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DUALITY AND SELF-DUALITY FOR CONIC CONVEX PROGRAMMING[1]

Zhi-Quan Luo[2], Jos F. Sturm[3] and Shuzhong Zhang[3]

ABSTRACT

This paper considers the problem of minimizing a linear function over the intersection of an affine space with a closed convex cone. In the first half of the paper, we give a detailed study of duality properties of this problem and present examples to illustrate these properties. In particular, we introduce the notions of weak/strong feasibility or infeasibility for a general primal-dual pair of conic convex programs, and then establish various relations between these notions and the duality properties of the problem. In the second half of the paper, we propose a self-dual embedding with the following properties: Any weakly centered sequence converging to a complementary pair either induces a sequence converging to a certificate of strong infeasibility, or induces a sequence of primal-dual pairs for which the amount of constraint violation converges to zero, and the corresponding objective values are in the limit not worse than the optimal objective value(s). In case of strong duality, these objective values in fact converge to the optimal value of the original problem. When the problem is neither strongly infeasible nor endowed with a complementary pair, we completely specify the asymptotic behavior of an indicator in relation to the status of the original problem, namely whether the problem (1) is weakly infeasible, (2) is feasible but with a positive duality gap, (3) has no duality gap nor complementary solution pair.

KEY WORDS. conic convex programming, semidefinite programming, duality, self-duality, interior point method.

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[2] Room 225/CRL, Department of Electrical and Computer Engineering, McMaster University, Hamilton, Ontario, L8S 4L7, Canada.

[3] Econometric Institute, Erasmus University Rotterdam, The Netherlands.

1 Introduction

A conic convex program is an optimization problem for which the objective is linear and the constraint set is given by the intersection of an affine space with a closed convex cone. As such, it contains as special cases the linear programming problem, the quadratic programming problem, and most notably, the semidefinite programming problem. Lately, the latter problem has been the focus of many studies due to its application to combinatorial optimizations and due to the applicability of interior point methods to solve this class of problems, see e.g. [8, 9, 10, 11, 12, 13, 16].

A common assumption used in the interior point methods for conic convex programming is that a feasible interior starting point exists and is known. This requirement can be partially alleviated when an infeasible interior point method is used, as in Kojima-Shindoh-Hara [8] and Lin-Saigal [9], or when a self-dual embedding is used as in Potra-Sheng [13]. Nonetheless, these references only consider the complementarity version of the problem, which in general is not equivalent to the original conic convex optimization problem unless the Slater condition holds. This is unfortunate since in many applications the Slater condition may fail to hold, and in such circumstances it is usually not sufficient to merely conclude whether or not a complementary solution exists. Instead, we must determine either that the problem is infeasible/unbounded, or else generate a sequence whose objective values approach the infimum of the original conic convex program. Clearly, this latter task requires a careful study of duality theory for conic convex programming in the absence of constraint qualifications.

There have been several studies of duality theory for conic convex programming, most of which assume a constraint qualification (in particular Slater's condition), see Duffin [3], Nesterov and Nemirovsky [11] and Alizadeh [1]. Wolkowicz [20] and Ramana et al. [14] manage to dispense with any constraint qualification by working with a nonstandard primal-dual pair. It is currently not known though, whether the interior point methods can cope with this nonstandard duality. The difficulty with the study of duality theory for conic convex programming lies in the nonlinearity of the conic constraint. Without any constraint qualification, the problem can exhibit many new duality features which are not present in linear programming. The same is true with the theory of self-duality. The latter theory forms the basis of a homogeneous self-dual model for conic convex programming. Such a model is particularly useful when interior point methods are used to solve the conic convex programming problem.

The contributions of this paper are three fold. First, we give a detailed study of duality theory for conic convex programming in the absence of constraint qualifications. Our study is based on generalizations of Farkas' lemma which provides characterizations for strong and weak infeasibility. Second, we present a self-dual embedding of the conic convex programming problem that is well suited for interior-point solution methods. In particular, we thoroughly analyze the limiting behavior of a weakly centered sequence for the self-dual embedding. Such a sequence is generated by most interior point methods. It is shown that a weakly centered sequence can provide information regarding the status of the original conic convex programming problem, namely whether the problem

- has a complementary pair, or
- is strongly infeasible, or
- is weakly infeasible, or
- is feasible but with a positive duality gap, or
- has no duality gap nor complementary solution pair.

Third, we present many interesting examples from semidefinite programming which reveal some delicate features of duality and self-duality theory for the general conic convex programming problem.

This paper is organized as follows. In Section 2, we formally state the conic convex programming problem and the relevant terminology. Section 3 is devoted to the study of duality relations for the standard conic primal-dual pair in the absence of constraint qualifications. In Section 4 we start with a generalization of the Goldman-Tucker type self-dual feasibility model. We then define an extended self-dual programming problem, and analyze the limiting behavior of so-called weakly centered solution sequences for this extended self-dual programming problem. These results are illustrated by some examples from semidefinite programming. Finally, we conclude the paper in Section 5.

Notation. Given a set $\mathcal{S}$, we let $\mathrm{cl}\,(\mathcal{S})$, $\mathrm{int}\,\mathcal{S}$ and $\mathrm{rel}\,\mathcal{S}$ denote the closure of $\mathcal{S}$, the interior of $\mathcal{S}$ and the relative interior of $\mathcal{S}$ respectively. If $\mathcal{S}$ is a subset of $\Re^n$ and A is an $m \times n$ matrix, then the image of $\mathcal{S}$ under the linear mapping A is denoted by $A\mathcal{S}$, i.e.

$$A\mathcal{S} = \{y \in \Re^m \mid y = Ax, \quad \text{for some } x \in \mathcal{S}\}.$$

2 Terminologies and Preliminaries

In this paper, we consider the general conic convex programming problem

$$\begin{array}{lll} (P) & \inf & c^{\mathrm{T}} x \\ & \text{s.t.} & x - c \in \mathcal{A} \\ & & x \in \mathcal{K}, \end{array}$$

where $c \in \Re^n$, $\mathcal{A}$ is a linear subspace of $\Re^n$ and $\mathcal{K}$ is a closed convex cone in $\Re^n$. Let $\mathcal{K}^*$ denote the dual cone of $\mathcal{K}$, i.e.

$$\mathcal{K}^* := \{s \in \Re^n \mid s^{\mathrm{T}} \mathcal{K} \subseteq \Re_+\}.$$

The cone $-\mathcal{K}^*$ is also known as the polar cone of $\mathcal{K}$ [15]. Denote the orthogonal complement of $\mathcal{A}$ (i.e. the dual cone of $\mathcal{A}$) by $\mathcal{A}^\perp$. It is well known that $(\mathcal{K}^*)^* = \mathcal{K}$ and $(\mathcal{A}^\perp)^\perp = \mathcal{A}$, because $\mathcal{A}$ and $\mathcal{K}$ are closed [15].

A seemingly more general form of conic convex programming is

$$\begin{array}{lll} (P') & \inf & f^{\mathrm{T}} x \\ & \text{s.t.} & x - g \in \mathcal{A} \\ & & x \in \mathcal{K}, \end{array}$$

where $f, g \in \Re^n$. However, if we denote the orthogonal projection matrices onto $\mathcal{A}$ and $\mathcal{A}^\perp$ by $P_{\mathcal{A}}$ and $P_{\mathcal{A}^\perp}$ respectively, it follows that (P') is equivalent to (P) with

$$c = P_{\mathcal{A}} f + P_{\mathcal{A}^\perp} g.$$

Associated with (P) is a dual problem,

$$\begin{array}{lll} (D) & \inf & c^{\mathrm{T}} s \\ & \text{s.t.} & s - c \in \mathcal{A}^\perp \\ & & s \in \mathcal{K}^*. \end{array}$$

We see that this dual is also a conic convex programming problem, and that dualizing the dual problem (D) yields the primal problem (P). As a side remark, we mention that the dual of (P') is

$$\begin{array}{lll} (D') & \inf & g^{\mathrm{T}} s \\ & \text{s.t.} & s - f \in \mathcal{A}^\perp \\ & & s \in \mathcal{K}^*, \end{array}$$

which is clearly equivalent to (D) for arbitrary $f \in c + \mathcal{A}^\perp$ and $g \in c + \mathcal{A}$.

Denote the feasible set of (P) by

$$\mathcal{F}_P := \mathcal{K} \cap (c + \mathcal{A})$$

and the feasible set of (D) by

$$\mathcal{F}_D := \mathcal{K}^* \cap (c + \mathcal{A}^\perp).$$

Let

$$p^* := \inf c^{\mathrm{T}} \mathcal{F}_P, \quad d^* := \inf c^{\mathrm{T}} \mathcal{F}_D$$

denote the optimal solution values of (P) and (D) respectively.

We denote the primal and dual optimal solution sets by

$$\mathcal{F}_P^* := \{x \in \mathcal{F}_P \mid c^{\mathrm{T}} x = p^*\}, \quad \mathcal{F}_D^* := \{s \in \mathcal{F}_D \mid c^{\mathrm{T}} s = d^*\}.$$

Definition 1 *Problem (P) is said to be* solvable *if*

$$\mathcal{F}_P^* \neq \emptyset.$$

A special case of nonsolvability occurs when $p^* = -\infty$. In this case, we say that (P) is *unbounded*.

We say that (P) is *feasible* if the set of feasible solutions $\mathcal{F}_P$ is nonempty. The interior solution sets for (P) and (D) are

$$\overset{o}{\mathcal{F}}_P = \mathcal{F}_P \cap \operatorname{int} \mathcal{K}$$

and

$$\overset{o}{\mathcal{F}}_D = \mathcal{F}_D \cap \operatorname{int} \mathcal{K}^*$$

respectively. The term *strong feasibility* denotes the case that (P) has so-called interior feasible points. Strong feasibility is also known as Slater's constraint qualification.

Definition 2 *Problem (P) is said to be* strongly feasible *if*

$$\overset{o}{\mathcal{F}}_P \neq \emptyset.$$

Definition 3 *Problem (P) is said to be* weakly feasible *if*

$$\mathcal{F}_P \neq \emptyset \textit{ but } \overset{o}{\mathcal{F}}_P = \emptyset.$$

If $x \in \mathcal{F}_P$ and $s \in \mathcal{F}_D$, then

$$x^{\mathrm{T}} s = c^{\mathrm{T}} x + c^{\mathrm{T}} s - \|c\|^2 . \tag{1}$$

This relation immediately implies the following weak duality result.

Lemma 1 (Weak duality) *If $\mathcal{F}_P \neq \emptyset$ and $\mathcal{F}_D \neq \emptyset$ then*

$$p^* + d^* \geq \|c\|^2 .$$

We remark that for $f \in c + \mathcal{A}^\perp$, $g \in c + \mathcal{A}$ and $f \perp g$ there holds

$$x^{\mathrm{T}} s = f^{\mathrm{T}} x + g^{\mathrm{T}} s. \tag{2}$$

Definition 4 *The primal-dual pair (P), (D) is said to be* strongly dual *if*

$$p^* + d^* = \|c\|^2 .$$

Definition 5 *The primal-dual pair (P), (D) is said to be* weakly dual *if*

$$p^* + d^* > \|c\|^2 .$$

We introduce a set of ϵ-feasible solutions as follows:

$$\mathcal{F}_P(\epsilon) := \{x \in \mathcal{K} \mid \|P_{\mathcal{A}^\perp}(x - c)\| \le \epsilon\}.$$

Similarly, we define

$$\mathcal{F}_D(\epsilon) := \{s \in \mathcal{K}^* \mid \|P_{\mathcal{A}}(s - c)\| \le \epsilon\}.$$

Clearly, we have $\mathcal{F}_P = \mathcal{F}_P(0) \subseteq \mathcal{F}_P(\epsilon)$ and $\mathcal{F}_D = \mathcal{F}_D(0) \subseteq \mathcal{F}_D(\epsilon)$, for all $\epsilon \ge 0$. Notice that $\mathcal{F}_P(\epsilon) \neq \emptyset$ for all $\epsilon > 0$ if and only if the distance between $\mathcal{A} + c$ and $\mathcal{K}$ is zero. Unlike the (finite) linear programming case, it is possible that $\mathcal{F}_P(\epsilon) \neq \emptyset$, for all $\epsilon > 0$ but $\mathcal{F}_P = \{\mathcal{A} + c\} \cap \mathcal{K} = \emptyset$. In this case, we say that (P) is *weakly* infeasible. We will provide some examples of weakly infeasible problems in Section 3.3 and Section 4.3.

Definition 6 *Problem (P) is said to be* weakly *infeasible if*

$$\mathcal{F}_P = \emptyset, \quad \textit{but} \quad \mathcal{F}_P(\epsilon) \neq \emptyset \ \ \forall \epsilon > 0.$$

Definition 7 *Problem (P) is said to be* strongly *infeasible if*

$$\mathcal{F}_P(\epsilon) = \emptyset, \qquad \textit{for some } \epsilon > 0.$$

Clearly, (P) is not strongly infeasible is equivalent to $\mathrm{dist}(\mathcal{A} + c, \mathcal{K}) = 0$. The notions of weak and strong infeasibility of dual problem (D) can be defined symmetrically. The following lemma characterizes strong infeasibility for (P).

Lemma 2 *Problem (P) is* not *strongly infeasible, i.e.*

$$\mathcal{F}_P(\epsilon) \neq \emptyset \quad \forall \epsilon > 0$$

if and only if

$$P_{\mathcal{A}^\perp} c \in \mathrm{cl}(P_{\mathcal{A}^\perp} \mathcal{K}).$$

Proof:

We have $\mathcal{F}_P(\epsilon) \neq \emptyset \, \forall \epsilon > 0$ if and only if there exists a sequence $x^{(1)}, x^{(2)}, \ldots$ in $\mathcal{K}$ such that

$$\lim_{i \to \infty} \left\| P_{\mathcal{A}^\perp}(x^{(i)} - c) \right\| = 0,$$

which is equivalent to

$$P_{\mathcal{A}^\perp} c \in \mathrm{cl}\,(P_{\mathcal{A}^\perp} \mathcal{K}).$$

□

In the remainder of this section, we collect some useful facts that will be used later in the development of duality and self-duality theory.

Below is a well-known elementary property of a closed convex cone $\mathcal{K}$.

Lemma 3 int $\mathcal{K}^* \neq \emptyset$ *if and only if* $\mathcal{K}$ *is pointed, i.e.* $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$. *Moreover, if* $s^0 \in \operatorname{int} \mathcal{K}^*$ *and* $x \in \mathcal{K}$ *then*

$$(s^0)^T x = 0 \quad \Rightarrow \quad x = 0. \tag{3}$$

The next lemma will be used in the study of self-duality in Section 4.

Lemma 4 *If* $\mathcal{K}$ *is a pointed closed convex cone with nonempty interior in the sense that*

$$\mathcal{K} \cap (-\mathcal{K}) = \{0\}, \quad \operatorname{int} \mathcal{K} \neq \emptyset,$$

then $\mathcal{K} \cap \mathcal{K}^*$ *is also a pointed closed convex cone with nonempty interior.*

Proof:

That the cone $\mathcal{K} \cap \mathcal{K}^*$ is pointed, closed and convex follows immediately from the fact that $\mathcal{K}$ and $\mathcal{K}^*$ are pointed closed convex cones. To show that int $(\mathcal{K} \cap \mathcal{K}^*) \neq \emptyset$, we will use a result of Rockafellar [15] (Corollary 16.4.2 therein) which states that if $\mathcal{K}_1$ and $\mathcal{K}_2$ are nonempty convex cones, then

$$\mathcal{K}_1^* \cap \mathcal{K}_2^* = (\mathcal{K}_1 + \mathcal{K}_2)^*.$$

Applying this result with $\mathcal{K}_1 = \mathcal{K}$ and $\mathcal{K}_2 = \mathcal{K}^*$, we have

$$\mathcal{K} \cap \mathcal{K}^* = (\mathcal{K} + \mathcal{K}^*)^*.$$

Using Lemma 3, it remains only to show that $\mathcal{K} + \mathcal{K}^*$ is pointed. Suppose it is not, then there exist $f \in \mathcal{K}$ and $g \in \mathcal{K}^*$ such that

$$-(f+g) \in \mathcal{K} + \mathcal{K}^*, \quad f + g \neq 0.$$

Assume without loss of generality that $f \neq 0$. The above relation implies that $-f = g - (f+g) \in \mathcal{K} + \mathcal{K}^*$ and hence

$$-f - f' \in \mathcal{K}^* \qquad \text{for some } f' \in \mathcal{K}.$$

Since $\mathcal{K}$ is pointed, we have $f' \neq -f$ so that

$$(f + f')^{\mathrm{T}}(-f - f') < 0, \;\; f + f' \in \mathcal{K}, \;\; -f - f' \in \mathcal{K}^*$$

which contradicts the definition of a dual cone.

$\square$

Finally, we will need the following lemma to calculate the orthogonal complement of a linear subspace.

Lemma 5 *Suppose* $\mathcal{A}$ *is a linear subspace in* $\Re^n$. *Let* $M \in \Re^{n \times n}$ *be an invertible matrix. Then*

$$(M\mathcal{A})^\perp = M^{-T} \mathcal{A}^\perp,$$

where $M^{-T} := (M^T)^{-1}$.

Proof:

Let $y \in \Re^n$ be any fixed vector. We note the following relations:

$$\begin{aligned} y \in (M\mathcal{A})^\perp &\iff y^{\mathrm{T}}(Mx) = 0, \quad \forall x \in \mathcal{A} \\ &\iff (M^{\mathrm{T}}y)^{\mathrm{T}}x = 0, \quad \forall x \in \mathcal{A} \\ &\iff M^{\mathrm{T}}y \in \mathcal{A}^\perp \\ &\iff y \in M^{-\mathrm{T}}\mathcal{A}^\perp. \end{aligned}$$

The proof is complete.

□

3 Relations of the Conic Primal-Dual Pair

Semidefinite and conic convex programming duality has been studied by Duffin [3], Wolkowicz [20], Ramana et al. [14], Nesterov and Nemirovsky [11] and Alizadeh [1], among others. A Slater condition or similar constraint qualification is usually assumed in order to arrive at duality results that show great similarity with the linear programming duality [3, 11, 1]. However, in [20] it is shown that constraint qualifications can be circumvented by requiring $\mathcal{K}$ to be a minimal cone for $c + \mathcal{A}$, i.e.

$$c + \mathcal{A} \cap \mathrm{rel}(\mathcal{K}) \neq \emptyset.$$

Such a minimal cone formulation always exists, but it is often not possible to compute the minimal cone for (P) in practice. The approach of [14] does not explicitly use minimal cones, but makes use of an extended Lagrange-Slater dual program. However, we are not aware of any solution method that uses the extended dual rather than the standard dual (D). Duality relations for the standard pair (P), (D) are studied in [3, 11]. In this section, we will derive the duality results by means of a generalized Farkas' lemma. The generalized Farkas' lemma will play a significant role in the analysis of self-dual embeddings in Section 4.

3.1 Farkas-type lemmas

The following lemma is a generalization of Farkas' lemma to conic convex programming. In previous generalizations of Farkas' lemma, it is usually assumed that $P_{\mathcal{A}^\perp}\mathcal{K}$ is closed, see Hurwicz [6], Ben-Israel [2] and Alizadeh [1]. Alternatively, Wolkowicz [20] extended Farkas' lemma under the assumption that $\mathcal{K}^*$ is a minimal cone for $\mathcal{A}^\perp$, i.e. $\mathcal{A}^\perp \cap \mathrm{rel}\mathcal{K}^* \neq \emptyset$. The formulation below however, does not have any requirements on (P). This fact will become essential in the remainder of this paper.

Lemma 6 (First Farkas' lemma) *$\mathcal{F}_P$ is not strongly infeasible in the sense that*

$$\mathcal{F}_P(\epsilon) \neq \emptyset \qquad \forall \epsilon > 0$$

if and only if

$$c^T(\mathcal{A}^\perp \cap \mathcal{K}^*) \subseteq \Re_+.$$

Proof:

It is easy to see that a linearly transformed convex cone is also a convex cone. Therefore, we can apply the bi-polar theorem (see Rockafellar [15], Theorem 14.1), which states that

$$\mathrm{cl}\,(P_{\mathcal{A}^\perp}\mathcal{K}) = (P_{\mathcal{A}^\perp}\mathcal{K})^{**}. \tag{4}$$

By Lemma 2 and the relation (4) we have

$$\begin{aligned} \mathcal{F}_P(\epsilon) \neq \emptyset \quad \forall \epsilon > 0 \iff \; & P_{\mathcal{A}^\perp} c \in \mathrm{cl}\,(P_{\mathcal{A}^\perp}\mathcal{K}) \\ \iff \; & c^{\mathrm{T}} P_{\mathcal{A}^\perp}(P_{\mathcal{A}^\perp}\mathcal{K})^* \subseteq \Re_+, \end{aligned} \tag{5}$$

where

$$\begin{aligned} (P_{\mathcal{A}^\perp}\mathcal{K})^* &= \{\sigma \in \Re^n \mid \sigma^{\mathrm{T}} P_{\mathcal{A}^\perp}\mathcal{K} \subseteq \Re_+\} \\ &= \{\sigma \in \Re^n \mid P_{\mathcal{A}^\perp}\sigma \in \mathcal{K}^*\} \\ &= (\mathcal{K}^* \cap \mathcal{A}^\perp) + \mathcal{A}. \end{aligned}$$

Combining the above relation with (5) we obtain

$$\mathcal{F}_P(\epsilon) \neq \emptyset \quad \forall \epsilon > 0 \iff c^{\mathrm{T}}(\mathcal{K}^* \cap \mathcal{A}^\perp) \subseteq \Re_+.$$

□

We let $\mathcal{D}_D$ be the set of dual improving directions, viz.

$$\mathcal{D}_D := \{s \in \mathcal{K}^* \cap \mathcal{A}^\perp \mid c^{\mathrm{T}} s \leq -1\}. \tag{6}$$

We know from Lemma 6 that (P) is strongly infeasible if and only if $\mathcal{D}_D \neq \emptyset$. We define an ϵ-relaxed set by

$$\mathcal{D}_D(\epsilon) := \{s \in \mathcal{K}^* \mid \|P_{\mathcal{A}} s\| \leq \epsilon,\ c^{\mathrm{T}} s \leq -1 + \epsilon\}. \tag{7}$$

Next we proceed to derive a characterization of primal infeasibility in terms of the set $\mathcal{D}_D(\epsilon)$. To this end, we need to homogenize the region $\mathcal{F}_P$ to form the cone

$$\left\{(x, x_0) \;\middle|\; \frac{x}{x_0} \in \mathcal{F}_P,\ x_0 > 0\right\}.$$

This cone is contained in the linear subspace $\mathcal{B}$,

$$\mathcal{B} := \begin{bmatrix} I & c \\ 0 & 1 \end{bmatrix} (\mathcal{A} \times \Re). \tag{8}$$

In particular, we have

$$x \in \mathcal{F}_P \iff (x_0 x^{\mathrm{T}}, x_0)^{\mathrm{T}} \in \mathcal{B} \cap (\mathcal{K} \times \Re_{++}).$$

This implies that

$$\begin{aligned} \mathcal{F}_P = \emptyset &\iff \mathcal{B} \cap (\mathcal{K} \times \Re_+) = (\mathcal{A} \cap \mathcal{K}) \times \{0\} \\ &\iff \begin{bmatrix} 0 & -1 \end{bmatrix} (\mathcal{B} \cap (\mathcal{K} \times \Re_+)) = \{0\} \\ &\iff \begin{bmatrix} 0 & -1 \end{bmatrix} (\mathcal{B} \cap (\mathcal{K} \times \Re_+)) \subseteq \Re_+. \end{aligned} \tag{9}$$

By Lemma 5, there holds

$$\mathcal{B}^\perp = \begin{bmatrix} I & 0 \\ -c^{\mathrm{T}} & 1 \end{bmatrix} (\mathcal{A}^\perp \times \{0\}) = \begin{bmatrix} I \\ -c^{\mathrm{T}} \end{bmatrix} \mathcal{A}^\perp. \tag{10}$$

By Lemma 6 and the relation (9), we obtain the second version of Farkas' lemma which provides a characterization of the primal infeasibility $\mathcal{F}_P = \emptyset$.

Lemma 7 (Second Farkas' lemma) *We have*

$$\mathcal{F}_P = \emptyset$$

if and only if

$$\mathcal{D}_D(\epsilon) \neq \emptyset, \qquad \forall \epsilon > 0.$$

Proof:

By (9) we have

$$\mathcal{F}_P = \emptyset \iff \begin{bmatrix} 0 & -1 \end{bmatrix} (\mathcal{B} \cap (\mathcal{K} \times \Re_+)) \subseteq \Re_+.$$

In light of Lemma 6, this implies that $\mathcal{F}_P = \emptyset$ if and only if the distance from $\mathcal{B}^\perp + (0, -1)^{\mathrm{T}}$ to $(\mathcal{K} \times \Re_+)^*$ is zero. Using the definition of dual cone, we can verify that $(\mathcal{K} \times \Re_+)^* = \mathcal{K}^* \times \Re_+$. Thus, $\mathcal{F}_P = \emptyset$ is equivalent to

$$\left\{ \begin{pmatrix} s \\ s_0 \end{pmatrix} \in \mathcal{K}^* \times \Re_+ \,\middle|\, \left\| P_{\mathcal{B}} \left(\begin{pmatrix} s \\ s_0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \right\| \leq \epsilon \right\} \neq \emptyset, \qquad \forall \epsilon > 0.$$

Using the expression (10) for $\mathcal{B}^\perp$, we conclude that $\mathcal{F}_P = \emptyset$ is equivalent to

$$\left\{ \begin{pmatrix} s \\ s_0 \end{pmatrix} \in \mathcal{K}^* \times \Re_+ \,\middle|\, \|P_{\mathcal{A}}(s)\| \leq \epsilon,\ |c^{\mathrm{T}} s + s_0 + 1| \leq \epsilon \right\} \neq \emptyset, \qquad \forall \epsilon > 0.$$

By the definition (7) of $\mathcal{D}_D(\epsilon)$ and a rescaling of variable s if necessary, we see that the above condition is further equivalent to

$$\mathcal{D}_D(\epsilon) \neq \emptyset, \qquad \forall \epsilon > 0.$$

This completes the proof.

□

We summarize the results of Lemma 6 and Lemma 7 in the following theorem.

Theorem 1 (Characterization of feasibility and weak/strong infeasibility)
Let $\mathcal{F}_P = \mathcal{K} \cap (c + \mathcal{A})$ and let $\mathcal{D}_D$ be given by (6). Then

- *$\mathcal{F}_P$ is strongly infeasible if and only if $\mathcal{D}_D$ is feasible.*
- *$\mathcal{F}_P$ is weakly infeasible if and only if $\mathcal{D}_D$ is weakly infeasible.*
- *$\mathcal{F}_P$ is feasible if and only if $\mathcal{D}$ is strongly infeasible.*

The proof of Theorem 1 follows immediately from Lemma 6 and Lemma 7. Theorem 1 will be used to derive duality theorems in the next subsection.

3.2 Duality theorems

It is well known that if (P) is a linear programming problem and p^* is finite, then strong duality holds, i.e. $p^* + d^* - \|c\|^2 = 0$. Our objective is to generalize the strong duality result of linear programming to conic convex programming. Notice that, for general conic convex programming, it is possible that p^* is finite but (D) is weakly infeasible (see e.g. Example 8 in Section 4.3). This means that we should allow an arbitrarily small constraint violation for the dual and define

$$d^*(\epsilon) := \inf c^{\mathrm{T}} \mathcal{F}_D(\epsilon).$$

Before stating our strong duality result, we derive the following weak duality.

Lemma 8 (Weak duality) *If $-\infty < p^* < \infty$ then*

$$p^* + \lim_{\epsilon \downarrow 0} d^*(\epsilon) - \|c\|^2 \geq 0.$$

Proof:

Let $s(\epsilon) \in \mathcal{F}_D(\epsilon)$, $\epsilon > 0$, be a sequence with

$$\lim_{\epsilon \downarrow 0} c^{\mathrm{T}} s(\epsilon) = \lim_{\epsilon \downarrow 0} d^*(\epsilon).$$

Define

$$u(\epsilon) := P_{\mathcal{A}}(s(\epsilon) - c).$$

By definition of $\mathcal{F}_D(\epsilon)$, we have $\|u(\epsilon)\| \leq \epsilon$. By definition of p^*, we know that for any $\epsilon' > 0$, there exists $x(\epsilon') \in \mathcal{F}_P$ with

$$c^{\mathrm{T}} x(\epsilon') < p^* + \epsilon'. \tag{11}$$

As $x(\epsilon') \in \mathcal{K}$ and $s(\epsilon) \in \mathcal{K}^*$, we have

$$0 \leq x(\epsilon')^{\mathrm{T}} s(\epsilon) = c^{\mathrm{T}} s(\epsilon) + (x(\epsilon') - c)^{\mathrm{T}} s(\epsilon).$$

Using the fact that $(x(\epsilon') - c) \in \mathcal{A}$ and $s(\epsilon) \in c + u(\epsilon) + \mathcal{A}^{\perp}$, we further obtain

$$(c + u(\epsilon))^{\mathrm{T}} x(\epsilon') \geq c^{\mathrm{T}}(c + u(\epsilon)) - c^{\mathrm{T}} s(\epsilon).$$

If we let $\epsilon \downarrow 0$, we see that

$$\begin{aligned} c^{\mathrm{T}} x(\epsilon') &= \lim_{\epsilon \downarrow 0} (c + u(\epsilon))^{\mathrm{T}} x(\epsilon') \\ &\geq \lim_{\epsilon \downarrow 0} (c^{\mathrm{T}}(c + u(\epsilon)) - c^{\mathrm{T}} s(\epsilon)) \\ &= \|c\|^2 - \lim_{\epsilon \downarrow 0} d^*(\epsilon). \end{aligned}$$

Combining this with (11), the lemma follows.

□

Based on the extended Farkas' lemma and the above weak duality result, we obtain a strong duality theorem:

Theorem 2 *There holds*

- *If $p^* = \infty$, then either (D) is strongly infeasible and*

$$\lim_{\epsilon \downarrow 0} d^*(\epsilon) = \infty,$$

  or (D) is not strongly infeasible and

$$\lim_{\epsilon \downarrow 0} d^*(\epsilon) = -\infty.$$

- *If $-\infty < p^* < \infty$ then*

$$p^* + \lim_{\epsilon \downarrow 0} d^*(\epsilon) - \|c\|^2 = 0.$$

- *If $p^* = -\infty$, then (D) is infeasible and*

$$\lim_{\epsilon \downarrow 0} d^*(\epsilon) = \infty.$$

Proof:

It is obvious that if (D) is strongly infeasible, then $\lim_{\epsilon\downarrow 0} d^*(\epsilon) = \infty$. Now consider the case that (D) is not strongly infeasible. We obtain from Lemma 6 that

$$c^{\mathrm{T}}(\mathcal{A} \cap \mathcal{K}) \subseteq \Re_+. \tag{12}$$

If $p^* = \infty$, then (P) is infeasible. By the definition (8) of $\mathcal{B}$, this is equivalent with

$$(\mathcal{B} \cap (\mathcal{K} \times \Re_+)) = (\mathcal{A} \cap \mathcal{K}) \times \{0\}. \tag{13}$$

Combining (12) and (13), it follows that

$$\begin{bmatrix} c^{\mathrm{T}} & \gamma \end{bmatrix} (\mathcal{B} \cap (\mathcal{K} \times \Re_+)) = c^{\mathrm{T}}(\mathcal{A} \cap \mathcal{K}) \subseteq \Re_+$$

for any $\gamma \in \Re$. Applying Lemma 6 to the above relation, and using (10), we obtain that for any $\epsilon > 0$ and $\epsilon' > 0$ there exists $s \in \mathcal{F}_D(\epsilon)$ and $s_0 \in \Re_+$ with

$$s_0 - \gamma + c^{\mathrm{T}}(s - c) \leq \epsilon'.$$

Since $s_0 \geq 0$ and $\gamma \in \Re$ is arbitrary, we have

$$d^*(\epsilon) = -\infty, \qquad \text{for all } \epsilon > 0.$$

This concludes the case $p^* = \infty$.

Now consider the case $p^* = -\infty$. Let $x \in \mathcal{F}_P$ be an arbitrary vector, and let $s(\epsilon) \in \mathcal{F}_D(\epsilon)$ be a sequence such that

$$c^{\mathrm{T}} s(\epsilon) \leq d^*(\epsilon) + \epsilon, \quad \text{for all } \epsilon > 0.$$

By the definition of $\mathcal{F}_D(\epsilon)$, there exists some $u(\epsilon)$ with $\|u(\epsilon)\| \leq \epsilon$ such that $s(\epsilon) - c + u(\epsilon) \in \mathcal{A}^\perp$ and $s(\epsilon) \in \mathcal{K}^*$. Since $(x - c) \in \mathcal{A}$ and $x \in \mathcal{K}$, we obtain

$$0 \leq x^{\mathrm{T}} s(\epsilon) = c^{\mathrm{T}} x + c^{\mathrm{T}} s(\epsilon) - (x - c)^{\mathrm{T}} u(\epsilon) - \|c\|^2.$$

Letting $\epsilon \downarrow 0$ in the above relation yields

$$0 \leq c^{\mathrm{T}} x + \lim_{\epsilon \downarrow 0} d^*(\epsilon) - \|c\|^2.$$

Since $p^* = -\infty$, we can choose $x \in \mathcal{F}_P$ such that $c^{\mathrm{T}} x \to -\infty$. This implies $\lim_{\epsilon\downarrow 0} d^*(\epsilon) = \infty$. Now the infeasibility of (D) follows immediately.

It only remains to consider the case $-\infty < p^* < \infty$. Since in this case (P) is feasible, but not unbounded ($p^* > -\infty$), we know that there cannot exist a primal improving direction, i.e.

$$c^{\mathrm{T}}(\mathcal{A} \cap \mathcal{K}) \subseteq \Re_+, \tag{14}$$

and as $p^* := \inf c^{\mathrm{T}} \mathcal{F}_P$ we have

$$c^{\mathrm{T}} \mathcal{F}_P \subseteq p^* + \Re_+. \tag{15}$$

Combining (14) and (15) yields

$$c^{\mathrm{T}}x \geq x_0 p^*$$

for all $(x, x_0) \in \mathcal{K} \times \Re_+$ with $(x - x_0 c) \in \mathcal{A}$. Stated differently, we have

$$\left[\begin{array}{cc} c^{\mathrm{T}} & -p^* \end{array}\right] (\mathcal{B} \cap (\mathcal{K} \times \Re_+)) \subseteq \Re_+.$$

Applying Lemma 6 and (10), we conclude that for any $\epsilon > 0$ and $\epsilon' > 0$ there exists $s \in \mathcal{F}_D(\epsilon)$ and $s_0 \in \Re_+$ with

$$s_0 + p^* + c^{\mathrm{T}}(s - c) \leq \epsilon'.$$

This shows that

$$p^* + d^*(\epsilon) - \|c\|^2 \leq 0, \qquad \text{for all } \epsilon > 0.$$

Combining this with Lemma 8 proves the theorem.

□

For the case that p^* is finite, we see from the above theorem that $p^* + d^* - \|c\|^2 = 0$ if $d^*(\epsilon)$ is lower semicontinuous. The examples in Section 4.3 show that this is in general not the case. Our next goal is to derive a sufficient condition for $d^*(\epsilon)$ to be lower semicontinuous. We need several lemmas.

Lemma 9 *Let $s^{(0)} \in \operatorname{int} \mathcal{K}^*$ and $\gamma \in \Re_+$ then*

$$\mathcal{K} \cap \{x \in \Re^n \mid (s^{(0)})^T x \leq \gamma\}$$

is compact.

Proof:

Suppose to the contrary that there exists a sequence

$$x^{(i)} \in \mathcal{K} \cap \{x \in \Re^n \mid (s^{(0)})^{\mathrm{T}} x \leq \gamma\}, \quad i = 1, 2, \ldots$$

with $\lim_{i \to \infty} \left\| x^{(i)} \right\| = \infty$. Let y be a cluster point of the bounded sequence

$$\left\{ \frac{x^{(i)}}{\|x^{(i)}\|}, \quad i = 1, 2, \ldots \right\}.$$

By construction, $\|y\| = 1$ and we have $y \in \mathcal{K}$ due to the closedness of $\mathcal{K}$. Moreover, there holds

$$(s^{(0)})^{\mathrm{T}} y \leq \lim_{i \to \infty} \frac{\gamma}{\|x^{(i)}\|} = 0.$$

But this contradicts the fact that $y \in \mathcal{K}, \ y \neq 0$.

□

Corollary 1 *Let $s^{(0)} \in \operatorname{int} \mathcal{K}^*$ and $\gamma \in \Re_+$. The following two statements are equivalent:*

1. *There exists $x \in \mathcal{F}_P$ such that $(s^{(0)})^T x \le \gamma$.*
2. *For each $\epsilon > 0$ there exists $x \in \mathcal{F}_P(\epsilon)$ such that $(s^{(0)})^T x \le \gamma + \epsilon$.*

Proof:

"1 $\Rightarrow$ 2" is obvious. To prove the other direction, suppose that the second statement holds. Then there exists a sequence $x^{(i)} \in \mathcal{K}$, $i = 1, 2, \ldots$, such that

$$\lim_{i\to\infty} \left\| P_{\mathcal{A}^\perp}(x^{(i)} - c) \right\| = 0 \tag{16}$$

and

$$\limsup_{i\to\infty} (s^{(0)})^{\mathrm{T}} x^{(i)} \le \gamma. \tag{17}$$

Let $\gamma_1 := \sup_{i\ge 1} (s^{(0)})^{\mathrm{T}} x^{(i)}$. We know from Lemma 9 that the sequence $x^{(1)}, x^{(2)}, \ldots$ is contained in a compact set, viz.

$$\mathcal{K} \cap \{x \in \Re^n \mid (s^{(0)})^{\mathrm{T}} x \le \gamma_1\}.$$

Hence, this sequence has a cluster point in $\mathcal{K}$, say $x \in \mathcal{K}$. From (16) it follows that $x - c \in \mathcal{A}$, hence

$$x \in \mathcal{F}_P.$$

Finally, we know from (17) that

$$(s^{(0)})^{\mathrm{T}} x \le \gamma.$$

□

The above corollary essentially states that if $\overset{o}{\mathcal{F}}_D \ne \emptyset$ (the Slater condition), then any lower level set of the primal must be either nonempty or strongly infeasible.

Corollary 2 *Let $\gamma \in \Re$ and suppose $\overset{o}{\mathcal{F}}_D \ne \emptyset$. Then*

$$\exists x \in \mathcal{F}_P \textit{ such that } c^T x \le \gamma$$

if and only if for any $\epsilon > 0$

$$\exists x \in \mathcal{F}_P(\epsilon) \textit{ such that } c^T x \le \gamma + \epsilon.$$

Proof:

Let $s^{(0)} \in \overset{o}{\mathcal{F}}_D$ be fixed. Then the identity (1), viz.

$$(s^{(0)})^{\mathrm{T}} x = c^{\mathrm{T}} x + c^{\mathrm{T}} s^{(0)} - \|c\|^2, \qquad \forall x \in \mathcal{F}_P$$

implies the one-to-one correspondence between the level sets of $c^{\mathrm{T}}x$ and $(s^{(0)})^{\mathrm{T}}x$. Now we invoke Corollary 1.

□

Hence, we can strengthen the duality result for the case in which the Slater condition holds. We thus arrive at the following well known strong duality theorem [1, 3, 11].

Theorem 3 (Strong duality) *Suppose that* $\overset{o}{\mathcal{F}}_D \neq \emptyset$. *Then either*

$$d^* = -\infty \quad \text{and} \quad \mathcal{F}_P = \emptyset$$

or

$$p^* + d^* = \|c\|^2 \quad \text{and} \quad \mathcal{F}_P^* \neq \emptyset.$$

Proof:

If $d^* = -\infty$, we obtain from Lemma 1 that $\mathcal{F}_P = \emptyset$. Now consider $d^* > -\infty$. We want to show that $d^*(\epsilon)$ is lower semicontinuous. Let $\bar{s} \in \overset{o}{\mathcal{F}}_D$ and let $s(\epsilon) \in \mathcal{F}_D(\epsilon)$ be a sequence with

$$\lim_{\epsilon \downarrow 0} c^{\mathrm{T}} s(\epsilon) = \lim_{\epsilon \downarrow 0} d^*(\epsilon).$$

Let $u(\epsilon) := P_{\mathcal{A}}(s(\epsilon) - c)$, and notice that $\|u(\epsilon)\| \leq \epsilon$.

Since $\bar{s} \in \overset{o}{\mathcal{F}}_D$, there exists some $\rho > 0$ such that

$$\|y - \bar{s}\| \leq \rho \quad \Rightarrow \quad y \in \mathcal{K}^*.$$

We claim that

$$s'(\epsilon) := \left(1 - \frac{\epsilon}{\rho}\right)(s(\epsilon) - u(\epsilon)) + \frac{\epsilon}{\rho}\bar{s} \in \mathcal{F}_D, \quad \forall \epsilon \in (0, \rho).$$

Indeed, since $(s(\epsilon) - u(\epsilon)) \in \mathcal{A}^{\perp} + c$ and $\bar{s} \in \mathcal{A}^{\perp} + c$, it follows that $s'(\epsilon) \in \mathcal{A}^{\perp} + c$. Furthermore, we can write

$$s'(\epsilon) = \left(1 - \frac{\epsilon}{\rho}\right) s(\epsilon) + \frac{\epsilon}{\rho}\left(\bar{s} - \left(\frac{\rho}{\epsilon} - 1\right) u(\epsilon)\right).$$

Notice that the vector

$$\left(\bar{s} - \left(\frac{\rho}{\epsilon} - 1\right) u(\epsilon)\right)$$

is within distance ρ from $\bar{s}$, thus it is in the cone $\mathcal{K}^*$. Since $s(\epsilon) \in \mathcal{K}^*$, it follows $s'(\epsilon)$ must also belong to $\mathcal{K}^*$. This shows $s'(\epsilon) \in \mathcal{F}_D$. Now we have that

$$d^* \leq \lim_{\epsilon \downarrow 0} c^{\mathrm{T}} s'(\epsilon) = \lim_{\epsilon \downarrow 0} c^{\mathrm{T}} s(\epsilon) = \lim_{\epsilon \downarrow 0} d^*(\epsilon).$$

On the other hand, it is obvious that $\lim_{\epsilon \downarrow 0} d^*(\epsilon) \leq d^*$, and hence $d^*(\epsilon)$ is lower semicontinuous. We thus obtain from Theorem 2 that

$$p^* + d^* - \|c\|^2 = 0.$$

Finally, we have from Corollary 2 that

$$c^{\mathrm{T}} x = p^*, \qquad \text{for some } x \in \mathcal{F}_P.$$

□

We obtain from Theorem 3 the well known fact that if $\overset{o}{\mathcal{F}}_P \times \overset{o}{\mathcal{F}}_D \neq \emptyset$, then $\mathcal{F}_P^* \times \mathcal{F}_D^* \neq \emptyset$ and

$$(x^*)^{\mathrm{T}} s^* = c^{\mathrm{T}}(x^* + s^* - c) = 0, \quad \text{for all } (x^*, s^*) \in \mathcal{F}_P^* \times \mathcal{F}_D^*.$$

We say that in this case the conic convex programming problem has a *complementary* solution:

Definition 8 *A complementary solution is a pair $(x, s) \in \mathcal{F}_P \times \mathcal{F}_D$ such that*

$$x^T s = 0.$$

Obviously, any complementary solution is an optimal solution. The converse however, is in general not true unless $\overset{o}{\mathcal{F}}_P \neq \emptyset$ or $\overset{o}{\mathcal{F}}_D \neq \emptyset$. This is because without the latter condition, there may exist a positive duality gap and as a result there cannot exist a complementary solution. Moreover, strong duality is necessary, but not sufficient for the existence of a complementary solution. The duality relations are summarized in the following table.

			primal feasible		primal infeasible	
			strong	weak	weak	strong
			1	2	3	4
dual feasible	strong	A	s.d., (P)+(D) solvable	s.d., (P) solvable	(D) unbounded	(D) unbounded
	weak	B		possible	possible	(D) unbounded
dual infeasible	weak	C			possible	possible
	strong	D				possible

The acronym 's.d.' in the above table stands for 'strongly dual'. All entries in the table represent possible combinations of the status of the primal and dual problem. Only if we cannot conclude anything more, we explicitly mention that the entry represents a possible state. Due to the complete symmetry of the conic convex programming duality, the table is symmetric, so we only need to consider the upper-right block. The entries in the first row of the table are denoted by 'A1', 'A2', 'A3' and 'A4', in the second row by 'B1', 'B2', and so on. The entries 'A1', 'A2', 'A3' and 'A4', are due to Theorem 3. Lemma 6 implies entry 'B4'. The possibility of states 'A3' and 'B3' will be demonstrated in Section 4.3 by Examples 4 and 7 respectively. The entries 'C3' and 'C4' will be illustrated by Examples 3 and 2 respectively.

Finally, the possibility of the entries in the table where weak infeasibility is not involved, is easily demonstrated by a 2-dimensional linear programming problem.

Example 1 *Let* $n = 2$, $c \in \Re^2$, $\mathcal{K} = \mathcal{K}^* = \Re^2_+$ *and*

$$\mathcal{A} = \{(x_1, x_2) \mid x_1 = 0\}, \quad \mathcal{A}^\perp = \{(s_1, s_2) \mid s_2 = 0\}.$$

We see that (P) is strongly feasible if $c_1 > 0$, *weakly feasible if* $c_1 = 0$ *and strongly infeasible if* $c_1 < 0$. *Similarly, (D) is strongly feasible if* $c_2 > 0$, *weakly feasible if* $c_2 = 0$ *and strongly infeasible if* $c_2 < 0$.

Weak infeasibility does not exist in linear programming. However, weakly infeasible problems are easily constructed in semidefinite programming, which is an important class of conic convex programming.

3.3 Semidefinite programming

Before we discuss some examples of semidefinite programming problems, we have to introduce some notation. For given positive integer $\bar{n}$, we denote the space of $\bar{n} \times \bar{n}$ symmetric matrices by $\mathcal{S}$. Given X and Y in $\mathcal{S}$, the standard inner-product is defined as

$$X \bullet Y = \mathrm{tr}\, XY.$$

The Frobenius norm of a matrix $X \in \mathcal{S}$ is $\|X\|_F = \sqrt{X \bullet X}$. If $X \in \mathcal{S}$ is positive (semi-) definite, we write $X \succ 0$ ($X \succeq 0$).

Since the dimension of the space of $\bar{n} \times \bar{n}$ symmetric matrices is n,

$$n := \frac{\bar{n}(\bar{n}+1)}{2},$$

we can construct for $\mathcal{S}$ an orthonormal basis U of cardinality n,

$$U = \{U_1, U_2, \ldots, U_n\}.$$

By orthonormality we mean that $U_i \bullet U_j = 0$ for $i \neq j$ and $\|U_i\|_F = 1$ for $i = 1, 2, \ldots, n$.

The conic convex problem (P) is a *semidefinite programming* problem (SDP) if

$$\mathcal{K} = \left\{ x \in \Re^n \mid \sum_{i=1}^{n} x_i U_i \succeq 0 \right\}.$$

We let $x = \mathrm{vec}(X, U)$ denote the coordinate vector of a matrix $X \in \mathcal{S}$ with respect to the basis U, i.e.

$$x_i = U_i \bullet X \text{ for } i = 1, 2, \ldots, n.$$

It is easily verified that for $X, Y \in \mathcal{S}$ we have

$$X \bullet Y = \mathrm{tr}\, XY = \mathrm{vec}(X, U)^{\mathrm{T}} \mathrm{vec}(Y, U).$$

Suppose that $A \in \Re^{m \times n}$ is a matrix with kernel $\mathcal{A}$, i.e.

$$\mathcal{A} = \{x \in \Re^n \mid Ax = 0\}.$$

There holds

$$\mathcal{A}^\perp = \{s \in \Re^n \mid s = A^{\mathrm{T}} y \quad \text{for some } y \in \Re^m\}, \quad \mathcal{K}^* = \mathcal{K}.$$

We define

$$C := \sum_{i=1}^{n} c_i U_i$$

and

$$A^{(i)} := \sum_{j=1}^{n} a_{ij} U_j, \qquad \text{for } i = 1, 2, \ldots, m.$$

We can now rewrite the SDP (P) in matrix form as

$$\begin{array}{lll} (SDPP) & \inf & C \bullet X \\ & \text{s.t.} & A_i \bullet (X - C) = 0, \quad \text{for } i = 1, 2, \ldots, m \\ & & X \succeq 0 \end{array}$$

with dual

$$\begin{array}{lll} (SDPD) & \inf & C \bullet S \\ & \text{s.t.} & S = C + \sum_{i=1}^{m} y_i A^{(i)} \quad \text{for some } y \in \Re^m \\ & & S \succeq 0. \end{array}$$

If we allow multiple semidefiniteness constraints, say $X^{(1)} \succeq 0, X^{(2)} \succeq 0$, we arrive at a seemingly more general form of semidefinite programming. Yet, these problems can be cast in the form (SDPP), see e.g. Vandenberghe and Boyd [19]. This is true in particular if semidefiniteness constraints are combined with nonnegativity constraints.

We will now give an example where the primal is weakly infeasible and the dual is strongly infeasible.

Example 2 *Consider $m = 2, \bar{n} = 2$ and*

$$C = \begin{bmatrix} 0 & 1 \\ & -1 \end{bmatrix}, \qquad A^{(1)} = \begin{bmatrix} 1 & 0 \\ & 0 \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix}.$$

The primal is weakly infeasible,

$$p^* = \inf \left\{ 2x_{12} - x_{22} \;\middle|\; X = \begin{bmatrix} 0 & 1 \\ & x_{22} \end{bmatrix} \succeq 0 \right\} = \infty,$$

and the dual is strongly infeasible,

$$d^* = \inf \left\{ 2s_{12} - s_{22} \;\middle|\; S = \begin{bmatrix} s_{11} & s_{12} \\ & -1 \end{bmatrix} \succeq 0 \right\} = \infty.$$

It is also possible that the primal and the dual are both weakly infeasible.

Example 3 *Consider* $m = 2, \bar{n} = 3$ *and*

$$C = \begin{bmatrix} 0 & 1 & 0 \\ & 0 & 1 \\ & & 0 \end{bmatrix}, \quad A^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & 0 \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{bmatrix}.$$

The primal is weakly infeasible,

$$p^* = \inf \left\{ 2x_{12} + 2x_{23} \;\middle|\; X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ & 0 & 1 \\ & & x_{33} \end{bmatrix} \succeq 0 \right\} = \infty$$

and the dual is also weakly infeasible,

$$d^* = \inf \left\{ 2s_{12} + 2s_{23} \;\middle|\; S = \begin{bmatrix} 0 & 1 & 0 \\ & s_{22} & s_{23} \\ & & 0 \end{bmatrix} \succeq 0 \right\} = \infty.$$

In Section 4.3 we will encounter combinations of primal weak infeasibility with dual weak feasibility and dual strong feasibility.

4 Self-Duality

Almost all early interior point algorithms for linear programming assume strong feasibility of (P) and (D) and even the availability of an initial interior solution. A seminal approach for solving linear programs without any regularity assumptions is the self-dual embedding, developed by Ye-Todd-Mizuno [21]. In this approach, the primal and dual linear programs are combined in a single self-dual linear program, with an obvious initial feasible interior point, thus fulfilling a key requirement of the interior point methods. When applied to the self-dual formulation, the interior point methods generate a *weakly centered* sequence (to be defined later) that converges to a strictly complementary solution. Such a solution of the self-dual linear program can be used either to recover a solution of the original linear programming problem or to demonstrate its infeasibility. Another advantage of this self-dual formulation is that it only increases the problem dimension by 1.

At the time of writing, the self-dual scheme was generalized to semidefinite programming by Potra-Sheng [13]. However, this reference only solves the complementarity version of semidefinite programming, which is not equivalent to semidefinite optimization, except if (P) and (D) are assumed to be strongly feasible.

Before we extend the Ye-Todd-Mizuno self-dual embedding to conic convex programming, we need to discuss a generalization of the Goldman-Tucker self-dual feasibility model.

4.1 A self-dual feasibility model

In the previous section, we have obtained a number of results for the *primal* problem (P). Due to the complete symmetry in duality, the results hold true for the dual (D) as well. In the sequel, we will consider the *pair* of problems (P) and (D) and we will refer to this primal-dual pair by the triplet $(c, \mathcal{A}, \mathcal{K})$. We say that a conic convex program $(c, \mathcal{A}, \mathcal{K})$ is infeasible if

$$\mathcal{F}_P \times \mathcal{F}_D = \emptyset,$$

i.e. if either the primal or the dual is infeasible. We let

$$\mathcal{D}_{PD}(\epsilon) := \left\{ (x, s) \in \mathcal{K} \times \mathcal{K}^* \mid \|P_{\mathcal{A}^\perp} x + P_{\mathcal{A}} s\| \leq \epsilon,\ c^{\mathrm{T}}(x+s) \leq -1 + \epsilon \right\},$$

and we define

$$\mathcal{D}_{PD} := \mathcal{D}_{PD}(0).$$

By Lemma 7, $(c, \mathcal{A}, \mathcal{K})$ is infeasible if and only if

$$\mathcal{D}_{PD}(\epsilon) \neq \emptyset, \quad \forall \epsilon > 0.$$

Furthermore, Lemma 6 implies that $(c, \mathcal{A}, \mathcal{K})$ is strongly infeasible if and only if $\mathcal{D}_{PD} \neq \emptyset$. We will now generalize the self-dual feasibility model of Goldman and Tucker [4, 18] to conic convex programming.

Definition 9 *A conic convex program* $(c, \mathcal{A}, \mathcal{K})$ *is* self-dual *if there exists a permutation matrix* Π_{SD2} *such that*

$$\Pi_{SD2} c = c,\ \ \Pi_{SD2} \mathcal{A} = \mathcal{A}^\perp,\ \ \Pi_{SD2} \mathcal{K} = \mathcal{K}^*.$$

We let

$$\mathcal{A}_{SD1} := \begin{bmatrix} I & 0 & c \\ 0 & I & c \\ 0 & 0 & 1 \\ -c^{\mathrm{T}} & -c^{\mathrm{T}} & -\|c\|^2 \end{bmatrix} (\mathcal{A} \times \mathcal{A}^\perp \times \Re).$$

Obviously, $\mathcal{A}_{SD1}$ is an $(n+1)$-dimensional subspace of $\Re^{2n+2}$ and by Lemma 5 its orthogonal complement is

$$\mathcal{A}_{SD1}^\perp = \begin{bmatrix} I & 0 & c \\ 0 & I & c \\ -c^{\mathrm{T}} & -c^{\mathrm{T}} & -\|c\|^2 \\ 0 & 0 & 1 \end{bmatrix} (\mathcal{A}^\perp \times \mathcal{A} \times \Re).$$

We introduce the closed convex cone $\mathcal{K}_{SD1}$,

$$\mathcal{K}_{SD1} := \mathcal{K} \times \mathcal{K}^* \times \Re_+ \times \Re_+,$$

whose dual cone is

$$\mathcal{K}_{SD1}^* := \mathcal{K}^* \times \mathcal{K} \times \Re_+ \times \Re_+.$$

Now consider the following homogeneous feasibility model:

$$\begin{array}{lll} (SD1) & \min & 0 \\ & \text{s.t.} & x - x_0 c \in \mathcal{A} \\ & & s - x_0 c \in \mathcal{A}^\perp \\ & & c^{\mathrm{T}} x + c^{\mathrm{T}} s - x_0 \left\| c \right\|^2 + s_0 = 0 \\ & & x \in \mathcal{K},\ s \in \mathcal{K}^*,\ x_0 \geq 0,\ s_0 \geq 0, \end{array}$$

or equivalently,

$$\min \left\{ 0 \;\middle|\; \begin{bmatrix} x^{\mathrm{T}} & s^{\mathrm{T}} & x_0 & s_0 \end{bmatrix}^{\mathrm{T}} \in \mathcal{A}_{SD1} \cap \mathcal{K}_{SD1} \right\}.$$

Obviously, we have

$$\begin{bmatrix} x^{\mathrm{T}} & s^{\mathrm{T}} & x_0 & s_0 \end{bmatrix}^{\mathrm{T}} \in \mathcal{A}_{SD1} \cap \mathcal{K}_{SD1}$$

if and only if

$$\begin{bmatrix} s^{\mathrm{T}} & x^{\mathrm{T}} & s_0 & x_0 \end{bmatrix}^{\mathrm{T}} \in \mathcal{A}_{SD1}^\perp \cap \mathcal{K}_{SD1}^*.$$

Hence, (SD1) is self-dual and

$$x^{\mathrm{T}} s + x_0 s_0 = 0, \qquad \text{for } \begin{bmatrix} x^{\mathrm{T}} & s^{\mathrm{T}} & x_0 & s_0 \end{bmatrix}^{\mathrm{T}} \in \mathcal{A}_{SD1} \cap \mathcal{K}_{SD1}. \tag{18}$$

Theorem 4 *There holds*

- *(SD1) has a solution with $x_0 > 0$ if and only if $(c, \mathcal{A}, \mathcal{K})$ has a complementary solution.*
- *(SD1) has a solution with $s_0 > 0$ if and only if $(c, \mathcal{A}, \mathcal{K})$ is strongly infeasible.*

Proof:

If $x_0 > 0$ then $(\frac{x}{x_0}, \frac{s}{x_0}) \in \mathcal{F}_P \times \mathcal{F}_D$ and we know from (18) that $(\frac{x}{x_0}, \frac{s}{x_0})$ is a complementary solution. Conversely, if (x^*, s^*) is a complementary solution to (P), (D), then

$$(x^*, s^*, 1, 0) \in \mathcal{A}_{SD1} \cap \mathcal{K}_{SD1}.$$

If $s_0 > 0$, then $x_0 = 0$ and

$$-s_0 = c^{\mathrm{T}} x + c^{\mathrm{T}} s < 0,$$

so that

$$\frac{1}{s_0}(x, s) \in \mathcal{D}_{PD}.$$

Applying Lemma 6, it follows that $(c, \mathcal{A}, \mathcal{K})$ is strongly infeasible. Conversely, if $(c, \mathcal{A}, \mathcal{K})$ is strongly infeasible, then we know from Lemma 6 that there exists $(x, s) \in \mathcal{D}_{PD}$. For such a pair $(x, s) \in \mathcal{D}_{PD}$ we obviously have

$$(x, s, 0, 1) \in \mathcal{A}_{SD1} \cap \mathcal{K}_{SD1}.$$

□

4.2 The extended self-dual program

The self-dual feasibility model introduced in the previous subsection does not have a strongly feasible point, thus making it difficult to apply the interior point methods directly. The purpose of this subsection is to augment the aforementioned self-dual model by allowing a simple starting point for the interior point methods. Dispensing with any constraint qualification, we will discuss to what extent the original problem $(c, \mathcal{A}, \mathcal{K})$ can be solved, when the interior point method is applied to such an extended self-dual model.

Throughout this subsection, we will make the following assumption on the cone $\mathcal{K}$.

Assumption 1 *The cone $\mathcal{K}$ is pointed and has a nonempty interior, i.e.*

$$\mathcal{K} \cap (-\mathcal{K}) = \{0\}, \quad \text{int } \mathcal{K} \neq \emptyset.$$

By Lemma 4, there exists some vector

$$\iota \in \text{int } (\mathcal{K} \cap \mathcal{K}^*).$$

Let us consider the following conic convex programming problem:

$$\begin{array}{lll} (SD2) & \min & (2 + \|\iota\|^2) y_0 \\ & \text{s.t.} & x \in (x_0 - y_0)c + y_0 \iota + \mathcal{A} \qquad (19) \\ & & s \in (x_0 - y_0)c + y_0 \iota + \mathcal{A}^\perp \qquad (20) \\ & & c^{\mathrm{T}}(x + s - (x_0 - y_0)c - 2y_0 \iota) + s_0 = y_0 \qquad (21) \\ & & \iota^{\mathrm{T}}(x + s) + x_0 + s_0 + y_0 + z_0 = (1 + y_0)(2 + \|\iota\|^2) \qquad (22) \\ & & x \in \mathcal{K}, \;\; s \in \mathcal{K}^*, \;\; (x_0, y_0, s_0, z_0) \in \Re^4_+. \qquad (23) \end{array}$$

We denote the vector of decision variables for (SD2) by w,

$$w := (x, s, x_0, y_0, s_0, z_0),$$

and we let $\mathcal{F}_{SD2}$ denote the feasible set of (SD2), with interior

$$\overset{o}{\mathcal{F}}_{SD2} := \mathcal{F}_{SD2} \cap \text{int } (\mathcal{K} \times \mathcal{K}^* \times \Re^4_+).$$

We see immediately that the vector

$$c_{SD2} := (\iota, \iota, 1, 1, 1, 1) \in \overset{o}{\mathcal{F}}_{SD2} \tag{24}$$

and that (SD2) has a trivial optimal solution

$$w^* := (0, 0, 0, 0, 0, 2 + \|\iota\|^2).$$

The optimal set of (SD2) is

$$\mathcal{F}^*_{SD2} := \{w \in \mathcal{F}_{SD2} \mid y_0 = 0\}.$$

Remark that if $w \in \mathcal{F}^*_{SD2}$, then

$$(x, s, x_0, s_0) \in \mathcal{A}_{SD1} \cap \mathcal{K}_{SD1}.$$

This means that every optimal solution of (SD2) induces an optimal solution of (SD1). The reverse is also true if the solutions are normalized according to the constraint (22). Self-dual programs that are similar to (SD2) were proposed and studied for linear programming by Ye, Todd and Mizuno [21] and Jansen, Roos and Terlaky [7].

Lemma 10 *The conic convex problem (SD2) is self-dual. Defining the permutation Π_{SD2} by*

$$\Pi_{SD2}(x, s, x_0, y_0, s_0, z_0) = (s, x, s_0, z_0, x_0, y_0),$$

we have

$$(2 + \|\iota\|^2)y_0 = \frac{1}{2} w^T \Pi_{SD2} w = x^T s + x_0 s_0 + y_0 z_0$$

for any $w \in \mathcal{F}_{SD2}$.

Proof:

Let $w \in \mathcal{F}_{SD2}$ and introduce the variables

$$\Delta x := x - (x_0 - y_0)c - y_0 \iota, \quad \Delta s := s - (x_0 - y_0)c - y_0 \iota.$$

Then, $\Delta x \in \mathcal{A}$ and $\Delta s \in \mathcal{A}^\perp$. From (21), we have

$$s_0 = -c^{\mathrm{T}}(\Delta x + \Delta s) - \|c\|^2 x_0 + (1 + \|c\|^2)y_0 \tag{25}$$

and from (22),

$$z_0 = 2 + \|\iota\|^2 + (1 - \|\iota\|^2 + 2\iota^{\mathrm{T}} c)y_0 - (1 + 2\iota^{\mathrm{T}} c)x_0 - \iota^{\mathrm{T}}(\Delta x + \Delta s) - s_0.$$

Now substituting (25) for s_0 yields

$$z_0 = 2 + \|\iota\|^2 - \|\iota - c\|^2 y_0 - (1 + 2\iota^{\mathrm{T}} c - \|c\|^2)x_0 - (\iota - c)^{\mathrm{T}}(\Delta x + \Delta s). \tag{26}$$

Now we define $r := \iota - c$ and let

$$\mathcal{A}_{SD2} := \begin{bmatrix} I & 0 & c & r \\ 0 & I & c & r \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -c^{\mathrm{T}} & -c^{\mathrm{T}} & -\|c\|^2 & 1 + \|c\|^2 \\ -r^{\mathrm{T}} & -r^{\mathrm{T}} & \|c\|^2 - (1 + 2\iota^{\mathrm{T}} c) & -\|r\|^2 \end{bmatrix} (\mathcal{A} \times \mathcal{A}^{\perp} \times \Re \times \Re)$$

and

$$\mathcal{K}_{SD2} := \mathcal{K} \times \mathcal{K}^* \times \Re^4_+.$$

We see from (19)–(20) and (23)–(26) that

$$\mathcal{F}_{SD2} = (w^* + \mathcal{A}_{SD2}) \cap \mathcal{K}_{SD2} = (c_{SD2} + \mathcal{A}_{SD2}) \cap \mathcal{K}_{SD2}.$$

By (24) and (22) we further have that (SD2) is the conic convex problem $(c_{SD2}, \mathcal{A}_{SD2}, \mathcal{K}_{SD2})$. Now define the permutation Π_{SD2} by

$$\Pi_{SD2}(x, s, x_0, y_0, s_0, z_0) = (s, x, s_0, z_0, x_0, y_0).$$

By Lemma 5, the orthogonal complement of $\mathcal{A}_{SD2}$ is

$$\mathcal{A}^{\perp}_{SD2} = \begin{bmatrix} I & 0 & c & r \\ 0 & I & c & r \\ -c^{\mathrm{T}} & -c^{\mathrm{T}} & -\|c\|^2 & 1 + \|c\|^2 \\ -r^{\mathrm{T}} & -r^{\mathrm{T}} & \|c\|^2 - (1 + 2\iota^{\mathrm{T}} c) & -\|r\|^2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} (\mathcal{A}^{\perp} \times \mathcal{A} \times \Re \times \Re),$$

we have $\mathcal{A}^{\perp}_{SD2} = \Pi_{SD2}\mathcal{A}_{SD2}$, $\mathcal{K}^*_{SD2} = \Pi_{SD2}\mathcal{K}_{SD2}$, $c_{SD2} = \Pi_{SD2} c_{SD2}$ so that the program $(c_{SD2}, \mathcal{A}_{SD2}, \mathcal{K}_{SD2})$ is self-dual. Finally, it follows from $(w - w^*)(\Pi_{SD2} w - \Pi_{SD2} w^*) = 0$ that

$$(2 + \|\iota\|^2) y_0 = x^{\mathrm{T}} s + x_0 s_0 + y_0 z_0.$$

□

We will now consider a sequence

$$\{w^{(1)}, w^{(2)}, \ldots\} \subseteq \mathcal{F}_{SD2} \quad \text{with} \quad \lim_{k \to \infty} y_0^{(k)} = 0.$$

Applying Lemma 9 (or alternatively relation (22)), it follows that this sequence is contained in a compact set and hence it has a limit. Moreover, any limit is an optimal solution to (SD2) and hence to (SD1).

Definition 10 *A sequence* $\{w^{(1)}, w^{(2)}, \ldots\} \subseteq \mathcal{F}_{SD2}$ *is said to be* weakly centered *if there exists a scalar constant* $\omega > 0$ *such that*

$$x_0^{(k)} s_0^{(k)} \geq \omega y_0^{(k)} > 0 \quad for \quad k = 1, 2, \ldots \tag{27}$$

and $\lim_{k\to\infty} y_0^{(k)} = 0$.

We remark that condition (27) is the minimal centrality condition of Todd and Ye [17]. This condition holds true for all path-following algorithms and for some potential reduction methods. In particular, Nesterov and Todd [12] developed a framework of primal-dual interior point algorithms for self-scaled conic convex problems that generate a weakly centered sequence of iterates. Self-scaled conic convex programming is a rich subclass of conic convex programming and it includes linear programming and semidefinite programming among others. It should be noted here that if $(c, \mathcal{A}, \mathcal{K})$ is a linear (semidefinite, self-scaled) programming problem, then (SD2) is also a linear (semidefinite, self-scaled) programming problem.

Following the argument of Güler and Ye [5], we can easily show that if (SD1) has an optimal solution with the property $x_0 + s_0 > 0$, then any limit point of a weakly centered sequence for (SD2) must have the same property.

Theorem 5 *Let* $w^{(1)}, w^{(2)}, \ldots$ *be a weakly centered sequence for (SD2). Then*

- $\liminf_{k\to\infty} x_0^{(k)} > 0$ *if and only if* $(c, \mathcal{A}, \mathcal{K})$ *has a complementary solution.*
- $\liminf_{k\to\infty} s_0^{(k)} > 0$ *if and only if* $(c, \mathcal{A}, \mathcal{K})$ *is strongly infeasible.*
- *(P) is strongly infeasible if* $\liminf_{k\to\infty} s_0^{(k)} > 0$ *and* $\liminf_{k\to\infty} c^T s^{(k)} < 0$.
- *(D) is strongly infeasible if* $\liminf_{k\to\infty} s_0^{(k)} > 0$ *and* $\liminf_{k\to\infty} c^T x^{(k)} < 0$.

Moreover, the sequence $w^{(1)}, w^{(2)}, \ldots$ *has a limit point. For any such limit point* $w^{(\infty)}$ *there holds*

- *If* $(c, \mathcal{A}, \mathcal{K})$ *has a complementary solution, then*

$$\frac{1}{x_0^{(\infty)}}(x^{(\infty)}, s^{(\infty)}) \in \mathcal{F}_P^* \times \mathcal{F}_D^*.$$

- *If* $(c, \mathcal{A}, \mathcal{K})$ *is strongly infeasible, then*

$$\frac{1}{s_0^{(\infty)}}(x^{(\infty)}, s^{(\infty)}) \in \mathcal{D}_{PD}.$$

Proof:

Following the argument of Güler and Ye [5] and using Lemma 10, we have for any $w' \in \mathcal{F}_{SD2}$, and for any k,

$$\begin{aligned} 0 &= (w^{(k)} - w')^{\mathrm{T}} \Pi_{SD2} (w^{(k)} - w') \\ &= (w^{(k)})^{\mathrm{T}} \Pi_{SD2} w^{(k)} - 2(w')^{\mathrm{T}} \Pi_{SD2} w^{(k)} + (w')^{\mathrm{T}} \Pi_{SD2} w' \\ &\leq 2(1 + \|\iota\|^2)(y_0^{(k)} + y_0') - 2(x_0' s_0^{(k)} + s_0' x_0^{(k)}). \end{aligned}$$

Multiplying with $\frac{x_0^{(k)} + s_0^{(k)}}{2(y_0^{(k)} + y_0')}$ and using (27), we obtain

$$0 \leq (1 + \|\iota\|^2)(x_0^{(k)} + s_0^{(k)}) - \frac{\omega y_0^{(k)}}{y_0^{(k)} + y_0'}(x_0' + s_0').$$

In particular, if we let $w' \in \mathcal{F}^*_{SD2}$, we have $y_0' = 0$ and hence

$$\liminf_{k \to \infty}(x_0^{(k)} + s_0^{(k)}) \geq \frac{\omega}{1 + \|\iota\|^2}(x_0' + s_0').$$

This shows that if (SD1) has a solution with $x_0 + s_0 > 0$, then any limit point $w^{(\infty)}$ of the sequence

$$\{w^{(k)} \mid k = 1, 2, \ldots\}$$

satisfies $x_0^{(\infty)} + s_0^{(\infty)} > 0$. Using Theorem 4, we can then use such a limit point to decide whether $(c, \mathcal{A}, \mathcal{K})$ has a complementary solution and whether it is strongly infeasible. The remaining statements of the theorem are now easily verified using (19)-(23).

□

Let $w^{(1)}, w^{(2)}, \ldots$ be a weakly centered sequence for (SD2). Theorem 5 characterizes the case that

$$\liminf_{k \to \infty} \left(x_0^{(k)} + s_0^{(k)}\right) > 0.$$

The remainder of this section is dedicated to the remaining case where

$$\lim_{k \to \infty} \left(x_0^{(k)} + s_0^{(k)}\right) = 0.$$

Here, we need to distinguish three possibilities, viz. problem $(c, \mathcal{A}, \mathcal{K})$ is

- weakly infeasible, or
- feasible but there is no strong duality, or
- feasible and strong duality applies, but there exists no complementary solution pair.

We will develop an indicator on the basis of which we can often exclude one or two of the above possibilities. Before we introduce this indicator, we remark that since $\lim_{k\to\infty}\left(x_0^{(k)}+s_0^{(k)}\right)=0$, there must exist k_1 such that

$$x_0^{(k)}+s_0^{(k)}<\omega \quad \forall k\geq k_1.$$

Consequently, there holds (cf. (27))

$$x_0^{(k)}\geq \frac{\omega y_0^{(k)}}{s_0^{(k)}}>y_0^{(k)}>0 \quad \text{and} \quad s_0^{(k)}\geq \frac{\omega y_0^{(k)}}{x_0^{(k)}}>y_0^{(k)}>0, \quad \text{for all } k\geq k_1. \tag{28}$$

For $k\geq k_1$, let us introduce the sequences

$$\hat{w}^{(k)}:=\frac{1}{x_0^{(k)}-y_0^{(k)}}w^{(k)}, \quad \bar{w}^{(k)}:=\frac{1}{s_0^{(k)}-y_0^{(k)}}w^{(k)}$$

and

$$\hat{\epsilon}^{(k)}:=\hat{y}_0^{(k)}\left\|\iota\right\|, \quad \bar{\epsilon}^{(k)}:=(\bar{x}_0^{(k)}+\bar{y}_0^{(k)})(1+\|\iota\|+\|c\|)^2.$$

We see from (19)-(23) that

$$(\hat{x}^{(k)},\hat{s}^{(k)})\in\mathcal{F}_P(\hat{\epsilon}^{(k)})\times\mathcal{F}_D(\hat{\epsilon}^{(k)}), \tag{29}$$

$$(\bar{x}^{(k)},\bar{s}^{(k)})\in\mathcal{D}_{PD}(\bar{\epsilon}^{(k)}). \tag{30}$$

Moreover, we have from (28) that

$$\lim_{k\to\infty}\hat{y}_0^{(k)}=0, \quad \lim_{k\to\infty}\bar{y}_0^{(k)}=0, \quad \lim_{k\to\infty}\hat{\epsilon}^{(k)}=0. \tag{31}$$

The following lemma shows that $\hat{w}^{(k)}$ provides information on the level sets of (P) and (D).

Lemma 11 *Let $\{w^{(k)},k=1,2,\ldots\}$ be a weakly centered sequence with $\lim_{k\to\infty}(x_0^{(k)}+s_0^{(k)})=0$. Then for any fixed $k\geq k_1$ there holds*

$$\inf(c+\hat{y}_0^{(k)}\iota)^T\mathcal{F}_P\geq c^T(c+\hat{y}_0^{(k)}\iota)-c^T\hat{s}^{(k)}.$$

Proof:

Let $x\in\mathcal{F}_P$ be an arbitrary primal feasible solution. As $x\in\mathcal{K}$ and $\hat{s}^{(k)}\in\mathcal{K}^*$, we have

$$0\leq x^{\mathrm{T}}\hat{s}^{(k)}=c^{\mathrm{T}}\hat{s}^{(k)}+(x-c)^{\mathrm{T}}\hat{s}^{(k)},$$

so that using $x\in c+\mathcal{A}$ and $\hat{s}^{(k)}\in c+\hat{y}_0^{(k)}\iota+\mathcal{A}^{\perp}$,

$$0\leq c^{\mathrm{T}}\hat{s}^{(k)}+(x-c)^{\mathrm{T}}(c+\hat{y}_0^{(k)}\iota).$$

Hence,

$$(c+\hat{y}_0^{(k)}\iota)^{\mathrm{T}}x\geq c^{\mathrm{T}}(c+\hat{y}_0^{(k)}\iota)-c^{\mathrm{T}}\hat{s}^{(k)}.$$

□

We now define a crucial quantity

$$\theta := \limsup_{k\to\infty} \hat{s}_0^{(k)}$$

which will be used as an indicator in the following theorem. We will show that a weakly centered sequence for (SD2) provides useful information for the original conic convex problem $(c, \mathcal{A}, \mathcal{K})$ even if this latter problem is neither strongly infeasible nor endowed with a complementary solution pair.

Theorem 6 *Let $w^{(1)}, w^{(2)}, \ldots$ be a weakly centered sequence with*

$$\lim_{k\to\infty} (x_0^{(k)} + s_0^{(k)}) = 0.$$

Then

$$\limsup_{k\to\infty} c^T \hat{x}^{(k)} + \theta \le p^*, \quad \limsup_{k\to\infty} c^T \hat{s}^{(k)} + \theta \le d^* \tag{32}$$

and

$$(\hat{x}^{(k)}, \hat{s}^{(k)}) \in \mathcal{F}_P(\hat{\epsilon}^{(k)}) \times \mathcal{F}_D(\hat{\epsilon}^{(k)}), \quad \lim_{k\to\infty} \hat{\epsilon}^{(k)} = 0. \tag{33}$$

Moreover,

- *if $(c, \mathcal{A}, \mathcal{K})$ is feasible, then*

$$\theta < \infty, \tag{34}$$

- *if $(c, \mathcal{A}, \mathcal{K})$ is feasible and strong duality applies, then*

$$\lim_{k\to\infty} c^T \hat{x}^{(k)} = p^*, \quad \lim_{k\to\infty} c^T \hat{s}^{(k)} = d^*, \quad \theta = 0, \tag{35}$$

- *if $\theta = \infty$, then $(c, \mathcal{A}, \mathcal{K})$ is weakly infeasible and*

$$(\bar{x}^{(k)}, \bar{s}^{(k)}) \in \mathcal{D}_{PD}(\bar{\epsilon}^{(k)}), \quad \lim_{k\to\infty} \bar{\epsilon}^{(k)} = 0. \tag{36}$$

Proof:

The relation (33) follows immediately from (29) and (31). We will now show (32). By the definition of p^*, we know that for any $\epsilon > 0$ there exist $x(\epsilon) \in \mathcal{F}_P$ such that

$$c^{\mathrm{T}} x(\epsilon) \le p^* + \epsilon.$$

Also, from (31) and Lemma 11 we have

$$\begin{aligned} c^{\mathrm{T}} x(\epsilon) &= \lim_{k\to\infty} (c + \hat{y}_0^{(k)} \iota)^{\mathrm{T}} x(\epsilon) \\ &\ge \limsup_{k\to\infty} (\|c\|^2 - c^{\mathrm{T}} \hat{s}^{(k)}). \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, this shows that

$$p^* \geq \limsup_{k\to\infty}(\|c\|^2 - c^{\mathrm{T}}\hat{s}^{(k)}). \tag{37}$$

However, from (21) we have

$$c^{\mathrm{T}}(\hat{x}^{(k)} + \hat{s}^{(k)} - c) + \hat{s}_0^{(k)} = (1 + 2\iota^{\mathrm{T}}c)\hat{y}_0^{(k)},$$

so that using (37),

$$p^* \geq \limsup_{k\to\infty}(\|c\|^2 - c^{\mathrm{T}}\hat{s}^{(k)}) = \limsup_{k\to\infty}(c^{\mathrm{T}}\hat{x}^{(k)} + \hat{s}_0^{(k)}).$$

By the primal-dual symmetry,

$$d^* \geq \limsup_{k\to\infty}(\|c\|^2 - c^{\mathrm{T}}\hat{x}^{(k)}) = \limsup_{k\to\infty}(c^{\mathrm{T}}\hat{s}^{(k)} + \hat{s}_0^{(k)}),$$

which proves (32). If strong duality holds, then (37) and the above relation imply

$$\liminf_{k\to\infty} c^{\mathrm{T}}\hat{s}^{(k)} \geq \|c\|^2 - p^* = d^* \geq \limsup_{k\to\infty}(c^{\mathrm{T}}\hat{s}^{(k)} + \hat{s}_0^{(k)}),$$

which shows that $\lim_{k\to\infty} c^{\mathrm{T}}\hat{s}^{(k)} = d^*$ and $\theta = \lim_{k\to\infty}\hat{s}_0^{(k)} = 0$. We thus obtain (35).

It remains to show (34) and (36). Suppose $\theta = \lim_{k\to\infty}\hat{s}_0^{(k)} = \infty$. Because

$$\bar{x}_0^{(k)} = \frac{x_0^{(k)}s_0^{(k)}}{(x_0^{(k)} - y_0^{(k)})(s_0^{(k)} - y_0^{(k)})}\frac{1}{\hat{s}_0^{(k)}},$$

we obtain for this case that

$$\lim_{k\to\infty}\bar{x}_0^{(k)} = \lim_{k\to\infty}\frac{1}{\hat{s}_0^{(k)}} = 0, \tag{38}$$

which together with (31) implies that

$$\lim_{k\to\infty}\bar{\epsilon}^{(k)} = 0.$$

The relation (36) now follows from (30). Applying Lemma 7 we obtain (34).

□

4.3 Examples in semidefinite programming

Many recent papers discuss primal-dual interior point algorithms for semidefinite programming (SDP), see [8, 9, 10, 12, 16], among others. All these algorithms generate a sequence of weakly centered iterates, so that the results of Section 4.2 are applicable.

We will illustrate the theory of weakly centered sequences for (SD2) with some semidefinite programming problems. We choose $\iota := \mathrm{vec}(I, U)$ where I is the $\bar{n}\times\bar{n}$ identity matrix. In the following examples, we will use the matrix representation rather than the coordinate vectors; the examples specify the matrices $C,\ A^{(1)},\ A^{(2)},\dots,A^{(m)}$ of the standard (SDPP) form. Moreover, the weakly centered sequence will be parameterized by a continuous parameter $\epsilon > 0$ such that

$$\lim_{\epsilon\downarrow 0} y_0(\epsilon) = 0.$$

We will only discuss those difficult cases where $\lim_{\epsilon\downarrow 0}(x_0(\epsilon)+s_0(\epsilon)) = 0$.

First, we consider a weakly infeasible problem.

Example 4 (Weakly infeasible) *Let $m = 2, \bar{n} = 2$ and*

$$C = \begin{bmatrix} 0 & 1 \\ & 1 \end{bmatrix}, \quad A^{(1)} = \begin{bmatrix} 1 & 0 \\ & 0 \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix}.$$

The primal is weakly infeasible,

$$p^* = \inf \left\{ 2x_{12} + x_{22} \;\middle|\; X = \begin{bmatrix} 0 & 1 \\ & x_{22} \end{bmatrix} \succeq 0 \right\} = \infty$$

and the dual is strongly feasible and unbounded,

$$d^* = \inf \left\{ 2s_{12} + s_{22} \;\middle|\; S = \begin{bmatrix} s_{11} & s_{12} \\ & 1 \end{bmatrix} \succeq 0 \right\} = -\infty.$$

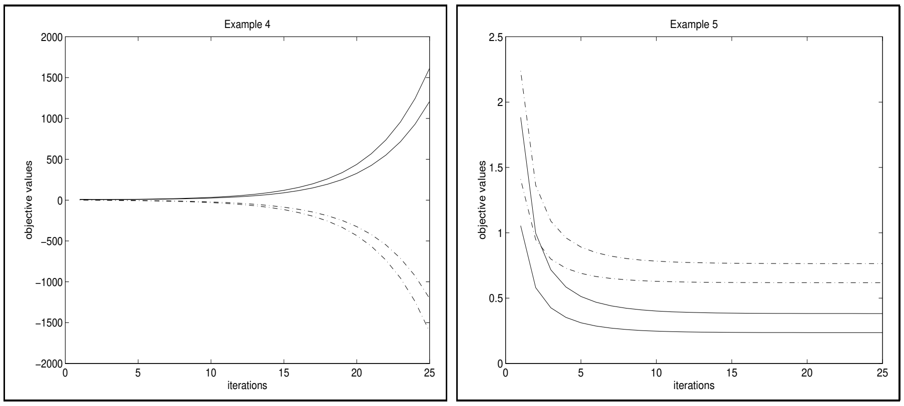
We construct a weakly centered sequence for $0 < \epsilon \leq 1/2$ as follows:

$$X(\epsilon) = \epsilon^3 I + \begin{bmatrix} 0 & \epsilon^2 \\ & \epsilon \end{bmatrix}, \quad S(\epsilon) = \epsilon^3 I + \begin{bmatrix} 1 & -\epsilon \\ & \epsilon^2 \end{bmatrix},$$

$$y_0(\epsilon) = \epsilon^3, \quad x_0(\epsilon) = \epsilon^2 + \epsilon^3, \quad s_0(\epsilon) = \epsilon + \epsilon^3, \quad z_0(\epsilon) = 3 - (2\epsilon + 2\epsilon^2 + 3\epsilon^3).$$

We see that $\theta = \lim_{\epsilon\to 0}\hat{s}_0(\epsilon) = \infty$, which indeed implies weak infeasibility, see Theorem 6.

We have applied the predictor-corrector algorithm for semidefinite programming with Nesterov-Todd directions [12, 16] to the self-dual embeddings of the SDP examples in this section. The plots below show the numerical results of this algorithm. The solid lines represent the primal values $c^{\mathrm{T}}\hat{x}^{(k)}$ and $c^{\mathrm{T}}\hat{x}^{(k)} + \hat{s}_0^{(k)}$, whereas the dual values $c^{\mathrm{T}}\hat{s}^{(k)}$ and $c^{\mathrm{T}}\hat{s}^{(k)} + \hat{s}_0^{(k)}$ are represented by dashed lines.



The next example, which is from Vandenberghe and Boyd [19], gives a feasible problem, where strong duality fails to hold.

Example 5 (Weakly dual) *Let $m = 2, \bar{n} = 3$ and*

$$C = \begin{bmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{bmatrix}, \quad A^{(1)} = \begin{bmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 1 \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & 0 \end{bmatrix}$$

so that the primal is solvable and weakly feasible,

$$p^* = \inf \left\{ x_{33} \;\middle|\; X = \begin{bmatrix} x_{11} & (1-x_{33})/2 & x_{13} \\ & 0 & x_{23} \\ & & x_{33} \end{bmatrix} \succeq 0 \right\} = 1$$

and the dual is solvable and weakly feasible,

$$d^* = \inf \left\{ s_{33} \;\middle|\; S = \begin{bmatrix} 0 & s_{33}-1 & 0 \\ & s_{22} & 0 \\ & & s_{33} \end{bmatrix} \succeq 0 \right\} = 1$$

Remark that $p^* + d^* - \|C\|_F^2 = 1 > 0$ so that strong duality fails. A weakly centered sequence for $0 < \epsilon \leq 1/2$ is given by

$$X(\epsilon) = \epsilon^2 I + \begin{bmatrix} 1 & \epsilon/2 & 0 \\ & 0 & 0 \\ & & 0 \end{bmatrix}, \quad S(\epsilon) = \epsilon^2 I + \begin{bmatrix} 0 & -\epsilon & 0 \\ & 1 & 0 \\ & & 0 \end{bmatrix},$$

$$y_0(\epsilon) = \epsilon^2, \quad x_0(\epsilon) = \epsilon + \epsilon^2, \quad s_0(\epsilon) = \epsilon + \epsilon^2, \quad z_0(\epsilon) = 3 - (2\epsilon + 4\epsilon^2),$$

so that $\theta = \lim_{\epsilon \to 0} \hat{s}_0(\epsilon) = 1$.

The third case is a problem where strong duality holds, but there exists no complementary solution pair (see Vandenberghe and Boyd [19]).

Example 6 (Strongly dual) *Let $m = 2, \bar{n} = 2$ and*

$$C = \begin{bmatrix} 0 & 1 \\ & 1 \end{bmatrix}, \quad A^{(1)} = \begin{bmatrix} 1 & 0 \\ & 0 \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix} 0 & 0 \\ & 1 \end{bmatrix},$$

so that the primal is solvable and weakly feasible,

$$p^* = \inf \left\{ 2x_{12} + x_{22} \;\middle|\; X = \begin{bmatrix} 0 & x_{12} \\ & 1 \end{bmatrix} \succeq 0 \right\} = 1$$

and the dual is strongly feasible but not solvable,

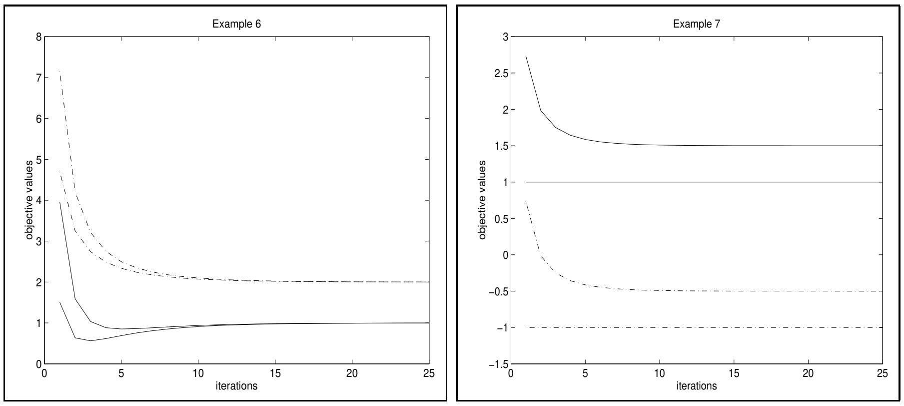
$$d^* = \inf \left\{ 2s_{12} + s_{22} \;\middle|\; S = \begin{bmatrix} s_{11} & 1 \\ & s_{22} \end{bmatrix} \succeq 0 \right\} = 2.$$

Notice that $p^* + d^* = \|C\|_F^2$*, but the dual has no optimal solution. A weakly centered sequence for* $0 < \epsilon \le 1/2$ *is*

$$X(\epsilon) = \epsilon^3 I + \begin{bmatrix} 0 & -\epsilon^2 \\ & \epsilon \end{bmatrix}, \quad S(\epsilon) = \epsilon^3 I + \begin{bmatrix} 1 & \epsilon \\ & \epsilon^2 \end{bmatrix},$$

$$y_0(\epsilon) = \epsilon^3, \quad x_0(\epsilon) = \epsilon + \epsilon^3, \quad s_0(\epsilon) = 2\epsilon^2 + \epsilon^3, \quad z_0(\epsilon) = 3 - (2\epsilon + 3\epsilon^2 + 3\epsilon^3).$$

Hence, $\theta = \lim_{\epsilon \to 0} s_0(\epsilon) = 0$ *as we already knew beforehand from Theorem 6.*



So far, we have seen a weakly infeasible problem with $\theta = \infty$, a feasible problem with only weak duality and $\theta \in (0, \infty)$ and a strongly dual problem with $\theta = 0$. The reader may wonder whether the value of the indicator θ completely characterizes the three cases that we consider. Unfortunately, this is not the case, as the next example shows.

Example 7 (Weakly infeasible) *Consider*

$$C = \begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix}, \quad A^{(1)} = \begin{bmatrix} 1 & 0 \\ & 0 \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix}.$$

The primal is weakly infeasible,

$$p^* = \inf \left\{ 2x_{12} \;\middle|\; X = \begin{bmatrix} 0 & 1 \\ & x_{22} \end{bmatrix} \succeq 0 \right\} = \infty,$$

and the dual is solvable and weakly feasible,

$$d^* = \inf \left\{ 2s_{12} \;\middle|\; S = \begin{bmatrix} s_{11} & s_{12} \\ & 0 \end{bmatrix} \succeq 0 \right\} = 0.$$

We construct a weakly centered sequence for $0 < \epsilon \leq 1/3$ as

$$X(\epsilon) = \epsilon^2 I + \begin{bmatrix} 0 & \epsilon \\ & 1 \end{bmatrix}, \quad S(\epsilon) = \epsilon^2 I + \begin{bmatrix} 1 & -\epsilon \\ & 0 \end{bmatrix},$$

$$y_0(\epsilon) = \epsilon^2, \quad x_0(\epsilon) = \epsilon + \epsilon^2, \quad s_0(\epsilon) = 2\epsilon + \epsilon^2, \quad z_0(\epsilon) = 2 - 3(\epsilon + \epsilon^2).$$

We see that $\theta = \lim_{\epsilon \to 0} \hat{s}_0(\epsilon) = 2$.

The reader may still wonder whether we can distinguish weak infeasibility from strong duality. It appears somewhat difficult indeed, to construct an example of an infeasible problem with $\theta = 0$, but it does exist.

Example 8 (Weakly infeasible) *Let $m = 2, \bar{n} = 3$ and*

$$C = \begin{bmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{bmatrix}, \quad A^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & 0 \end{bmatrix}.$$

The primal is solvable and weakly feasible,

$$p^* = \inf \left\{ 2x_{12} \;\middle|\; X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ & 0 & -x_{11}/2 \\ & & x_{33} \end{bmatrix} \succeq 0 \right\} = 0$$

and the dual is weakly infeasible,

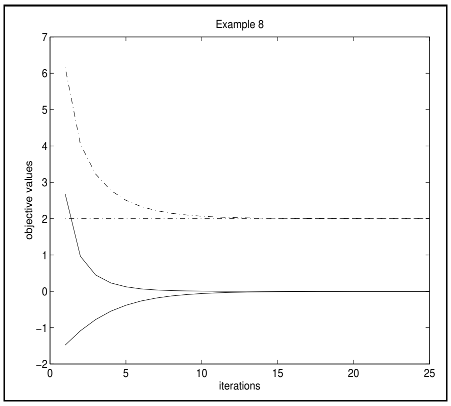
$$d^* = \inf \left\{ 2s_{12} \;\middle|\; S = \begin{bmatrix} s_{11} & 1 & 0 \\ & s_{22} & s_{11} \\ & & 0 \end{bmatrix} \succeq 0 \right\} = \infty.$$

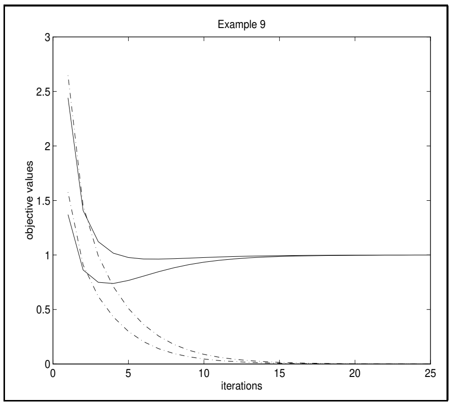
We construct a weakly centered sequence for $0 < \epsilon \leq 1/2$ by

$$X(\epsilon) = \epsilon^4 I + \begin{bmatrix} \epsilon^2 & -\epsilon^3/2 & 0 \\ & 0 & -\epsilon^2/2 \\ & & 1 \end{bmatrix}, \quad S(\epsilon) = \epsilon^4 I + \begin{bmatrix} \epsilon^2 & \epsilon & 0 \\ & 2 & \epsilon^2 \\ & & 0 \end{bmatrix},$$

$$y_0(\epsilon) = \epsilon^4, \quad x_0(\epsilon) = \epsilon + \epsilon^4, \quad s_0(\epsilon) = \epsilon^3 + \epsilon^4, \quad z_0(\epsilon) = 2 - (\epsilon + 3\epsilon^3 + 4\epsilon^4),$$

so that $\theta = \lim_{\epsilon \to 0} \hat{s}_0(\epsilon) = 0$.





After Example 8 there is little hope that feasibility with lack of strong duality would imply $\theta > 0$. Indeed, we can construct a feasible problem with only weak duality but $\theta = 0$.

Example 9 (Weakly dual) *Let* $m = 3, \bar{n} = 4$ *and*

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 1 \end{bmatrix}, \quad A^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 0 & 1 & 0 \\ & & 0 & 0 \\ & & & 0 \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{bmatrix},$$

$$A^{(3)} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 1 \end{bmatrix}.$$

It can be verified that $p^* + d^* - \|C\|_F^2 = 1 + 1 - 1 > 0$, *but one can construct a weakly centered sequence, similar to the one in Example 8, for which* $\theta = \lim_{\epsilon \to 0} \hat{s}_0(\epsilon) = 0$.

Our results on the indicator θ are summarized in the following table.

	θ		
	0	$(0, \infty)$	∞
weakly infeasible	Example 8	Example 7	Example 4
weakly dual	Example 9	Example 5	impossible
strongly dual	Example 6	impossible	impossible

The possible combinations in the table are illustrated by Examples 4-9. The impossibility of the remaining combinations follows from Theorem 6.

5 Concluding Remarks

Even for linear programming, solving a single self-dual embedding may not be enough to solve the original problem (P), see Ye, Todd and Mizuno [21]. In particular, if it is concluded that the primal-dual pair $(c, \mathcal{A}, \mathcal{K})$ is infeasible, it is in general not yet known whether (P) is infeasible or (D) is infeasible or both. However, we can determine whether or not (P) is feasible by solving a self-dual program for a feasibility formulation. To be more specific, let us define a vector $f_{\mathcal{D}} \in \Re^{n+1}$ by

$$f_{\mathcal{D}} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

and consider the following problem:

$$(P_{\mathcal{D}}) \quad \min \left\{ f_{\mathcal{D}}^{\mathrm{T}} x \;\middle|\; x \in \mathcal{B} \cap \mathcal{K} \right\}$$

$$(D_{\mathcal{D}}) \quad \min \left\{ 0 \;\middle|\; s \in (f_{\mathcal{D}} + \mathcal{B}^{\perp}) \cap \mathcal{K}^* \right\}.$$

We see that problem $(P_{\mathcal{D}})$ is always feasible and we know from Lemma 7 that $(D_{\mathcal{D}})$ is strongly infeasible if and only if the original problem (P) is feasible. Suppose we embed the primal-dual pair $(P_{\mathcal{D}}), (D_{\mathcal{D}})$ in a self-dual problem (SD2). Given a weakly centered sequence $w^{(k)}, k = 1, 2, \ldots$ for this self-dual embedding, the feasibility and weak/strong infeasibility of (P) is completely characterized by the limiting behavior of $x_0^{(k)}$ and $s_0^{(k)}$, due to Theorem 1 and Theorem 5. Moreover, we know from Theorem 5 that the strong infeasibility of $(D_{\mathcal{D}})$ will be demonstrated by a solution

$$\bar{x} \in \mathcal{B} \cap \mathcal{K}, \qquad f_{\mathcal{D}}^{\mathrm{T}} \bar{x} < 0.$$

Partitioning $\bar{x} = \begin{pmatrix} \xi \\ \xi_0 \end{pmatrix}$, we have $\xi/\xi_0 \in \mathcal{F}_P$. Hence, we obtain a feasible solution to (P) if it exists.

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