stops at step 1, and - by an additional exchange pivot - tableau 3 as the criss-cross method stops at step 2.2. □

In the sequel we will assume that for any complementary solution (x, s) of LCP(q, M), a corresponding tableau is available. Applying Lemma 5.1 to a tableau which corresponds to a complementary solution (x, s) of LCP(q, M) we can discern two cases (the first tableau or one of the the other two). These cases are addressed to in the following lemmas. We will give a definition first, on improving complementarity of a complementary solution of LCP(q, M).

Definition 5.1 *Let $(x, s) \geq 0$ denote a complementary solution of LCP(q, M). Let t be a vector of appropriate size. We say that using vector t, the complementarity of (x, s) can be* improved by k *if there is an appropriate $\theta \in \mathbb{R}$ such that*

$$(x, s) + \theta t \geq 0,$$

is again a complementary solution with k new strictly complementary elements. Thus, k complementary variables that were both zero before the addition become strictly complementary when the addition is performed.

Lemma 5.2 *Suppose that a complementary solution (x, s) of LCP(q, M) is given and the corresponding complementary tableau $\tilde{M}$. Furthermore, assume that the following subtableau results from applying Lemma 5.1 to a complementary part of the subtableau corresponding to (x, s), where both $x_i = 0$ and $s_i = 0$.*

$$
\begin{array}{|c|ccc|ccc|}
\hline
0 & + & \cdots & + & 0 & \cdots & 0 \\
\hline
- & & & & & & \\
\vdots & & & & & & \\
- & & & & & & \\
\hline
0 & & & & & & \\
\vdots & & & & & & \\
0 & & & & & & \\
\hline
\end{array}
$$

Let us denote this tableau by T_1. The upper left tableau containing the negative numbers and the zero diagonal element is denoted by $\bar{T}_1$. Let the number of negatives in the first column of $\bar{T}_1$ equal ℓ. Assume that the elements of (x, s) corresponding to the first column of T_1 are zero. Then, the complementarity of (x, s) can be improved by $\ell + 1$.

Proof:

Let j denote the index of the variable corresponding to the first column of T_1. In Section 3.2 we defined vectors $t_{(j)}$, $j \in \bar{J}_B$. Tableau T_1 can be regarded as a subtableau of $\tilde{M}$, which contains the zero set of the complementary solution (where $x + s = 0$). As a consequence we have in the full tableau $\tilde{M}$, and in $t_{(j)}$, $j \in \bar{J}_B$, such a sign-pattern that $t_{(j)_i} \leq 0$ when both $x_i = 0$ and $s_i = 0$ We can subtract the vector $t_{(j)}$ from (x, s), however we should care about a ratio-test

$$(x, s) - \theta t_{(j)},$$

where θ is an appropriate scalar. We obtain strict complementarity for $\ell + 1$ new elements of the solution. The additional strict complementary is obtained by definition of $t_{(j)}$ (see Section 3.2). Thus, complementarity improves by $\ell + 1$. $\square$

The above lemma corresponds to the last two tableaus of Lemma 5.1. The first tableau is considered in the next lemma, which provides the tool to identify the indices in $\mathcal{T}$.

Lemma 5.3 *Consider the following sufficient LCP.*

$$-\begin{bmatrix} \hat{m} & \hat{d}^T \\ \hat{q} & \hat{M} \end{bmatrix} \begin{pmatrix} \hat{\xi} \\ \hat{x} \end{pmatrix} + \begin{pmatrix} \hat{\sigma} \\ \hat{s} \end{pmatrix} = 0, \tag{3}$$

where $\hat{q}$, $\hat{d} \in \mathbb{R}^n$, $\hat{m} \in \mathbb{R}$, $\hat{M}$ is sufficient and $\hat{m} > 0$ and $\hat{q} > 0$. If LCP$(\hat{q}, \hat{M})$ has a complementary solution, then the augmented LCP (3) has no complementary solution with either $\hat{\sigma} > 0$ or $\hat{\xi} > 0$ (i.e. their index belongs to $\mathcal{T}$ in the optimal tripartition).

Proof:

First assume to the contrary that there exists a complementary solution $(\hat{\xi}, \hat{x}, \hat{\sigma}, \hat{s})$ with $\hat{\xi} > 0$ (hence $\hat{\sigma} = 0$). Since (3) is homogeneous, we can assume without loss of generality that $\hat{\xi} = 1$. Writing out the LCP in an equations system we get

$$\begin{aligned} -\hat{d}^T\hat{x} + \hat{\sigma} &= \hat{m} \qquad (4) \\ -\hat{M}\hat{x} + \hat{s} &= \hat{q}. \qquad (5) \end{aligned}$$

Multiplying (5) by $\hat{X}$, we obtain

$$-\hat{X}\hat{M}\hat{x} + \hat{X}\hat{s} = \hat{X}\hat{q}.$$

Since $\hat{X}, \hat{s}$ are complementary we have $\hat{X}\hat{s} = 0$. Hence, $\hat{X}\hat{M}\hat{x} = -\hat{X}\hat{q} \leq 0$ because $\hat{q} > 0$. From the definition of sufficiency, we have $\hat{X}\hat{M}\hat{x} = 0$, hence, $\hat{X}\hat{q} = 0$, which implies $\hat{x} = 0$ because $\hat{q} > 0$. Now, from (4) we get $\hat{\sigma} = \hat{m}$ which contradicts complementarity. Similarly we can obtain that $\hat{\sigma} > 0$ is also impossible. If we pivot on '$\hat{m}$' (1×1 principal pivot), the role of $\hat{\xi}$ and $\hat{\sigma}$ is interchanged, while the same sign structure follows for $\hat{\sigma}$, thus by the above argument it follows that $\hat{\sigma} = 0$ in any complementary solution. In other words the index corresponding to $\hat{\sigma}$ and $\hat{\xi}$ belongs to the set $\mathcal{T}$ of the optimal tripartition. □

Lemma 5.4 *Suppose that a complementary solution (x, s) of LCP(q, M) and the corresponding complementary tableau $\tilde{M}$ are given. Furthermore, assume that the following subtableau results from applying Lemma 5.1 to a complementary part of the subtableau corresponding to (x, s), where both $x_i = 0$ and $s_i = 0$.*

$-$	
$\ominus$ $\vdots$ $\ominus$	

Let us denote this tableau by T_2. The upper left negative diagonal element is denoted by $\bar{T}_2$. The pair corresponding to $\bar{T}_2$ is in $\mathcal{T}$.

Proof:

Let us rewrite tableau T_2 as follows

$-\hat{m}$	$-\hat{d}^T$
$-\hat{q}$	$-\hat{M}$

where $\hat{m} > 0$ a scalar, $\hat{q} > 0$ and $\hat{d}$ vectors of the same size and $\hat{M}$ is sufficient. By Lemma 5.3, there exists no complementary solution for this sub-LCP with $\hat{\xi} > 0$ and $\hat{\sigma} > 0$. We conclude that the corresponding index of $\hat{\xi}$ and $\hat{\sigma}$ is in $\mathcal{T}$. □

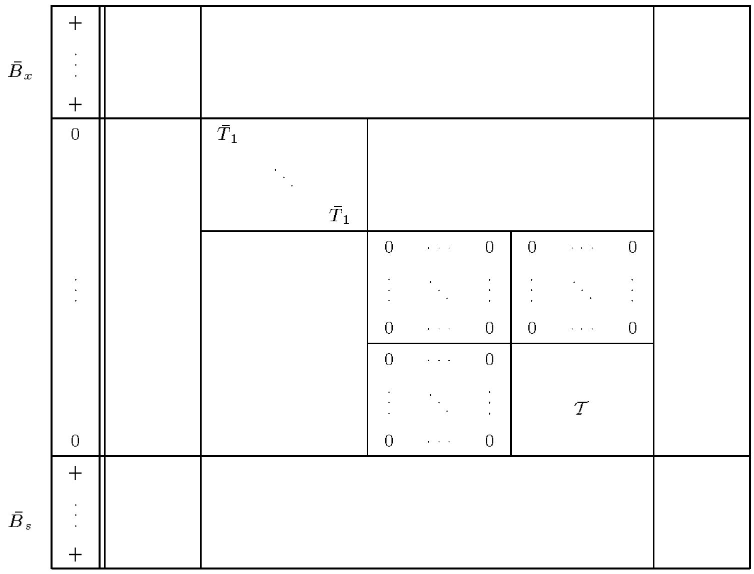

Table 2: A Balinski-Tucker tableau for LCPs.

Using the lemmas above we give the next definition, which is a generalization of the tableaus that Balinksi and Tucker [3] considered.

Definition 5.2 *A Balinski-Tucker tableau for LCPs is a complementary basis tableau where the sub-tableaus corresponding to the zero components of the complementary basis have structure $\bar{T}_1$ (corresponding to Lemma 5.2), a block of zero elements, and a block corresponding to all indices in $\mathcal{T}$. Table 2 is an illustration of a Balinski-Tucker tableau for LCPs.*

Theorem 5.1 *Given a complementary solution (x, s) for LCP(q, M), where M is sufficient, and a corresponding complementary tableau, we can construct a Balinski-Tucker tableau for LCPs in a finite number of steps.*

Proof:

The proof consists of repeatedly applying Lemma 5.1 to a sequence of sub-LCPs. The set of basic variables corresponding to x that are zero are denoted by B_x. Similarly we denote the set of zero basic variables corresponding to s by B_s. Rearrange columns of the tableau corresponding to (x, s) to get the following tableau.

				$+$ $\vdots$ $+$	
	$*$	$*$		0 $\vdots$ 0	B_x
	$*$	$*$		0 $\vdots$ 0	B_s
				$+$ $\vdots$ $+$	

The proof is constructive. We start by picking an arbitrary element of either B_x or B_s. Let us denote this element by j. After applying Lemma 5.1, one of the two cases distinguished in the Lemmas 5.2 and 5.4 occur. Since only principal pivoting is used in the criss-cross method [17], each tableau and subtableau remains sufficient in the subsequent steps. If the first case of Lemma 5.1 occurs with a negative diagonal element, then the corresponding pair belongs to $\mathcal{T}$. This pair is rearranged to the last pair of (B_x, B_s). The procedure is repeated for the remaining subproblem. In the other case, a subset of variables with a subtableau T_1 is identified. These elements (pairs) are removed from the subproblem and the procedure is repeated. Note that for the first case (the situation of Lemma 5.4) the sub-LCPs will not reduce in size. All columns corresponding to the indices in $\mathcal{T}$ should stay in the subproblem. However, choosing another driving column, which does not correspond to the indices in $\mathcal{T}$, the procedure can be continued. Therefore, the number of variables, whose status is still undecided, reduces. Note that principal rearrangements of sufficient matrices are sufficient (Proposition 2.3). Since only a finite number of pivots is needed to obtain the two distinguished cases, and the number of variables still undecided decreases by at least one at each step, constructing a Balinski-Tucker tableau for LCPs can be done in a finite number of steps. Repeatedly applying Lemma 5.1 gives a Balinski-Tucker tableau for LCPs. □

Remark 5.1 *Since we are free to choose the indices 'j' we can conclude that a Balinski-Tucker tableau (for LCPs) is not unique in general.*

Remark 5.2 *The cardinality of the set $\mathcal{T}$ is equal to the number of $\bar{T}_2$ blocks that occur during the process, or the size of the last block in a Balinski-Tucker tableau for LCPs.*

5.2 Tripartition Identification

In this section we will outline the optimal tripartition identification algorithm. This algorithm is mainly based on the constructive proof of Theorem 5.1 and that of the next theorem.

Theorem 5.2 *If a Balinski-Tucker tableau for LCPs is available a maximal complementary solution and its corresponding tripartition can be derived in strongly polynomial time.*

Proof:

The proof is constructive. We are only interested in the basic variables in the optimal basis which are zero, the other variables are strictly complementary. In order to obtain a maximal complementary solution, we need to improve the complementarity of the pair of variables that are zero. Let (x, s) denote a complementary solution of LCP(q, M), such that the variables for which the solution already satisfies strict complementarity correspond to the indices of the sets $\bar{B}_x$ and $\bar{B}_s$. If (x, s) is maximal complementary we are finished. Else we will show that complementarity can be improved. Subsequently applying Lemma 5.2, we conclude that complementarity of (x, s) improves for the $\bar{T}_1$ blocks. For $\bar{T}_1$ the scalar θ (see the proof of Lemma 5.2) needs to be chosen to assure that the positive part of (x, s) stays positive. For the zero block in the Balinski-Tucker tableaus for LCPs complementarity can be further improved. Let us denote this zero block by $\bar{T}_3$. We can perform an addition for each column of $t_{(j)}$ of $\bar{T}_3$. Let j denote the index of the corresponding column of the Balinski-Tucker tableau for LCPs. We subtract a small multiple of the column $t_{(j)}$ from the current complementary solution (x, s). By definition of $t_{(j)}$ it is clear that the complementarity of (x, s) improves by one. Applying this for all columns of $\bar{T}_3$ we obtain strict complementarity of the corresponding elements of (x, s). In this way, strict complementarity is reached for all the elements not in the identified $\mathcal{T}$ block. The complementarity cannot be improved, for the indices corresponding to the elements belonging to the set $\mathcal{T}$ (see the proof of Lemma 5.4). With the above procedure we have now obtained a maximal complementary solution. □

The TI-algorithm is outlined as follows

Algorithm 5.2 TRIPARTITION IDENTIFICATION ALGORITHM

1. Start with an optimal basis and its corresponding tableau.
2. We keep the basic solution fixed and construct a Balinski-Tucker tableau for LCPs by using the procedure described in the proof of Theorem 5.1.
3. From the basic solution we construct a maximal complementary solution by using the procedure described in the proof of Theorem 5.2

■

Note that the algorithm is applicable for LP, QP and LCPs with sufficient matrices. LP is a special case of the above procedure. For LP the situation of Lemma 5.4 in the proof of Theorem 5.2 cannot occur. All diagonal elements are zero for LP. Indeed we conclude from the algorithm that the set $\mathcal{T}$ is empty and strict complementarity is obtained (cf. Section 3.1).

The finiteness of Algorithm 5.2 is a corollary of Theorems 5.1 and 5.2.

Corollary 5.1 *The TI-algorithm is finite and finds an optimal tripartition.*

Proof:

Constructing a Balinski-Tucker tableau for LCPs can be done in a finite number of pivots. In constructing a maximal complementary solution only a finite number of additions is needed. We conclude that the TI-algorithm is finite. Since, the solution obtained after applying the procedure of Theorem 5.2 is maximal complementary the proof is completed. □

If we use pivoting to obtain a Balinski-Tucker tableau for LCPs, the TI-algorithm is not polynomial. However, finding a Balinski-Tucker tableau for LCPs consists of solving sub-LCPs. Using the BI-algorithm from Section 4 we can construct a polynomial Balinski-Tucker tableau identification algorithm, and hence another TI-algorithm. This is stated in the next theorem.

Theorem 5.3 *There exists a polynomial algorithm to identify a Balinski-Tucker tableau, and hence a polynomial TI-algorithm.*

Proof:

Finding a Balinski-Tucker tableau for LCPs consists of solving sub-QP problems or sub-LCPs. Using an interior point method (see, Anstreicher et al. [2] for QP or the algorithm Kojima et al. [26] for LCP) we can solve these subproblems in polynomial time. However, as we know in general we will obtain a maximal complementary solution and not a basic solution as is needed for constructing a Balinski-Tucker tableau for LCPs. Applying the BI-algorithm from Section 4 we can find basic solutions of the sub-QP problems and sub-LCPs. The BI-algorithm is strongly polynomial (see, Theorem 4.1), thus we can find a Balinski-Tucker tableau for LCPs in polynomial time. It remains to be shown that from a Balinski-Tucker tableau for LCPs a maximal complementary solution and its corresponding tripartition can be found in polynomial time. Applying Theorem 5.2 a maximal complementary solution and its corresponding tripartition can be found in strongly polynomial time. This completes the proof. □

6 Summary and Subjects of Further Research

In this paper we have presented generalizations of the results of Megiddo [32] and Balinski and Tucker [3] for LP to QP and LCP with sufficient matrices. First, we have described an algorithm to find a complementary basis starting from a complementary solution (for example the maximal complementary solution provided by an interior point algorithm). Second, we have provided a method to find a maximal complementary solution and its corresponding tripartition starting from a complementary basis. A subject of further research is to incorporate the presented algorithms in efficient implementations for combined interior point and pivot algorithms. We believe that the discussed algorithms provide a clearer insight in the difference and connection between optimal bases and optimal tripartitions.

Zhang's algorithms [48] are closely related to Megiddo's algorithm for optimal basis identification. Independently, Tijssen and Sierksma [41] obtained similar results. Using the BI-algorithm proposed in this paper and a Balinski-Tucker tableau for LCPs, it is interesting to examine if a similar result as Zhang's, holds for QP and LCPs with sufficient matrices. That is, to examine whether it is possible to find a Balinski-Tucker tableau for LCPs in strongly polynomial time, starting from an optimal tripartition and its corresponding maximal complementary solution for QP and LCPs.

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