On regenerative processes and inventory control

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Abstract

In this paper we discuss a general framework for single item inventory control models. This framework is based on the regenerative structure of these models. Using results from the theory of regenerative processes a unified presentation of those models is presented. Although most of the results are already known for special cost structures this unified presentation yields us the possibility to show that the same techniques can be applied to each instance.

Keywords: Inventory, regenerative processes.

1 Introduction

Ever since in february 1913 Ford Harris [12] published the well-known economic order quantity (EOQ) model, the number of publications on inventory theory in scientific journals for management science and operations research has been rapidly growing. Because there are a lot of assumptions to be made in inventory models, e.g. with respect to the demand process, the inventory policy and the cost structure, a huge number of different inventory models have been analysed. In an excellent overview from 1990 by Chikán [4] 336 *different* models are discussed, based on about 160 publications.

What is still missing in the literature, in our opinion, is a general approach towards inventory models. It appears that most models can be analysed using the theory of regenerative processes. In particular, for order-up-to level models with a fixed reorder level or a fixed reorder interval, it can easily be shown that the inventory process is a regenerative process. Using this nice structure, one can derive the average costs and service levels from general expressions, leading to different results under different assumptions.

In this paper we will introduce a general framework in which a lot of inventory models are captured. In particular, inventory models where the demand process is an increasing Levy process (Protter [18]) and unsatisfied demand is completely backordered fit into this framework. In Section 2 we will derive general relations between the net stock process, inventory position process and demand process, and provide general expressions for performance characteristics such as average cost and service levels.

For the well-known (s, S) model, many results have appeared in the literature. This particular model has been studied extensively since it was proven that this policy is optimal under certain assumptions (see e.g. Iglehart [14], Veinott [26]), and since it allows for a nice analysis using renewal theory. A much less analysed model is the order-up-to level inventory model with fixed reorder intervals, the so-called (R, S) model. Although this model is in general suboptimal, it has a number of advantages. For example, it is easy to understand and to implement by practitioners, it results in a predictable workload and it is extensively used in practice (Hax & Candea [13]). Moreover, it is easy to extend this model to a multi-item situation, in terms of coordinating the

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ordering of different products (see e.g. Goyal & Satir [10]). In Section 3 we will analyse in detail the (R, S) inventory model, and show how to use the general framework to find expressions for the average cost and service levels. Also the optimisation problem associated with determining the optimal (R, S) policy is discussed. In Section 4 the (s, S) model is discussed, and the last section summarises the main results.

2 A general framework for single item inventory models

In inventory control the decision maker is dealing with two objectives. First of all, he likes to control the *cost* of keeping inventory, and secondly, he likes to achieve a certain *service level*. In order to control the cost of inventory and the service level the decision maker faces two main questions. These questions are when to order and how much to order. Clearly, the question of how much to order depends on the demand process the decision maker is expecting in the future. In particular, if at some time in the future the system is out of stock and during that period demand is arriving the decision maker needs to know whether this demand is lost or can be backlogged. To model this we will therefore distinguish between the so-called *lost sales* case and the *backlog* case. In the lost sales case it is assumed that any demand arising when the system is out of stock will be lost. In the backlog case it is assumed that any demand occurring when the system is out of stock will be backlogged and is filled as soon as a new replenishment arrives. In this paper we will only consider the backlog case since the lost sales case is more difficult to analyse. Clearly, the amount of backlog also depends on the time it takes before an order arrives and so we introduce the following assumption with respect to the arrival process of orders. In principle we can distinguish between so-called deterministic and stochastic lead times.

Property 2.1 *(deterministic lead times)*
If an order is placed at some time t this order arrives at the facility at time $t + L$ with $L \geq 0$ a fixed constant.

The constant L in Property 2.1 is called the lead time and when L is taken to be 0 this corresponds to instantaneous replenishments. Moreover, by the above property it follows that an order placed earlier than another order will arrive sooner at the facility and so no overtaking of orders takes place. This property plays a very important role in the mathematical analysis of the basic inventory models. A generalisation of the above property is now given by the next one.

Property 2.2 *(stochastic lead times)*
If the ith order is placed at time t_i then this order arrives at the facility at time $t_i + \boldsymbol{L}_i$ with $\boldsymbol{L}_i$ a nonnegative random variable. The random variables $\boldsymbol{L}_i$, $i \geq 1$, are independent and identically distributed.

In this paper we only consider deterministic lead times. However, the analysis easily extends to stochastic lead times under the assumption that orders do not overtake. To describe the behaviour of the inventory level we need to introduce the demand process for a single item. In this paper we will consider a *stochastic* demand process $\boldsymbol{D} = \{\boldsymbol{D}(t) : t \geq 0\}$ given by

$$\boldsymbol{D}(t) := \text{total demand for the item up to time } t$$

For the stochastic demand process we will restrict ourselves to a compound renewal process or an increasing Levy process. To define a compound renewal process we introduce a renewal process $\boldsymbol{N} = \{\boldsymbol{N}(t) : t \geq 0\}$ with independent and identically distributed interarrival times $\boldsymbol{T}_i$, $i \geq 1$, having a right continuous distribution F_T satisfying $F_T(0) = 0$ and $F_T(\infty) = 1$ (Karlin & Taylor [15]). This renewal process represents the arrival process of customers. Moreover, the nth arriving customer has demand $\boldsymbol{Y}_n$ with $\boldsymbol{Y}_n$, $n \geq 1$, denoting a sequence of nonnegative, independent and identically distributed random variables with finite first moment $\mu_1 > 0$ and right continuous distribution F_Y. Observe the random variable $\boldsymbol{Y}_n$, $n \geq 1$, is either concentrated on $\{0,1,2,\ldots\}$ or attains any value on $[0,\infty)$. It is also assumed that the renewal process $\boldsymbol{N}$ is independent of the sequence $\boldsymbol{Y}_n$, $n \geq 1$, and the compound renewal process $\boldsymbol{D} = \{\boldsymbol{D}(t) : t \geq 0\}$ is now defined as

$$\boldsymbol{D}(t) := \sum_{n=0}^{\boldsymbol{N}(t)} \boldsymbol{Y}_n, \quad \boldsymbol{Y}_0 := 0 \tag{1}$$

Since the renewal process $\boldsymbol{N}$ is càdlàg, i.e. it $I\!\!P$-a.s. ($I\!\!P$-almost surely) has sample paths which are right continuous with left limits, it follows that the compound renewal process $\boldsymbol{D}$ is also càdlàg. A special case of a compound renewal process is given by a compound Poisson process with arrival rate $\lambda > 0$ and by definition (see e.g. Çinlar [5]) this process has independent and stationary increments. Such a process is called a Levy process (Protter [18]) and is defined as follows.

Definition 2.1 *A stochastic process $\boldsymbol{X} = \{\boldsymbol{X}(t) : t \geq 0\}$ with $\boldsymbol{X}(0) = 0$ and state space $I\!\!R$ or $\mathbb{Z}$ is called a Levy process if*

1. *The stochastic process $\boldsymbol{X}$ has increments independent of the past. This means that the random variable $\boldsymbol{X}(t) - \boldsymbol{X}(s)$ is independent of $\{\boldsymbol{X}(u) : u \leq s\}$ for any $0 < s < t$.*
2. *The stochastic process $\boldsymbol{X}$ has stationary increments. This means that the random variable $\boldsymbol{X}(t) - \boldsymbol{X}(s)$ has the same distribution as $\boldsymbol{X}(t-s)$ for any $0 < s < t$. (notation: $\boldsymbol{X}(t) - \boldsymbol{X}(s) \stackrel{d}{=} \boldsymbol{X}(t-s)$)*
3. *The stochastic process $\boldsymbol{X}$ is càdlàg.*

Since $I\!\!P$-a.s. any demand process $\boldsymbol{D}$ has increasing sample paths, we call a Levy process satisfying this monotonicity property an *increasing Levy process*. It can be shown (Feller [8]) if the state space of the process $\boldsymbol{D}$ is given by $\{0,1,2,\ldots\}$ that the class of increasing Levy processes coincides with the class of compound Poisson processes with nonnegative integer valued demand. Moreover, if the state space is given by $[0,\infty)$ then the class of increasing Levy processes is much larger than the class of compound Poisson processes with nonnegative individual demand. An example of an increasing Levy process on $[0,\infty)$ which is not a compound Poisson process is given by a Gamma process (Feller [9]).
To introduce the cost structure of a single item model governed by some inventory control rule, we first need to define the following different inventory processes. First of all, consider the stochastic process $\boldsymbol{I} = \{\boldsymbol{I}(t) : t \geq 0\}$ with $\boldsymbol{I}(0) = S$ and

$$\boldsymbol{I}(t) := \text{actual stock on the shelves at time } t$$

This process is called the on-hand stock process and since we always assume that the demand process is càdlàg it is clear that the on-hand stock process is also càdlàg. Moreover, if we introduce the càdlàg stochastic process $\boldsymbol{B} = \{\boldsymbol{B}(t) : t \geq 0\}$ with $\boldsymbol{B}(0) = 0$ and

$$\boldsymbol{B}(t) := \text{amount of items backlogged at time } t$$

then the so-called net stock or net inventory process $\boldsymbol{IN} = \{\boldsymbol{IN}(t) : t \geq 0\}$ is defined by

$$\boldsymbol{IN}(t) := \boldsymbol{I}(t) - \boldsymbol{B}(t), t \geq 0$$

Clearly, $\boldsymbol{IN}(0) = \boldsymbol{I}(0) - \boldsymbol{B}(0) = S$ and by the definition of backlogging it follows that $\boldsymbol{IN}(t) > 0$ implies $\boldsymbol{B}(t) = 0$ and $\boldsymbol{IN}(t) \leq 0$ implies $\boldsymbol{I}(t) = 0$. Again, since the stochastic processes $\boldsymbol{I}$ and $\boldsymbol{B}$ are càdlàg, we obtain that the stochastic process $\boldsymbol{IN}$ is càdlàg. Moreover, if $\boldsymbol{\sigma}_1$ denotes the (possibly random) time that the first order is triggered, it follows for the backordering case that

$$\boldsymbol{IN}(t) = \boldsymbol{IN}(0) - \boldsymbol{D}(t) = S - \boldsymbol{D}(t)$$

for every $t < \boldsymbol{\sigma}_1 + L$. Finally, if the stochastic process $\boldsymbol{O} = \{\boldsymbol{O}(t) : t \geq 0\}$ with $\boldsymbol{O}(0) = 0$ is defined by

$$\boldsymbol{O}(t) := \text{ number of ordered and not yet delivered items at time } t$$

then the inventory position process $\boldsymbol{IP} = \{\boldsymbol{IP}(t) : t \geq 0\}$ is given by

$$\boldsymbol{IP}(t) := \boldsymbol{IN}(t) + \boldsymbol{O}(t), t \geq 0$$

Observe now, if the lead time L equals zero, then the inventory position equals the net stock, since in that case the outstanding orders are immediately delivered. Also, by the above definitions it follows that $\boldsymbol{I}(t)$ is always nonnegative, while $\boldsymbol{IN}(t)$ and $\boldsymbol{IP}(t)$ can also attain negative values. The next result relates for the backordering case the inventory position to the net stock. Observe that this result holds for any demand process $\boldsymbol{D}$ which is càdlàg.

Theorem 2.1 *If backordering occurs, $\boldsymbol{D}$ is a càdlàg demand process and Property 2.1 holds, then it follows for every $t \geq 0$ that*

$$\boldsymbol{IN}(t+L) = \boldsymbol{IP}(t) - (\boldsymbol{D}(t+L) - \boldsymbol{D}(t)) \qquad \mathbb{P} - a.s.$$

Proof: Since all orders have a fixed lead time L it follows that in the interval $(t, t+L]$ all orders $\boldsymbol{O}(t)$ outstanding at time t did arrive. Moreover, orders placed after time t did not arrive and so we obtain that

$$\begin{aligned} \boldsymbol{O}(t) &= \text{outstanding orders at time } t \\ &= \text{addition to net stock in } (t, t+L] \\ &=: \boldsymbol{A}(t, t+L] \end{aligned}$$

By the definition of the net stock process it follows in case of backordering that

$$\boldsymbol{IN}(t+L) = \boldsymbol{IN}(t) + \boldsymbol{A}(t, t+L] - (\boldsymbol{D}(t+L) - \boldsymbol{D}(t+))$$

with $\boldsymbol{D}(t+) := \lim_{s \downarrow t} D(s)$. Since $\boldsymbol{O}(t) = \boldsymbol{A}(t, t+L]$ and $\boldsymbol{IP}(t) = \boldsymbol{IN}(t) + \boldsymbol{O}(t)$ this yields by the right continuous sample paths of the demand process that

$$\begin{aligned} \boldsymbol{IN}(t+L) &= \boldsymbol{IN}(t) + \boldsymbol{O}(t) - (\boldsymbol{D}(t+L) - \boldsymbol{D}(t+)) \\ &= \boldsymbol{IP}(t) - (\boldsymbol{D}(t+L) - \boldsymbol{D}(t+)) \\ &= \boldsymbol{IP}(t) - (\boldsymbol{D}(t+L) - \boldsymbol{D}(t)) \qquad \mathbb{P} - \text{a.s.} \end{aligned}$$

which shows the desired result. □

By the above proof it is easy to verify that the same relation also holds for a random time $\boldsymbol{\sigma}$ and so it follows that

$$\boldsymbol{IN}(\boldsymbol{\sigma}+L) = \boldsymbol{IP}(\boldsymbol{\sigma}) - (\boldsymbol{D}(\boldsymbol{\sigma}+L) - \boldsymbol{D}(\boldsymbol{\sigma})) \qquad \mathbb{P} - \text{a.s.}$$

for any nonnegative random variable $\boldsymbol{\sigma}$. This observation will be used in the proof of the next theorem. Moreover, due to $\boldsymbol{D}$ and $\boldsymbol{IN}$ are càdlàg, it is clear by Theorem 2.1 that also the inventory position process is càdlàg. This sample path property is needed to show that the inventory position process, under some mild additional condition, has a limiting distribution if $\boldsymbol{IP}$ is a regenerative process. As will be shown in the next sections, this is indeed the case for some important inventory models and by means of Theorem 2.1 it enables us to derive (asymptotic) results for the net stock process. Since the inventory holding and shortage cost of an inventory system clearly depends on the net stock process, the last observation yields us the possibility to evaluate the so-called average cost for a given inventory control rule. To continue, we now introduce the definition of a regenerative process (Asmussen [3]).

Definition 2.2 *A stochastic process $\boldsymbol{X} = \{\boldsymbol{X}(t) : t \geq 0\}$ with metric state space $\mathcal{E}$ is called a regenerative process if there exists an increasing sequence $\boldsymbol{\sigma}_n$, $n \geq 0$, with $\boldsymbol{\sigma}_0 := 0$ of random points, such that*

1. *The random variables* $\boldsymbol{\sigma}_{n+1} - \boldsymbol{\sigma}_n$, $n \geq 0$, *are independent and identically distributed with right continuous distribution* F *satisfying* $F(0) = 0$ *and* $F(\infty) = 1$.

2. *For each* $n \geq 0$ *the post-*$\boldsymbol{\sigma}_n$ *process*

$$\{\boldsymbol{X}(t + \boldsymbol{\sigma}_n) : t \geq 0\}$$

is independent of $\boldsymbol{\sigma}_0, \ldots, \boldsymbol{\sigma}_n$.

3. *The distribution of* $\{\boldsymbol{X}(t + \boldsymbol{\sigma}_n) : t \geq 0\}$ *is independent of* n.

To show that $\boldsymbol{IN}$ is a regenerative process in case $\boldsymbol{IP}$ is a regenerative process, we need the following definition (Protter [18]).

Definition 2.3 *Let* $\boldsymbol{X} = \{\boldsymbol{X}(t) : t \geq 0\}$ *be a stochastic process with metric state space* $\mathcal{E}$ *and* $\boldsymbol{\sigma}$ *a nonnegative random variable. The random variable* $\boldsymbol{\sigma}$ *is called a stopping time with respect to* $\boldsymbol{X}$ *if for every* $t \geq 0$ *the occurrence of the event* $\{\boldsymbol{\sigma} \leq t\}$ *only depends on* $\{\boldsymbol{X}(s) : s \leq t\}$.

In the remainder we will assume that the inventory control rule is based on the demand process. This means that the decision to order at time t depends on the realisation of the demand process up to time t. This also holds for the (random) size of the corresponding order. By this assumption it follows that the realisation of the inventory position process up to time t is a function of the realisation of the demand process up to time t and this observation will be used in the proof of the next theorem. Observe that all well-known inventory control rules satisfy the above property and some of these policies will be discussed at the end of this section. It is now possible to prove the following result.

Theorem 2.2 *If* $\boldsymbol{D}$ *is an increasing Levy process and the inventory position process* $\boldsymbol{IP}$ *is regenerative with increasing sequence* $\boldsymbol{\sigma}_n$, $n \geq 0$, *of random points, and for every* $n \geq 0$ *the random variable* $\boldsymbol{\sigma}_n$ *is a stopping time with respect to* $\boldsymbol{D}$ *then the process* $\tilde{\boldsymbol{IN}} := \{\boldsymbol{IN}(t + L) : t \geq 0\}$ *is also a regenerative process with the same sequence of random points.*

Proof: By the remark after Theorem 2.1 it follows for every $n \geq 0$ that

$$\boldsymbol{IN}(t + \boldsymbol{\sigma}_n + L) = \boldsymbol{IP}(t + \boldsymbol{\sigma}_n) - (\boldsymbol{D}(t + \boldsymbol{\sigma}_n + L) - \boldsymbol{D}(t + \boldsymbol{\sigma}_n)) \qquad I\!\!P - \text{a.s.} \tag{2}$$

Since $\boldsymbol{D}$ is an increasing Levy process and $\boldsymbol{\sigma}_n$ is a stopping time with respect to $\boldsymbol{D}$ it is shown by Theorem 32, Chapter I.4 of Protter [18] that the process $\boldsymbol{D}_{\boldsymbol{\sigma}_n} := \{\boldsymbol{D}(t + \boldsymbol{\sigma}_n) - \boldsymbol{D}(\boldsymbol{\sigma}_n) : t \geq 0\}$ is again an increasing Levy process with $\boldsymbol{D}_{\boldsymbol{\sigma}_n}$ having the same distribution as $\boldsymbol{D}$ and $\boldsymbol{D}_{\boldsymbol{\sigma}_n}$ independent of $\{\boldsymbol{D}(s) : s \leq \boldsymbol{\sigma}_n\}$. Because $\boldsymbol{\sigma}_0, \ldots, \boldsymbol{\sigma}_n$ are increasing stopping times of $\boldsymbol{D}$ and hence determined by $\{\boldsymbol{D}(s) : s \leq \boldsymbol{\sigma}_n\}$ it follows that $\{\boldsymbol{D}(t + \boldsymbol{\sigma}_n + L) - \boldsymbol{D}(t + \boldsymbol{\sigma}_n) : t \geq 0\}$ is independent of $\boldsymbol{\sigma}_0, \ldots, \boldsymbol{\sigma}_n$. Applying now relation (2) and $\boldsymbol{IP}$ is a regenerative process with random points $\boldsymbol{\sigma}_n$, $n \geq 0$, we obtain by condition 2 of Definition 2.2 that

$$\{\boldsymbol{IN}(t + L + \boldsymbol{\sigma}_n) : t \geq 0\}$$

is independent of $\boldsymbol{\sigma}_0, \ldots, \boldsymbol{\sigma}_n$. Moreover, since $\boldsymbol{D}$ is an increasing Levy process it follows due to $t + \boldsymbol{\sigma}_n$ is also a stopping time with respect to $\boldsymbol{D}$ that again by Theorem 32, Chapter I.4 of Protter [18] the random variable $\boldsymbol{D}(t+\boldsymbol{\sigma}_n+L)-\boldsymbol{D}(t+\boldsymbol{\sigma}_n)$ is independent of $\{\boldsymbol{D}(s) : s \leq t+\boldsymbol{\sigma}_n\}$ for every $t \geq 0$. This yields by the observation before Theorem 2.2 that $\boldsymbol{D}(t+\boldsymbol{\sigma}_n+L)-\boldsymbol{D}(t+\boldsymbol{\sigma}_n)$ is independent of $\{\boldsymbol{IP}(s) : s \leq t + \boldsymbol{\sigma}_n\}$ and so using $\boldsymbol{D}(t + \boldsymbol{\sigma}_n + L) - \boldsymbol{D}(t + \boldsymbol{\sigma}_n) \stackrel{d}{=} \tilde{\boldsymbol{D}}(L)$ with $\tilde{\boldsymbol{D}}$ having the same distribution as $\boldsymbol{D}$ we obtain by the regenerative property of $\boldsymbol{IP}$ and relation (2) that

$$\boldsymbol{IN}(t + \boldsymbol{\sigma}_n + L) \stackrel{d}{=} \boldsymbol{IP}(t + \boldsymbol{\sigma}_n) - \tilde{\boldsymbol{D}}(L)$$

with $\tilde{\boldsymbol{D}}$ independent of $\boldsymbol{IP}(t + \boldsymbol{\sigma}_n)$ and $\boldsymbol{IP}(t + \boldsymbol{\sigma}_n) \stackrel{d}{=} \boldsymbol{IP}(t)$. Hence, we have verified the third condition of Definition 2.2 and since the first condition trivially holds by the assumption that $\boldsymbol{IP}$ is a regenerative process with random points $\boldsymbol{\sigma}_n$, $n \geq 0$, the desired result is proved. □

In case $\boldsymbol{D}$ is a compound renewal process one can use the following result to show that the process $\tilde{\boldsymbol{IN}} = \{\boldsymbol{IN}(t+L) : t \geq 0\}$ is a regenerative process. Since this result can be verified by elementary probability theory we omit its proof.

Theorem 2.3 *If the stochastic process $\{(\boldsymbol{IP}(t), \boldsymbol{D}(t+L)-\boldsymbol{D}(t)) : t \geq 0\}$ is a regenerative process with increasing sequence $\boldsymbol{\sigma}_n$, $n \geq 0$, then the process $\tilde{\boldsymbol{IN}} = \{\boldsymbol{IN}(t+L) : t \geq 0\}$ is also regenerative with the same sequence of random points.*

Observe that Theorem 2.2 can also be derived from Theorem 2.3 by the observation that the joint stochastic process $\{(\boldsymbol{IP}(t), \boldsymbol{D}(t+L) - \boldsymbol{D}(t)) : t \geq 0\}$ is a regenerative process if $\boldsymbol{IP}$ is a regenerative process with an increasing sequence $\boldsymbol{\sigma}_n$, $n \geq 0$, of random points, which are also stopping times with respect to $\boldsymbol{D}$, and $\boldsymbol{D}$ is an increasing Levy process.
To introduce a cost structure measuring the inventory holding and shortage cost we now consider a nonnegative Borel measurable function $f : \mathbb{R} \to \mathbb{R}_+$. This function is called a costrate function and since it is clear that the (càdlàg) net stock process $\boldsymbol{IN}$ determines the cost of the inventory system we assume that the following property always holds.

Property 2.3 *The cumulative stochastic process $\{\int_0^t f(\boldsymbol{IN}(y))dy : t \geq 0\}$ is well defined and for every $t \geq 0$ the random variable $\int_0^t f(\boldsymbol{IN}(y))dy$ has a finite expectation.*

Clearly, the random variable $\int_0^t f(\boldsymbol{IN}(y))dy$ represents the (random) cost up to time t of the inventory system. Moreover, if we introduce the function $\tilde{v} : [0, \infty) \to \mathbb{R}_+$ given by

$$\tilde{v}(t) := \mathbb{E}(\int_0^t f(\tilde{\boldsymbol{IN}}(y))dy) = \mathbb{E}(\int_0^t f(\boldsymbol{IN}(y+L))dy)$$

then we obtain by Property 2.3 and f nonnegative that the function $\tilde{v}$, representing the expected cost of the process $\tilde{\boldsymbol{IN}}$ up to time t, is nonnegative, increasing and finite valued for every $t \geq 0$. For $\tilde{\boldsymbol{IN}}$ a regenerative process the next result relates the expected cost up to time t to the expected cost occurring within the first cycle. Observe for any nonnegative random variable $\boldsymbol{\sigma}$ we define $t \wedge \boldsymbol{\sigma} := \min\{t, \boldsymbol{\sigma}\}$.

Theorem 2.4 *If $\tilde{\boldsymbol{IN}} := \{\boldsymbol{IN}(t+L) : t \geq 0\}$ is a regenerative process with an increasing sequence $\boldsymbol{\sigma}_n$, $n \geq 0$, of random points then it follows that*

$$\tilde{v}(t) = \tilde{v}_0(t) + \int_0^t \tilde{v}_0(t-x)M(dx)$$

with $M(x) := \sum_{k=1}^{\infty} F^{k*}(x)$ the well-known renewal function associated with the distribution F of the cycle lengths $\boldsymbol{\sigma}_{n+1} - \boldsymbol{\sigma}_n$, $n \geq 0$, and $\tilde{v}_0 : [0, \infty) \to \mathbb{R}$ given by

$$\tilde{v}_0(t) := \mathbb{E}(\int_0^{t \wedge \boldsymbol{\sigma}_1} f(\tilde{\boldsymbol{IN}}(y))dy)$$

Proof: For every $t \geq 0$ it follows by the additive property of the integral that

$$\begin{aligned} \tilde{v}(t) &= \mathbb{E}(\int_0^{t \wedge \boldsymbol{\sigma}_1} f(\tilde{\boldsymbol{IN}}(y))dy) + \mathbb{E}(\int_{t \wedge \boldsymbol{\sigma}_1}^t f(\tilde{\boldsymbol{IN}}(y))dy) \\ &= \tilde{v}_0(t) + \mathbb{E}(\int_{t \wedge \boldsymbol{\sigma}_1}^t f(\tilde{\boldsymbol{IN}}(y))dy) \end{aligned}$$

To analyse the last term we observe that

$$\mathbb{E}(\int_{t \wedge \boldsymbol{\sigma}_1}^t f(\tilde{\boldsymbol{IN}}(y))dy) = \int_0^t \mathbb{E}(\int_{t \wedge \boldsymbol{\sigma}_1}^t f(\tilde{\boldsymbol{IN}}(y))dy | \boldsymbol{\sigma}_1 = x)F(dx)$$

Since $\tilde{\boldsymbol{IN}}$ is a regenerative process this yields for every $x \leq t$ by first applying condition 2 and then condition 3 of Definition 2.2 that

$$\begin{aligned}
&I\!\!E(\int_{t\wedge\boldsymbol{\sigma}_1}^{t} f(\tilde{\boldsymbol{IN}}(y))dy|\boldsymbol{\sigma}_1 = x) \\
&= I\!\!E(\int_{x}^{t} f(\tilde{\boldsymbol{IN}}(y))dy|\boldsymbol{\sigma}_1 = x) \quad = \quad I\!\!E(\int_{0}^{t-x} f(\tilde{\boldsymbol{IN}}(y+\boldsymbol{\sigma}_1))dy|\boldsymbol{\sigma}_1 = x) \\
&= I\!\!E(\int_{0}^{t-x} f(\tilde{\boldsymbol{IN}}(y+\boldsymbol{\sigma}_1))dy) \quad = \quad I\!\!E(\int_{0}^{t-x} f(\tilde{\boldsymbol{IN}}(y))dy) \\
&= \tilde{v}(t-x)
\end{aligned}$$

Hence, the function $\tilde{v}$ satisfies the so-called renewal equation

$$\tilde{v}(t) = \tilde{v}_0(t) + \int_0^t \tilde{v}(t-x)F(dx), \quad t \geq 0$$

and since by Property 2.3 the function $\tilde{v}_0$ is bounded on finite intervals this implies by Theorem 2.4, Chapter 4 of Asmussen [3] the desired result. □

It is easy to show by standard techniques and using Theorem 2.4 and $\tilde{v}_0$ is an increasing nonnegative function that by the weak renewal theorem (Ross [20]) it follows that

$$\lim_{t\uparrow\infty} \frac{\tilde{v}(t)}{t} = \frac{\tilde{v}_0(\infty)}{I\!\!E\boldsymbol{\sigma}_1}$$

if $\tilde{\boldsymbol{IN}}$ is a regenerative process with an increasing sequence $\boldsymbol{\sigma}_n$, $n \geq 0$, of random points satisfying $0 < I\!\!E\boldsymbol{\sigma}_1 < \infty$. Observe by the definition of $\tilde{v}_0$ that

$$\tilde{v}_0(\infty) = I\!\!E(\int_0^{\boldsymbol{\sigma}_1} f(\tilde{\boldsymbol{IN}}(y))dy) = I\!\!E(\int_L^{\boldsymbol{\sigma}_1+L} f(\boldsymbol{IN}(y))dy)$$

and this may attain the value ∞. By this observation the following corollary immediately follows.

Corollary 2.1 *If* $\tilde{\boldsymbol{IN}} := \{\boldsymbol{IN}(t+L) : t \geq 0\}$ *is a regenerative process with an increasing sequence* $\boldsymbol{\sigma}_n$, $n \geq 0$, *of random points satisfying* $0 < I\!\!E\boldsymbol{\sigma}_1 < \infty$ *then it follows that*

$$\lim_{t\uparrow\infty} \frac{v(t)}{t} = \frac{I\!\!E(\int_L^{\boldsymbol{\sigma}_1+L} f(\boldsymbol{IN}(y))dy)}{I\!\!E\boldsymbol{\sigma}_1}$$

with the function $v : [0,\infty) \to I\!\!R$ *given by*

$$v(t) := I\!\!E(\int_0^t f(\boldsymbol{IN}(y))dy)$$

Proof: Since $\tilde{v}(t) = v(t+L) - v(L)$ for every $t \geq 0$ and $v(L)$ is finite by Property 2.3 the desired result follows by the previous remarks. □

The value $\lim_{t\uparrow\infty} v(t)/t$ is called the expected average cost of the inventory system and by Corollary 2.1 it exists and equals the expected cost of the first cycle divided by the expected length of the first cycle. Applying now the above results one only needs to show for a given inventory system, governed by some control rule, that the inventory position process is regenerative for $\boldsymbol{D}$ an increasing Levy process or the process $\{(\boldsymbol{IP}(t), \boldsymbol{D}(t+L) - \boldsymbol{D}(t)) : t \geq 0\}$ is regenerative for $\boldsymbol{D}$ a compound renewal process. This is easy to verify for the so-called order-up-to level inventory models and this will be the topic of the next sections. To conclude our discussion on regenerative processes we notice for f a continuous costrate function with finite supnorm that the stochastic process $\{f(\tilde{\boldsymbol{IN}}(t)) : t \geq 0\}$ is càdlàg and regenerative if $\tilde{\boldsymbol{IN}}$ is a (càdlàg) regenerative process. It is now possible to show by applying much more elaborate proof techniques as used for the expected average case that the pointwise limit of $\mathbb{E}f(\tilde{\boldsymbol{IN}}(t))$ exists as t converges to infinity if additionally the distribution F of the cycle length is nonlattice and $0 < \mathbb{E}\boldsymbol{\sigma}_1 < \infty$ (see Theorem 1.2, Chapter 5 of Asmussen [3]). By Corollary 2.1 it follows that

$$\lim_{t\uparrow\infty} \mathbb{E}f(\boldsymbol{IN}(t)) = \lim_{t\uparrow\infty} \mathbb{E}f(\tilde{\boldsymbol{IN}}(t)) = \frac{\mathbb{E}(\int_L^{\boldsymbol{\sigma}_1+L} f(\boldsymbol{IN}(y))dy)}{\mathbb{E}\boldsymbol{\sigma}_1}$$

for any continuous f with a finite supnorm, and this implies by approximating a step function with one discontinuity from above and below by a sequence of continuous functions with uniformly bounded supnorm that

$$\lim_{t\uparrow\infty} \mathbb{P}\{\boldsymbol{IN}(t) \leq x\} = \frac{\mathbb{E}(\int_L^{\boldsymbol{\sigma}_1+L} 1_{\{\boldsymbol{IN}(y)\leq x\}} dy)}{\mathbb{E}\boldsymbol{\sigma}_1}$$

An example of an often used costrate function is given by

$$f(x) = \begin{cases} hx & \text{if } x \geq 0 \\ -px & \text{if } x < 0 \end{cases} \tag{3}$$

Clearly, for this costrate function $h > 0$ represents the inventory cost per unit of inventory in stock per unit of time. Moreover, $p > 0$ represents the shortage cost per backordered unit per unit of time. It is sometimes difficult to estimate in practice the shortage cost (unless specified by some contract!) and so if it is not possible to give an accurate estimation of this cost we can circumvent this by imposing a so-called service level constraint. Although different types of service level constraints exist in the literature (Silver & Peterson [24]), we only mention the so-called β-service level representing the restriction that the ratio of the long-run expected demand satisfied directly from stock on hand and the long-run expected demand is at least β with β some prespecified number between 0 and 1. If we introduce the (càdlàg) stochastic process $\boldsymbol{V} = \{\boldsymbol{V}(t) : t \geq 0\}$ with

$$\boldsymbol{V}(t) := \text{total amount of items backordered up to time } t$$

then the β-service level equals

$$\lim_{t\uparrow\infty} \left(1 - \frac{\mathbb{E}\boldsymbol{V}(t)}{\mathbb{E}\boldsymbol{D}(t)}\right)$$

provided that this limit exists. Clearly, if $\boldsymbol{D}$ is an increasing Levy process we obtain that $\mathbb{E}\boldsymbol{D}(t) = \gamma t$ for some $\gamma > 0$ and so the existence of the above limit is equivalent to the existence of $\lim_{t\uparrow\infty} \frac{1}{t}\mathbb{E}\boldsymbol{V}(t)$. Moreover, if $\boldsymbol{D}$ is a compound renewal process it follows that $\mathbb{E}\boldsymbol{D}(t) = \mu_1 \mathbb{E}\boldsymbol{N}(t)$ and since we always assume that the interarrival times $\boldsymbol{T}_i$, $i \geq 1$, of the customers have a positive finite expectation this yields again, using the weak renewal theorem (Ross [20]), that the existence of the above limit is equivalent to the existence of $\lim_{t\uparrow\infty} \frac{1}{t}\mathbb{E}\boldsymbol{V}(t)$. In the next sections we will show that this limit indeed exists for the most important inventory control models by using the regenerative structure of the inventory position process. Moreover, we will also identify this limit. Observe for inventory models with a service level constraint we set the costrate function f equal to zero on $(-\infty, 0)$. In the next sections we will analyse in detail the following basic inventory rules.

1. (R, S) rule.

   In this rule every R time units an order is placed if the inventory position at that time is below S. The size of the order is such that the inventory position is raised to order-up-to level S. The variables $R > 0$ and $S \geq 0$ are decision variables and need to be chosen optimally dependent on the cost structure and possibly a service level constraint.

2. (s, S) rule.

   In this rule an order is triggered at the moment the inventory position hits the reorder level $s < S$ and the size of the order is such that the inventory position is raised to order-up-to level S. Clearly, the decision variables s and S have to be chosen optimally given a certain cost structure and possibly a service level constraint.

We are now interested in the determination of the average cost under the different decision rules with fixed decision parameters if we assume additionally that placing an order will incur a setup cost $K > 0$. This yields the objective function of our optimisation problem over all feasible decision variables. The above decision rules with fixed decision parameters induce a regenerative structure on the net stock process and so it is possible to analyse the above inventory models by the theory of regenerative processes. This final remark concludes our discussion of the general framework of inventory models.

3 The (R, S) inventory model

In this section we will analyse a stochastic inventory model governed by an (R, S) rule with a fixed lead time $L > 0$ and a general costrate function f satisfying Property 2.3. In Section 3.1 a general analysis with respect to the average cost and related topics is presented and in Section 3.2 the determination of the optimal policy is discussed. Finally, in Section 3.3 we specialise the results for the piecewise linear costrate function introduced in (3).

3.1 General analysis

Before discussing the most fundamental observation for the (R, S) inventory model we assume without loss of generality that for a given (R, S) policy we have $\boldsymbol{IN}(0) = \boldsymbol{IP}(0) = S$. It is now possible to show the following result.

Theorem 3.1 *The inventory position process* $\boldsymbol{IP}$ *governed by an* (R, S) *rule and an increasing Levy demand process is a regenerative process with the increasing sequence of points given by* $0, R, 2R, \ldots$*. Moreover, the net stock process* $\tilde{\boldsymbol{IN}}$ *is regenerative with the same sequence of points.*

Proof: By the definition of the (R, S) policy it follows that

$$\begin{aligned} \boldsymbol{IP}(t + nR) &= \boldsymbol{IP}(nR) - (\boldsymbol{D}(t + nR) - \boldsymbol{D}(nR)) \\ &= S - (\boldsymbol{D}(t + nR) - \boldsymbol{D}(nR)) \end{aligned}$$

for every $0 \leq t < R$ and $n \geq 0$. This implies due to $\boldsymbol{\sigma}_n = nR$ (see Definition 2.2) for every $n \geq 0$ that condition 1 of Definition 2.2 is trivially satisfied, while conditions 2 and 3 of Definition 2.2 are a direct consequence of the above equality and $\boldsymbol{D}$ an increasing Levy process. This shows the first part and the second part is an immediate consequence of Theorem 2.2 and the observation that $\boldsymbol{\sigma}_n$ is a (trivial) stopping time with respect to $\boldsymbol{D}$. □

By the memory property of an arbitrary renewal process it is clear that the inventory position process is not regenerative with regeneration points $\boldsymbol{\sigma}_n = nR$. In this case condition 3 of Definition 2.2 is not satisfied. However, if the arrival process is in equilibrium, i.e. it is a delayed renewal process with delay distribution given by the equilibrium distribution (Asmussen [3]), it is not difficult to verify that the distribution of $\boldsymbol{IP}(t + nR)$, $t \geq 0$, does not depend on n and so in this case $\boldsymbol{IP}$ is a regenerative process with regeneration points $\boldsymbol{\sigma}_n = nR$, $n \geq 0$. The next result is an easy consequence of Theorem 3.1 and Corollary 2.1.

Theorem 3.2 *If the demand process* $\boldsymbol{D}$ *is an increasing Levy process then it follows that the average cost* $\Phi(R, S)$ *associated with a fixed* (R, S) *policy* ($R > 0$, $S \geq 0$) *is given by*

$$\Phi(R, S) = \frac{1}{R}\left(K\mathbb{P}\{\boldsymbol{D}(R) > 0\} + \int_0^R \mathbb{E}f(S - \boldsymbol{D}(y + L))dy\right)$$

Proof: By Theorem 3.1 and Corollary 2.1 it follows that the average holding and shortage cost associated with the costrate function f is given by

$$\begin{aligned}\frac{1}{R}\mathbb{E}\left(\int_L^{R+L} f(\boldsymbol{IN}(y))dy\right) &= \frac{1}{R}\mathbb{E}\left(\int_0^R f(\boldsymbol{IN}(y+L))dy\right) \\ &= \frac{1}{R}\mathbb{E}\left(\int_0^R f(S-\boldsymbol{D}(y+L))dy\right)\end{aligned}$$

Moreover, by the renewal reward theorem (Ross [20]) and the definition of an (R,S) policy the average ordering cost is given by

$$\frac{1}{R}K\mathbb{P}\{\boldsymbol{D}(R)>0\}$$

and adding these two components yields the desired result. □

To analyse the stochastic process $\boldsymbol{V}$, introduced at the end of Section 2, we consider first the related (càdlàg) stochastic process $\boldsymbol{B}=\{\boldsymbol{B}(t):t\geq 0\}$ with

$$\boldsymbol{B}(t) := \text{amount of items backordered at time } t$$

Since in the interval $[(n-1)R+L, nR+L)$, $n\geq 1$, it can only happen that no order arrives or an order arrives at time $(n-1)R+L$, we obtain that the number $\boldsymbol{B}_n$ of items backordered within the interval $[(n-1)R+L, nR+L)$ is given by

$$\boldsymbol{B}_n = \boldsymbol{B}((nR+L)-) - \boldsymbol{B}((n-1)R+L)$$

with $\boldsymbol{B}(t-) := \lim_{s\uparrow t}\boldsymbol{B}(s)$. By the definition of the net stock process it follows that $\boldsymbol{B}(t) = (-\boldsymbol{IN}(t))^+$ with $(x)^+ := \max\{0,x\}$ and so

$$\boldsymbol{B}_n = (-\tilde{\boldsymbol{IN}}((nR)-))^+ - (-\tilde{\boldsymbol{IN}}((n-1)R))^+ \tag{4}$$

It is now possible to show the following result.

Theorem 3.3 *If the demand process* $\boldsymbol{D}$ *is an increasing Levy process it follows for a given* (R,S) *policy* $(R>0, S\geq 0)$ *that*

$$\lim_{t\uparrow\infty}\frac{\mathbb{E}\boldsymbol{V}(t)}{t} = \frac{1}{R}\left(\mathbb{E}\left((\boldsymbol{D}((R+L)-)-S)^+\right) - \mathbb{E}\left((\boldsymbol{D}(L)-S)^+\right)\right)$$

Proof: By Theorem 3.1 we know that $\tilde{\boldsymbol{IN}}$ is a regenerative process with increasing sequence $\boldsymbol{\sigma}_n$, $n\geq 0$, given by $\boldsymbol{\sigma}_n = nR$. This implies by (4) that the random variable $\boldsymbol{B}_n$ is independent of $\boldsymbol{\sigma}_0,\ldots,\boldsymbol{\sigma}_{n-1}$ and

$$\boldsymbol{B}_n \stackrel{d}{=} (-\tilde{\boldsymbol{IN}}(R-))^+ - (-\tilde{\boldsymbol{IN}}(0))^+$$

Since $\tilde{\boldsymbol{IN}}(t) = \boldsymbol{IN}(t+L) = S - \boldsymbol{D}(t+L)$ for every $t<\boldsymbol{\sigma}_1$ this yields that

$$\boldsymbol{B}_n \stackrel{d}{=} (\boldsymbol{D}((R+L)-)-S)^+ - (\boldsymbol{D}(L)-S)^+$$

Denoting now by $\boldsymbol{N}_{\boldsymbol{\sigma}}$ the renewal process associated with the sequence $\boldsymbol{\sigma}_n$, $n\geq 0$, it follows by the definition of the stochastic process $\boldsymbol{V}$ and the random variables $\boldsymbol{B}_n$, $n\geq 1$, that

$$\sum_{n=0}^{\boldsymbol{N}_{\boldsymbol{\sigma}}(t)}\boldsymbol{B}_n \leq \boldsymbol{V}(t+L)-\boldsymbol{V}(L) \leq \sum_{n=0}^{\boldsymbol{N}_{\boldsymbol{\sigma}}(t)+1}\boldsymbol{B}_n$$

for every $t\geq 0$ and $\boldsymbol{B}_0 := 0$. Hence, by a standard application of the renewal reward theorem (Ross [20]) we obtain that

$$\lim_{t\uparrow\infty}\frac{\mathbb{E}\boldsymbol{V}(t)}{t} = \lim_{t\uparrow\infty}\frac{\mathbb{E}\boldsymbol{V}(t+L)-\mathbb{E}\boldsymbol{V}(L)}{t+L} = \frac{1}{R}\mathbb{E}\boldsymbol{B}_1$$

and this shows the desired result. □

The above result concludes our general discussion of the (R,S) inventory model. In the next section we consider the associated optimisation problem with and without a β-service level constraint.

3.2 Optimisation

By Theorem 3.2 it follows for an (R,S) inventory model with an increasing Levy demand process and without a service level constraint, that the optimal (R,S) policy is a solution of the optimisation problem

$$\inf\{\frac{1}{R}(K\mathbb{P}\{\boldsymbol{D}(R)>0\}+\int_0^R \mathbb{E}f(S-\boldsymbol{D}(t+L))dt):R>0,S\in\mathcal{T}\} \tag{P}$$

with the set $\mathcal{T}$ either given by $[0,\infty)$ or by $\{0,1,2,\ldots\}$. In case $\mathcal{T}=[0,\infty)$ we assume that the demand process can attain any value on $[0,\infty)$ while for $\mathcal{T}=\{0,1,2,\ldots\}$ we only have integer valued demand. By the separability of the objective function the above optimisation problem reduces to

$$\inf\{\frac{1}{R}\left(K\mathbb{P}\{\boldsymbol{D}(R)>0\}+\varphi(R)\right):R>0\}$$

with the function $\varphi:(0,\infty)\to\mathbb{R}_+$ given by the optimisation problem

$$\varphi(R):=\inf\{\int_0^R \mathbb{E}f(S-\boldsymbol{D}(t+L))dt:S\in\mathcal{T}\} \tag{$\mathrm{P}_{\varphi(R)}$}$$

If $\mathcal{T}=[0,\infty)$ and the costrate function f is convex then the objective function of $(\mathrm{P}_{\varphi(R)})$ for R fixed is clearly convex on $(0,\infty)$ and so for this case the optimisation problem $(\mathrm{P}_{\varphi(R)})$ is a convex optimisation problem. Considering the first part of the optimisation problem (P) we obtain for a compound Poisson process that

$$\mathbb{P}\{\boldsymbol{D}(R)>0\}=\mathbb{P}\{\boldsymbol{N}(R)\geq 1\}=1-\exp(-\lambda R)$$

while for a Gamma process it follows that $\mathbb{P}\{\boldsymbol{D}(R)>0\}=1$. If $S(R)\in\mathcal{T}$ denotes an optimal solution of $(\mathrm{P}_{\varphi(R)})$ then by the above observations the optimisation problem (P) reduces to

$$\inf\{\frac{1}{R}(K\mathbb{P}\{\boldsymbol{D}(R)>0\}+\int_0^R \mathbb{E}f(S(R)-\boldsymbol{D}(t+L))dt):R>0\}$$

Next, the same model as above is considered, with a β-service level constraint. Introducing for a given (R,S) policy

$\beta(R,S)$:= the long-run expected demand satisfied directly from stock on hand divided by the long-run expected demand

the optimisation problem (P_β) with the β-service level constraint $\beta(R,S)\geq\beta$ is given by

$$\inf\left\{\begin{array}{l}\frac{1}{R}(K\mathbb{P}\{\boldsymbol{D}(R)>0\}+\int_0^R \mathbb{E}f(S-\boldsymbol{D}(t+L))dt): \\ R>0,S\in\mathcal{T},\beta(R,S)\geq\beta\}\end{array}\right. \tag{P_β}$$

To compute $\beta(R,S)$ we observe for an increasing Levy demand process with rate $\gamma>0$ that

$$\beta(R,S)=\lim_{t\uparrow\infty}\left(1-\frac{\mathbb{E}\boldsymbol{V}(t)}{\mathbb{E}\boldsymbol{D}(t)}\right)=1-\frac{1}{\gamma}\lim_{t\uparrow\infty}\frac{\mathbb{E}\boldsymbol{V}(t)}{t}$$

This implies by Theorem 3.3 that

$$\beta(R,S)=1-\frac{1}{\gamma R}\left(\mathbb{E}((\boldsymbol{D}((R+L)-)-S)^+)-\mathbb{E}((\boldsymbol{D}(L)-S)^+)\right)$$

The restriction $\beta(R,S) \geq \beta$ with $0 < \beta < 1$ can now be rewritten as

$$\mathbb{E}((\boldsymbol{D}((R+L)-) - S)^+) - \mathbb{E}((\boldsymbol{D}(L) - S)^+) \leq (1-\beta)\gamma R$$

To analyse the above restriction we observe since the function

$$S \to (\boldsymbol{D}((R+L)-) - S)^+ - (\boldsymbol{D}(L) - S)^+$$

is a (continuous) piecewise linear decreasing function on $[0,\infty)$ for every realisation of the demand process $\boldsymbol{D}$ that also the function

$$S \to \mathbb{E}((\boldsymbol{D}((R+L)-) - S)^+) - \mathbb{E}((\boldsymbol{D}(L) - S)^+)$$

is a (continuous) decreasing function on $[0,\infty)$. This implies that the restriction $\beta(R,S) \geq \beta$ can be replaced by $S \geq S(\beta, R)$ with

$$S(\beta,R) := \inf\{S \in \mathcal{T} : \mathbb{E}((\boldsymbol{D}((R+L)-) - S)^+) - \mathbb{E}((\boldsymbol{D}(L) - S)^+) \leq (1-\beta)\gamma R\}$$

Since the costrate function f satisfies $f(x) = 0$ for every $x < 0$ in case a service level constraint is used and in all cases the nonnegative function f is increasing on $(0,\infty)$ we obtain that the function

$$S \to \int_0^R \mathbb{E}f(S - \boldsymbol{D}(t+L))dt$$

is increasing on $[0,\infty)$. By this observation it now follows that (P_β) reduces to

$$\inf\{\frac{1}{R}(K\mathbb{P}\{\boldsymbol{D}(R) > 0\} + \int_0^R \mathbb{E}f(S(\beta,R) - \boldsymbol{D}(t+L))dt) : R > 0\}$$

and this shows the optimisation problem associated with a β-service level constraint.

3.3 Piecewise linear costrate function

To return to the first problem and analyse the optimisation problem ($\text{P}_{\varphi(R)}$) in more detail we consider now the piecewise linear costrate function f given by (3). Introducing for every $x \in \mathbb{R}$ the functions $(x)^+ := \max\{0,x\}$ and $(x)^- := \min\{0,x\}$ it follows due to $(x)^+ + (x)^- \equiv x$ that

$$\begin{aligned} f(S - \boldsymbol{D}(t+L)) &= h(S - \boldsymbol{D}(t+L))^+ - p(S - \boldsymbol{D}(t+L))^- \\ &= -pS + p\boldsymbol{D}(t+L) + (p+h)(S - \boldsymbol{D}(t+L))^+ \end{aligned} \tag{5}$$

Since $\mathbb{E}\boldsymbol{D}(t+L) = \gamma(t+L)$ for some $\gamma > 0$ (remember $\boldsymbol{D}$ is an increasing Levy process) we thus obtain that

$$\mathbb{E}f(S - \boldsymbol{D}(t+L)) = -pS + p\gamma(t+L) + (p+h)\mathbb{E}((S - \boldsymbol{D}(t+L))^+)$$

Hence, it follows that

$$\begin{aligned} \int_0^R \mathbb{E}f(S - \boldsymbol{D}(t+L))dt &= -pSR + \frac{1}{2}p\gamma((R+L)^2 - L^2) \\ &\quad + (p+h)\int_0^R \mathbb{E}((S - \boldsymbol{D}(t+L))^+)dt \end{aligned}$$

and this yields for every $R > 0$ that

$$\begin{aligned} \varphi(R) &= \frac{1}{2}p\gamma((R+L)^2 - L^2) \\ &\quad + \inf\{-pSR + (p+h)\int_0^R \mathbb{E}((S - \boldsymbol{D}(t+L))^+)dt : S \in \mathcal{T}\} \end{aligned}$$

To compute the optimal solution of the optimisation problem $(\mathrm{P}_{\varphi(R)})$ we observe by partial integration that

$$\mathbb{E}((S-\boldsymbol{D}(t+L))^{+}) = \int_0^S \mathbb{P}\{\boldsymbol{D}(t+L) \leq x\}dx$$

and so by Fubini's theorem (Mikolás [17]) we obtain that

$$\begin{aligned}\varphi(R) &= \frac{1}{2}p\gamma((R+L)^2-L^2)\\ &\quad +\inf\{-pSR+(p+h)\int_0^S\int_0^R \mathbb{P}\{\boldsymbol{D}(t+L)\leq x\}dtdx : S\in\mathcal{T}\}\end{aligned}$$

An alternative interpretation of the above integrand is given by the following result.

Lemma 3.1 *If for a given (R,S) policy the value $G_R(x)$ denotes the fraction of time that the net stock process is above level $S-x$, i.e.*

$$G_R(x) := \lim_{t\uparrow\infty}\frac{1}{t}\mathbb{E}\int_0^t 1_{\{\boldsymbol{IN}(y)\geq S-x\}}dy$$

with 1_A denoting the indicator function of the event A, then it follows that

$$G_R(x) = \frac{1}{R}\int_0^R \mathbb{P}\{\boldsymbol{D}(t+L)\leq x\}dt$$

Proof: Taking the costrate function f equal to $f(z)=1$ if $z\geq S-x$ and $f(z)=0$ otherwise it follows by Corollary 2.1 and Theorem 3.1 that

$$G_R(x) = \frac{1}{R}\mathbb{E}(\int_L^{R+L} f(\boldsymbol{IN}(t))dt)$$

Since $\boldsymbol{IN}(t) = S-\boldsymbol{D}(t)$ for every $t<R+L$ this implies that

$$\begin{aligned}G_R(x) &= \frac{1}{R}\mathbb{E}(\int_L^{R+L} f(S-\boldsymbol{D}(t))dt)\\ &= \frac{1}{R}\int_L^{R+L}\mathbb{E}f(S-\boldsymbol{D}(t))dt\\ &= \frac{1}{R}\int_0^R \mathbb{P}\{\boldsymbol{D}(t+L)\leq x\}dt\end{aligned}$$

and so the desired result is proved. □

By Lemma 3.1 the integrand corresponding with the optimisation problem $(\mathrm{P}_{\varphi(R)})$ is given by

$$-pSR+(p+h)R\int_0^S G_R(x)dx$$

Hence, if $\mathcal{T}=[0,\infty)$ the right-hand derivative with respect to S equals $-pR+(p+h)RG_R(S)$. Moreover, if $\mathcal{T}=\{0,1,2,\ldots\}$ then it follows for any $S\in\mathcal{T}$ due to $G_R(x)=G_R(S)$ if $S\leq x<S+1$ that

$$\begin{aligned}&-p(S+1)R+(p+h)R\int_0^{S+1}G_R(x)dx-(-pSR+(p+h)R\int_0^S G_R(x)dx)\\ &= -pR+(p+h)R\int_S^{S+1}G_R(x)dx\\ &= -pR+(p+h)RG_R(S)\end{aligned}$$

By these observations the next result follows immediately.

Lemma 3.2 *An optimal solution $S(R)$ of $(P_{\varphi(R)})$ is given by*

$$S(R) = \inf\{x \in \mathcal{T} : G_R(x) \geq \frac{p}{p+h}\}$$

Proof: The result follows by the observations before Lemma 3.2 and the necessary and sufficient first-order optimality conditions. □

Observe that the optimal order-up-to level must satisfy a newsboy type equation (see e.g. Hadley & Whitin [11]). In principle, it is now possible to determine $S(R)$ for R fixed by a classical bisection method given that it is possible to compute or approximate the distribution G_R. This will be the topic of the remainder of this section.

3.4 Determining the order-up-to level

In case the random variables $\boldsymbol{Y}_n$, $n \geq 1$, take values in $\{0, 1, \ldots\}$ one may introduce the "density" $g_R(n)$, given by

$$g_R(n) := \lim_{t \uparrow \infty} \frac{1}{t} \mathbb{E} \int_0^t 1_{\{\boldsymbol{IN}(y) = S - n\}} dy \tag{6}$$

Applying a similar argument as in the proof of Lemma 3.1 it follows that

$$g_R(n) = \frac{1}{R} \int_0^R \mathbb{P}\{\boldsymbol{D}(t+L) = n\} dt$$

Since any increasing Levy process with state space $\{0, 1, \ldots\}$ is necessarily a compound Poisson process it is possible to compute the distribution $\{g_R(n) : n = 0, 1, \ldots\}$ by means of a recurrent scheme. To derive such a scheme we need to compute the generating function $P(z)$ of the distribution $\{g_R(n) : n = 0, 1, \ldots\}$. Clearly, it follows that

$$\begin{aligned} P(z) &= \sum_{n=0}^{\infty} z^n g_R(n) \\ &= \frac{1}{R} \sum_{n=0}^{\infty} z^n \int_0^R \mathbb{P}\{\boldsymbol{D}(t+L) = n\} dt \\ &= \frac{1}{R} \int_0^R \mathbb{E}(z^{\boldsymbol{D}(t+L)}) dt \end{aligned}$$

Since for a compound Poisson process with state space $\{0, 1, \ldots\}$ it is easy to verify that

$$\mathbb{E}(z^{\boldsymbol{D}(t)}) = \exp(-\lambda t(1 - P_Y(z)))$$

with $P_Y(z)$ the generating function of the random variable $\boldsymbol{Y}_1$, we obtain that

$$\begin{aligned} P(z) &= \frac{1}{R} \int_0^R \exp(-\lambda(t+L)(1 - P_Y(z))) dt \\ &= \frac{1}{R} \int_L^{R+L} \exp(-\lambda t(1 - P_Y(z))) dt \\ &= \frac{1}{\lambda R(1 - P_Y(z))} (\exp(-\lambda L(1 - P_Y(z))) - \exp(-\lambda(R+L)(1 - P_Y(z)))) \end{aligned} \tag{7}$$

We will now derive an efficient recurrent scheme for the generating function

$$(1 - P_Y(z))^{-1} \exp(-\eta\ (1 - P_Y(z)))$$

for an arbitrary $\eta > 0$. By the above formula this yields a recurrent scheme for computing $g_R(n)$, $n = 1, 2, \ldots$. Before constructing this recurrent scheme we need the following result.

Lemma 3.3 *If the function* $C : [0, 1) \to \mathbb{R}$ *is given by* $C(z) = -\log(1 - P_Y(z))$ *then this function is the generating function of the sequence* $\{c_n : n = 0, 1, \ldots\}$ *with*

$$c_n := \sum_{k=1}^{\infty} \frac{1}{k} \mathbb{P}\{\mathbf{Y}_1 + \cdots + \mathbf{Y}_k = n\}$$

Moreover, it follows that $c_0 = -\log(1 - \mathbb{P}\{\mathbf{Y}_1 = 0\})$ *and for every* $k \geq 1$ *we obtain that*

$$\sum_{j=0}^{k-1} (F_Y(k) - F_Y(j)) = \sum_{j=0}^{k-1} (1 - F_Y(j))(k - j)c_{k-j}$$

Proof: By the Taylor expansion of the function $z \to -\log(1 - z)$ for $|z| < 1$ it follows that

$$\begin{aligned} C(z) &= -\log(1 - P_Y(z)) \\ &= \sum_{k=1}^{\infty} \frac{1}{k} (P_Y(z))^k \\ &= \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=0}^{\infty} \mathbb{P}\{\mathbf{Y}_1 + \cdots + \mathbf{Y}_k = n\} z^n \\ &= \sum_{n=0}^{\infty} c_n z^n \end{aligned}$$

and this shows the first part. Moreover, if we introduce the generating function $H(z)$ of the sequence $\{h_n : n = 1, 2, \ldots\}$ with $h_n := \theta^{-1}(n^{-1} - c_n)$ and $\theta = \log(\mu)$, $\mu = \mathbb{E}\mathbf{Y}_1$, then we obtain that

$$\begin{aligned} H(z) &= \frac{1}{\theta} \sum_{n=1}^{\infty} (\frac{1}{n} - c_n) z^n \\ &= \frac{1}{\theta} (\sum_{n=1}^{\infty} \frac{1}{n} z^n - C(z)) \\ &= \frac{1}{\theta} (\log(1 - P_Y(z)) - \log(1 - z)) \\ &= \frac{1}{\theta} \log\left(\frac{1 - P_Y(z)}{1 - z}\right) \end{aligned}$$

and this yields

$$\exp(-\theta(1 - H(z)) = \exp(-\theta) \exp(\theta H(z)) = \exp(-\theta) \frac{1 - P_Y(z)}{1 - z}$$

Since $\exp(-\theta) = 1/\mu$ we obtain that $\exp(-\theta(1 - H(z))$ is the generating function of the equilibrium distribution $\{\mu^{-1}(1 - F_Y(n)) : n = 0, 1, \ldots\}$. Applying now Adelson's recursion scheme [1] for $\exp(-\theta(1 - H(z)))$ we obtain for every $k \geq 1$ that

$$\frac{1}{\mu}(1 - F_Y(k)) = \frac{\theta}{k} \sum_{j=0}^{k-1} \frac{1}{\mu} (1 - F_Y(j))(k - j) h_{k-j}$$

and this yields by the definition of the sequence $\{h_n : n = 1, 2, \ldots..\}$ that the recurrent relation

$$\sum_{j=0}^{k-1} (F_Y(k) - F_Y(j)) = \sum_{j=0}^{k-1} (1 - F_Y(j))(k - j)c_{k-j}$$

holds. Moreover, since $\mathbf{Y}_n$, $n \geq 1$, are independent and identically distributed, it is easy to verify that $c_0 = -\log(1 - \mathbb{P}\{\mathbf{Y}_1 = 0\})$ and thus the desired result follows. □

In case the distribution of the demand $\boldsymbol{Y}_1$ is given by a Poisson distribution with parameter μ then it follows since the convolution of independent and Poisson distributed random variables is again Poisson distributed that

$$\begin{aligned} c_n &= \sum_{k=1}^{\infty} \frac{1}{k} \mathbb{P}\{\boldsymbol{Y}_1 + \cdots + \boldsymbol{Y}_k = n\} \\ &= \sum_{k=1}^{\infty} \frac{1}{k} \exp(-k\mu) \frac{(k\mu)^n}{n!} \\ &= \frac{\mu^n}{n!} \sum_{k=1}^{\infty} \exp(-k\mu) k^{n-1} \end{aligned}$$

and so in this case the value c_n, $n = 1, 2, \ldots$, has a nice analytical expression. One can now show the following recurrent relation for the generating function

$$R(z) = (1 - P_Y(z))^{-1} \exp(-\eta(1 - P_Y(z)))$$

Theorem 3.4 *If the generating function $R(z)$ of the sequence $\{r_n : n = 0, 1, \ldots\}$ equals $(1 - P_Y(z))^{-1} \exp(-\eta(1 - P_Y(z)))$ for some $\eta > 0$ then it follows that r_0 is given by*

$$r_0 = (1 - \mathbb{P}\{\boldsymbol{Y}_1 = 0\})^{-1} \exp(-\eta(1 - \mathbb{P}\{\boldsymbol{Y}_1 = 0\}))$$

while for $k \geq 1$ we obtain that

$$r_k = \frac{1}{k} \sum_{j=0}^{k-1} (k-j)(\eta \mathbb{P}\{\boldsymbol{Y}_1 = k-j\} + c_{k-j}) r_j$$

with $\{c_n : n = 1, 2, \ldots\}$ satisfying the recurrent relation of Lemma 3.3.

Proof: Clearly, the value of r_0 is given by the generating function $R(z)$ evaluated in 0. Moreover, if we denote by $R^{(j)}(z)$ the jth derivative of the generating function $R(z)$ at the point z then it follows for every $|z| < 1$ that

$$\begin{aligned} R^{(1)}(z) &= (\eta P_Y^{(1)}(z) + P_Y^{(1)}(z)(1 - P_Y(z))^{-1}) R(z) \\ &= (\eta P_Y^{(1)}(z) + C^{(1)}(z)) R(z) \end{aligned}$$

with $C(z) = -\log(1 - P_Y(z))$ the generating function discussed in Lemma 3.3. Applying now the product rule of differentiation it follows for every $k \geq 1$ that

$$R^{(k)}(z) = \sum_{j=0}^{k-1} \binom{k-1}{j} R^{(j)}(z) \left(\eta P_Y^{(k-j)}(z) + C^{(k-j)}(z)\right)$$

and this implies that

$$\begin{aligned} \frac{1}{k!} R^{(k)}(z) &= \sum_{j=0}^{k-1} \frac{(k-1)!(k-j)!}{k!(k-j-1)!} \frac{1}{j!} R^{(j)}(z) \left(\frac{\eta}{(k-j)!} P_Y^{(k-j)}(z) + \frac{1}{(k-j)!} C^{(k-j)}(z)\right) \\ &= \frac{1}{k} \sum_{j=0}^{k-1} (k-j) \frac{1}{j!} R^{(j)}(z) \left(\frac{\eta}{(k-j)!} P_Y^{(k-j)}(z) + \frac{1}{(k-j)!} C^{(k-j)}(z)\right) \end{aligned}$$

Taking now $z \downarrow 0$ we obtain the above recurrent relation. □

Step 0 Set $c_0 := -\log(1-F_Y(0))$, $r_0^L := (1-F_Y(0))^{-1}\exp(-\lambda L(1-F_Y(0)))$,
$r_0^{R+L} := (1-F_Y(0))^{-1}\exp(-\lambda(R+L)(1-F_Y(0)))$, $k := 0$
and $G := (r_0^L - r_0^{R+L})/\lambda R$
Step 1 while $G < p/(p+h)$ do:
$k := k+1$
$c_k := \frac{1}{k}(1-F_Y(0))^{-1}\left(\sum_{j=0}^{k}(F_Y(k)-F_Y(j)) - \sum_{j=1}^{k-1}(1-F_Y(j))(k-j)c_{k-j}\right)$
$r_k^L := \frac{1}{k}\sum_{j=0}^{k-1}(k-j)(\lambda L \mathbb{P}\{\boldsymbol{Y}_1 = k-j\} + c_{k-j})r_j^L$
$r_k^{R+L} := \frac{1}{k}\sum_{j=0}^{k-1}(k-j)(\lambda(R+L)\mathbb{P}\{\boldsymbol{Y}_1 = k-j\} + c_{k-j})r_j^{R+L}$
$G := G + (r_k^L - r_k^{R+L})/\lambda R$
Step 2 Take $S = k$ as an optimal solution.

Algorithm 3.1: Algorithm to determine optimal order-up-to level.

By the above result and (7) it is clear how to evaluate the distribution $\{g_R(n) : n = 0, 1, \ldots\}$ by means of a set of easy recurrent relations. The above results are summarised in Algorithm 3.1. It is also possible to compute the Laplace-Stieltjes transform $\mathcal{G}_R(\alpha)$ of the distribution G_R for an arbitrary increasing Levy process with state space $[0, \infty)$. Observe by Lemma 3.1 that

$$\begin{aligned}\mathcal{G}_R(\alpha) &:= \int_{0-}^{\infty}\exp(-\alpha x)dG_R(x)\\ &= \frac{1}{R}\int_0^R\int_{0-}^{\infty}\exp(-\alpha x)d\mathbb{P}\{\boldsymbol{D}(t+L)\le x\}dt\end{aligned}$$

Since it is well-known (Feller [9]) that the Laplace-Stieltjes transform of $\boldsymbol{D}(t)$ with $\boldsymbol{D}$ an increasing Levy process is given by $\exp(-t\gamma(\alpha))$ with $\gamma(0) = 0$ and the derivative $\gamma^{(1)}$ of γ a completely monotone function, we obtain by this observation that

$$\begin{aligned}\mathcal{G}_R(\alpha) &= \frac{1}{R}\int_0^R \exp(-(t+L)\gamma(\alpha))dt\\ &= \frac{1}{R}\int_L^{R+L}\exp(-t\gamma(\alpha))dt\\ &= \frac{1}{R\gamma(\alpha)}(\exp(-L\gamma(\alpha)) - \exp(-(R+L)\gamma(\alpha)))\end{aligned}$$

In case $\boldsymbol{D}$ is a compound Poisson process it follows that $\gamma(\alpha) = \lambda(1-\mathcal{F}_Y(\alpha))$ with $\mathcal{F}_Y$ the Laplace-Stieltjes transform of the demand distribution F_Y and $\lambda > 0$ the arrival rate of the Poisson process. Hence, in this case we obtain that

$$\mathcal{G}_R(\alpha) = \frac{1}{\lambda(1-\mathcal{F}_Y(\alpha))R}(\exp(-\lambda L(1-\mathcal{F}_Y(\alpha))) - \exp(-\lambda(R+L)(1-\mathcal{F}_Y(\alpha))))$$

By the unicity of the Laplace-Stieltjes transform the distribution G_R has the alternative representation

$$G_R(x) = \frac{1}{\lambda R}(\mathbb{E}U_Y(x-\boldsymbol{D}(L)) - \mathbb{E}U_Y(x-\boldsymbol{D}(R+L)))$$

where U_Y denotes the renewal function (with a renewal in 0!) associated with the demands $\boldsymbol{Y}_n$, $n \ge 1$, and this function is given by

$$U_Y(x) := \sum_{n=0}^{\infty} F_Y^{n*}(x)$$

for $x \ge 0$ and $U_Y(x) = 0$ for $x < 0$. We finally observe with respect to the optimisation problem $(\mathrm{P}_{\varphi(R)})$ that by the availability of the Laplace-Stieltjes transform it is possible to compute the first and second moment of the distribution G_R and so we could approximate G_R by a Gamma two moment fit. As observed by De Kok [16], approximating G_R by a Gamma distribution $G_R^{(app)}$ with the same first and second moment and solving $G_R^{(app)}(S) = p/(p+h)$ yields good approximative solutions.

4 The (s, S) inventory model

In this section we will analyse a stochastic inventory model governed by an (s, S) rule with a fixed lead time $L > 0$ and a general costrate function f satisfying Property 2.3. In Section 4.1 a general analysis with respect to the average cost and related topics is presented and in Section 4.2 the determination of the optimal policy is discussed. Finally, in Section 4.3 we specialise the results for the piecewise linear costrate function introduced in (3).

4.1 General analysis

Before discussing in Theorem 4.1 the most fundamental observation for the (s, S) inventory model we assume without loss of generality that for a given (s, S) policy we have $\boldsymbol{IN}(0) = \boldsymbol{IP}(0) = S$. By the definition of the (s, S) policy one can identify an increasing sequence $\boldsymbol{\sigma}_n$, $n \geq 0$, of stopping times ($\boldsymbol{\sigma}_0 := 0$) with respect to the càdlàg inventory position process $\boldsymbol{IP}$ and the càdlàg demand process $\boldsymbol{D}$, given by

$$\boldsymbol{\sigma}_n := \inf\{t > 0 : \boldsymbol{IP}(t) = S, \boldsymbol{IP}(t-) < S, t > \boldsymbol{\sigma}_{n-1}\}, n \geq 1$$

with $\boldsymbol{IP}(t-) := \lim_{s \uparrow t} \boldsymbol{IP}(s)$. These random points represent the reorder points, i.e. the random time $\boldsymbol{\sigma}_n$, $n \geq 1$, denotes the time of placing the nth order. Since the inventory position process $\boldsymbol{IP}$ is càdlàg it follows that the state space of this process is given by $[s, S]$ if the demand process $\boldsymbol{D}$ attains any value in $[0, \infty)$, and by $\{s, s+1, \ldots, S\}$ if the demand process is integer valued. Moreover, if the demand process $\boldsymbol{D}$ is an increasing Levy process or a compound renewal process an alternative representation of $\boldsymbol{\sigma}_n$ holds. Clearly,

$$\boldsymbol{\sigma}_n = \sum_{k=1}^{n} (\boldsymbol{\sigma}_k - \boldsymbol{\sigma}_{k-1})$$

and by the definition of the (s, S) policy it follows that

$$\boldsymbol{\sigma}_k - \boldsymbol{\sigma}_{k-1} = \inf\{t > 0 : \boldsymbol{D}(t + \boldsymbol{\sigma}_{k-1}) - \boldsymbol{D}(\boldsymbol{\sigma}_{k-1}) \geq \Delta\}$$

with $\Delta := S - s$. By the interpretation of the stopping time $\boldsymbol{\sigma}_{k-1}$ we obtain for any increasing Levy process and any compound renewal process that $\boldsymbol{D}_{\boldsymbol{\sigma}_k} = \{\boldsymbol{D}(t + \boldsymbol{\sigma}_k) - \boldsymbol{D}(\boldsymbol{\sigma}_k) : t \geq 0\}$ is again an increasing Levy process or a compound renewal process with $\boldsymbol{D}_{\boldsymbol{\sigma}_k}$ independent of $\{\boldsymbol{D}(t) : t \leq \boldsymbol{\sigma}_k\}$ and $\boldsymbol{D}_{\boldsymbol{\sigma}_k}$ has the same distribution as $\boldsymbol{D}$. By this observation it follows that $\boldsymbol{\sigma}_k - \boldsymbol{\sigma}_{k-1}$, $k \geq 1$, are independent and identically distributed with distribution given by the distribution of $\boldsymbol{\tau}(\Delta)$ where

$$\boldsymbol{\tau}(x) := \inf\{t > 0 : \boldsymbol{D}(t) \geq x\}, \quad x > 0$$

The most fundamental observation for the (s, S) inventory model is given by the next result.

Theorem 4.1 *The inventory position process $\boldsymbol{IP}$ governed by an (s, S) policy and an increasing Levy demand process or a compound renewal demand process is a regenerative process with an increasing sequence of random points given by $\boldsymbol{\sigma}_0, \boldsymbol{\sigma}_1, \ldots$. Moreover, the net stock process $\tilde{\boldsymbol{IN}}$ is regenerative with the same sequence of points.*

Proof: By the above remark it follows that condition 1 of Definition 2.2 holds. Moreover, by the definition of the (s, S) policy we obtain that

$$\begin{aligned} \boldsymbol{IP}(t + \boldsymbol{\sigma}_n) &= \boldsymbol{IP}(\boldsymbol{\sigma}_n) - (\boldsymbol{D}(t + \boldsymbol{\sigma}_n) - \boldsymbol{D}(\boldsymbol{\sigma}_n)) \\ &= S - (\boldsymbol{D}(t + \boldsymbol{\sigma}_n) - \boldsymbol{D}(\boldsymbol{\sigma}_n)) \end{aligned} \tag{8}$$

for every $0 \leq t < \boldsymbol{\sigma}_{n+1} - \boldsymbol{\sigma}_n$. Since the process $\boldsymbol{D}_{\boldsymbol{\sigma}_n}$ is independent of $\{\boldsymbol{D}(t) : t \leq \boldsymbol{\sigma}_n\}$ for any increasing Levy process or any compound renewal process and by the observation before Theorem 4.1 the random variables $\boldsymbol{\sigma}_0, \ldots, \boldsymbol{\sigma}_n$ are completely determined by $\{\boldsymbol{D}(t) : t \leq \boldsymbol{\sigma}_n\}$ it follows by (8) that condition 2 of Definition 2.2 holds. Finally, due to $\boldsymbol{D}_{\boldsymbol{\sigma}_n}$ has the same distribution as $\boldsymbol{D}$ we again obtain by (8) that condition 3 of Definition 2.2 holds. This shows the first part of the above result and the second part is an immediate consequence of Theorem 2.2 and the observation that $\boldsymbol{\sigma}_n$, $n \geq 0$, is a stopping time with respect to $\boldsymbol{D}$. □

By exactly the same arguments as used for the (R, S) inventory model (see Theorem 3.2) one can show the following result.

Theorem 4.2 *If the demand process $\boldsymbol{D}$ is an increasing Levy process or a compound renewal process then it follows for $\mathbb{E}\boldsymbol{\sigma}_1 > 0$ finite that the average cost $\Phi(\Delta, S)$, $\Delta := S - s$, associated with a given (s, S) policy, is given by*

$$\begin{aligned}\Phi(\Delta, S) &= \frac{1}{\mathbb{E}\boldsymbol{\sigma}_1}\left(K + \mathbb{E}\Big(\int_0^{\boldsymbol{\sigma}_1} f(S - \boldsymbol{D}(y+L))dy\Big)\right) \\ &= \frac{1}{\mathbb{E}\boldsymbol{\tau}(\Delta)}\left(K + \mathbb{E}\Big(\int_0^{\boldsymbol{\tau}(\Delta)} f(S - \boldsymbol{D}(y+L))dy\Big)\right)\end{aligned}$$

Proof: Similar as proof of Theorem 3.2. □

To verify whether the random variable $\boldsymbol{\sigma}_1$ has a finite expectation we observe due to $\boldsymbol{\sigma}_1 \stackrel{d}{=} \boldsymbol{\tau}(\Delta)$ that

$$\mathbb{E}\boldsymbol{\sigma}_1 = \int_0^\infty \mathbb{P}\{\boldsymbol{\sigma}_1 > t\}dt = \int_0^\infty \mathbb{P}\{\boldsymbol{\tau}(\Delta) > t\}dt = \int_0^\infty \mathbb{P}\{\boldsymbol{D}(t) < \Delta\}dt$$

Observe if the state space of the demand process $\boldsymbol{D}$ is given by $\{0, 1, \ldots\}$ then clearly $\Delta := S - s$ belongs to $\{1, 2, \ldots\}$ and in this case the above formula reduces to

$$\mathbb{E}\boldsymbol{\sigma}_1 = \int_0^\infty \mathbb{P}\{\boldsymbol{D}(t) \leq \Delta - 1\}dt$$

The next result provides a more simplified expression in case the demand process $\boldsymbol{D}$ is a compound renewal process.

Lemma 4.1 *If the demand process $\boldsymbol{D}$ is a compound renewal process with the distribution F_Y of the independent and identically distributed random variables $\boldsymbol{Y}_n$, $n \geq 1$, continuous, then it follows that*

$$\mathbb{E}\boldsymbol{\sigma}_1 = \mathbb{E}\boldsymbol{T}_1(1 + M_Y(\Delta))$$

where $M_Y(x) := \sum_{k=1}^\infty F_Y^{k*}(x)$ is the well-known renewal function associated with the distribution F_Y. For a lattice distribution F_Y concentrated at $\{0, 1, \ldots\}$ a similar result holds with $\Delta \in \mathbb{N}$ replaced by $\Delta - 1$.

Proof: Since $\boldsymbol{\sigma}_1 \stackrel{d}{=} \boldsymbol{\tau}(\Delta)$ the above result follows by showing that for every $x > 0$ we have $\mathbb{E}\boldsymbol{\tau}(x) = \mathbb{E}\boldsymbol{T}_1(1 + M_Y(x))$. Observe for F_Y continuous and $x > 0$ that ($\boldsymbol{Y}_0 := 0$!)

$$\begin{aligned}\mathbb{E}\boldsymbol{\tau}(x) &= \int_0^\infty \mathbb{P}\{\boldsymbol{\tau}(x) > t\}dt \\ &= \int_0^\infty \mathbb{P}\{\boldsymbol{D}(t) < x\}dt \\ &= \sum_{k=0}^\infty \int_0^\infty \mathbb{P}\{\boldsymbol{Y}_0 + \cdots + \boldsymbol{Y}_k < x, \boldsymbol{N}(t) = k\}dt\end{aligned}$$

Since the renewal process $\boldsymbol{N} := \{\boldsymbol{N}(t) : t \geq 0\}$ is independent of $\boldsymbol{Y}_n$, $n \geq 0$, this implies

$$\begin{aligned} \mathbb{E}\boldsymbol{\tau}(x) &= \sum_{k=0}^{\infty} \mathbb{P}\{\boldsymbol{Y}_0 + \cdots + \boldsymbol{Y}_k < x\} \int_0^{\infty} \mathbb{P}\{\boldsymbol{N}(t) = k\} dt \\ &= \sum_{k=0}^{\infty} \mathbb{P}\{\boldsymbol{Y}_0 + \cdots + \boldsymbol{Y}_k \leq x\} \int_0^{\infty} \mathbb{P}\{\boldsymbol{N}(t) = k\} dt \end{aligned}$$

where the last equality follows by the continuity of F_Y. Due to

$$\begin{aligned} &\int_0^{\infty} \mathbb{P}\{\boldsymbol{N}(t) = k\} dt = \int_0^{\infty} \mathbb{E}(1_{\{\boldsymbol{N}(t)=k\}}) dt \\ &= \mathbb{E}(\int_0^{\infty} 1_{\{\boldsymbol{N}(t)=k\}} dt) = \mathbb{E}\boldsymbol{T}_{k+1} = \mathbb{E}\boldsymbol{T}_1 \end{aligned}$$

we obtain the first part. If the lattice distribution F_Y is concentrated at $\{0, 1, \ldots\}$ it is easy to check in the above proof for $x \in \{1, 2, \ldots\}$ that $\mathbb{E}\boldsymbol{\tau}(x) = \mathbb{E}\boldsymbol{T}_1(1 + M_Y(x-1))$ and this shows the desired result. □

By exactly the same proof as for the (R, S) inventory model (see Theorem 3.3) and using that $\boldsymbol{\sigma}_n$, $n \geq 0$, is a stopping time with respect to the demand process $\boldsymbol{D}$ one can show the following result for the càdlàg stochastic process $\boldsymbol{V}$ introduced in Section 2.

Theorem 4.3 *If the demand process $\boldsymbol{D}$ is an increasing Levy process or a compound renewal process then it follows for a given (s, S) policy and $0 < \mathbb{E}\boldsymbol{\tau}(\Delta) < \infty$, $\Delta = S - s$, that*

$$\lim_{t \uparrow \infty} \frac{\mathbb{E}\boldsymbol{V}(t)}{t} = \frac{1}{\mathbb{E}\boldsymbol{\tau}(\Delta)} \left(\mathbb{E}\left((\boldsymbol{D}((\boldsymbol{\tau}(\Delta) + L)-) - S)^+\right) - \mathbb{E}\left((\boldsymbol{D}(L) - S)^+\right)\right)$$

Proof: See the proof of Theorem 3.3. □

In case the demand process $\boldsymbol{D}$ is a compound renewal process it is possible to give an alternative representation for $\mathbb{E}((\boldsymbol{D}((\boldsymbol{\tau}(\Delta) + L)-) - S)^+)$. If we denote by $\boldsymbol{\Gamma} = \{\boldsymbol{\Gamma}(t) : t \geq 0\}$ the so-called forward recurrence process associated with the demands $\boldsymbol{Y}_n$, $n \geq 1$, (Asmussen [3]) then it follows that

$$\boldsymbol{D}((\boldsymbol{\tau}(\Delta) + L)-) = \Delta + \boldsymbol{\Gamma}(\Delta) + \tilde{\boldsymbol{D}}(L-) \qquad \mathbb{P} - a.s.$$

with $\tilde{\boldsymbol{D}}(L)$ independent of $\boldsymbol{\Gamma}(\Delta)$ and $\tilde{\boldsymbol{D}}$ has the same distribution as $\boldsymbol{D}$. By the above equality this yields

$$\boldsymbol{D}((\boldsymbol{\tau}(\Delta) + L)-) - S \stackrel{d}{=} \boldsymbol{\Gamma}(\Delta) - s + \tilde{\boldsymbol{D}}(L-)$$

and hence

$$\mathbb{E}\left((\boldsymbol{D}((\boldsymbol{\tau}(\Delta) + L)-) - S)^+\right) = \mathbb{E}\left((\boldsymbol{\Gamma}(\Delta) - s + \tilde{\boldsymbol{D}}(L-))^+\right)$$

Moreover, if the distribution F_T of the interarrival times $\boldsymbol{T}_i$, $i \geq 1$, is continuous we obtain

$$\mathbb{E}\left((\boldsymbol{D}((\boldsymbol{\tau}(\Delta) + L)-) - S)^+\right) = \mathbb{E}\left((\boldsymbol{\Gamma}(\Delta) - s + \tilde{\boldsymbol{D}}(L))^+\right)$$

The above observation concludes our general discussion of the (s, S) inventory model. In the next section we consider the optimisation problem with and without a β-service level constraint.

4.2 Optimisation

By Theorem 4.2 it follows for an (s, S) inventory model with an increasing Levy demand process or a compound renewal demand process and without a service level constraint, that the optimal (s, S) policy is a solution of the optimisation problem

$$\inf\{\frac{1}{\mathbb{E}\boldsymbol{\tau}(\Delta)}\left(K+\int_0^{\boldsymbol{\tau}(\Delta)}\mathbb{E}f(S-\boldsymbol{D}(t+L))dt\right):\Delta\in\mathcal{T}\setminus\{0\},S\in\mathcal{T}\} \tag{P}$$

with the set $\mathcal{T}$ either given by $[0,\infty)$ or by $\{0,1,2,\ldots\}$. Again by the separability of the objective function the above optimisation problem reduces to

$$\inf\{\frac{1}{\mathbb{E}\boldsymbol{\tau}(\Delta)}\left(K+\varphi(\Delta)\right):\Delta\in\mathcal{T}\setminus\{0\}\}$$

with the function $\varphi:\mathcal{T}\setminus\{0\}\to\mathbb{R}_+$ given by the optimisation problem

$$\varphi(\Delta):=\inf\{\int_0^{\boldsymbol{\tau}(\Delta)}\mathbb{E}f(S-\boldsymbol{D}(t+L))dt:S\in\mathcal{T}\} \tag{$\mathrm{P}_{\varphi(\Delta)}$}$$

Similarly as for the (R,S) inventory model one can now derive the optimisation problem with a β-service level constraint. We leave the details to the reader.

4.3 Piecewise linear costrate function

To analyse the optimisation problem ($\mathrm{P}_{\varphi(\Delta)}$) in more detail we consider now the piecewise linear costrate function given by (3). Applying relation (5) we obtain that the objective function of the optimisation problem ($\mathrm{P}_{\varphi(\Delta)}$) equals

$$\begin{aligned}\mathbb{E}(\int_0^{\boldsymbol{\tau}(\Delta)}f(S-\boldsymbol{D}(t+L))dt) &= -pS\mathbb{E}\boldsymbol{\tau}(\Delta)+p\mathbb{E}(\int_0^{\boldsymbol{\tau}(\Delta)}\boldsymbol{D}(t+L)dt)\\ &\quad +(p+h)\mathbb{E}(\int_0^{\boldsymbol{\tau}(\Delta)}(S-\boldsymbol{D}(t+L))^+dt)\end{aligned}$$

and this yields for every $\Delta\in\mathcal{T}\setminus\{0\}$ that

$$\begin{aligned}\varphi(\Delta) &= p\mathbb{E}(\int_0^{\boldsymbol{\tau}(\Delta)}\boldsymbol{D}(t+L)dt)\\ &\quad +\inf\{-pS\mathbb{E}\boldsymbol{\tau}(\Delta)+(p+h)\mathbb{E}(\int_0^{\boldsymbol{\tau}(\Delta)}(S-\boldsymbol{D}(t+L))^+dt):S\in\mathcal{T}\}\end{aligned}$$

To analyse the second term in the above objective function we observe that

$$\begin{aligned}(S-\boldsymbol{D}(t+L))^+ &= \int_0^S 1_{\{S-\boldsymbol{D}(t+L)\geq x\}}dx\\ &= \int_0^S 1_{\{S-\boldsymbol{D}(t+L)\geq S-x\}}dx\\ &= \int_0^S 1_{\{\boldsymbol{D}(t+L)\leq x\}}dx \qquad \mathbb{P}-a.s.\end{aligned}$$

By applying Fubini's theorem (Mikolás [17]) this yields

$$\begin{aligned}\mathbb{E}(\int_0^{\boldsymbol{\tau}(\Delta)}(S-\boldsymbol{D}(t+L))^+dt) &= \mathbb{E}(\int_0^{\boldsymbol{\tau}(\Delta)}\int_0^S 1_{\{\boldsymbol{D}(t+L)\leq x\}}dxdt)\\ &= \mathbb{E}(\int_0^S\int_0^{\boldsymbol{\tau}(\Delta)}1_{\{\boldsymbol{D}(t+L)\leq x\}}dtdx)\\ &= \int_0^S\mathbb{E}(\int_0^{\boldsymbol{\tau}(\Delta)}1_{\{\boldsymbol{D}(t+L)\leq x\}}dt)dx\end{aligned}$$

and so

$$\begin{aligned}\varphi(\Delta) &= p\mathbb{E}(\int_0^{\boldsymbol{\tau}(\Delta)} \boldsymbol{D}(t+L)dt) + \\ & \inf\{-pS\mathbb{E}\boldsymbol{\tau}(\Delta) + (p+h)\int_0^S \mathbb{E}(\int_0^{\boldsymbol{\tau}(\Delta)} 1_{\{\boldsymbol{D}(t+L)\leq x\}}dt)dx : S \in \mathcal{T}\}\end{aligned} \tag{9}$$

An alternative interpretation of the above integrand is given by the following result.

Lemma 4.2 *If for a given (s,S) policy the value $G_\Delta(x)$ denotes the fraction of time that the net stock process is above the level $S-x$, i.e.*

$$G_\Delta(x) := \lim_{t\uparrow\infty} \frac{1}{t}\mathbb{E}(\int_0^t 1_{\{\boldsymbol{IN}(y)\geq S-x\}}dy)$$

then it follows that

$$G_\Delta(x) = \frac{1}{\mathbb{E}\boldsymbol{\tau}(\Delta)}\mathbb{E}(\int_0^{\boldsymbol{\tau}(\Delta)} 1_{\{\boldsymbol{D}(t+L)\leq x\}}dt)$$

Proof: Since the demand process is either an increasing Levy process or a compound renewal process, we know by Theorem 4.1 that the process $\tilde{\boldsymbol{IN}}$ is a regenerative process. Applying now the same arguments as in the proof of Lemma 3.1 the desired result follows. □

By Lemma 4.2 the integrand corresponding with the optimisation problem $(\mathrm{P}_{\varphi(\Delta)})$ is given by

$$-pS\mathbb{E}\boldsymbol{\tau}(\Delta) + (p+h)\mathbb{E}\boldsymbol{\tau}(\Delta)\int_0^S G_\Delta(x)dx$$

Hence, if $\mathcal{T} = [0,\infty)$ the right-hand derivative with respect to S equals $-p\mathbb{E}\boldsymbol{\tau}(\Delta) + (p+h)\mathbb{E}\boldsymbol{\tau}(\Delta)G_\Delta(S)$. Moreover, if $\mathcal{T} = \{0,1,2,\ldots\}$ then it follows for any $S \in \mathcal{T}$ due to $G_\Delta(x) = G_\Delta(S)$ if $S \leq x < S+1$ that

$$\begin{aligned}& -p(S+1)\mathbb{E}\boldsymbol{\tau}(\Delta) + (p+h)\mathbb{E}\boldsymbol{\tau}(\Delta)\int_0^{S+1} G_\Delta(x)dx \\ & -(-pS\mathbb{E}\boldsymbol{\tau}(\Delta) + (p+h)\mathbb{E}\boldsymbol{\tau}(\Delta)\int_0^S G_\Delta(x)dx) \\ = \quad & -p\mathbb{E}\boldsymbol{\tau}(\Delta) + (p+h)\mathbb{E}\boldsymbol{\tau}(\Delta)\int_S^{S+1} G_\Delta(x)dx \\ = \quad & -p\mathbb{E}\boldsymbol{\tau}(\Delta) + (p+h)\mathbb{E}\boldsymbol{\tau}(\Delta)G_\Delta(S)\end{aligned}$$

By these observations the next result follows immediately.

Lemma 4.3 *An optimal solution $S(\Delta)$ of $(P_{\varphi(\Delta)})$ is given by*

$$S(\Delta) = \inf\{x \in \mathcal{T} : G_\Delta(x) \geq \frac{p}{p+h}\}$$

Proof: The result follows by the previous observations and the necessary and sufficient first-order optimality conditions. □

Similar as for the (R,S) inventory model the optimal order-up-to level for an (s,S) inventory model must satisfy a newsboy type equation. In order to calculate $\mathbb{E}(\int_0^{\boldsymbol{\tau}(\Delta)} \boldsymbol{D}(t+L)dt)$ showing up in (9) we assume now that the demand process $\boldsymbol{D}$ is a compound Poisson process. Moreover, the arrival rate of the associated Poisson arrival process is given by $\lambda > 0$.

Lemma 4.4 *If $\boldsymbol{D}$ is a compound Poisson process and the corresponding distribution F_Y of the independent and identically distributed demands $\boldsymbol{Y}_n$, $n \geq 1$, is continuous then it follows for every $\Delta > 0$ that*

$$\mathbb{E}(\int_0^{\boldsymbol{\tau}(\Delta)} \boldsymbol{D}(t+L)dt) = L\mu_1(1+M_Y(\Delta)) + \frac{1}{\lambda}\int_0^{\Delta} yM_y(dy)$$

with $M_Y(x) := \sum_{k=1}^{\infty} F_Y^{k*}(x)$ and $\mu_1 := \mathbb{E}\boldsymbol{Y}_1$. Moreover, if F_Y is concentrated on $\{0,1,\ldots\}$ then the same formula holds with $\Delta \in \mathbb{N}$ replaced by $\Delta - 1$.

Proof: In case F_Y is continuous it follows by the definition of the stopping time $\boldsymbol{\tau}(\Delta)$ that

$$\begin{aligned} \mathbb{E}(\int_0^{\boldsymbol{\tau}(\Delta)} \boldsymbol{D}(t+L)dt) &= \mathbb{E}(\int_0^{\infty} \boldsymbol{D}(t+L)1_{\{\boldsymbol{\tau}(\Delta)>t\}}dt) \\ &= \mathbb{E}(\int_0^{\infty} \boldsymbol{D}(t+L)1_{\{\boldsymbol{D}(t)<\Delta\}}dt) \end{aligned}$$

Since $\boldsymbol{D}$ is a compound Poisson process we obtain that $\boldsymbol{D}(t+L) \stackrel{d}{=} \boldsymbol{D}(t) + \tilde{\boldsymbol{D}}(L)$ with $\tilde{\boldsymbol{D}}$ again a compound Poisson process and $\tilde{\boldsymbol{D}}(L)$ independent of $\boldsymbol{D}(t)$. This implies using Lemma 4.1 that

$$\begin{aligned} \mathbb{E}(\int_0^{\boldsymbol{\tau}(\Delta)} \boldsymbol{D}(t+L)dt) &= \mathbb{E}(\int_0^{\infty} (\boldsymbol{D}(t)+\tilde{\boldsymbol{D}}(L))1_{\{\boldsymbol{D}(t)<\Delta\}}dt) \\ &= \mathbb{E}(\int_0^{\infty} \boldsymbol{D}(t)1_{\{\boldsymbol{D}(t)<\Delta\}}dt) + \mathbb{E}\tilde{\boldsymbol{D}}(L)\mathbb{E}(\int_0^{\infty} 1_{\{\boldsymbol{D}(t)<\Delta\}}dt) \\ &= \mathbb{E}(\int_0^{\infty} \boldsymbol{D}(t)1_{\{\boldsymbol{D}(t)<\Delta\}}dt) + \lambda L\mu_1\mathbb{E}\boldsymbol{\tau}(\Delta) \\ &= \mathbb{E}(\int_0^{\infty} \boldsymbol{D}(t)1_{\{\boldsymbol{D}(t)<\Delta\}}dt) + L\mu_1(1+M_Y(\Delta)) \end{aligned}$$

To analyse the first term we observe with $\boldsymbol{S}_k := \sum_{i=0}^{k} \boldsymbol{Y}_i$, $k \geq 0$, $\boldsymbol{Y}_0 := 0$, that

$$\begin{aligned} \mathbb{E}(\int_0^{\infty} \boldsymbol{D}(t)1_{\{\boldsymbol{D}(t)<\Delta\}}dt) &= \sum_{k=0}^{\infty}\mathbb{E}(\int_0^{\infty} \boldsymbol{D}(t)1_{\{\boldsymbol{D}(t)<\Delta,\boldsymbol{N}(t)=k\}}dt) \\ &= \sum_{k=0}^{\infty}\mathbb{E}(\int_0^{\infty} \boldsymbol{S}_k 1_{\{\boldsymbol{S}_k<\Delta,\boldsymbol{N}(t)=k\}}dt) \end{aligned}$$

Since F_Y is continuous and the Poisson arrival process $\boldsymbol{N}$ is independent of $\boldsymbol{Y}_n$, $n \geq 1$, this yields

$$\begin{aligned} \mathbb{E}(\int_0^{\infty} \boldsymbol{D}(t)1_{\{\boldsymbol{D}(t)<\Delta\}}dt) &= \sum_{k=0}^{\infty}\int_0^{\infty} \mathbb{E}(\boldsymbol{S}_k 1_{\{\boldsymbol{S}_k<\Delta\}}1_{\{\boldsymbol{N}(t)=k\}})dt \\ &= \sum_{k=0}^{\infty}\mathbb{E}(\boldsymbol{S}_k 1_{\{\boldsymbol{S}_k\leq\Delta\}})\int_0^{\infty} \mathbb{P}\{\boldsymbol{N}(t)=k\}dt \\ &= \frac{1}{\lambda}\sum_{k=0}^{\infty}\mathbb{E}(\boldsymbol{S}_k 1_{\{\boldsymbol{S}_k\leq\Delta\}}) \\ &= \frac{1}{\lambda}\sum_{k=1}^{\infty}\mathbb{E}(\boldsymbol{S}_k 1_{\{\boldsymbol{S}_k\leq\Delta\}}) \quad = \quad \frac{1}{\lambda}\int_0^{\Delta} yM_Y(dy) \end{aligned}$$

and hence the first part is verified. To verify the second part we only observe for $\Delta \in \mathbb{N}$ that

$$\begin{aligned} \mathbb{E}(\boldsymbol{S}_k 1_{\{\boldsymbol{S}_k<\Delta\}}1_{\{\boldsymbol{N}(t)=k\}}) &= \mathbb{E}(\boldsymbol{S}_k 1_{\{\boldsymbol{S}_k<\Delta\}})\mathbb{P}\{\boldsymbol{N}(t)=k\} \\ &= \mathbb{E}(\boldsymbol{S}_k 1_{\{\boldsymbol{S}_k\leq\Delta-1\}})\mathbb{P}\{\boldsymbol{N}(t)=k\} \end{aligned}$$

and by copying the proof of the first part it follows for F_Y concentrated at $\{0,1,\ldots\}$ that

$$\mathbb{E}(\int_0^{\boldsymbol{\tau}(\Delta)} \boldsymbol{D}(t+L)dt) = L\mu_1(1+M_Y(\Delta-1)) + \frac{1}{\lambda}\int_0^{\Delta-1} yM_Y(dy)$$

which shows the desired result. □

Finally, we mention the following result. Observe that this result can be shown by using similar techniques as in the previous lemma applied to the costrate function of Lemma 4.2.

Lemma 4.5 *If $\boldsymbol{D}$ is a compound Poisson process and the corresponding distribution F_Y of the independent and identically distributed demands $\boldsymbol{Y}_n$, $n \geq 1$, is continuous, then it follows that*

$$G_\Delta(x) = \mathbb{E}U_Y(\min\{x - \boldsymbol{D}(L), \Delta\})(U_Y(\Delta))^{-1}$$

with $U_Y(x) := \sum_{k=0}^{\infty} F_Y^{k*}(x) = 1 + M_Y(x)$.

Proof: Use Lemma 4.2 and $\boldsymbol{D}(t+L) \stackrel{d}{=} \boldsymbol{D}(t) + \tilde{\boldsymbol{D}}(L)$ where $\tilde{\boldsymbol{D}}$ is again a compound Poisson process and $\tilde{\boldsymbol{D}}(L)$ independent of $\boldsymbol{D}(t)$. By this observation we obtain after some elementary steps and denoting the distribution of $\tilde{\boldsymbol{D}}(L)$ by F_L that

$$G_\Delta(x) = (\mathbb{E}\boldsymbol{\tau}(\Delta))^{-1}\int_{0^-}^{\infty} Z(y)F_L(dy)$$

with

$$\begin{aligned} Z(y) &= \int_0^\infty \mathbb{P}\{\boldsymbol{D}(t) \leq x - y, \boldsymbol{D}(t) \leq \Delta\}dt \\ &= \int_0^\infty \mathbb{P}\{\boldsymbol{D}(t) \leq \min\{x-y, \Delta\}\}dt \end{aligned}$$

Observe now that $Z(y) = 0$ for $y > x$ and $Z(y) = \mathbb{E}\boldsymbol{\tau}(\min\{x-y,\Delta\})$ for $y \leq x$ and hence by the calculation of $\mathbb{E}\boldsymbol{\tau}(x)$ for every $x \geq 0$ in the proof of Lemma 4.1 the desired result follows. □

Although other results for the (s,S) inventory model can be derived using the framework of regenerative processes we will not pursue this. For some other results on more specialised versions of this model we refer to Archibald & Silver [2], Federgruen & Schechner [6], Federgruen & Zipkin [7], Richards [19], Sahin [21, 22, 23] and Sivazlian [25]. Finally, we would like to mention that efficient numerical procedures can be derived for the computation of $G_\Delta(x)$. This will be presented in another paper.

5 Conclusions

In this paper we presented a general framework for inventory models with an increasing Levy demand process or a compound renewal demand process and backordering. This framework is based on the regenerative structure of the inventory models and using results from the theory of regenerative processes it was possible to give a unified presentation of those models. In particular, we have shown how to use this framework to obtain expressions for the average cost and service levels in (R,S) and (s,S) models. For the first model, in Section 3 new results are reported on the determination of the optimal order-up-to level in case the costrate function is piecewise linear.

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