The break quantity rule in a 1–warehouse, N–retailers distribution system

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Abstract

In this paper the effect of the break quantity rule on the inventory costs in a 1–warehouse, N–retailers distribution system is analyzed. The break quantity rule is to deliver large orders from the warehouse, and small orders from the nearest retailer, where a so–called *break quantity* determines whether an order is small or large. Under the assumptions that the stock at the warehouse can only be used to satisfy large orders, and that demand during the leadtimes is normally distributed, an expression for the inventory costs is derived. The objective of this paper is to provide insight into the effect of the break quantity rule on the inventory holding costs, and therefore we present extensive computational results, showing that in many cases the rule leads to a significant cost reduction.

Keywords: Break quantity rule, Inventory, Multi–echelon distribution systems.

1 Introduction

In most 1–warehouse, N–retailers distribution systems, it is assumed that all customer demand takes place at the retailers (e.g. Axsäter [2, 3], Eppen and Schrage [6], Jackson [11], Jönsson and Silver [12], Federgruen and Zipkin [8, 9]). Hence, the warehouse is only used either to exploit quantity discounts available as a result of combining orders from several retailers, or for risk–pooling purposes. Warehouses can be used to pool risk in at least two different ways (Schwarz [17]). First, they can pool risk over the outside–supplier lead time, thereby reducing the variance of the retailers net inventory process. This type of risk–pooling, which is called the *joint ordering effect* by Eppen and Schrage [6], does not require the warehouse to hold inventory. Secondly, warehouses can pool risk by actually holding inventory

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and using it to rebalance retailer inventories which have become unbalanced because of very high or very low demands at one or more retailers. Eppen and Schrage [6] call this *the depot effect*. Imbalance of stocklevels at the retailers causes a major (mathematical) problem, as first observed by Clark and Scarf [4]. Therefore, in most of the literature a so–called *balance assumption* or *allocation assumption* (Eppen and Schrage [6]) is made, which rules out the possibility that the stocklevels at the retailers will be unbalanced. Langenhoff and Zijm [14] showed that in a pure distribution system, where holding costs are equal for all echelons, it is, under the balance assumption, optimal not to keep stock at the warehouse.

Imbalance of stocklevels is mainly caused by highly variable demand at the retailers. Therefore, it seems interesting to find ways to reduce this variability in order to decrease the probability of unbalanced stocklevels and thus increasing the service performance. In recent papers by Nass, Dekker and Van Sonderen–Huisman [15], Dekker *et al.* [5] and De Kok [13] the use of the break quantity rule in multi–echelon distribution systems was analyzed. This tactical rule is to deliver large orders from the warehouse, and small orders from the retailer, where a so–called *break quantity* determines whether an order is small or large. A motivation for handling large orders differently is because in many cases customers ordering large batches do not have an immediate need for delivery, or their order sizes are larger than necessary due to quantity discounts. Observe that when the break quantity rule is applied, the warehouse is also used to deliver orders directly to customers. Although from a practical point of view this situation is an improvement over the previous one, from a mathematical point of view it causes more problems. However, since Dekker *et al.* [5] showed that using the break quantity rule will reduce the variability of demand at the retailers, the probability of imbalance will also decrease, and by the above observations it seems reasonable to make the following assumption:

The stock at the warehouse can only be used to satisfy large orders.

Hence, upon arrival of an order from the outside supplier, part of the order will be held at the warehouse in order to satisfy large orders, and the remaining part will be allocated to the retailers. The stock at the retailers can only be replenished when the next order from the outside supplier arrives at the warehouse. Under the above assumption, the 1–warehouse, N–retailers distribution system with the break quantity rule can be modeled as a 1–warehouse, $(N+1)$–retailers distribution system, where retailers $1, \ldots, N$ are the original retailers and retailer 0 denotes the warehouse, which satisfies demand from customers with large orders.

Dekker *et al.* [5] showed that applying the break quantity rule leads to a reduction in the average costs. However, their expression for the average costs did not include the inventory costs at the warehouse, which will surely be influenced by the break quantity rule. Although the retailer inventory costs certainly decrease, it is difficult to say whether the warehouse inventory costs will increase or decrease. Since the demand at each retailer is separated into small and large demand, and the variability of small demand is less than the normal variability, it is easy to see that the variability of large demand exceeds the normal variability. However, since all large demand of N retailers is centralized, the total variability will decrease. The net effect is therefore difficult to predict. In this paper an expression for the total inventory costs, including the warehouse costs, is derived, under the assumption that the leadtime demand is (approximately) normally distributed. This result, presented in the next section, extends the analysis of Dekker *et al.* [5]. We do not aim at optimizing the break quantities, because delivering large orders directly from the warehouse also has a significant impact on the transportation costs. On the one hand, direct deliveries are always shorter and therefore they reduce the transportation costs. On the other hand, the average shipment size decreases, which leads to an increase of the costs. Moreover, the warehouse can combine the delivery of several large orders, in which case vehicle routing aspects also influence the distribution costs. The net cost effect thus depends, among others, on the locations of the warehouse, retailers and the customers, the transport tariffs, and the order sizes, and is in general too complex to analyze. Another motivation for not optimizing the break quantities, is because the delivery leadtime of large orders is affected, and thus marketing aspects should also be taken into account. Therefore, the objective of this paper is to provide insight into the effect of the break quantity rule on the inventory holding costs. In Section 3 the performance of the rule is tested for a wide range of parameter values. In Section 4 the main conclusions are reported.

2 Inventory costs

Consider a 1–warehouse, N–retailer distribution system, where inventory is reviewed periodically. At every review, retailer j ($j = 1, \ldots, N$) places an order at the warehouse to raise the inventory position to the retailer's order–up–to–level, and the warehouse places an order at the outside supplier, with a size equal to the sum of all the retailer orders. Observe that the ordersize of the warehouse is always equal to the system demand in the previous period. If the warehouse placed a replenishment order in period t, this order will arrive in period $t + L$. At this moment, the products are allocated among the retailers, such that the probability of a stockout at retailer j ($j = 1, \ldots, N$) at the end of period $t + L + \ell_j$ is equal for all retailers, where ℓ_j denotes the warehouse–retailer j leadtime. We assume that this so–called *equal fractile position* does not imply a negative allocation to one or more retailers, i.e. we adopt the *balance assumption.* The sequence of events in any period is: review, order, arrival of a replenishment order, demand occurs.

Demand that cannot be satisfied from stock on hand will be backlogged. At each review a holding cost h is charged for every unit of stock on hand, and for every unit backlogged a penalty cost p is charged. Since all stockpoints place a replenishment order at every review, and all demand is backlogged, the ordering costs do not influence the optimal policy, and therefore they will not be considered.

Customers at retailer j ($j = 1, \ldots, N$) arrive according to a Poisson process with arrival rate λ_j, and the sizes of the consecutive orders are i.i.d. with distribution function F_j. We consider this demand process because it enables us to distinguish between customers with small and with large orders. If the break quantity rule is applied, with a break quantity vector $\boldsymbol{q} := \{q_1, \ldots, q_N\}$, then all orders at retailer j with a size exceeding the retailer j break quantity q_j will be delivered from the warehouse. As mentioned in the introduction, we assume that all stock at the warehouse is dedicated for large demand. Hence, it is possible to model this situation as a 1–warehouse, $(N+1)$–retailers distribution system, where retailer 0 denotes the warehouse. Clearly, the warehouse–retailer 0 leadtime ℓ_0 equals zero.

If a customer arrives at retailer j with an order of size Y_j, then the ordersize with respect to this retailer equals $Y_j 1_{\{Y_j \le q_j\}}$, while for retailer 0 it equals $Y_j 1_{\{Y_j > q_j\}}$, with 1_X the indicator function of the event X. By this observation, we obtain that the mean and variance of the demand at retailer j ($j = 1, \ldots, N$) are given by

$$\mu_j = \lambda_j E[Y_j 1_{\{Y_j \le q_j\}}] \tag{1}$$

$$\sigma_j^2 = \lambda_j E[Y_j^2 1_{\{Y_j \le q_j\}}] \tag{2}$$

and at retailer 0 by

$$\mu_0 = \sum_{j=1}^{N} \lambda_j E[Y_j 1_{\{Y_j > q_j\}}] \tag{3}$$

$$\sigma_0^2 = \sum_{j=1}^{N} \lambda_j E[Y_j^2 1_{\{Y_j > q_j\}}] \tag{4}$$

Since the leadtime demand is a compound Poisson sum of i.i.d. random variables, we may apply the central limit theorem [10], and approximate the distribution of leadtime demand by the normal distribution. Hence, we can follow the approach of Eppen and Schrage [6] to obtain the following result, which is proved in Appendix 1.

Theorem 2.1 *If the system demand during L periods, and the retailer j ($j = 0, \ldots, N$) demand during $\ell_j + 1$ periods are normally distributed, then the minimum average costs, as a function of the break quantity vector $\boldsymbol{q}$, are given by*

$$\begin{aligned} C(\boldsymbol{q}) &= h \sum_{j=1}^{N} \ell_j \lambda_j E[Y_j 1_{\{Y_j \le q_j\}}] \\ &\quad + c_1 \left[\left(\sqrt{\sum_{j=1}^{N} \lambda_j E[Y_j^2 1_{\{Y_j > q_j\}}]} + \sum_{j=1}^{N} \sqrt{(\ell_j + 1) \lambda_j E[Y_j^2 1_{\{Y_j \le q_j\}}]} \right)^2 + L \sum_{j=1}^{N} E[Y_j^2] \right]^{1/2} \end{aligned}$$

with $c_1 := (p+h)\varphi(k)$, $k = \Phi^{-1}(\frac{p}{p+h})$ *with* $\varphi(\cdot)$ *and* $\Phi(\cdot)$ *the pdf and cdf of the standard normal distribution.*

The first term of $C(\boldsymbol{q})$ denotes the pipeline holding costs for all stock in transit between the warehouse and the retailers. If the break quantity rule is applied, part of the inventory is held back at the warehouse, and thus these costs decrease. The second term denotes the costs of holding inventory at the retailers and the warehouse. One can see that it is difficult to predict whether or not these costs will decrease if the break quantity rule is applied with an arbitrary $\boldsymbol{q}$. Therefore, in the next section we perform many numerical tests to analyze the reduction in average costs for different parameter combinations.
To conclude this section, we discuss the situation where all retailers are identical. In this case we can omit the subscript j. From Theorem 2.1 the next result follows directly.

Corollary 2.1 *If the system demand during L periods, the demand at retailer* j ($j = 1,\dots,N$) *during* $\ell+1$ *periods, and the one–period demand at retailer* 0 *are normally distributed,* and *all retailers are identical, then the minimum average costs, as a function of the break quantity* q *are given by*

$$\begin{aligned} C(q) &= hN\ell\lambda E[Y1_{\{Y\le q\}}] \\ &\quad + c_1\left[\left(\sqrt{N\lambda E[Y^2 1_{\{Y>q\}}]} + N\sqrt{\lambda(\ell+1)E[Y^2 1_{\{Y\le q\}}]}\right)^2 + LN\lambda E[Y^2]\right]^{1/2} \end{aligned}$$

3 Computational results

In this section we analyze the effect of the break quantity rule on the inventory costs, for a wide range of parameter values. However, we limit ourselves to the cases where respectively 90%, 95% and 99% of the customers are satisfied by the retailers. The reason for this is twofold. First, due to fixed overhead costs at the retailers and for competitive reasons (e.g. fast deliveries) it is desirable that a large percentage of customers is served through the retailers. Secondly, in case of small break quantities the transportation costs are very likely to increase considerably, due to a significant reduction in the average shipment size. In this section we only analyze the identical retailer case. In order to obtain that a fraction β of customers is satisfied by the retailers, we determine the break quantity by $q = \inf\{y > 0 : F(y) \ge \beta\}$. Moreover, it is assumed that the ordersizes are Gamma distributed. The motivation for this choice lies in the fact that this distribution only allows positive order sizes, it is easy to fit around the first two moments, and it allows for an easy calculation of $E[Y1_{\{Y\le q\}}]$ and $E[Y^2 1_{\{Y\le q\}}]$. To calculate the Gamma distribution function, an approximation by Abramowitz and Stegun [1] was used. For $\beta = 0.90$, $\beta = 0.95$ and $\beta = 0.99$, we analyzed the cost reduction obtained by applying the break quantity rule, for 5184 parameter combinations. In Table 1 the tested values of the parameters are presented.

parameter	symbol	values tested
unit holding cost	h	1
unit penalty cost	p	$5, 10, 15$
'external' leadtime	L	$1, 5, 10$
'internal' leadtime	ℓ	$1, 3, 5$
arrival rate	λ	$5, 10, 20$
mean order size	$E[Y]$	$5, 10, 15, 20$
squared coefficient of variation	$c_Y^2 := VAR[Y]/(E[Y])^2$	$0.5, 1, 1.5, 2$
number of retailers	N	$1, 5, 10, 15$

Table 1: Tested values of parameters

In Table 2 the minimum, average and maximum cost reduction are presented for all three values of β. The cost reduction was calculated as $100 \times \frac{C(\infty)-C(q)}{C(\infty)}$, where $C(\infty)$ equals the inventory costs when the break quantity rule is not applied. In the last row the percentage of cases where the break quantity rule was cost effective is reported. In Figure 1 the relative frequencies of all cost reductions are presented for all values of β. For all 5184 cases, the cost reductions were rounded to the nearest integer. For each value the relative frequency is plotted, and a line is drawn trough all the obtained points. From this figure one may conclude that overall the break quantity performs well. Note that the peaks in the figure are caused by the fact that we used discrete values for the parameters.

	$\beta = 0.90$	$\beta = 0.95$	$\beta = 0.99$
minimum cost reduction	-12.45	-13.44	-13.98
average cost reduction	19.53	10.84	1.10
maximum cost reduction	41.93	27.12	8.23
percentage of positive cost reductions	92.44	86.50	56.02

Table 2: Relative cost reductions obtained by the break quantity rule

On average, the break quantity rule leads to a reduction in the inventory holding costs. However, in some cases the rule leads to an *increase* in the expected costs, in particular when the break quantity is large (i.e. when $\beta = 0.99$ the expected costs increase in 44% of the cases).

After analyzing the results we observed that the values of N, c_Y^2 and ℓ had the largest impact on the performance of the break quantity rule. We now discuss each of them in detail. The minimum cost reductions were in all cases obtained when the number of retailers was equal

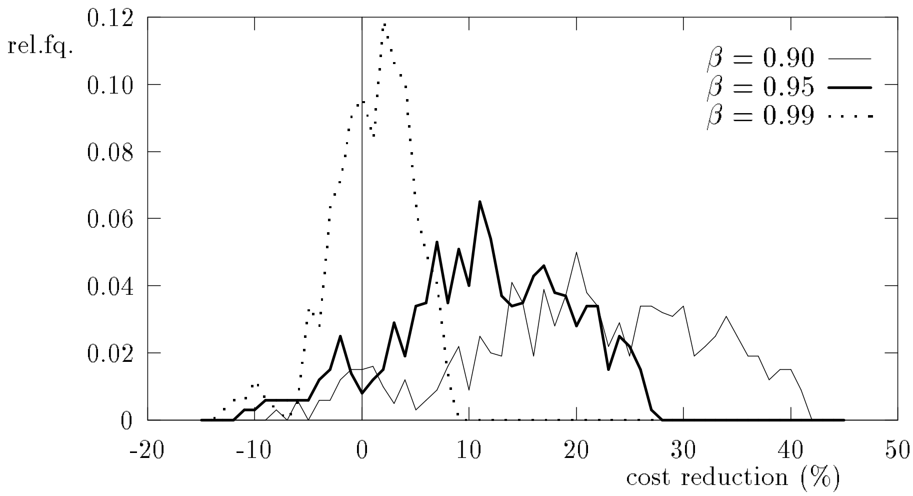

Figure 1: Relative frequencies (rel.fq.) of cost reductions

to one. To analyze the effect of the number of retailers, we considered the performance of the break quantity rule for $N = 1$ and $N = 15$ separately. From Table 3 and Figure 2 one observes that the number of retailers is very important for the results.

$N = 1$	$\beta = 0.90$	$\beta = 0.95$	$\beta = 0.99$
minimum cost reduction	-12.15	-13.44	-13.98
average cost reduction	5.87	0.67	-3.76
maximum cost reduction	27.82	15.40	1.68
percentage of positive cost reductions	69.75	48.77	1.23
$N = 15$	$\beta = 0.90$	$\beta = 0.95$	$\beta = 0.99$
minimum cost reduction	11.27	5.49	-0.21
average cost reduction	27.41	16.75	4.02
maximum cost reduction	41.93	27.12	8.23
percentage of positive cost reductions	100	100	91.67

Table 3: Relative cost reduction for $N = 1$ and $N = 15$

For $N = 15$ and $\beta = 0.90$ or $\beta = 0.95$ the break quantity rule leads to a cost reduction *in all cases*! For $\beta = 0.99$ and $N = 15$ the worst case leads to an increase in costs of 0.21%. On the other hand, for $N = 1$ and $\beta = 0.99$, the rule leads to an increase in inventory costs in almost 99% of the tested cases. Note from Table 2 and Table 3 that for all values of β the maximum cost reduction was obtained for $N = 15$. Hence, we conclude that delivering large

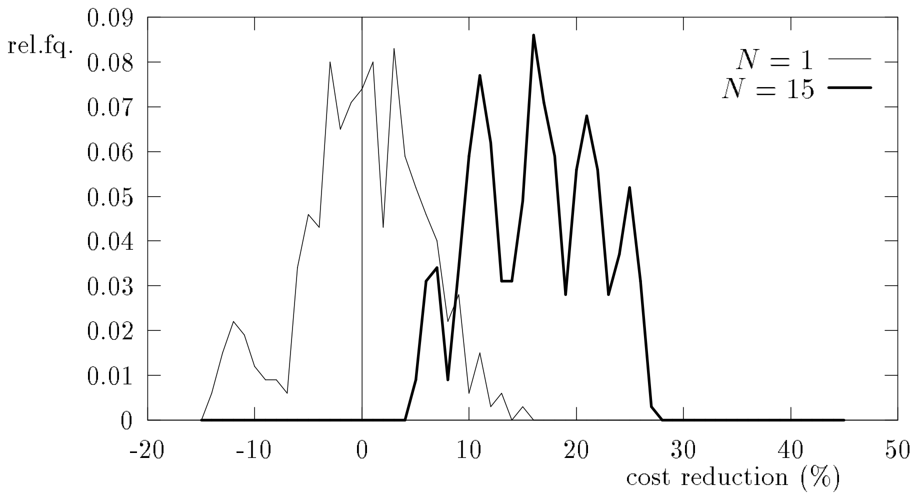

Figure 2: Relative frequencies of cost reductions for $N = 1$ and $N = 15$ ($\beta = 0.95$)

orders from the warehouse is most advantageous in distribution systems with many retailers. Since the main advantage of the break quantity rule is the reduction of the demand variability at the retailers, we next analyze the influence of the squared coefficient of variation of the order sizes (c_Y^2) on the performance of the rule. In Table 4 and Figure 3 results are presented for $c_Y^2 = 0.5$ and $c_Y^2 = 2.0$.

$c_Y^2 = 0.5$	$\beta = 0.90$	$\beta = 0.95$	$\beta = 0.99$
minimum cost reduction	-12.15	-13.44	-10.85
average cost reduction	12.10	5.63	-0.36
maximum cost reduction	22.81	13.10	3.07
percentage of positive cost reductions	90.12	79.32	35.19
$c_Y^2 = 2.0$	$\beta = 0.90$	$\beta = 0.95$	$\beta = 0.99$
minimum cost reduction	-5.37	-11.01	-13.98
average cost reduction	25.83	15.46	2.55
maximum cost reduction	41.93	27.12	8.23
percentage of positive cost reductions	94.14	90.74	67.54

Table 4: Relative cost reduction for $c_Y^2 = 0.5$ and $c_Y^2 = 2.0$

As expected, when demand is highly variable the rule performs better. Note that for $\beta = 0.90$ the percentage of cases where the break quantity rule leads to a cost reduction does not differ much for $c_Y^2 = 0.5$ or $c_Y^2 = 2.0$. However, the difference in magnitude of the cost reduction

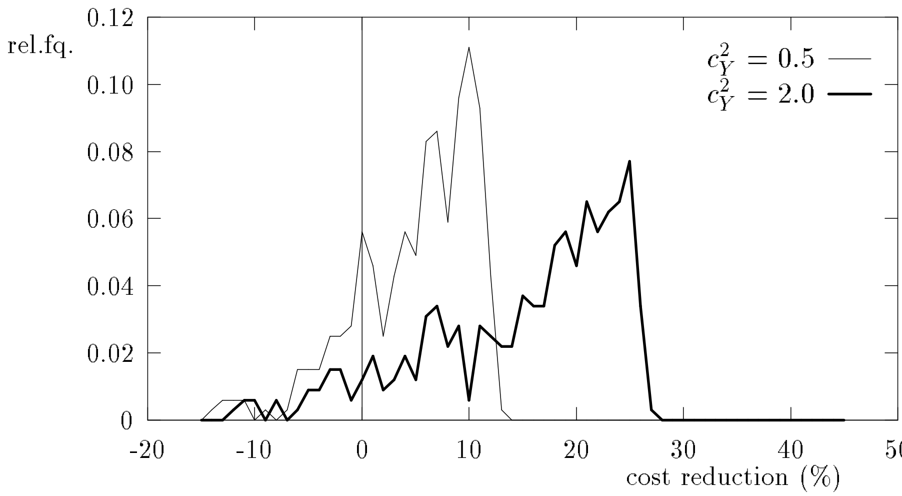

Figure 3: Relative frequencies of cost reductions for $c_Y^2 = 0.5$ and $c_Y^2 = 2.0$ ($\beta = 0.95$)

is very large. Observe from Table 2 and Table 4 that the maximum cost reductions were all obtained for $c_Y^2 = 2.0$.

In Section 2 it was mentioned that using break quantities leads to a reduction in the pipeline inventory costs. Since these costs are linearly dependent on the warehouse–retailer leadtime ℓ, it is obvious that this leadtime will have a significant impact on the results. In the final table and figure the cost reductions are presented for $\ell = 1$ and $\ell = 5$.

$\ell = 1$	$\beta = 0.90$	$\beta = 0.95$	$\beta = 0.99$
minimum cost reduction	-12.15	-13.44	-13.98
average cost reduction	13.21	6.22	-1.14
maximum cost reduction	35.45	22.12	5.25
percentage of positive cost reductions	78.70	72.92	31.71
$\ell = 5$	$\beta = 0.90$	$\beta = 0.95$	$\beta = 0.99$
minimum cost reduction	2.92	-1.41	-4.80
average cost reduction	24.48	14.46	2.87
maximum cost reduction	41.93	27.12	8.23
percentage of positive cost reductions	100	97.92	72.45

Table 5: Relative cost reduction for $\ell = 1$ and $\ell = 5$

For large values of ℓ, the break quantity rule leads in almost all cases to a reduction in the inventory holding costs. Again, the maximum and minimum cost reduction were obtained

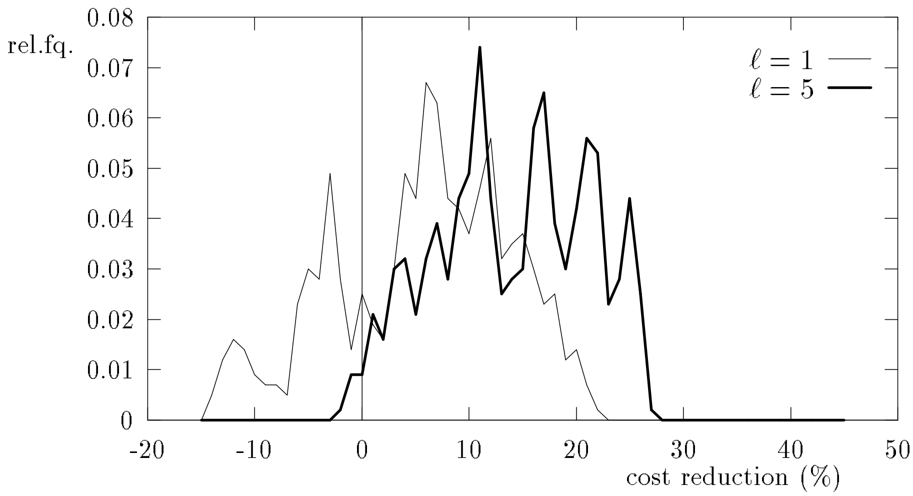

Figure 4: Relative frequencies of cost reductions for $\ell = 1$ and $\ell = 5$ ($\beta = 0.95$)

for $\ell = 5$ and $\ell = 1$, respectively. Also, for $\ell = 5$ and $\beta = 0.90$ the costs decreased in all situations.

4 Conclusions

In most 1–warehouse, N–retailers distribution systems, it is assumed that all customer demand is satisfied by the retailers. However, it was shown by Dekker *et al.* [5] that delivering large orders from the warehouse can lead to a considerable reduction in the retailer's inventory costs. We used a break quantity to distinguish between small orders and large orders. In this paper the results of Dekker *et al.* [5] were extended by also including the inventory costs at the warehouse. Using the normal approximation of the leadtime demand, we derived an expression for the average inventory costs, as a function of the break quantities.
For the identical retailer case we investigated the performance of the so–called break quantity rule for a wide range of parameter combinations. It followed that the relative cost reduction was highly influenced by the number of retailers, the warehouse–retailer leadtime, and the variability of the order sizes. For all three parameters it holds that: *the larger, the better*. If the number of retailers is very small, say 1 or 2, then with a high probability the rule leads to an increase in the total costs. However, for a moderate and large number of retailers, the costs decreased in almost all cases. The demand variability seemed to have a large impact on the size of the cost reduction, rather than on the fact whether or not the break quantity rule is cost effective. Finally, the warehouse–retailer leadtime determines to a large extent the pipeline holding costs, and therefore affects the performance of the rule. It was observed that for relatively large 'internal' leadtimes, the costs decreased in almost all cases.

References

[1] M. Abramowitz and I. Stegun. *Handbook of Mathematical Functions.* Dover, New York, 1965.

[2] S. Axsäter. Simple solution procedures for a class of two–echelon inventory problems. *Operations Research*, 38:64–69, 1990.

[3] S. Axsäter. Approximating general multi–echelon inventory systems by Poisson models. *International Journal of Production Economics*, 35:201–206, 1994.

[4] A.J. Clark and H. Scarf. Optimal policies for a multi–echelon inventory problem. *Management Science*, 6:475–490, 1960.

[5] R. Dekker, J.B.G. Frenk, M.J. Kleijn, A.G. de Kok, and N. Piersma. On the use of break quantities in multi–echelon distribution systems. Technical Report 9548/A, Erasmus University Rotterdam, The Netherlands, 1995.

[6] G. Eppen and L. Schrage. Centralized ordering policies in a multi–warehouse system with lead times and random demand. In L.B. Schwarz, editor, *Multilevel production/inventory control systems: Theory and practice.* North–Holland, Amsterdam, 1981.

[7] G.D. Eppen. Effect of centralization on expected costs in multi–location newsboy problem. *Management Science*, 25:498–501, 1979.

[8] A. Federgruen and P. Zipkin. Approximations of dynamic, multilocation production and inventory problems. *Management Science*, 30:69–84, 1984.

[9] A. Federgruen and P. Zipkin. Computational issues in an infinite–horizon, multi–echelon inventory model. *Operations Research*, 32:818–836, 1984.

[10] W. Feller. *An Introduction to Probability Theory and Its Applications, vol. 2.* Wiley, New York, 1971.

[11] P.L. Jackson. Stock allocation in a two–echelon distribution system or "what to do until your ship comes in". *Management Science*, 34:880–895, 1988.

[12] H. Jönsson and E.A. Silver. Analysis of a two–echelon inventory control system with complete redistribution. *Management Science*, 33:215–227, 1987.

[13] A.G. de Kok. Demand management in a multi–stage distribution chain. Technical Report TUE/BDK/LBS/93–35, Eindhoven University of Technology, The Netherlands, 1993.

[14] L.J.G. Langenhoff and W.H.M. Zijm. An analytical theory of multi–echelon production/distribution systems. *Statistica Neerlandica*, 44:149–174, 1990.

[15] R. Nass, R. Dekker, and W. van Sonderen-Huisman. Distribution optimization by means of break quantities: A case study. Technical Report 9375/A, Econometric Institute, Erasmus University Rotterdam, 1993.

[16] E.L. Porteus. Stochastic inventory theory. In D.P. Heyman and M.J. Sobel, editors, *Handbooks in OR & MS, vol. 2*. Elsevier Science Publishers B.V., North–Holland, 1990.

[17] L.B. Schwarz. A model for assessing the value of warehouse risk–pooling: risk–pooling over outside–supplier leadtimes. *Management Science*, 35:828–842, 1989.

Appendix 1: Proof of Theorem 2.1

Define V as the system demand during the first L periods, and W_j as the demand at retailer j during $l_j + 1$ periods $(j = 0, \ldots, N)$
One can verify that

$$\begin{aligned}
&\mu_v := E[V] = L\sum_{i=0}^{N}\mu_i, \quad \sigma_v^2 := Var[V] = L\sum_{i=0}^{N}\sigma_i^2 \\
&\mu_{w,j} := E[W_j] = (\ell_j + 1)\mu_j, \quad \sigma_{w,j}^2 := Var[W_j] = (\ell_j+1)\sigma_j^2, \quad j = 0, \ldots, N
\end{aligned}$$

Suppose that at the beginning of period 0 the warehouse places an order at an outside supplier to raise the system inventory position to the level y. Then at the beginning of period L the system stock is equal to $y - V$. Now the stock is allocated among the retailers such that the probability of a stockout at retailer j at the end of period $L + \ell_j$ is equal for all retailers. Let x_j be the stock allocated to retailer j. Then the inventory at retailer j at the end of period $L + \ell_j$ equals $x_j - W_j$. By the balance assumption and the assumption that the leadtime demand is normally distributed, it follows that there exists some $\nu > 0$ such that

$$x_j = \mu_{w,j} + \nu\sigma_{w,j} \tag{5}$$

for all j. Using the relation $\sum_i x_i = y - V$, one can easily determine ν. Substituting ν in (5) it follows that

$$x_j = \mu_{w,j} + \frac{y - V - \sum_i \mu_{w,i}}{\sum_i \sigma_{w,i}}\sigma_{w,j}$$

At the end of period $L + \ell_j$ the stock level S_j at retailer j will be

$$\begin{aligned}
S_j &= \mu_{w,j} + \frac{y - V - \sum_i \mu_{w,i}}{\sum_i \sigma_{w,i}}\sigma_{w,j} - W_j \\
&= s_j - \xi_j
\end{aligned}$$

where

$$s_j := \mu_{w,j} + \frac{y - \sum_i \mu_{w,i}}{\sum_i \sigma_{w,i}}\sigma_{w,j}$$

and

$$\xi_j := W_j + \frac{\sigma_{w,j}}{\sum_i \sigma_{w,i}} V$$

Since V and W_j are normally distributed, it follows that ξ_j is also normally distributed, with mean $\bar{\mu}_j$ and variance $\bar{\sigma}_j^2$ given by

$$\begin{aligned} \bar{\mu}_j &= \mu_{w,j} + \frac{\sigma_{w,j}}{\sum_i \sigma_{w,i}} \mu_v \\ \bar{\sigma}_j^2 &= \sigma_{w,j}^2 + \frac{\sigma_{w,j}^2}{(\sum_i \sigma_{w,i})^2} \sigma_v^2 \end{aligned} \tag{6}$$

The expected one–period holding and penalty costs at retailer j are determined by

$$\int_0^{s_j} h(s_j - x) dG(x) + \int_{s_j}^{\infty} p(x - s_j) dG(x)$$

with $G(\cdot)$ the distribution function of ξ_j. By standard newsboy arguments (e.g. Porteus [16]) it follows that the optimal value of s_j is equal to

$$s_j^* = \bar{\mu}_j + \Phi^{-1}(\frac{p}{p+h})\bar{\sigma}_j$$

with $\Phi(\cdot)$ the standard normal distribution. Let $k := \Phi^{-1}(\frac{p}{p+h})$. Solving

$$s_j^* = \mu_{w,j} + \frac{y^* - \sum_i \mu_{w,i}}{\sum_i \sigma_{w,i}} \sigma_{w,j}$$

for the optimal system order–up–to–level y^* yields

$$y^* = \mu_v + \sum_i \mu_{w,i} + k \left[\left(\sum_i \sigma_{w,i} \right)^2 + \sigma_v^2 \right]^{1/2}$$

Observe that y^* is independent of j, and thus it is optimal for all retailers.
Using straightforward calculations one can show (e.g. Eppen [7], Eppen and Schrage [6], Porteus [16]) that the expected costs per period C_j for retailer j $(j = 0, \ldots, N)$ are given by

$$\begin{aligned} C_j &= h\ell_j\mu_j + h(s_j^* - \bar{\mu}_j) + (h+p)\bar{\sigma}_j R\left(\frac{s_j^* - \bar{\mu}_j}{\bar{\sigma}_j}\right) \\ &= h\ell_j\mu_j + (hk + (p+h)R(k))\bar{\sigma}_j \end{aligned} \tag{7}$$

where $R(z) := \int_z^\infty (x-z)d\Phi(x)$ denotes the right–hand unit normal linear–loss integral.
The expected total one–period system costs are given by

$$C := \sum_i C_i = h \sum_i \ell_i \mu_i + c_1 \sum_i \bar{\sigma}_i \tag{8}$$

with

$$c_1 := hk + (p+h)R(k) = (p+h)\varphi(k)$$

It follows by (6) that

$$\begin{aligned}
\sum_i \bar{\sigma}_i &= \sum_i \sqrt{\sigma_{w,i}^2 + \frac{\sigma_{w,i}^2}{(\sum_i \sigma_{w,i})^2}\sigma_v^2} \\
&= \sum_i \sqrt{\frac{\sigma_{w,i}^2(\sum_i \sigma_{w,i})^2 + \sigma_{w,i}^2\sigma_v^2}{(\sum_i \sigma_{w,i})^2}} \\
&= \sqrt{(\sum_i \sigma_{w,i})^2 + \sigma_v^2} \\
&= \sqrt{(\sum_i \sqrt{(\ell_i + 1)}\sigma_i)^2 + L\sum_i \sigma_i^2}
\end{aligned}$$

After substituting this expression into (8), together with (1), (2), (3) and (4), the desired result follows.