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DUALITY RESULTS FOR CONIC CONVEX PROGRAMMING[1]

by

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ABSTRACT

This paper presents a unified study of duality properties for the problem of minimizing a linear function over the intersection of an affine space with a convex cone in finite dimension. Existing duality results are carefully surveyed and some new duality properties are established. Examples are given to illustrate these new properties. The topics covered in this paper include Gordon-Stiemke type theorems, Farkas type theorems, perfect duality, Slater condition, regularization, Ramana's duality, and approximate dualities. The dual representations of various convex sets, convex cones and conic convex programs are also discussed.

KEY WORDS: Conic convex programming, duality, semidefinite programming.

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1 Introduction

A conic convex program is an optimization problem for which the objective is linear and the constraint set is given by the intersection of an affine space with a convex cone. As such, it contains as special cases the linear programming problem, the quadratic programming problem, and most notably, the semidefinite programming problem. Recently, the latter problem has been the focus of many studies due to mainly two reasons. First, it has a wide range of applications in, among others, system and control theory [11] and combinatorial optimization [1]. Second, it appears that interior point methods are well suited for solving this type of optimization problems, see e.g. [27, 28, 30, 31, 32, 40].

In this paper, we consider the following conic convex programming problem

$$\begin{aligned} \inf \quad & c^{\mathrm{T}} x \\ \text{s.t.} \quad & x - b \in \mathcal{A} \\ & x \in \mathcal{K}, \end{aligned} \tag{P}$$

where $\mathcal{A}$ is a linear subspace of $\Re^n$, $\mathcal{K}$ is a convex cone in $\Re^n$, c and b are vectors in $\Re^n$. Throughout this paper we assume, without loss of generality, that c and b lie in $\mathcal{A}$ and its orthogonal complement respectively. We shall denote this conic convex program by $\mathrm{CP}(b, c, \mathcal{A}, \mathcal{K})$. Notice that in this setting, the convex cone $\mathcal{K}$ is not necessarily closed, and consequently, the domain of the conic convex program may not be closed either.

The importance of duality theory is well recognized in the context of convex programming. Among other things, it has played a central role in detecting infeasibility, lower-bounding the optimal objective value, and in the design and analysis of iterative algorithms for solving linear and quadratic programs. Indeed, if the optimal value p^* of (P) is finite and the infimum is attained, then an optimal solution of (P) should consist of a feasible solution x^* with $c^{\mathrm{T}} x^* = p^*$ and a dual certificate proving the claim that p^* is indeed the infimum. Similarly, infeasibility of (P) can be established by using a Farkas–type dual solution.

To a large extent, duality results for linear programming can be generalized to the setting of conic convex programming, as was pointed out by Duffin [15]. However, certificates in the context of conic convex programming, such as those proposed in [15], can be infinitely long. More recently, Borwein and Wolkowicz [10] proposed a regularization scheme which results in certificates of finite length. We will see however, that checking the feasibility (correctness) of regularized certificates can be a nontrivial task. Fortunately, the structure of regularized certificates is now well understood for an important class of conic convex programming, viz. semidefinite programming, due to the recent results of Ramana [34].

Since computational algorithms can only generate approximate solutions and certificates, we are led naturally to study the properties of approximate dual solutions for $\mathrm{CP}(b, c, \mathcal{A}, \mathcal{K})$. We will see that while exact dual solutions provide a lower bound on the optimal value of a conic convex optimization problem, approximate dual solutions provide a lower bound for the optimal value of 'reasonably' sized (primal) solutions. Our analysis of approximate solutions follows the approach initiated by Todd and Ye [42].

This paper presents a unified treatment of duality theory for finite dimensional conic convex programming. We consider the conic convex programming problem in its most general form in the sense that we do not make such assumptions as closedness, pointedness, solidness, or constraint qualifications. We carefully survey some existing results known for $\mathrm{CP}(b, c, \mathcal{A}, \mathcal{K})$, and show that various duality results that were previously known only for the case of closed, pointed and / or solid convex cones can be extended to this general setting. Many new and interesting proofs and examples make the survey self–contained.

This paper is organized as follows. Section 2 introduces the terminologies for describing feasibility and related issues in conic convex programming. Section 3 provides some relevant results from convex analysis. Characterizations of strong feasibility, boundedness and related issues are discussed in Section 4. In Section 5, we present several extensions of Farkas' lemma to conic convex programming, yielding characterizations of (in)feasibility and strong infeasibility. Section 6 contains comprehensive duality results of conic convex programming. In particular, it provides a derivation of the relationships between the standard primal and dual conic convex programming problems. In Section 7, we give a new and simplified treatment of regularization. In Section 8, we study the structure of regularized programs in the special case of semidefinite programming, and this leads to Ramana's semidefinite programming duality. The exact meaning of inexact solutions is revealed in Section 9. This paper is concluded with some final remarks in Section 10.

Notation. Given a set $\mathcal{S}$, we let $\mathrm{cl}\,\mathcal{S}$, $\mathrm{int}\,\mathcal{S}$ and $\mathrm{rel}\,\mathcal{S}$ denote the closure of $\mathcal{S}$, the interior of $\mathcal{S}$ and the relative interior of $\mathcal{S}$ respectively. If $\mathcal{S}$ is a subset of $\Re^n$ and A is an $m \times n$ matrix, then the image of $\mathcal{S}$ under the linear mapping A is denoted by $A\mathcal{S}$, i.e.

$$A\mathcal{S} = \{y \in \Re^m \mid y = Ax, \quad \text{for some } x \in \mathcal{S}\}.$$

The kernel, the image and the rank of A are denoted by $\mathrm{Ker}\,A$, $\mathrm{Img}\,A$ and $\mathrm{rank}\,A$ respectively. If $\mathcal{A}$ is a linear subspace, then $P_{\mathcal{A}}$ denotes the orthogonal projection matrix onto $\mathcal{A}$. The dimension of $\mathcal{A}$ is denoted $\dim \mathcal{A}$. In particular, there holds $\dim \mathrm{Img}\,A = \mathrm{rank}\,A$. If A is a symmetric matrix, we write $A \succeq 0$ if and only if A is positive semidefinite.

Given a vector $x \in \Re^n$, we let $\|x\|$ denote a norm of x, for which the dual norm is $\|x\|^*$, i.e.

$$\|x\|^* = \max_y \{y^{\mathrm{T}} x \mid \|y\| = 1\}.$$

The Euclidean norm of x is denoted by $\|x\|_2$. The distance from a vector $x \in \Re^n$ to a convex set $\mathcal{S} \subseteq \Re^n$ is

$$\mathrm{dist}(x, \mathcal{S}) = \inf_{s \in \mathcal{S}} \|x - s\|.$$

Similarly, we let

$$\mathrm{dist}(\mathcal{S}, \mathcal{S}') = \inf_{x \in \mathcal{S}} \mathrm{dist}(x, \mathcal{S}')$$

denote the distance between two convex sets $\mathcal{S}$ and $\mathcal{S}'$. The Minkowski sum of $\mathcal{S}$ and $\mathcal{S}'$ is

$$\mathcal{S} \oplus \mathcal{S}' = \{z \in \Re^n \mid z = x + y \text{ for some } x \in \mathcal{S},\ y \in \mathcal{S}'\},$$

and the (asymmetric) difference of $\mathcal{S}$ and $\mathcal{S}'$ is

$$\mathcal{S} \setminus \mathcal{S}' := \{x \in \mathcal{S} \mid x \notin \mathcal{S}'\}.$$

We let $\Re_+$ ($\Re_{++}$) denote the half–line of nonnegative (positive) real numbers.

2 Terminologies and Preliminaries

A convex cone is by definition a set $\mathcal{K}$ with the property that $\{0\} \in \mathcal{K}$ and

$$\alpha(\mathcal{K} \oplus \mathcal{K}) = \mathcal{K}, \quad \text{for all } \alpha > 0.$$

Instead of 'convex cone', some authors prefer the name 'nonempty convex cone' for the above notion. Let $\mathcal{K}^*$ denote the dual cone of $\mathcal{K}$, i.e.

$$\mathcal{K}^* := \{s \in \Re^n \mid s^{\mathrm{T}}\mathcal{K} \subseteq \Re_+\}.$$

The cone $-\mathcal{K}^*$ is also known as the polar cone of $\mathcal{K}$ [38]. It is easily verified that the dual cone $\mathcal{K}^*$ is convex and closed. We let $\operatorname{sub}\mathcal{K}$ denote the largest linear subspace that is contained in $\mathcal{K}$, i.e.

$$\operatorname{sub}\mathcal{K} := \mathcal{K} \cap (-\mathcal{K}).$$

Notice that

$$\operatorname{sub}\mathcal{K}^* = \{s \in \Re^n \mid s^{\mathrm{T}}\mathcal{K} = \{0\}\};$$

we define $\mathcal{K}^\perp := \operatorname{sub}\mathcal{K}^*$. A convex cone $\mathcal{K}$ is said to be *pointed* if $\operatorname{sub}\mathcal{K} = \{0\}$; $\mathcal{K}$ is said to be *solid* if $\operatorname{int}\mathcal{K} \neq \emptyset$. The linear subspace that is spanned by elements of $\mathcal{K}$ is

$$\operatorname{span}\mathcal{K} := \mathcal{K} \oplus -\mathcal{K}.$$

Example 1 *Let* $\mathcal{K} = \Re \times \Re_+ \times \{0\}$, *then* $\mathcal{K}^* = \{0\} \times \Re_+ \times \Re$, $\operatorname{span}\mathcal{K} = \Re^2 \times \{0\}$, $\operatorname{sub}\mathcal{K} = \Re \times \{0\} \times \{0\}$ *and* $\mathcal{K}^\perp = \{0\} \times \{0\} \times \Re$.

Consider now problem (P), i.e. the conic convex program $\mathrm{CP}(b, c, \mathcal{A}, \mathcal{K})$. If $\mathcal{K}$ is closed, we say that (P) is a *closed conic convex program.* The set of feasible solutions of (P) is

$$\mathcal{F}_P := (b + \mathcal{A}) \cap \mathcal{K}.$$

It is easy to see that

$$\mathcal{F}_P = \mathcal{F}_P \oplus (\mathcal{A} \cap \mathcal{K}). \tag{1}$$

If $x \in \mathcal{A} \cap \mathcal{K}$, then x is called a *direction* (or a recession direction in the terminology of Rockafellar [38]); x is an *interior direction* if it belongs to $\mathcal{A} \cap \operatorname{rel} \mathcal{K}$. If x is a direction and $-x$ is not, i.e. $x \in (\mathcal{A} \cap \mathcal{K}) \setminus \operatorname{sub}(\mathcal{A} \cap \mathcal{K})$, then x is a *one–sided direction*. If $x \in \mathcal{A} \cap \mathcal{K}$ is such that $c^{\mathrm{T}}x \leq 0$, then x is a *lower level direction*; such x is a *one–sided lower level direction* if $-x$ is not a lower level direction. An *improving direction* is a direction x with $c^{\mathrm{T}}x < 0$. An *improving direction sequence* is a sequence $x^{(1)}, x^{(2)}, \ldots$ in $\mathcal{K}$ such that $c^{\mathrm{T}}x^{(i)} \leq -1$ for all i and

$$\lim_{i \to \infty} \operatorname{dist}(x^{(i)}, \mathcal{A}) = 0.$$

Notice that if there exists an improving direction, then there certainly exists an improving direction sequence. We will see later that the converse is in general not true.

If $\operatorname{sub}(\mathcal{A} \cap \mathcal{K}) \neq \{0\}$, it is often convenient to restrict ourselves to solutions in $(\operatorname{sub}(\mathcal{A} \cap \mathcal{K}))^{\perp}$. Namely, it follows from (1) that

$$\mathcal{F}_P = \left(\mathcal{F}_P \cap (\operatorname{sub}(\mathcal{A} \cap \mathcal{K}))^{\perp}\right) \oplus \operatorname{sub}(\mathcal{A} \cap \mathcal{K}).$$

Based on this observation, we say that $x \in \Re^n$ is a *normalized feasible solution* of (P) if

$$x \in \mathcal{F}_P \cap (\operatorname{sub}(\mathcal{A} \cap \mathcal{K}))^{\perp}.$$

Obviously, if $\mathcal{A} \cap \mathcal{K}$ is pointed, then any feasible solution is a normalized feasible solution.

Example 2 *The standard LP problem*

$$\min\{\tilde{c}^T y \mid A^T y + s = \tilde{b},\ y \in \Re^m,\ s \geq 0\},$$

where A is an $m \times n$ matrix, can be cast as a conic convex program $CP(b, c, \mathcal{A}, \mathcal{K})$ in $\Re^{m+n}$ by letting

$$b^T := \begin{bmatrix} 0 & \tilde{b}^T \end{bmatrix}^T, \quad c^T = \begin{bmatrix} \tilde{c}^T & 0 \end{bmatrix}^T,$$

$$\mathcal{A} := \operatorname{Ker} \begin{bmatrix} A^T & I \end{bmatrix}, \quad \mathcal{K} := \Re^m \times \Re^n_+,$$

where I denotes the identity matrix of order n. Since in this case there holds

$$\operatorname{sub}(\mathcal{A} \cap \mathcal{K}) = (\operatorname{Ker} A^T) \times \{0\}^n,$$

the normalized feasible set is

$$\mathcal{F}_P \cap (\operatorname{sub}(\mathcal{A} \cap \mathcal{K}))^{\perp} = \{(y, s) \in \mathcal{F}_P \mid y \in \operatorname{Img} A\}.$$

It is customary in linear programming theory to assume that A has full row rank, i.e. $\operatorname{Img} A = \Re^m$, *which implies that $\mathcal{F}_P$ consists only of normalized feasible solutions.*

The set of *interior solutions* is defined as

$$\overset{o}{\mathcal{F}}_P := \mathcal{F}_P \cap \operatorname{rel} \mathcal{K}.$$

We say that (P) is *feasible* (or consistent) if $\mathcal{F}_P \neq \emptyset$ and (P) is *strongly feasible* (or super–consistent in the terminology of Duffin [15]) if $\overset{o}{\mathcal{F}}_P \neq \emptyset$. If (P) is feasible but not strongly feasible, then (P) is said to be *weakly feasible*.

Strong feasibility as defined above is also known as the generalized Slater's constraint qualification.

Obviously, if (P) is feasible, i.e. if $(b + \mathcal{A}) \cap \mathcal{K} \neq \emptyset$, then $\operatorname{dist}(b + \mathcal{A}, \mathcal{K}) = 0$. The converse is in general not true, even if $\mathcal{K}$ is closed; see Example 3 at the end of this section. This observation gives rise to the definition of *weak infeasibility*, which is sometimes referred to as sub–consistency [15] or asymptotic consistency [5]. Problem (P) is said to be *weakly infeasible* if

$$\operatorname{dist}(b + \mathcal{A}, \mathcal{K}) = 0 \quad \text{but} \quad \mathcal{F}_P = \emptyset.$$

If

$$\operatorname{dist}(b + \mathcal{A}, \mathcal{K}) > 0,$$

then (P) is called *strongly infeasible*.

Let

$$p^* := \inf c^{\mathrm{T}} \mathcal{F}_P$$

denote the optimal value of (P). The set of feasible solutions for which the optimal value is attained is

$$\mathcal{F}_P^* := \{x \in \mathcal{F}_P \mid c^{\mathrm{T}} x = p^*\},$$

and the *normalized optimal set* is

$$\mathcal{F}_P^* \cap (\operatorname{sub}(\mathcal{A} \cap \mathcal{K}))^{\perp}.$$

Problem (P) is said to be *solvable* (or convergent in the terminology of Duffin [15]) if $\mathcal{F}_P^* \neq \emptyset$. A special case of unsolvability occurs when $p^* = -\infty$. In this case, we say that (P) is *unbounded*. Notice that if (P) is feasible and there exists an improving direction, then (P) is unbounded.

Associated with (P) is a *dual program* (D), viz.

$$\begin{aligned} \inf \quad & b^{\mathrm{T}} s \\ \text{s.t.} \quad & s - c \in \mathcal{A}^{\perp} \\ & s \in \mathcal{K}^*. \end{aligned} \tag{D}$$

In other words, the dual of the conic convex program $\mathrm{CP}(b, c, \mathcal{A}, \mathcal{K})$ is by definition the closed conic convex program $\mathrm{CP}(c, b, \mathcal{A}^{\perp}, \mathcal{K}^*)$. In analogy to the definitions of $\mathcal{F}_P$, $\overset{o}{\mathcal{F}}_P$, p^* and $\mathcal{F}_P^*$ for the primal program, we define

$$\mathcal{F}_D := (c + \mathcal{A}^{\perp}) \cap \mathcal{K}^*, \quad \overset{o}{\mathcal{F}}_D := (c + \mathcal{A}^{\perp}) \cap \operatorname{rel} \mathcal{K}^*,$$

and

$$d^* := \inf b^{\mathrm{T}} \mathcal{F}_D, \quad \mathcal{F}_D^* := \{s \in \mathcal{F}_D \mid b^{\mathrm{T}} s = d^*\},$$

for the dual program. If (D) is weakly (in)feasible, strongly (in)feasible or solvable, then (P) is said to be *dual* weakly (in)feasible, dual strongly (in)feasible or dual solvable, respectively. Similarly, (P) is said to have a dual level direction, dual improving direction, etc., if (D) has a level direction, improving direction, etc. We will see later that if $\mathcal{K}$ is closed, then the dual of (D) is again (P).

The following example illustrates some of the terminologies introduced above.

Example 3 *Consider the program CP*$(b, c, \mathcal{A}, \mathcal{K})$ *in* $\Re^3$*, with*

$$b = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T, \quad c = \begin{bmatrix} 0 & c_2 & 0 \end{bmatrix}^T,$$

$$\mathcal{A} = \{x \in \Re^3 \mid x_1 = 0, x_3 = 0\},$$

and

$$\mathcal{K} = \left\{ x \in \Re^3 \;\middle|\; \begin{bmatrix} x_1 & x_3/\sqrt{2} \\ x_3/\sqrt{2} & x_2 \end{bmatrix} \succeq 0 \right\}.$$

Then $\mathcal{K}^* = \mathcal{K}$ *and* $\mathcal{A}^\perp = \{s \in \Re^3 \mid s_2 = 0\}$*. The primal is weakly infeasible,*

$$p^* = \inf \left\{ c_2 x_2 \;\middle|\; \begin{bmatrix} 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & x_2 \end{bmatrix} \succeq 0 \right\} = \infty.$$

The dual is

$$d^* = \inf \left\{ s_3 \;\middle|\; \begin{bmatrix} s_1 & s_3/\sqrt{2} \\ s_3/\sqrt{2} & c_2 \end{bmatrix} \succeq 0 \right\}.$$

Hence, the dual is strongly infeasible if $c_2 < 0$*, weakly feasible and solvable with optimal value* $d^* = 0$ *if* $c_2 = 0$*, and strongly feasible and unbounded if* $c_2 > 0$*.*

Example 3 is a semidefinite programming problem. We refer to Vandenberghe and Boyd [45] for an introduction to semidefinite programming and its applications.

3 Basic properties of convex cones

The result below is quoted from Corollary 16.4.2 of Rockafellar [38]. We give a direct proof for completeness.

Lemma 1 *Let* $\mathcal{K}_1$ *and* $\mathcal{K}_2$ *be two convex cones, then*

$$\mathcal{K}_1^* \cap \mathcal{K}_2^* = (\mathcal{K}_1 \oplus \mathcal{K}_2)^*.$$

Proof: By definition, we have

$$x \in (\mathcal{K}_1 \oplus \mathcal{K}_2)^*$$

if and only if

$$x^{\mathrm{T}}(\mathcal{K}_1 \oplus \mathcal{K}_2) \subseteq \Re_+.$$

Since $0 \in \mathcal{K}_1 \cap \mathcal{K}_2$, the above relation is equivalent with

$$x^{\mathrm{T}}\mathcal{K}_1 \subseteq \Re_+, \quad x^{\mathrm{T}}\mathcal{K}_2 \subseteq \Re_+,$$

i.e. $x \in \mathcal{K}_1^* \cap \mathcal{K}_2^*$.

□

Based on Lemma 1, one may guess that the cones $\mathcal{K}_1^* \oplus \mathcal{K}_2^*$ and $(\mathcal{K}_1 \cap \mathcal{K}_2)^*$ are identical. However, this is in general not true even if $\mathcal{K}_1$ and $\mathcal{K}_2$ are both closed, since the Minkowski sum $\mathcal{K}_1^* \oplus \mathcal{K}_2^*$ may not be closed. For instance in Example 3, we have $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}} \in \mathrm{cl}\,(\mathcal{A} \oplus \mathcal{K})$, but $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}} \notin \mathcal{A} \oplus \mathcal{K}$.

Corollary 1 *Let $\mathcal{K}$ be a convex cone. Then* $\mathrm{span}\,\mathcal{K} = (\mathrm{sub}\,\mathcal{K}^*)^\perp = \mathcal{K}^{\perp\perp}$*, i.e.* $\mathrm{span}\,\mathcal{K}$ *is the smallest linear subspace containing $\mathcal{K}$.*

Proof: Apply Lemma 1 with $\mathcal{K}_1 = \mathcal{K}$ and $\mathcal{K}_2 = -\mathcal{K}$.

□

Recall that a solid convex cone in $\Re^n$ is by definition a convex cone $\mathcal{K}$ for which $\mathrm{int}\,\mathcal{K} \neq \emptyset$, or equivalently, for which the smallest subspace containing $\mathcal{K}$ is $\Re^n$. Hence, we obtain from Corollary 1 that $\mathcal{K}$ is solid if and only if $\mathcal{K}^*$ is pointed. Notice however, that if $\mathcal{K}$ is not closed then $\mathcal{K}$ may be pointed whereas $\mathcal{K}^*$ is not solid. (For instance, consider $\mathcal{K} = (\Re \times \Re_{++}) \cup \{0\}$.)

Notice from Lemma 1 and Corollary 1 that

$$\begin{aligned}\left(\mathrm{sub}\,(\mathcal{A}^\perp \cap \mathcal{K}^*)\right)^\perp &= (\mathrm{sub}\,(\mathcal{A}^* \cap \mathcal{K}^*))^\perp \\ &= (\mathrm{sub}\,(\mathcal{A} \oplus \mathcal{K})^*)^\perp \\ &= \mathrm{span}\,(\mathcal{A} \oplus \mathcal{K}).\end{aligned}$$

Hence, it follows that

$$\mathcal{F}_D \cap \mathrm{span}\,(\mathcal{A} \oplus \mathcal{K})$$

is the normalized dual feasible set.

A well known result is the *bipolar theorem* (see Duffin [15], Ben-Israel [4] and Rockafellar [38], among others).

Theorem 1 (bipolar theorem) *Let $\mathcal{K}$ be a convex cone in $\Re^n$. There holds*

$$\mathrm{cl}\,\mathcal{K} = \mathcal{K}^{**}.$$

The bipolar theorem shows the nice symmetry between the dual pair (P) and (D): if $\mathcal{K}$ is closed, then $\mathrm{CP}(b, c, \mathcal{A}, \mathcal{K})$ is the dual of the conic convex program $\mathrm{CP}(c, b, \mathcal{A}^\perp, \mathcal{K}^*)$.

The bipolar theorem gives a dual characterization of $\mathrm{cl}\,\mathcal{K}$. Theorem 2 below gives a dual characterization of $\mathrm{rel}\,\mathcal{K}$. To the best of our knowledge, this characterization is new.

Theorem 2 *Let $\mathcal{K}$ be a convex cone in $\Re^n$. Then*

$$x \in \mathrm{rel}\,\mathcal{K}$$

if and only if

$$x \in \mathrm{span}\,\mathcal{K}, \quad x^T(\mathcal{K}^* \setminus \mathcal{K}^\perp) \subseteq \Re_{++}.$$

Proof: Let s be an arbitrary nonzero vector in $(\mathcal{K}^* \setminus \mathcal{K}^\perp)$, and let $\hat{s}$ denote the nonzero orthogonal projection of s onto the subspace $K^\perp$. By definition, $x \in \mathrm{rel}\,\mathcal{K}$ implies that there exists a positive number $\delta(s)$ such that $x - \delta(s)\hat{s} \in \mathcal{K}$. This yields

$$0 \le s^{\mathrm{T}}(x - \delta(s)\hat{s}) = s^{\mathrm{T}}x - \delta(s)\|\hat{s}\|_2^2 < s^{\mathrm{T}}x.$$

Moreover, since $\mathrm{rel}\,\mathcal{K} \subseteq \mathrm{span}\,\mathcal{K}$, we have $x \in \mathrm{span}\,\mathcal{K}$.

Conversely, suppose that $x \in \mathrm{span}\,\mathcal{K}$ is such that $x^{\mathrm{T}}(\mathcal{K}^* \setminus \mathcal{K}^\perp) \subseteq \Re_{++}$. Since $x^{\mathrm{T}}\mathcal{K}^\perp = \{0\}$ (see Corollary 1), it follows that $x^{\mathrm{T}}\mathcal{K}^* \subseteq \Re_+$, i.e. $x \in \mathcal{K}^{**}$. Let

$$\epsilon := \inf_s \{x^{\mathrm{T}}s \mid s \in \mathcal{K}^* \cap \mathrm{span}\,\mathcal{K},\ \|s\| = 1\}.$$

Then $\epsilon > 0$, because $\mathcal{K}^*$ and $\mathrm{span}\,\mathcal{K}$ are closed. By construction, we have for all $y \in \mathrm{span}\,\mathcal{K}$, $y \neq 0$, and $s \in \mathcal{K}^*$ that

$$s^{\mathrm{T}}\left(x + \frac{\epsilon}{\|y\|^*}y\right) \ge 0,$$

which implies that $x \in \mathrm{rel}\,\mathcal{K}^{**}$, Using the bipolar theorem, it follows that $x \in \mathrm{rel}\,\mathcal{K}$.

□

The following lemma gives a formula for the relative interior of a Minkowski sum of cones. It follows from Corollary 6.1.1 in Rockafellar [38], but we give a direct proof for completeness.

Lemma 2 *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be convex cones in $\Re^n$. Then*

$$\mathrm{rel}\,(\mathcal{K}_1 \oplus \mathcal{K}_2) = (\mathrm{rel}\,\mathcal{K}_1) \oplus \mathrm{rel}\,\mathcal{K}_2.$$

Proof: From Corollary 1, we know that span $(\mathcal{K}_1 \oplus \mathcal{K}_2)$ is the smallest linear subspace containing $\mathcal{K}_1 \oplus \mathcal{K}_2$. Since

$$\operatorname{span}(\mathcal{K}_1 \oplus \mathcal{K}_2) = (\operatorname{span}\mathcal{K}_1) \oplus \operatorname{span}\mathcal{K}_2,$$

it follows that

$$x \in (\operatorname{rel}\mathcal{K}_1) \oplus \operatorname{rel}\mathcal{K}_2 \;\Rightarrow\; x \in \operatorname{rel}(\mathcal{K}_1 \oplus \mathcal{K}_2). \tag{2}$$

On the other hand, we have $\operatorname{cl}\operatorname{rel}\mathcal{K}_1 = \operatorname{cl}\mathcal{K}_1$ and $\operatorname{cl}\operatorname{rel}\mathcal{K}_2 = \operatorname{cl}\mathcal{K}_2$ because $\mathcal{K}_1$ and $\mathcal{K}_2$ are convex, and hence

$$\mathcal{K}_1 \oplus \mathcal{K}_2 \subseteq \operatorname{cl}\left((\operatorname{rel}\mathcal{K}_1) \oplus \operatorname{rel}\mathcal{K}_2\right), \tag{3}$$

where we used the fact that the closure of a set is the union of that set with its limit points. Relation (3) implies that

$$\operatorname{rel}(\mathcal{K}_1 \oplus \mathcal{K}_2) \subseteq (\operatorname{rel}\mathcal{K}_1) \oplus \operatorname{rel}\mathcal{K}_2. \tag{4}$$

Combining (2) and (4) yields

$$(\operatorname{rel}\mathcal{K}_1) \oplus \operatorname{rel}\mathcal{K}_2 = \operatorname{rel}(\mathcal{K}_1 \oplus \mathcal{K}_2).$$

□

In fact, Lemma 2 holds not only for convex cones but also for more general sets known as *robust sets*. (A set $\mathcal{S}$ is said to be robust if it satisfies $\operatorname{cl}\operatorname{rel}\mathcal{S} = \operatorname{cl}\mathcal{S}$.) This fact can be shown by the same proof as used in Lemma 2.

The following lemma shows how an invertible linear transformation of a cone affects its dual.

Lemma 3 *Let $\mathcal{K}$ be a convex cone in $\Re^n$ and let $M \in \Re^{n\times n}$ be an invertible matrix. Then*

$$(M^T\mathcal{K})^* = M^{-1}\mathcal{K}^*.$$

Proof: We note the following relations

$$\begin{aligned} y \in (M^{\mathrm{T}}\mathcal{K})^* &\iff y^{\mathrm{T}}M^{\mathrm{T}}\mathcal{K} \subseteq \Re_+ \\ &\iff My \in \mathcal{K}^* \\ &\iff y \in M^{-1}\mathcal{K}^*. \end{aligned}$$

□

4 Characterization of strong feasibility

Combining Lemma 1, Theorem 2, and Lemma 2, we obtain the following result.

Theorem 3 *There exists a primal interior solution if and only if there exists no one–sided dual level direction.*

Proof: By definition, a conic convex program $\mathrm{CP}(b, c, \mathcal{A}, \mathcal{K})$ has an interior solution if and only if $(b + \mathcal{A}) \cap \mathrm{rel}\, \mathcal{K} \neq \emptyset$, i.e.

$$b \in \mathcal{A} \oplus \mathrm{rel}\, \mathcal{K} = \mathrm{rel}\,(\mathcal{A} \oplus \mathcal{K}),$$

where we used Lemma 2. From Theorem 2, we know that the above relation holds if and only if

$$b \in \mathrm{span}\,(\mathcal{A} \oplus \mathcal{K}), \quad b^{\mathrm{T}}((\mathcal{A} \oplus \mathcal{K})^* \setminus (\mathcal{A} \oplus \mathcal{K})^{\perp}) \subseteq \Re_{++},$$

which, using Lemma 1, is equivalent with

$$b^{\mathrm{T}}(\mathcal{A}^{\perp} \cap \mathcal{K}^*) = \{0\}, \quad b^{\mathrm{T}}((\mathcal{A}^{\perp} \cap \mathcal{K}^*) \setminus \mathrm{sub}\,(\mathcal{A}^{\perp} \cap \mathcal{K}^*)) \subseteq \Re_{++},$$

i.e. there exist no one–sided dual level directions.

□

The above characterization of strong feasibility was established by Carver [12] for the case that $\mathcal{K} = \Re^n_+$. For general solid closed convex cones, the result can be found in Fan [17], Duffin [15], and Berman and Ben-Israel [6]. Notice however, that Theorem 3 above is applicable also if $\mathcal{K}$ is not solid.

Special cases of Theorem 3 are the arbitrage and pricing result in the theory of financial markets [22] and well known theorems of Lyapunov, Stein and Taussky in matrix theory (see the discussion in Berman and Ben-Israel [6] and Berman [5]). Applying Theorem 3 to the conic convex program $\mathrm{CP}(0, 0, \mathcal{A}, \mathcal{K})$ yields a characterization of the existence of primal interior directions:

Corollary 2 *Consider a conic convex program $CP(b, c, \mathcal{A}, \mathcal{K})$. There exists a primal interior direction, i.e.*

$$\mathcal{A} \cap \mathrm{rel}\, \mathcal{K} \neq \emptyset$$

if and only if there is no one–sided dual direction.

If $\mathcal{K} = \Re^n_+$ (the polyhedral case), Corollary 2 reduces to a classical result of Gordan [21] and Stiemke [39].

Combining Theorem 3 and Corollary 2, it follows that if there exists an interior direction ($\mathcal{A} \cap \mathrm{rel}\, \mathcal{K} \neq \emptyset$), then there must also exist an interior solution ($\overset{o}{\mathcal{F}}_P \neq \emptyset$).

Based on Theorem 3, we derive a characterization of the existence of *improving* interior directions:

Corollary 3 *Consider a conic convex program CP$(b, c, \mathcal{A}, \mathcal{K})$. There exists a primal improving interior direction, i.e.*

$$c^T(\mathcal{A} \cap \operatorname{rel} \mathcal{K}) \not\subseteq \Re_+$$

if and only if the dual is infeasible and there is no one–sided dual direction.

Proof: Notice that if $c = 0$ then there exist no primal improving directions, and the dual has a feasible solution, viz. $0 \in \mathcal{A}^\perp \cap \mathcal{K}^*$.

Suppose now that $c \neq 0$. Below, we will construct an artificial conic convex program, for which the interior solutions correspond to primal improving directions of the original program. The corollary will then follow as an application of Theorem 3. First, since $c \in \mathcal{A}$, there holds

$$-\frac{c}{\|c\|_2^2} + (\mathcal{A} \cap \operatorname{Ker} c^{\mathrm{T}}) = \{x \in \mathcal{A} \mid c^{\mathrm{T}} x = -1\}.$$

Hence, x is a primal improving interior direction if and only if there exists some $\alpha > 0$ such that

$$\alpha x \in \left(-\frac{c}{\|c\|_2^2} + (\mathcal{A} \cap \operatorname{Ker} c^{\mathrm{T}}) \right) \cap \operatorname{rel} \mathcal{K}.$$

Applying Theorem 3, it follows that there exist primal improving interior directions if and only if the conic convex program $\mathrm{CP}(-c/\|c\|_2^2, 0, \mathcal{A} \cap \operatorname{Ker} c^{\mathrm{T}}, \mathcal{K})$ has no one–sided dual level directions. The dual of $\mathrm{CP}(-c/\|c\|_2^2, 0, \mathcal{A} \cap \operatorname{Ker} c^{\mathrm{T}}, \mathcal{K})$ is $\mathrm{CP}(0, -c/\|c\|_2^2, \mathcal{A}^\perp \oplus \operatorname{Img} c, \mathcal{K}^*)$, and if it has a one–sided level direction s, it must be contained in $(\mathcal{A}^\perp \oplus \operatorname{Img} c) \cap \mathcal{K}^*$. Since $c \in \mathcal{A}$, it follows that either $c^{\mathrm{T}} s > 0$ and there is $\alpha > 0$ such that $\alpha s \in (c + \mathcal{A}^\perp) \cap \mathcal{K}$, or $c^{\mathrm{T}} s = 0$ and

$$s \in (\mathcal{A}^\perp \cap \mathcal{K}^*) \setminus -\mathcal{K}^*.$$

Hence, $\mathrm{CP}(-c/\|c\|_2^2, 0, \mathcal{A} \cap \operatorname{Ker} c^{\mathrm{T}}, \mathcal{K})$ has no one–sided dual level directions if and only if the conic convex program $\mathrm{CP}(b, c, \mathcal{A}, \mathcal{K})$ is dual infeasible and has no one–sided dual directions.

□

We remark that Nesterov, Todd and Ye [33] called a closed conic convex program *strictly infeasible* if it has a dual improving interior direction. Corollary 3 shows that a program is strictly infeasible in the sense of [33] if and only if it is infeasible and has no one–sided directions. We will see in Corollary 4 that strict infeasibility implies strong infeasibility.

The relation between dual directions and primal strong feasibility has now been fully investigated. We now proceed to study the relationships between dual directions and boundedness of the dual feasible set, the dual lower level sets and the dual optimal set.

Lemma 4 *Consider a conic convex program CP*$(b, c, \mathcal{A}, \mathcal{K})$ *for which the primal is strongly feasible. Let* $s^{(1)}, s^{(2)}, \ldots$ *in* $\mathcal{K}^* \cap \text{span}\,(\mathcal{A} \oplus \mathcal{K})$ *be a sequence with*

$$\lim_{i\to\infty} \text{dist}(s^{(i)}, c + \mathcal{A}^\perp) = 0, \quad \limsup_{i\to\infty} b^T s^{(i)} < \infty.$$

Then $s^{(i)}$, $i = 1, 2, \ldots$, *is a bounded sequence.*

Proof: Suppose to the contrary that $\mathcal{K}^* \cap \text{span}\,(\mathcal{A} \oplus \mathcal{K})$ contains some sequence $s^{(1)}, s^{(2)}, \ldots$ such that

$$\lim_{i\to\infty} \left\| s^{(i)} \right\| = \infty, \tag{5}$$

whereas $\lim_{i\to\infty} \text{dist}(s^{(i)}, c + \mathcal{A}^\perp) = 0$ and $\limsup_{i\to\infty} b^{\mathrm{T}} s^{(i)} < \infty$.

Without loss of generality, we assume that $\left\| s^{(i)} \right\| > 0$ for all i and that the limit

$$y := \lim_{i\to\infty} \frac{s^{(i)}}{\|s^{(i)}\|}$$

exists. Since the sequence $s^{(i)} / \left\| s^{(i)} \right\|$, $i = 1, 2, \ldots$, is contained in the closed cone $\mathcal{K}^* \cap \text{span}\,(\mathcal{A} \oplus \mathcal{K})$, it follows that

$$y \in \mathcal{K}^* \cap \text{span}\,(\mathcal{A} \oplus \mathcal{K}) = \mathcal{K}^* \cap (\text{sub}\,(\mathcal{A}^\perp \cap \mathcal{K}^*))^\perp, \tag{6}$$

where we used Lemma 1. Moreover, using (5) we have

$$y = y - \lim_{i\to\infty} \frac{1}{\|s^{(i)}\|} c = \lim_{i\to\infty} \frac{s^{(i)} - c}{\|s^{(i)}\|} \in \mathcal{A}^\perp, \tag{7}$$

and, since $\limsup_{i\to\infty} b^{\mathrm{T}} s^{(i)} < \infty$,

$$b^{\mathrm{T}} y = \lim_{i\to\infty} \frac{b^{\mathrm{T}} s^{(i)}}{\|s^{(i)}\|} = 0. \tag{8}$$

By construction, $\|y\| = 1$, so that (6)—(8) implies

$$y \in (\mathcal{A}^\perp \cap \mathcal{K}^*) \setminus -\mathcal{K}^*, \quad b^{\mathrm{T}} y = 0,$$

i.e. y is a one–sided lower level direction, which contradicts the primal strong feasibility (see Theorem 3).

□

Theorem 4 *Consider a conic convex program CP*$(b, c, \mathcal{A}, \mathcal{K})$. *If the dual is weakly infeasible then the primal has no interior direction.*

Proof: Apply Lemma 4 to the dual feasibility problem $\mathrm{CP}(0, c, \mathcal{A}, \mathcal{K})$.

□

Combining Theorem 4 with the dual characterization of (primal) interior directions (Corollary 2) yields the following result.

Corollary 4 *Consider a conic convex program $CP(b, c, \mathcal{A}, \mathcal{K})$. If the dual is weakly infeasible then the dual has a one–sided direction.*

Notice that Corollary 4 is stated in terms of the dual program, since the closedness of $\mathcal{K}^*$ is essential for this result.

Theorem 5 *A dual feasible conic convex program is primal strongly feasible if and only if the normalized dual optimal set is nonempty and bounded.*

Proof: If the normalized dual optimal set is nonempty and bounded, then there is obviously no one–sided dual level direction. Using Theorem 3, this implies that $\overset{o}{\mathcal{F}}_P \neq \emptyset$.

Conversely, we know from Lemma 4 that if $\overset{o}{\mathcal{F}}_P \neq \emptyset$ then any sequence $s^{(1)}, s^{(2)}, \ldots$ of normalized dual feasible solutions with $\lim_{i\to\infty} b^{\mathrm{T}} s^{(i)} = d^*$ is bounded. Since $\mathcal{F}_D$ is nonempty and closed, it follows that the normalized dual optimal set is nonempty and bounded.

□

Corollary 5 *Consider a dual feasible conic convex program. The normalized dual feasible set is bounded if and only if there exists a primal interior direction.*

Proof: Apply Theorem 5 to the dual feasibility problem $\mathrm{CP}(0, c, \mathcal{A}, \mathcal{K})$.

□

5 Farkas–type lemmas

In the previous section, we have discussed a dual characterization of strong feasibility. We will now give a characterization of strong infeasibility.

Lemma 5 (First Farkas–type lemma) *Consider a conic convex program $CP(b, c, \mathcal{A}, \mathcal{K})$. The primal is strongly infeasible if and only if there exists a dual improving direction.*

Proof: By definition, the primal is not strongly infeasible if and only if $\text{dist}(b+\mathcal{A},\mathcal{K})=0$. Since $\text{dist}(b+\mathcal{A},\mathcal{K})=0$ if and only if there exists a sequence $x^{(1)}, x^{(2)}, \ldots$ in $\mathcal{K}$ such that

$$\lim_{i\to\infty} P_{\mathcal{A}^\perp} x^{(i)} = b,$$

we obtain the relation

$$\text{dist}(b+\mathcal{A},\mathcal{K}) = 0 \iff b \in \text{cl}\, P_{\mathcal{A}^\perp}\mathcal{K}. \tag{9}$$

It is easy to see that a linearly transformed convex cone is also a convex cone. Therefore, we can apply the bipolar theorem, which states that

$$\text{cl}\, P_{\mathcal{A}^\perp}\mathcal{K} = (P_{\mathcal{A}^\perp}\mathcal{K})^{**}. \tag{10}$$

Combining the relation (9)–(10) yields

$$\text{dist}(b+\mathcal{A},\mathcal{K}) = 0 \iff b^{\mathrm{T}}(P_{\mathcal{A}^\perp}\mathcal{K})^{*} \subseteq \Re_+, \tag{11}$$

where

$$\begin{aligned}(P_{\mathcal{A}^\perp}\mathcal{K})^{*} &= \{\sigma \in \Re^n \mid \sigma^{\mathrm{T}} P_{\mathcal{A}^\perp}\mathcal{K} \subseteq \Re_+\}\\ &= \{\sigma \in \Re^n \mid P_{\mathcal{A}^\perp}\sigma \in \mathcal{K}^*\}\\ &= (\mathcal{K}^* \cap \mathcal{A}^\perp) + \mathcal{A}.\end{aligned}$$

Combining the above relation with (11) and noting $b \in \mathcal{A}^\perp$, we obtain

$$\text{dist}(b+\mathcal{A},\mathcal{K}) = 0 \iff b^{\mathrm{T}}(\mathcal{K}^* \cap \mathcal{A}^\perp) \subseteq \Re_+.$$

□

For the case that $\mathcal{K} = \Re^n_+$, Lemma 5 reduces to the famous lemma of Farkas [18]. For general closed convex programming, the result has been established by Duffin [15] and Berman [5].

Applying Lemma 5 to the conic convex program $\text{CP}(c, b, \mathcal{A}^\perp, \mathcal{K}^*)$, we see that

$$\text{dist}(c+\mathcal{A}^\perp,\mathcal{K}^*) = 0 \iff c^{\mathrm{T}}(\mathcal{A} \cap \mathcal{K}^{**}) \subseteq \Re_+. \tag{12}$$

However, from the bipolar theorem we have $\mathcal{K}^{**} = \text{cl}\,\mathcal{K}$. This together with Theorem 4 leads to the following characterization of feasibility for conic convex programs satisfying a generalized Slater condition.

Corollary 6 *Consider a conic convex program $CP(b, c, \mathcal{A}, \mathcal{K})$ with $\mathcal{A} \cap \text{rel}\,\mathcal{K} \neq \emptyset$. There holds*

$$\mathcal{F}_D \neq \emptyset$$

if and only if

$$c^T(\mathcal{A} \cap \mathcal{K}) \subseteq \Re_+.$$

Proof: Since $\mathcal{A} \cap \operatorname{rel} \mathcal{K} \neq \emptyset$, we have

$$\mathcal{A} \cap \operatorname{cl} \mathcal{K} = \operatorname{cl}(\mathcal{A} \cap \mathcal{K}).$$

Hence, we can replace $\mathcal{K}^{**}$ with $\mathcal{K}$ in relation (12). Moreover, we know from Theorem 4 that (D) cannot be weakly infeasible. The corollary thus follows from relation (12).

□

The result of Corollary 6 is due to Wolkowicz [46]. For the special case that $\mathcal{K}$ is closed and pointed, Corollary 6 reduces to a generalization of Farkas' lemma as it can be found in many papers, including [6, 5, 4, 14, 1, 44].

Naturally, we are also interested in a characterization of feasibility without a Slater-type condition. We can easily obtain such a characterization from Lemma 5.

Lemma 6 (Second Farkas–type lemma) *A conic convex program $CP(b, c, \mathcal{A}, \mathcal{K})$ is dual feasible if and only if there does not exist any primal improving direction sequence.*

Proof: Notice that if $c = 0$ then $0 \in \mathcal{F}_D$ and $c^{\mathrm{T}}\mathcal{K} = \{0\}$. In other words, we have dual feasibility and no primal improving direction sequence if $c = 0$. It remains to consider the case that $c \neq 0$.

Suppose $c \neq 0$. We use a similar technique as in the proof of Corollary 3, namely we will construct an artificial conic convex program, for which the dual improving directions correspond to dual feasible solutions of the original program. The corollary will then follow from Lemma 5. First, since $c \in \mathcal{A}$, there holds

$$c^{\mathrm{T}} s > 0 \quad \text{and} \quad s \in (\mathcal{A}^{\perp} \oplus \operatorname{Img} c) \cap \mathcal{K}^*$$

if and only if

$$\alpha s \in (c + \mathcal{A}^{\perp}) \cap \mathcal{K}^* \quad \text{for some } \alpha > 0.$$

We conclude that the dual feasible set $(c + \mathcal{A}^{\perp}) \cap \mathcal{K}^*$ is nonempty if and only if the conic convex program $\mathrm{CP}(-c/\|c\|_2^2, 0, \left(\mathcal{A}^{\perp} \oplus \operatorname{Img} c\right)^{\perp}, \mathcal{K})$ has a dual improving direction. Applying Lemma 5, it follows that $\mathcal{F}_D \neq \emptyset$ if and only if

$$\operatorname{dist}\left(\frac{-c}{\|c\|_2^2} + \left(\mathcal{A}^{\perp} \oplus \operatorname{Img} c\right)^{\perp}, \mathcal{K}\right) > 0. \tag{13}$$

From Lemma 1, we have

$$\begin{aligned} \frac{-c}{\|c\|_2^2} + \left(\mathcal{A}^{\perp} \oplus \operatorname{Img} c\right)^{\perp} &= \frac{-c}{\|c\|_2^2} + (\mathcal{A} \cap \operatorname{Ker} c^{\mathrm{T}}) \\ &= \{x \in \mathcal{A} \mid c^{\mathrm{T}} x = -1\}. \end{aligned}$$

This implies that (13) holds if and only if there exists no primal improving direction sequence.

primal	dual
Interior direction	Bounded normalized feasible set
Strongly feasible	Nonempty and bounded normalized optimal set
Weakly feasible	One–sided level direction, but no improving direction sequence
Weakly infeasible	Improving direction sequence, but no improving direction
Strongly infeasible	Improving direction
Strongly infeasible and no one–sided direction	Interior improving direction

Table 1: Feasibility characterizations ('if–and–only–if') for dual feasible closed conic convex programs

□

The result of Lemma 6 can also be found in Duffin [15], with a different proof. It is remarkable that this is the only reference where we have found this beautiful characterization of feasibility.

Table 1 summarizes the feasibility characterizations for dual feasible closed conic convex programs. Since duality is completely symmetric for closed conic convex programs, we can make an analogous table of dual (in)feasibility characterizations for primal feasible programs. The characterizations that are listed in Table 1 are direct applications of Corollary 5, Theorem 5, Theorem 3, Lemma 6, Lemma 5 and Corollary 3.

6 Strong duality

It is well known that if (P) is a linear program and p^* is finite, then strong duality holds, i.e. $p^* + d^* = 0$. Our objective is to generalize the strong duality result for linear programming to conic convex programming. Notice that, for a general conic convex program, it is possible that d^* is finite but (P) is weakly infeasible (see e.g. Example 3 with $c_2 = 0$). This means that we should allow an arbitrarily small constraint violation for the primal and define its *subvalue* as:

$$p^- := \lim_{\epsilon \downarrow 0} \inf_x \{c^{\mathrm{T}} x \mid x \in \mathcal{K},\ \mathrm{dist}(x, b + \mathcal{A}) < \epsilon\}.$$

If (P) is strongly infeasible, then $p^- = \infty$, but for weakly infeasible programs, the subvalue is possibly finite. We also define a matrix M_c,

$$M_c := \begin{bmatrix} I & -c \\ 0 & 1 \end{bmatrix}, \tag{14}$$

where I denotes the identity matrix of order n.

Lemma 7 *Let $\gamma \in \Re$, and consider a conic convex program $CP(b, c, \mathcal{A}, \mathcal{K})$. Then there holds*

$$\operatorname{dist}\left(\begin{bmatrix} b \\ \gamma \end{bmatrix} + M_c^T(\mathcal{A} \times \{0\}), \mathcal{K} \times \Re_+\right) = 0$$

if and only if $p^- \leq \gamma$.

Proof: By definition, we have $p^- \leq \gamma$ if and only if there exists a sequence $x^{(1)}, x^{(2)}, \ldots$ in $\mathcal{K}$ such that

$$\lim_{i\to\infty} \operatorname{dist}(x^{(i)}, b + \mathcal{A}) = 0, \quad \lim_{i\to\infty} c^{\mathrm{T}} x^{(i)} \leq \gamma. \tag{15}$$

Letting

$$x_{n+1}^{(i)} := \max\{0, \gamma - c^{\mathrm{T}} x^{(i)}\} \quad \text{for } i = 1, 2, \ldots,$$

we obtain a sequence $(x^{(1)}, x_{n+1}^{(1)}), (x^{(2)}, x_{n+1}^{(2)}), \ldots$ in $\mathcal{K} \times \Re_+$ with

$$\lim_{i\to\infty} \operatorname{dist}\left(\begin{bmatrix} x^{(i)} \\ x_{n+1}^{(i)} \end{bmatrix}, \begin{bmatrix} b \\ \gamma \end{bmatrix} + M_c^{\mathrm{T}}(\mathcal{A} \times \{0\})\right) = 0. \tag{16}$$

Conversely, if $(x^{(i)}, x_{n+1}^{(i)}) \in \mathcal{K} \times \Re_+$, $i = 1, 2, \ldots$, is a sequence satisfying (16), then $x^{(1)}, x^{(2)}, \ldots$ is a sequence satisfying (15).

□

Lemma 8 *Let $\gamma \in \Re$, and consider a conic convex program $CP(b, c, \mathcal{A}, \mathcal{K})$ with* $\operatorname{dist}(b + \mathcal{A}, \mathcal{K}) = 0$. *Then there holds*

$$\begin{bmatrix} b^T & \gamma \end{bmatrix} \left((\mathcal{K} \times \Re_+) \cap M_c^{-1}(\mathcal{A}^\perp \times \Re)\right) \subseteq \Re_+$$

if and only if $d^* \geq -\gamma$.

Proof: Notice that

$$M_c^{-1} = \begin{bmatrix} I & c \\ 0 & 1 \end{bmatrix}. \tag{17}$$

By definition, we have $d^* < -\gamma$ if and only if there exists a vector $s \in \mathcal{F}_D$ such that

$$b^{\mathrm{T}} s + \gamma < 0.$$

Letting $s_{n+1} := 1$, we see that

$$\begin{bmatrix} s \\ s_{n+1} \end{bmatrix} \in (\mathcal{K}^* \times \Re_+) \cap M_c^{-1}(\mathcal{A}^\perp \times \Re), \quad b^{\mathrm{T}} s + \gamma s_{n+1} < 0. \tag{18}$$

Conversely, suppose that there exists (s, s_{n+1}) satisfying (18). Notice that if $s_{n+1} = 0$ then s is a dual improving direction which contradicts the assumption that $\mathrm{dist}(b+\mathcal{A}, \mathcal{K}) = 0$ (see Lemma 5). Hence, $s_{n+1} > 0$ and $s/s_{n+1} \in \mathcal{F}_D$. Moreover, we have

$$d^* \leq c^{\mathrm{T}} s/s_{n+1} < -\gamma,$$

which completes the proof.

□

Combining Lemmas 7 and 8 with the extended Farkas' lemma, we obtain a strong duality theorem:

Theorem 6 (Strong duality) *Consider a conic convex program $CP(b, c, \mathcal{A}, \mathcal{K})$. If the dual is infeasible and the primal is strongly infeasible, then*

$$p^- = d^* = \infty.$$

Otherwise, there holds

$$p^- = -d^*.$$

Proof: First consider the case of primal strong infeasibility, i.e. $\mathrm{dist}(b + \mathcal{A}, \mathcal{K}) > 0$. In that we case, $p^- = \infty$. From Lemma 5 it then follows that there exists a dual improving direction. Hence, we have $d^* = -\infty = -p^-$ if there exists a dual solution, and $d^* = \infty = p^-$ otherwise.

It remains to consider the case $\mathrm{dist}(b + \mathcal{A}, \mathcal{K}) = 0$. Given $\gamma \in \Re$, we know from Lemma 7 that

$$p^- \leq \gamma \iff \mathrm{dist}\left(\begin{bmatrix} b \\ \gamma \end{bmatrix} + M_c^{\mathrm{T}}(\mathcal{A} \times \{0\}), \mathcal{K} \times \Re_+\right) = 0.$$

The above relation implies, using Lemma 5 and Lemma 3, that

$$p^- \leq \gamma \iff \begin{bmatrix} b^{\mathrm{T}} & \gamma \end{bmatrix} \left((\mathcal{K} \times \Re_+) \cap M_c^{-1}(\mathcal{A}^\perp \times \Re)\right) \subseteq \Re_+.$$

Applying now Lemma 8 yields

$$p^- \leq \gamma \iff d^* \geq -\gamma.$$

Since γ is arbitrary, it follows that

$$p^- = -d^*.$$

□

Since $p^* \geq p^-$, we obtain from Theorem 6 the *weak duality relation*

$$p^* \geq -d^*. \tag{19}$$

For the case that the primal and the dual are not both infeasible, we see from the above theorem that $p^- = -d^*$. We will now show that if the primal has an interior solution (generalized Slater condition), then the subvalue coincides with the optimal value, i.e. $p^* = p^-$. Hence, we can strengthen the duality result for the case in which the generalized Slater condition holds. We thus arrive at the following strong duality theorem.

Theorem 7 (Slater duality) *Suppose that* $\overset{o}{\mathcal{F}}_P \neq \emptyset$. *Then*

$$p^* = p^- = -d^*.$$

Moreover, if $p^* > -\infty$ *then*

$$\mathcal{F}_D^* \neq \emptyset,$$

and the normalized dual optimal solution set is bounded.

Proof: Observe that since $\overset{o}{\mathcal{F}}_P \neq \emptyset$, there holds

$$\operatorname{cl} \mathcal{F}_P = (b + \mathcal{A}) \cap \operatorname{cl} \mathcal{K}.$$

Therefore, for the purpose of proving the theorem, we can assume without loss of generality that $\mathcal{K}$ is closed, i.e. $\mathcal{K} = \mathcal{K}^{**}$.

Analogous to the definition of p^-, we define the *dual subvalue* d^- as

$$d^- := \lim_{\epsilon \downarrow 0} \inf_s \{ b^{\mathrm{T}} s \mid s \in \mathcal{K}^*,\ \operatorname{dist}(s, c + \mathcal{A}^\perp) < \epsilon \}.$$

Applying Theorem 6 to the conic convex program $\mathrm{CP}(c, b, \mathcal{A}^\perp, \mathcal{K}^*)$, we obtain $p^* = -d^- \geq -d^*$. Hence, if $p^* = -\infty$ then $d^* = \infty$ and the theorem holds true. It remains to consider the case that $p^* = -d^- > -\infty$. By definition, the condition

$$d^- = -p^* < \infty$$

means that there exists a sequence $s^{(1)}, s^{(2)}, \ldots$ in $\mathcal{K}^* \cap \operatorname{span}(\mathcal{A} \oplus \mathcal{K})$ with

$$\lim_{i \to \infty} \operatorname{dist}(s^{(i)}, c + \mathcal{A}^\perp) = 0 \quad \text{and} \quad \limsup_{i \to \infty} b^{\mathrm{T}} s^{(i)} = -p^* < \infty.$$

It follows from Lemma 4 that this sequence has a cluster point, say $s^{(\infty)}$. Obviously

$$s^{(\infty)} \in \mathcal{F}_D, \quad b^{\mathrm{T}} s^{(\infty)} = -p^*.$$

It follows from the relation

$$-p^* = d^* \leq b^{\mathrm{T}} s^{(\infty)} = -p^*,$$

that

$$p^* + d^* = 0, \quad \mathcal{F}_D^* \neq \emptyset.$$

Finally, the boundedness of the normalized dual optimal solution set follows from Theorem 5.

□

For the case that $\mathcal{K}$ is closed and solid, the strong duality theorem with Slater condition is well known; see for example [1, 15, 31], among others. The result of Theorem 7, which holds for general convex cones, is due to Borwein and Wolkowicz [9].

Theorem 7 implies the following well known fact: if $\mathcal{K}$ is closed and $\overset{o}{\mathcal{F}}_P \times \overset{o}{\mathcal{F}}_D \neq \emptyset$, then $\mathcal{F}_P^* \times \mathcal{F}_D^* \neq \emptyset$ and

$$(x^*)^{\mathrm{T}} s^* = c^{\mathrm{T}} x^* + b^{\mathrm{T}} s^* = 0, \quad \text{for all } (x^*, s^*) \in \mathcal{F}_P^* \times \mathcal{F}_D^*.$$

For the above case, we say that a conic convex programming problem has a *complementary solution*. A complementary solution is a pair $(x, s) \in \mathcal{F}_P \times \mathcal{F}_D$ such that

$$x^{\mathrm{T}} s = 0.$$

A *face* of a cone $\mathcal{K}$ is a set

$$\mathrm{face}\,(\mathcal{K}, s) := \{x \in \mathcal{K} \mid x^{\mathrm{T}} s = 0\},$$

where $s \in \mathcal{K}^*$. Notice for $x, y \in \mathcal{K}$ that

$$x + y \in \mathrm{face}\,(\mathcal{K}, s) \implies x, y \in \mathrm{face}\,(\mathcal{K}, s),$$

which explains why "$\mathrm{face}\,(\mathcal{K}, s)$" is called a face of $\mathcal{K}$. We remark that Theorem 2 implies that $\mathcal{K} \cap \mathrm{sub}\,\mathcal{K}$ is the smallest face of $\mathcal{K}$. If (x, s) is a complementary solution, then

$$\mathcal{F}_P^* = \mathcal{F}_P \cap \mathrm{face}\,(\mathcal{K}, s), \quad \mathcal{F}_D^* = \mathcal{F}_D \cap \mathrm{face}\,(\mathcal{K}^*, x).$$

Therefore, $\mathcal{F}_P^*$ and $\mathcal{F}_D^*$ are also known as the *optimal faces* of (P) and (D) respectively. A *strictly complementary solution pair* of (P) is a pair $(x, s) \in \mathcal{F}_P \times \mathcal{F}_D$ such that

$$x \in \mathrm{rel}\,(\mathrm{face}\,(\mathcal{K}, s)), \quad s \in \mathrm{rel}\,(\mathrm{face}\,(\mathcal{K}^*, x)).$$

By definition, such a solution pair is also a complementary solution pair. It was shown by Tucker [43] and Goldman and Tucker [20] that any solvable linear programming problem has a strictly complementary solution pair. Unfortunately, a conic convex program may not have any strictly complementary solution pair, even if it satisfies primal and dual Slater conditions. One therefore also encounters the term *maximal complementary solution pair*, which is a complementary solution pair $(x, s) \in (\mathrm{rel}\,\mathcal{F}_P^*) \times \mathrm{rel}\,(\mathcal{F}_D^*)$.

Obviously, any complementary solution pair is an optimal solution pair. The converse however, is in general not true unless $\overset{o}{\mathcal{F}}_P \neq \emptyset$ or $\overset{o}{\mathcal{F}}_D \neq \emptyset$. This is because without the latter condition, there may exist a positive duality gap and as a result there cannot exist a complementary solution. Moreover, strong duality is necessary, but not sufficient for the existence of a complementary solution.

			primal feasible		primal infeasible	
			strong	weak	weak	strong
			1	2	3	4
dual feasible	strong	A	$p^* = -d^*$ (P)+(D) solvable	$p^* = -d^*$ (P) solvable	(D) unbounded	(D) unbounded
	weak	B		possible	possible	(D) unbounded
dual infeasible	weak	C			possible	possible
	strong	D				possible

Table 2: Duality for closed conic convex programs

The duality relations for conic convex programs $\mathrm{CP}(b, c, \mathcal{A}, \mathcal{K})$ with $\mathcal{K}$ closed are summarized in Table 2.

All entries in the table represent possible combinations of the status of the primal and dual problem. Only if we cannot conclude anything more, we explicitly mention that the entry represents a possible state. Due to the complete symmetry of the closed conic convex programming duality, the table is symmetric, so we only need to consider the upper-right block. The entries in the first row of the table are denoted by 'A1', 'A2', 'A3' and 'A4', in the second row by 'B1', 'B2', and so on. The entries 'A1', 'A2', 'A3' and 'A4', are due to Theorem 7. Lemma 5 implies entry 'B4'. The possibility of states 'A3' and 'B3' and 'D3' (and hence 'C4') is demonstrated by Example 3, while the entry 'C3' is illustrated by Example 4, which is a semidefinite programming problem.

Example 4 *Consider the program CP($b, c, \mathcal{A}, \mathcal{K}$) in $\Re^6$ with*

$$b = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T, \quad c = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T,$$

$$\mathcal{A} = \{x \in \Re^6 \mid x_2 = 0, x_5 = 0\},$$

and

$$\mathcal{K} = \left\{ x \in \Re^6 \;\middle|\; \begin{bmatrix} x_1 & x_4/\sqrt{2} & x_6/\sqrt{2} \\ x_4/\sqrt{2} & x_2 & x_5/\sqrt{2} \\ x_6/\sqrt{2} & x_5/\sqrt{2} & x_3 \end{bmatrix} \succeq 0 \right\}.$$

Then $\mathcal{K}^* = \mathcal{K}$ and $\mathcal{A}^\perp = \{s \in \Re^6 \mid s_1 = s_3 = s_4 = s_6 = 0\}$. The primal is weakly infeasible,

$$p^* = \inf \left\{ x_4 \;\middle|\; \begin{bmatrix} x_1 & x_4/\sqrt{2} & x_6/\sqrt{2} \\ x_4/\sqrt{2} & 0 & 1 \\ x_6/\sqrt{2} & 1 & x_3 \end{bmatrix} \succeq 0 \right\} = \infty,$$

and the dual is also weakly infeasible:

$$d^* = \inf\left\{ s_5 \;\middle|\; \begin{bmatrix} 0 & 1 & 0 \\ 1 & s_2 & s_5/\sqrt{2} \\ 0 & s_5/\sqrt{2} & 0 \end{bmatrix} \succeq 0 \right\} = \infty.$$

Finally, the possibility of the entries in Table 2 where weak infeasibility is not involved, can be demonstrated by a 2-dimensional linear programming problem:

Example 5 *Let* $n = 2$, $c \in \Re^2$, $\mathcal{K} = \mathcal{K}^* = \Re^2_+$ *and*

$$\mathcal{A} = \{(x_1, x_2) \mid x_1 = 0\}, \quad \mathcal{A}^\perp = \{(s_1, s_2) \mid s_2 = 0\}.$$

We see that (P) is strongly feasible if $c_1 > 0$*, weakly feasible if* $c_1 = 0$ *and strongly infeasible if* $c_1 < 0$*. Similarly, (D) is strongly feasible if* $c_2 > 0$*, weakly feasible if* $c_2 = 0$ *and strongly infeasible if* $c_2 < 0$*.*

Weak infeasibility does not exist in linear programming. However, Examples 3 and 4 illustrate that weakly infeasible problems do exist in semidefinite programming. The latter is an important class of conic convex programming problems.

7 Regularization

In Theorem 6, we have shown that $s \in \mathcal{F}_D$ is an optimal solution of (D) if and only if there exists a sequence $x^{(i)} \in \mathcal{K}$, $i = 1, 2, \ldots$, with

$$\lim_{i\to\infty} \operatorname{dist}(b + \mathcal{A}, x^{(i)}) = 0, \quad \lim_{i\to\infty} c^{\mathrm{T}} x^{(i)} = -b^{\mathrm{T}} s. \tag{20}$$

Such a sequence is called a *certificate* of the optimality of the dual solution s. Since this certificate is a sequence, it has a rather inconvenient property: its length is infinite. We will see in this section that finite length certificates can be obtained by means of *regularization*.

If $\overset{o}{\mathcal{F}}_P \neq \emptyset$ (a generalized Slater condition), then $p^* = -d^*$ and no regularization is needed, see Theorem 7. For a weakly feasible conic convex program $\mathrm{CP}(b, c, \mathcal{A}, \mathcal{K})$ however, we may try to replace $\mathcal{K}$ by a lower dimensional face, say face $(\mathcal{K}, s)$ for a certain $s \in \mathcal{K}^*$, such that

$$(b + \mathcal{A}) \cap \operatorname{face}(\mathcal{K}, s) = (b + \mathcal{A}) \cap \mathcal{K}, \quad (b + \mathcal{A}) \cap \operatorname{rel}\operatorname{face}(\mathcal{K}, s) \neq \emptyset.$$

If we succeed in finding such a face (which is then known as the *minimal cone* [10, 9, 46]), then we can regularize $\mathrm{CP}(b, c, \mathcal{A}, \mathcal{K})$ to $\mathrm{CP}(b, c, \mathcal{A}, \operatorname{face}(\mathcal{K}, s))$, which satisfies the generalized Slater condition. Such a regularization approach, which we call *primal regularization*, was proposed by Borwein and Wolkowicz [10, 9] and Wolkowicz [46].

In this section, we propose a *dual regularization* approach, which is based on the dual characterization of strong feasibility, Theorem 3. In this approach, we transform all one–sided, non–improving, dual level directions into two–sided directions (lines), thus enlarging the dimension of $\operatorname{sub}\mathcal{K}^*$. For notational convenience, we will now interchange the role of the primal and the dual: we assume that we want to solve the dual program $\mathrm{CP}(c, b, \mathcal{A}^\perp, \mathcal{K}^*)$, and to this end we transform one–sided primal level directions into lines, thus enlarging $\mathcal{K}$.

Let $\mathcal{K}$ be a convex cone in $\Re^n$, and let $\mathcal{A}$ be a linear subspace of $\Re^n$. We define an operator $\Gamma_{\mathcal{A}}$ on $\mathcal{K}$ as follows:

$$\Gamma_{\mathcal{A}}\mathcal{K} := \mathrm{cl}\,\left(\mathcal{K} \oplus \mathrm{span}\,(\mathcal{A} \cap \mathrm{cl}\,\mathcal{K})\right). \tag{21}$$

Observe from this definition that

$$\mathcal{A} \cap \mathrm{cl}\,\mathcal{K} \subseteq \mathrm{sub}\,(\Gamma_{\mathcal{A}}\mathcal{K}),$$

i.e. if $x \in \mathcal{A} \cap \mathcal{K} \setminus -\mathcal{K}$ is a one–sided direction with respect to $\mathcal{K}$, then this direction x is not one–sided with respect to $\Gamma_{\mathcal{A}}\mathcal{K}$. Observe also that $\Gamma_{\mathcal{A}}\mathcal{K} = \mathrm{cl}\,\mathcal{K}$ if and only if the convex program $\mathrm{CP}(b, c, \mathcal{A}, \mathcal{K})$ has no primal one–sided directions, i.e.

$$\Gamma_{\mathcal{A}}\mathcal{K} = \mathrm{cl}\,\mathcal{K} \iff \mathrm{span}\,(\mathcal{A} \cap \mathrm{cl}\,\mathcal{K}) \subseteq \mathrm{cl}\,\mathcal{K}. \tag{22}$$

Although $\mathrm{CP}(b, c, \mathcal{A}, \mathcal{K} \oplus \mathrm{span}\,(\mathcal{A} \cap \mathcal{K}))$ has no primal one–sided directions, it is quite possible for the *closed* conic convex program $\mathrm{CP}(b, c, \mathcal{A}, \Gamma_{\mathcal{A}}\mathcal{K})$ to have primal one–sided directions (see Example 6 below). Therefore, it makes sense to apply the operator $\Gamma_{\mathcal{A}}$ k times in succession, resulting in an operator $\Gamma_{\mathcal{A}}^k$. More precisely, we let

$$\begin{cases} \Gamma_{\mathcal{A}}^0\mathcal{K} := \mathcal{K}, \\ \Gamma_{\mathcal{A}}^k\mathcal{K} := \Gamma_{\mathcal{A}}\Gamma_{\mathcal{A}}^{k-1}\mathcal{K}, \quad \text{for } k = 1, 2, \ldots. \end{cases} \tag{23}$$

In addition, we define

$$\Gamma_{\mathcal{A}}^\infty\mathcal{K} := \Gamma_{\mathcal{A}}^{\dim\mathcal{A}}\mathcal{K}. \tag{24}$$

Each time that we apply the operator $\Gamma_{\mathcal{A}}$ to a cone $\Gamma_{\mathcal{A}}^k\mathcal{K}$, we move any one–sided direction in $\mathcal{A} \cap \Gamma_{\mathcal{A}}^k\mathcal{K}$ into $\mathrm{sub}\,\Gamma_{\mathcal{A}}^{k+1}\mathcal{K}$, so that it is not one–sided with respect to the larger cone $\Gamma_{\mathcal{A}}^{k+1}\mathcal{K}$. After applying the $\Gamma_{\mathcal{A}}$ operator $\dim\mathcal{A}$ times in succession, there will be no one–sided directions in $\mathcal{A} \cap \Gamma_{\mathcal{A}}^\infty\mathcal{K}$, as the following lemma shows.

Lemma 9 *Let $\mathcal{K}$ be a convex cone in $\Re^n$ and let $\mathcal{A}$ be a linear subspace of $\Re^n$. Then*

$$\Gamma_{\mathcal{A}}^k\mathcal{K} = \Gamma_{\mathcal{A}}^\infty\mathcal{K} \quad \textit{for all } k \geq \dim\mathcal{A}.$$

Proof: Since the sets $\{\Gamma_{\mathcal{A}}^k\mathcal{K} : k = 0, 1, 2, \ldots\}$ are nested, we only need to show that $\Gamma_{\mathcal{A}}^k\mathcal{K} = \Gamma_{\mathcal{A}}^{k+1}\mathcal{K}$ for some finite k. Suppose $\Gamma_{\mathcal{A}}^k\mathcal{K} \neq \Gamma_{\mathcal{A}}^{k+1}\mathcal{K}$ for some k so that

$$\mathrm{span}\,(\mathcal{A} \cap \Gamma_{\mathcal{A}}^k\mathcal{K}) \not\subseteq \Gamma_{\mathcal{A}}^k\mathcal{K}. \tag{25}$$

From the definition (21), we have

$$\mathrm{sub}\,(\mathcal{A} \cap \Gamma_{\mathcal{A}}^{k}\mathcal{K}) \subset \mathrm{span}\,(\mathcal{A} \cap \Gamma_{\mathcal{A}}^{k}\mathcal{K}) \subseteq \mathrm{sub}\,(\mathcal{A} \cap \Gamma_{\mathcal{A}}^{k+1}\mathcal{K}),$$

so that

$$\dim \mathrm{sub}\left(\mathcal{A} \cap \Gamma_{\mathcal{A}}^{k}\mathcal{K}\right) < \dim \mathrm{sub}\left(\mathcal{A} \cap \Gamma_{\mathcal{A}}^{k+1}\mathcal{K}\right) \leq \dim \mathcal{A}.$$

Thus, the dimension of $\mathrm{sub}\left(\mathcal{A} \cap \Gamma_{\mathcal{A}}^{k}\mathcal{K}\right)$ is increased by one whenever (25) holds. Using an inductive argument, it follows that $k+1 \leq \dim \mathcal{A}$. Consequently, there will be some $k \leq \dim \mathcal{A}$ for which (25) does not hold. Together with (22), this implies the lemma.

□

We will now show that the property of strong infeasibility is invariant under the operator $\Gamma_{\mathcal{A}}$.

Lemma 10 *Consider a conic convex program CP$(b, c, \mathcal{A}, \mathcal{K})$ and let $\mathcal{A}' \subseteq \mathcal{A}$ be a linear subspace. There holds*

$$\mathrm{dist}\left(b + \mathcal{A}, \Gamma_{\mathcal{A}'}^{k+1}\mathcal{K}\right) = \mathrm{dist}\left(b + \mathcal{A}, \Gamma_{\mathcal{A}'}^{k}\mathcal{K}\right)$$

for all $k = 0, 1, \ldots$.

Proof: Since $\Gamma_{\mathcal{A}'}^{k}\mathcal{K} \subseteq \Gamma_{\mathcal{A}'}^{k+1}\mathcal{K}$, we obviously have

$$\mathrm{dist}\left(b + \mathcal{A}, \Gamma_{\mathcal{A}'}^{k+1}\mathcal{K}\right) \leq \mathrm{dist}\left(b + \mathcal{A}, \Gamma_{\mathcal{A}'}^{k}\mathcal{K}\right). \tag{26}$$

To prove the converse, we fix any vector x in $\Gamma_{\mathcal{A}'}^{k+1}\mathcal{K}$. It follows from the definition (21) that there exists a sequence $\{(u^{(i)}, v^{(i)})\}$ with

$$u^{(i)} \in \Gamma_{\mathcal{A}'}^{k}\mathcal{K}, \quad v^{(i)} \in \mathrm{span}\left(\mathcal{A}' \cap \Gamma_{\mathcal{A}'}^{k}\mathcal{K}\right), \qquad i = 1, 2, \ldots,$$

such that

$$x = \lim_{i \to \infty} (u^{(i)} + v^{(i)}).$$

As $v^{(i)} \in \mathrm{span}\left(\mathcal{A}' \cap \Gamma_{\mathcal{A}'}^{k}\mathcal{K}\right) \subseteq \mathcal{A}' \subseteq \mathcal{A}$, we have

$$\mathrm{dist}(u^{(i)} + v^{(i)}, b + \mathcal{A}) = \mathrm{dist}(u^{(i)}, b + \mathcal{A}) \geq \mathrm{dist}\left(b + \mathcal{A}, \Gamma_{\mathcal{A}'}^{k}\mathcal{K}\right),$$

where the last step is due to $u^{(i)} \in \Gamma_{\mathcal{A}'}^{k}\mathcal{K}$. Letting $i \to \infty$ yields

$$\mathrm{dist}(x, b + \mathcal{A}) \geq \mathrm{dist}\left(b + \mathcal{A}, \Gamma_{\mathcal{A}'}^{k}\mathcal{K}\right).$$

Since x is an arbitrary element of $\Gamma_{\mathcal{A}'}^{k+1}\mathcal{K}$, we obtain

$$\mathrm{dist}\left(b + \mathcal{A}, \Gamma_{\mathcal{A}'}^{k+1}\mathcal{K}\right) \geq \mathrm{dist}\left(b + \mathcal{A}, \Gamma_{\mathcal{A}'}^{k}\mathcal{K}\right).$$

Combining this with (26) proves the lemma.

□

The following lemma shows that regularization with the subspace $(\mathcal{A} \cap \operatorname{Ker} c^{\mathrm{T}})$ does not change the dual feasible set.

Lemma 11 *Consider a conic convex program CP$(b, c, \mathcal{A}, \mathcal{K})$. There holds*

$$\mathcal{F}_D = (c + \mathcal{A}^{\perp}) \cap \left(,^{k}_{\mathcal{A} \cap \operatorname{Ker} c^T} \mathcal{K} \right)^*$$

for all $k \in \{0, 1, 2, \ldots\}$.

Proof: Let $s \in \mathcal{F}_D$. It suffices to prove that this implies

$$s \in \left(,^{k}_{\mathcal{A} \cap \operatorname{Ker} c^{\mathrm{T}}} \mathcal{K} \right)^*, \quad k = 0, 1, 2, ... \tag{27}$$

Since $\left(,^{0}_{\mathcal{A} \cap \operatorname{Ker} c^{\mathrm{T}}} \mathcal{K} \right)^* = \mathcal{K}^*$, relation (27) holds trivially for $k = 0$. Now assume that (27) holds for some $k \in \{0, 1, 2, \ldots\}$. We need to show that (27) holds for $k + 1$ in the sense that $x^{\mathrm{T}} s \geq 0$ for any $x \in ,^{k+1}_{\mathcal{A} \cap \operatorname{Ker} c^{\mathrm{T}}} \mathcal{K}$. By definition, $x \in ,^{k+1}_{\mathcal{A} \cap \operatorname{Ker} c^{\mathrm{T}}} \mathcal{K}$ means that there exists some sequence $(u^{(i)}, v^{(i)})$, $i = 1, 2, \ldots$, satisfying

$$u^{(i)} \in ,^{k}_{\mathcal{A} \cap \operatorname{Ker} c^{\mathrm{T}}} \mathcal{K}, \quad v^{(i)} \in \operatorname{span} \left(\mathcal{A} \cap \operatorname{Ker} c^{\mathrm{T}} \cap ,^{k}_{\mathcal{A} \cap \operatorname{Ker} c^{\mathrm{T}}} \mathcal{K} \right),$$

such that

$$x = \lim_{i \to \infty} (u^{(i)} + v^{(i)}).$$

However, since $s \in c + \mathcal{A}^{\perp}$ we have $s^{\mathrm{T}} v^{(i)} = 0$, whereas (27) implies $s^{\mathrm{T}} u(i) \geq 0$, for all i. Consequently, there holds $s^{\mathrm{T}} x \geq 0$.

□

Although regularization does not affect the dual feasible set, it can change the nature of dual (in)feasibility.

Lemma 12 *For a conic convex program CP$(b, c, \mathcal{A}, \mathcal{K})$, the regularized program*

$$\mathrm{CP}\,(b, c, \mathcal{A}, ,^{\infty}_{\mathcal{A} \cap \operatorname{Ker} c^T} \mathcal{K})$$

is either dual strongly feasible or dual strongly infeasible.

Proof: Recall from Lemma 9 that $,_{\mathcal{A} \cap \operatorname{Ker} c^{\mathrm{T}}} ,^{\infty}_{\mathcal{A} \cap \operatorname{Ker} c^{\mathrm{T}}} \mathcal{K} = ,^{\infty}_{\mathcal{A} \cap \operatorname{Ker} c^{\mathrm{T}}} \mathcal{K}$. Hence, if the regularized primal has a one–sided level direction, it must be an improving direction. It thus follows from

Lemma 5 that the regularized dual is strongly infeasible if and only if the regularized primal has one–sided level directions. Using Theorem 3, we conclude that the regularized dual is either dual strongly feasible or dual strongly infeasible.

□

Together, Lemma 11 and Lemma 12 imply that the regularization of a dual weakly feasible problem results in a dual strongly feasible problem. Similarly, the regularization of a dual weakly infeasible problem results in a dual strongly infeasible problem. However, the *set* of dual feasible solutions is not affected by regularization. These conclusions are summarized in the following theorem.

Theorem 8 *Consider a conic convex program $CP(b, c, \mathcal{A}, \mathcal{K})$ and let*

$$\mathcal{K}' := \,,\, {}^{\infty}_{\mathcal{A}\cap \mathrm{Ker}\, c^T}\mathcal{K}.$$

There holds

- *The dual feasible sets of $CP(b, c, \mathcal{A}, \mathcal{K})$ and its regularization $CP(b, c, \mathcal{A}, \mathcal{K}')$ coincide, i.e.*

$$\mathcal{F}_D = (c + \mathcal{A}^{\perp}) \cap (\mathcal{K}')^*.$$

- *The regularized program $CP(b, c, \mathcal{A}, \mathcal{K}')$ is dual strongly feasible if and only if $\mathcal{F}_D \neq \emptyset$.*
- *The regularized program $CP(b, c, \mathcal{A}, \mathcal{K}')$ is dual strongly infeasible if and only if $\mathcal{F}_D = \emptyset$.*

Combining Theorem 8 with Table 2, we see that the regularized conic convex program is in perfect duality:

Corollary 7 *Assume the same setting as in Theorem 8. Then there holds*

- *If $d^* = \infty$, then the regularized primal $CP(b, c, \mathcal{A}, \mathcal{K}')$ is either infeasible or unbounded.*
- *If $-\infty < d^* < \infty$, then the regularized primal is solvable with optimal value equal to $-d^*$, i.e.*

$$d^* = -\min c^T \left((b + \mathcal{A}) \cap \mathcal{K}'\right).$$

- *If $d^* = -\infty$ then the regularized primal $CP(b, c, \mathcal{A}, \mathcal{K}')$ is infeasible.*

Applying Corollary 7 to the conic convex program $\mathrm{CP}(0, c, \mathcal{A}, \mathcal{K})$, we obtain a generalization of Farkas' lemma:

Corollary 8 *A conic convex program $CP(b, c, \mathcal{A}, \mathcal{K})$ is dual feasible if and only if*

$$c^T \left(\mathcal{A} \cap \, , {}^{\infty}_{\mathcal{A} \cap \mathrm{Ker}\, c^T} \mathcal{K}) \right) \subseteq \Re_+ .$$

We have seen that the regularization of a dual feasible conic program results in a conic program with strong dual feasibility. Due to this property, the dual regularized cone is called the *minimal cone* for (D). The regularization scheme presented here is a dual version of the minimal cone duality of Borwein and Wolkowicz [9, 10] and Wolkowicz [46]. Namely, the dual conic convex program (D) is regularized in [9, 10, 46] by replacing the original cone $\mathcal{K}^*$ by a smaller cone, such that the resulting program will be strongly feasible whenever (D) is feasible. In the preceding, we regularized (P) by transforming all its one–sided, non-improving, level directions into two-sided directions (lines), thus enlarging the cone $\mathcal{K}$. In this way, new primal solutions are created that play the role of sequences that approach feasibility for the original problem (P), as can be seen from Lemma 9 and Lemma 10.

An illustration of regularization for a semidefinite programming problem is given in Example 6.

Example 6 *Consider the program $CP(b, c, \mathcal{A}, \mathcal{K})$ in $\Re^6$ with*

$$\mathcal{K} = \mathcal{S} \times \mathcal{S},$$

where we let

$$\mathcal{S} := \left\{ x \in \Re^3 \;\middle|\; \begin{bmatrix} x_1 & x_3/\sqrt{2} \\ x_3/\sqrt{2} & x_2 \end{bmatrix} \succeq 0 \right\}.$$

Moreover, we let

$$b = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \sqrt{2} \end{bmatrix}^T, \quad c = \begin{bmatrix} 0 & 0 & c_4 & c_4 & 0 & 0 \end{bmatrix}^T,$$

$$\mathcal{A} = \{ x \in \Re^6 \mid x_2 = 0, x_3 = x_4, x_5 = x_6 = 0 \}.$$

Then $\mathcal{K}^* = \mathcal{K}$ and $\mathcal{A}^\perp = \{ s \in \Re^6 \mid s_1 = 0, s_3 = -s_4 \}$. In other words, the primal is

$$p^* = \inf \left\{ 2 c_4 x_4 \;\middle|\; \begin{bmatrix} x_1 & x_4/\sqrt{2} \\ x_4/\sqrt{2} & 0 \end{bmatrix} \succeq 0, \begin{bmatrix} x_4 & 1 \\ 1 & 0 \end{bmatrix} \succeq 0 \right\} = \infty$$

with dual

$$d^* = \inf \left\{ \sqrt{2} s_6 \;\middle|\; \begin{bmatrix} 0 & s_3/\sqrt{2} \\ s_3/\sqrt{2} & s_2 \end{bmatrix} \succeq 0, \begin{bmatrix} 2c_4 - s_3 & s_6/\sqrt{2} \\ s_6/\sqrt{2} & s_5 \end{bmatrix} \succeq 0 \right\}.$$

Notice that the primal is weakly infeasible, whereas (D) is

- *weakly infeasible if $c_4 < 0$,*
- *weakly feasible and solvable with optimal value $d^* = 0$ if $c_4 = 0$, and*
- *weakly feasible and unbounded if $c_4 > 0$.*

Since $\mathcal{A} \cap \mathcal{K} = \Re_+ \times \{0\}^5 \subset \operatorname{Ker} c^T$, *we obtain*

$$\Gamma_{\mathcal{A}\cap \operatorname{Ker} c^T}\mathcal{K} = \Re \times \Re_+ \times \Re \times \mathcal{S}, \quad (\Gamma_{\mathcal{A}\cap \operatorname{Ker} c^T}\mathcal{K})^* = \{0\} \times \Re_+ \times \{0\} \times \mathcal{S},$$

for all c_4. *Notice that if* $c_4 \neq 0$, *then* $\dim(\mathcal{A} \cap \operatorname{Ker} c^T) = 1$ *and* $CP(b, c, \mathcal{A}, \Gamma_{\mathcal{A}\cap \operatorname{Ker} c^T}\mathcal{K})$ *is the regularized program. Indeed, the regularized dual is strongly infeasible for* $c_4 < 0$ *and strongly feasible for* $c_4 > 0$.

However, if $c_4 = 0$ *then* $\dim(\mathcal{A} \cap \operatorname{Ker} c^T) = \dim \mathcal{A} = 2$, *and we have to take one step more. It can easily be verified that*

$$\Gamma^2_{\mathcal{A}}\mathcal{K} = (\Re \times \Re_+ \times \Re) \times (\Re \times \Re_+ \times \Re),$$

and

$$\left(\Gamma^2_{\mathcal{A}}\mathcal{K}\right)^* = (\{0\} \times \Re_+ \times \{0\}) \times (\{0\} \times \Re_+ \times \{0\}).$$

Consequently, the regularization makes the dual strongly feasible, and makes the primal solvable with optimal value 0.

Example 6 reveals a drawback of the regularization scheme: although the regularized certificates are finite, it may not be easy to check their feasibility, since this involves the cone $\Gamma^\infty_{\mathcal{A}\cap \operatorname{Ker} c^{\mathrm{T}}}\mathcal{K}$. For semidefinite programming (as in Example 6) however, we will see in Section 8 that $\Gamma^\infty_{\mathcal{A}\cap \operatorname{Ker} c^{\mathrm{T}}}\mathcal{K}$ can be completely described by semidefiniteness constraints, after adding artificial variables. The resulting regularized semidefinite program coincides with the regularized dual of Ramana [34], which was originally derived in a very different way. The relation between *primal* regularization and the so-called extended Lagrange–Slater dual of Ramana [34] was already recognized by Ramana, Tunçel and Wolkowicz [37]. The way in which Zhao, Karisch, Rendl and Wolkowicz [47] make the regularization explicit, is more or less the opposite of the technique of Ramana. Namely, in [47], the regularized semidefinite relaxation of a quadratic assignment problem is transformed into a strongly feasible semidefinite programming problem by *eliminating* variables, instead of adding variables.

8 Regularization of semidefinite programs

We will now further analyze the structure of the regularized conic convex program, for the special case that $\mathcal{K}$ is the cone of positive semidefinite matrices. We consider two types of semidefinite cones: the semidefinite cone for symmetric matrices and the semidefinite cone for Hermitian matrices.

We let $\mathcal{S}^{(\bar{n})}$ denote the real linear space of $\bar{n} \times \bar{n}$ symmetric matrices, with dimension

$$\dim \mathcal{S}^{(\bar{n})} = \frac{1}{2}\bar{n}(\bar{n}+1).$$

The standard inner product $X \bullet Y$ for two symmetric matrices $X,\ Y \in \mathcal{S}^{(\bar{n})}$ is defined as

$$X \bullet Y = \operatorname{tr} XY.$$

Similarly, we let $\mathcal{H}^{(\bar{n})}$ denote the real linear space of $\bar{n} \times \bar{n}$ Hermitian matrices, with dimension

$$\dim \mathcal{H}^{(\bar{n})} = \bar{n}^2.$$

As a real valued inner product $X \bullet Y$ for two Hermitian matrices $X,\ Y \in \mathcal{H}^{(\bar{n})}$, we define

$$X \bullet Y = \operatorname{tr} XY,$$

exactly as in the symmetric case.

In terms of the above inner product, we can define an orthonormal basis of $\mathcal{H}^{(\bar{n})}$ ($\mathcal{S}^{(\bar{n})}$). In this way, we obtain a one–to–one correspondence between Hermitian (symmetric) matrices in $\mathcal{H}^{(\bar{n})}$ ($\mathcal{S}^{(\bar{n})}$) and their coordinate vectors in $\Re^n$, where $n = \dim \mathcal{H}^{(\bar{n})}$ ($n = \dim \mathcal{S}^{(\bar{n})}$). In particular, if we have an orthonormal basis $U^{(1)},\ U^{(2)}, \ldots,\ U^{(n)}$ of $\bar{n} \times \bar{n}$ Hermitian (symmetric) matrices, then $x \in \Re^n$ is the coordinate vector of $X \in \mathcal{H}^{(\bar{n})}$ ($X \in \mathcal{S}^{(\bar{n})}$) if and only if

$$X = \sum_{i=1}^{n} x_i U^{(i)}.$$

Moreover, if $x,\ y \in \Re^n$ are the coordinate vectors of $X,\ Y \in \mathcal{H}^{(\bar{n})}$ ($X,\ Y \in \mathcal{S}^{(\bar{n})}$), then

$$X \bullet Y = \left(\sum_{i=1}^{n} x_i U^{(i)} \right) \bullet \left(\sum_{i=1}^{n} y_i U^{(i)} \right) = x^{\mathrm{T}} y.$$

We can therefore treat elements of $\mathcal{H}^{(\bar{n})}$ ($\mathcal{S}^{(\bar{n})}$) both as $\bar{n} \times \bar{n}$ Hermitian (symmetric) matrices, and as real vectors of order n. We refer to Alizadeh, Heaberly and Overton [2] and Todd, Toh and Tütüncü [41] for a specific orthonormal basis of $\mathcal{S}^{(\bar{n})}$. Below, we will treat the Hermitian case only. However, all derivations can be immediately translated to the symmetric case; the main difference is the dimension n.

We let $\mathcal{H}_+^{(\bar{n})}$ denote the convex cone of positive semidefinite matrices. For a semidefinite program $\mathrm{CP}(b, c, \mathcal{A}, \mathcal{H}_+^{(\bar{n})})$, we let $B \in \mathcal{H}^{(\bar{n})}$ and $C \in \mathcal{H}^{(\bar{n})}$ denote the matrix representations of the coordinate vectors $b \in \Re^n$ and $c \in \Re^n$, respectively.

Consider an $l \times \bar{n}$ matrix R satisfying $RR^{\mathrm{H}} = I$, where $1 \leq l \leq \bar{n}$ and R^{H} denotes the complex conjugate transpose (or adjoint) of R. Notice that for such R, there must exist a $(\bar{n} - l) \times \bar{n}$ matrix Q such that $\begin{bmatrix} Q^{\mathrm{H}} & R^{\mathrm{H}} \end{bmatrix}$ is a unitary matrix. We define the following linear subspace of $\mathcal{H}^{(\bar{n})}$,

$$\mathrm{HKer}\,(R) := \{X \in \mathcal{H}^{(\bar{n})} \mid RXR^{\mathrm{H}} = 0\}.$$

(In terms of the symmetric Kronecker product [2, 41], $\mathrm{HKer}\,(R)$ corresponds to $\mathrm{Ker}\,(R \otimes_s R)$ in $\Re^n$, for the symmetric case.) There holds

$$\mathcal{H}_+^{(\bar{n})} \oplus \mathrm{HKer}\,(R) = \{X \in \mathcal{H}^{(\bar{n})} \mid RXR^{\mathrm{H}} \succeq 0\}, \tag{28}$$

which is a closed convex cone in $\mathcal{H}^{(\bar{n})}$. Here, it is convenient to interpret the unitary matrix $\left[\begin{array}{cc} Q^{\mathrm{H}} & R^{\mathrm{H}} \end{array}\right]$ as a basis of the complex Euclidean space $\mathcal{C}^n$, since

$$\mathrm{HKer}\,(R) = \left\{ X \left| \left[\begin{array}{c} Q \\ R \end{array}\right] X \left[\begin{array}{cc} Q^{\mathrm{H}} & R^{\mathrm{H}} \end{array}\right] = \left[\begin{array}{cc} X_{11} & X_{12} \\ X_{12}^{\mathrm{H}} & 0 \end{array}\right] \text{ for some } X_{11},\, X_{12} \right. \right\},$$

and

$$\mathcal{H}_{+}^{(\bar{n})} \oplus \mathrm{HKer}\,(R) = \left\{ X \left| \left[\begin{array}{c} Q \\ R \end{array}\right] X \left[\begin{array}{cc} Q^{\mathrm{H}} & R^{\mathrm{H}} \end{array}\right] = \left[\begin{array}{cc} X_{11} & X_{12} \\ X_{12}^{\mathrm{H}} & X_{22} \end{array}\right],\ X_{22} \succeq 0 \right. \right\}.$$

We will derive in this section that the regularized cones , ${}^{k}_{\mathcal{A}}\mathcal{H}_{+}^{(\bar{n})}$ are of the form (28). First of all, we notice that this is indeed the case for $k = 0$, viz.

$$\mathcal{H}_{+}^{(\bar{n})} = \mathcal{H}_{+}^{(\bar{n})} \oplus \mathrm{HKer}\,(I),$$

where I is the $\bar{n} \times \bar{n}$ identity matrix.

Let $\mathcal{C}^{\bar{n}}$ denote the space of complex $\bar{n}$-tuples. We will see below that $\mathrm{Img}\,R^{\mathrm{H}} = R^{\mathrm{H}}\mathcal{C}^{l}$ plays a crucial role. We want to make clear that $\mathrm{Img}\,R^{\mathrm{H}}$ is a *complex* linear subspace of $\mathcal{C}^{\bar{n}}$, where we use the standard complex valued inner product $y^{\mathrm{H}}x$ for $x, y \in \mathcal{C}^{\bar{n}}$. This is in contrast with the space of Hermitian matrices which is real: although the off–diagonal entries of Hermitian matrices are complex, the inner product $X \bullet Y$ is *real* valued for $X,\, Y \in \mathcal{H}^{(\bar{n})}$.

Lemma 13 *Let $\mathcal{A}$ be a linear subspace of $\mathcal{H}^{(\bar{n})}$, and let $\mathcal{K} = \mathcal{H}_{+}^{(\bar{n})} \oplus \mathrm{HKer}\,(R)$. If $Y \in \mathrm{rel}\,(\mathcal{A} \cap \mathcal{K})$, then*

$$(\mathrm{Ker}\,Y) \cap \mathrm{Img}\,R^{H} \subseteq (\mathrm{Ker}\,\tilde{Y}) \cap \mathrm{Img}\,R^{H} \quad \forall \tilde{Y} \in \mathcal{A} \cap \mathcal{K}.$$

Proof: Suppose to the contrary that there exists a $\tilde{Y} \in \mathcal{A} \cap \mathcal{K}$ such that $\tilde{Y}u \neq 0$ for some $u \in (\mathrm{Ker}\,Y) \cap \mathrm{Img}\,R^{\mathrm{H}}$. Since $\tilde{Y} \in \mathcal{K}$, it holds

$$u^{\mathrm{H}}\tilde{Y}u > 0.$$

This implies that for any $\epsilon > 0$,

$$u^{\mathrm{H}}(Y - \epsilon\tilde{Y})u = -\epsilon\, u^{\mathrm{H}}\tilde{Y}u < 0,$$

which contradicts the fact that $Y \in \mathrm{rel}\,(\mathcal{A} \cap \mathcal{K})$.

□

Lemma 14 *Let R and $\mathcal{K}$ be as in Lemma 13, and let $Y \in \mathcal{A} \cap \mathcal{K}$. If*

$$W \in (Y + \mathrm{HKer}\,(R)) \cap \mathcal{H}_{+}^{(\bar{n})}$$

then

$$\mathrm{Ker}\,W \supseteq (\mathrm{Ker}\,Y) \cap \mathrm{Img}\,R^{H}, \tag{29}$$

with equality holding if and only if $W \in \mathrm{rel}\,((Y + \mathrm{HKer}\,(R)) \cap \mathcal{H}_{+}^{(\bar{n})})$.

Proof: First, we remark that since W is positive semidefinite, we have

$$u \in \operatorname{Ker} W \iff u^{\mathrm{H}} W u = 0.$$

Since $W \in Y + \operatorname{HKer}(R)$ and $Y \in \mathcal{K}$, it holds

$$u^{\mathrm{H}} W u = u^{\mathrm{H}} Y u \geq 0 \quad \forall u \in \operatorname{Img} R^{\mathrm{H}}, \tag{30}$$

with $u^{\mathrm{H}} Y u = 0$ if and only if $u \in (\operatorname{Ker} Y) \cap \operatorname{Img} R^{\mathrm{H}}$. This proves the conclusion (29). Now consider $v \in (\operatorname{Img} R^{\mathrm{H}})^{\perp} = \operatorname{Ker} R$, $v \neq 0$, and notice that

$$vv^{\mathrm{H}} \in \operatorname{HKer}(R).$$

Hence,

$$W + \lambda vv^{\mathrm{H}} \in (Y + \operatorname{HKer}(R)) \cap \mathcal{H}_+^{(\bar{n})} \text{ for all } \lambda \geq 0.$$

For $W \in \operatorname{rel}((Y + \operatorname{HKer}(R)) \cap \mathcal{H}_+^{(\bar{n})})$, it thus follows that $v^{\mathrm{H}} W v > 0$. Consequently, $\operatorname{Ker} W \subseteq \operatorname{Img} R^{\mathrm{H}}$. Together with (30), we obtain

$$\operatorname{Ker} W = (\operatorname{Ker} Y) \cap \operatorname{Img} R^{\mathrm{H}}.$$

□

We arrive now at the central result of this section, viz., if $\mathcal{K}$ is of the form (28), then so is $\Gamma_{\mathcal{A}}\mathcal{K}$.

Theorem 9 *If* $\mathcal{K} = \mathcal{H}_+^{(\bar{n})} \oplus \operatorname{HKer}(R)$, *then*

$$\Gamma_{\mathcal{A}}\mathcal{K} = \mathcal{H}_+^{(\bar{n})} \oplus \operatorname{HKer}(\tilde{R}),$$

where $\tilde{R}$ *is any matrix satisfying*

$$\begin{cases} \operatorname{Img} \tilde{R}^H = (\operatorname{Img} R^H) \cap \operatorname{Ker} Y, \ \textit{for some } Y \in \operatorname{rel}(\mathcal{A} \cap \mathcal{K}), \\ \tilde{R}\tilde{R}^H = I. \end{cases}$$

Proof: Suppose that $X \in \Gamma_{\mathcal{A}}\mathcal{K}$, i.e. $X = \lim_{i\to\infty} X^{(i)} - Y^{(i)}$ for some sequences $X^{(1)}, X^{(2)}, \ldots$ and $Y^{(1)}, Y^{(2)}, \ldots$ in $\mathcal{K}$ and $\mathcal{A} \cap \mathcal{K}$ respectively. We know from Lemma 13 and the definition of $\tilde{R}$ that $\operatorname{Img} \tilde{R}^{\mathrm{H}} \subseteq (\operatorname{Img} R^{\mathrm{H}}) \cap \operatorname{Ker} Y^{(i)}$, so that

$$\tilde{R}(X^{(i)} + Y^{(i)})\tilde{R}^{\mathrm{H}} = \tilde{R}X^{(i)}\tilde{R}^{\mathrm{H}} \succeq 0$$

for all $i \in \{1, 2, \ldots\}$. The above relation shows that $X \in \mathcal{H}_+^{(\bar{n})} \oplus \operatorname{HKer}(\tilde{R})$, from which we conclude that $\Gamma_{\mathcal{A}}\mathcal{K} \subseteq \mathcal{H}_+^{(\bar{n})} \oplus \operatorname{HKer}(\tilde{R})$. To prove the converse inclusion, consider a matrix $X \in \mathcal{H}_+^{(\bar{n})} \oplus \operatorname{HKer}(\tilde{R})$. Without loss of generality, we may assume that there exists a matrix Q such that

$$R^{\mathrm{H}} = \begin{bmatrix} Q^{\mathrm{H}} & \tilde{R}^{\mathrm{H}} \end{bmatrix}.$$

We partition RYR^{H}, $Y \in \mathrm{rel}\,(\mathcal{A} \cap \mathcal{K})$ as follows:

$$RYR^{\mathrm{H}} = \begin{bmatrix} Q \\ \tilde{R} \end{bmatrix} Y \begin{bmatrix} Q^{\mathrm{H}} & \tilde{R}^{\mathrm{H}} \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^{\mathrm{H}} & Y_{22} \end{bmatrix}.$$

By definition of $\tilde{R}$ and using the fact that $RYR^{\mathrm{H}} \succeq 0$, it follows that

$$Y_{11} \succ 0, \quad Y_{12} = 0, \quad Y_{22} = 0.$$

Similarly, we partition the matrix RXR^{H} as follows:

$$RXR^{\mathrm{H}} = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^{\mathrm{H}} & X_{22} \end{bmatrix}.$$

Since $X \in \mathcal{H}_+^{(\bar{n})} \oplus \mathrm{HKer}\,(\tilde{R})$, we have $X_{22} \succeq 0$. Let α_1 and α_2 be positive numbers such that

$$\alpha_1 Y_{11} + X_{11} \succeq 0, \quad \alpha_2 Y_{11} \succeq X_{12} X_{12}^{\mathrm{H}}.$$

(Such numbers exist, because Y_{11} is positive definite.) Then for any $\epsilon > 0$ there holds

$$R(X + \epsilon I + (\alpha_1 + \alpha_2/\epsilon)Y)R^{\mathrm{H}} \succeq 0.$$

Letting

$$X(\epsilon) := X + \epsilon I + (\alpha_1 + \alpha_2/\epsilon)Y,$$

it follows that $X(\epsilon) \in \mathcal{K}$ and

$$X = \lim_{\epsilon \downarrow 0} (X(\epsilon) - (\alpha_1 + \alpha_2/\epsilon)Y),$$

so that $X \in , {}_{\mathcal{A}}\mathcal{K}$.

□

We already observed that

$$, {}^{0}_{\mathcal{A}}\mathcal{H}_+^{(\bar{n})} = \mathcal{H}_+^{(\bar{n})} = \mathcal{H}_+^{(\bar{n})} \oplus \mathrm{HKer}\,(I).$$

With an inductive argument, it thus follows from Theorem 9 that there exist matrices $R^{(1)}$, $R^{(2)}, \ldots$ such that

$$, {}^{k}_{\mathcal{A}}\mathcal{H}_+^{(\bar{n})} = \mathcal{H}_+^{(\bar{n})} \oplus \mathrm{HKer}\,(R^{(k)}),$$

for $k = 1, 2, \ldots$. Notice also from Theorem 9 that $, {}_{\mathcal{A}}\mathcal{K} \neq \mathcal{K}$ if and only if $\mathrm{rank}\,\tilde{R} < \mathrm{rank}\,R$. Together with Lemma 9, this implies that

$$, {}^{k}_{\mathcal{A}}\mathcal{K} = , {}^{\infty}_{\mathcal{A}}\mathcal{K} \quad \text{for all } k \geq \min\{\bar{n}, \dim \mathcal{A}\}. \tag{31}$$

It should be noted that $\bar{n}$ can be considerably smaller than $\dim \mathcal{A}$.

The following lemma shows the interesting fact that the linear subspace $\mathrm{HKer}\,(\tilde{R})$, where $\tilde{R}$ is defined as in Theorem 9, can be modeled by semidefinite constraints.

Lemma 15 *Let R and $\tilde{R}$ as in Theorem 9. There holds*

$$\mathrm{HKer}\,(\tilde{R}) = \left\{ W_{12} + W_{12}^H \,\middle|\, \begin{bmatrix} W_{11} & W_{12} \\ W_{12}^H & I \end{bmatrix} \succeq 0, W_{11} + U \in \mathcal{A}, U \in \mathrm{HKer}\,(R) \right\}. \tag{32}$$

Proof: We first notice that

$$\begin{bmatrix} W_{11} & W_{12} \\ W_{12}^{\mathrm{H}} & I \end{bmatrix} \succeq 0 \iff W_{12} W_{12}^{\mathrm{H}} \preceq W_{11}. \tag{33}$$

Consider W_{11}, W_{12}, W_{22} and U satisfying the right hand side of (32). Let $Y = W_{11} + U$, then

$$W_{11} \in (Y + \mathrm{HKer}\,(R)) \cap \mathcal{H}_+^{(\bar{n})}, \quad Y \in \mathcal{H}_+^{(\bar{n})} \oplus \mathrm{HKer}\,(R).$$

Using respectively Lemma 13, Lemma 14 and (33), we obtain

$$\mathrm{Img}\,\tilde{R}^{\mathrm{H}} \subseteq (\mathrm{Img}\,R^{\mathrm{H}}) \cap \mathrm{Ker}\,Y \subseteq \mathrm{Ker}\,W_{11} \subseteq \mathrm{Ker}\,W_{12}^{\mathrm{H}},$$

so that $W_{12}^{\mathrm{H}} \tilde{R}^{\mathrm{H}} = (\tilde{R} W_{12})^{\mathrm{H}} = 0$, and

$$W_{12} + W_{12}^{\mathrm{H}} \in \mathrm{HKer}\,(\tilde{R}).$$

Conversely, suppose that $X \in \mathrm{HKer}\,(\tilde{R})$. Since $\tilde{R}\tilde{R}^{\mathrm{H}} = I$ there exists some matrix Q such that $\begin{bmatrix} Q^{\mathrm{H}} & \tilde{R}^{\mathrm{H}} \end{bmatrix}$ is unitary. By definition, $X \in \mathrm{HKer}\,(\tilde{R})$ means that

$$\begin{bmatrix} Q \\ \tilde{R} \end{bmatrix} X \begin{bmatrix} Q^{\mathrm{H}} & \tilde{R}^{\mathrm{H}} \end{bmatrix} = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^{\mathrm{H}} & 0 \end{bmatrix},$$

for some X_{11} and X_{12}. Letting

$$W_{12} := \begin{bmatrix} Q^{\mathrm{H}} & \tilde{R}^{\mathrm{H}} \end{bmatrix} \begin{bmatrix} X_{11}/2 & X_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q \\ \tilde{R} \end{bmatrix},$$

it follows that

$$X = W_{12} + W_{12}^{\mathrm{H}}, \quad \mathrm{Img}\,\tilde{R}^{\mathrm{H}} \subseteq \mathrm{Ker}\,W_{12}^{\mathrm{H}}.$$

Since $\mathrm{Img}\,\tilde{R}^{\mathrm{H}} = (\mathrm{Img}\,R) \cap \mathrm{Ker}\,Y$ for some $Y \in \mathrm{rel}\,(\mathcal{A} \cap (\mathcal{H}_+^{\bar{n}} \oplus \mathrm{HKer}\,(R)))$, we know from Lemma 14 and the above inclusion that

$$\mathrm{Ker}\,\bar{W}_{11} = \mathrm{Img}\,\tilde{R}^{\mathrm{H}} \subseteq \mathrm{Ker}\,W_{12}^{\mathrm{H}}$$

for any $\bar{W}_{11} \in \mathrm{rel}\,((Y + \mathrm{HKer}\,(R)) \cap \mathcal{H}_+^{(\bar{n})})$. Letting $\bar{U} = Y - \bar{W}_{11}$, it follows that

$$\begin{bmatrix} \alpha\bar{W}_{11} & W_{12} \\ W_{12}^{\mathrm{H}} & I \end{bmatrix} \succeq 0, \quad \bar{W}_{11} + \bar{U} \in \mathcal{A}, \quad \bar{U} \in \mathrm{HKer}\,(R)$$

for sufficiently large $\alpha > 0$. Letting $W_11 := \alpha\bar{W}_{11}$ and $U := \alpha\bar{U}$, we see that (32) is satisfied.

$\square$

Consider the kth regularized semidefinite program

$$\inf\{C \bullet X \mid X \in (B + \mathcal{A}) \cap \Gamma^{k}_{\mathcal{A}\cap \mathrm{Ker}\, c^{\mathrm{T}}} \mathcal{H}^{(\bar{n})}_{+}\}. \tag{34}$$

We already know from (31) that for all $k = 0, 1, 2, \ldots$, there exist $R^{(k)}$ such that

$$\Gamma^{k}_{\mathcal{A}\cap \mathrm{Ker}\, c^{\mathrm{T}}} \mathcal{H}^{(\bar{n})}_{+} = \mathcal{H}^{(\bar{n})}_{+} \oplus \mathrm{HKer}\,(R^{(k)}).$$

Using Lemma 15, it follows that (34) is equivalent to

$$\begin{array}{ll} \inf & C \bullet (X + W^{(k)}_{12} + (W^{(k)}_{12})^{\mathrm{H}}) \\ \text{s.t.} & X + W^{(k)}_{12} + (W^{(k)}_{12})^{\mathrm{H}} \in B + \mathcal{A} \\ & \begin{bmatrix} W^{(k)}_{11} & W^{(k)}_{12} \\ (W^{(k)}_{12})^{\mathrm{H}} & I \end{bmatrix} \succeq 0, \quad X \succeq 0, \\ & W^{(k)}_{11} + U \in \mathcal{A} \cap \mathrm{Ker}\, c^{\mathrm{T}}, \quad U \in \mathrm{HKer}\,(R^{(k-1)}). \end{array}$$

With a recursive argument, we obtain

$$\begin{array}{ll} \inf & C \bullet (X + W^{(k)}_{12} + (W^{(k)}_{12})^{\mathrm{H}}) \\ \text{s.t.} & X + W^{(k)}_{12} + (W^{(k)}_{12})^{\mathrm{H}} \in B + \mathcal{A} \\ & X \succeq 0, \\ & \begin{bmatrix} W^{(i)}_{11} & W^{(i)}_{12} \\ (W^{(i)}_{12})^{\mathrm{H}} & I \end{bmatrix} \succeq 0 \quad \text{for } i = 1, 2, \ldots, k \\ & W^{(i)}_{11} + W^{(i-1)}_{12} + (W^{(i-1)}_{12})^{\mathrm{H}} \in \mathcal{A} \cap \mathrm{Ker}\, c^{\mathrm{T}} \quad \text{for } i = 2, 3, \ldots, k \\ & W^{(1)}_{11} \in \mathcal{A} \cap \mathrm{Ker}\, c^{\mathrm{T}}, \end{array} \tag{PRAM}$$

which is again a semidefinite program. This regularized program was proposed by Ramana [34] for the real symmetric case, see also [37, 36]. Moreover, Ramana uses $k = \dim \mathcal{A}^{\perp} = \bar{n}^2 - \dim \mathcal{A}$, whereas we show that $k = \min(\bar{n}, \dim \mathcal{A})$ is sufficient. It is important to note that checking the feasibility of a solution for Ramana's regularized semidefinite program is easy, because it involves only linear and positive semidefiniteness constraints.

Remark 1 *The introduction of auxiliary variables $W^{(k)}$ into the regularized program has also disadvantages. In particular, the duality relation of (D) and its Ramana dual (PRAM) is asymmetric, since Ramana's dualization scheme increases the dimension of the problem. In order to regain symmetricity, Ramana and Freund [35] propose to consider the primal–dual pair of (PRAM) and its standard dual semidefinite programming problem. Since the subvalue of (PRAM) is equal to its* *optimal value, it follows from (6) that this primal–dual pair is again in perfect duality, and this fact is known from Ramana and Freund [35]. However, as recently noticed by De Klerk, Roos and Terlaky [26], it is possible that the dual of (PRAM) is weakly (in)feasible, and we can therefore not obtain results as in Theorem 8 for the primal–dual pair of Ramana and Freund.*

9 Inexact dual solutions

As pointed out by Nesterov and Nemirovsky [31], interior point methods are well suited for solving conic convex programs. Although interior point methods typically require the existence of primal and dual interior solutions, it is possible to solve conic programs that are not strongly feasible by using the self–dual embedding technique [29]. With (P) being a nonlinear program, it is not surprising that the interior point methods (or indeed any other methods) require an infinite number of iterations to obtain an exact solution. Within a finite number of iterations these iterative methods can only compute an approximate solution of (P). Naturally such an approximate solution of (P) can be interpreted as an exact solution of a perturbed problem (backward error analysis). However, this interpretation is of little practical use. In what follows, we show that an approximate solution of (P) can be used to infer many useful properties of the original conic program (P) such as 'approximate infeasibility'.

In the analysis of approximate solutions, it is convenient to add a variable which measures the constraint violation. A good way to construct such a variable is by making use of the *norm* cone, which is defined as follows:

$$\mathcal{K}_{\text{norm}} := \{(x_0, x) \in \Re_+ \times \Re^n \mid x_0 \geq \|x\|\}.$$

Using the basic properties of norms, it is easily seen that $\mathcal{K}_{\text{norm}}$ is a closed, pointed and solid convex cone. Moreover, it follows from the definition of dual norms that

$$\mathcal{K}^*_{\text{norm}} = \{(x_0, x) \in \Re_+ \times \Re^n \mid x_0 \geq \|x\|^*\}.$$

The theorem below shows that if we have an approximate primal improving direction, viz. some $x \in \mathcal{A}$ such that $c^{\mathrm{T}}x = -1$ and x 'almost' in $\mathcal{K}$, then the dual cannot have any 'reasonably' sized feasible solution.

Theorem 10 *Consider a conic convex program CP$(b, c, \mathcal{A}, \mathcal{K})$. There holds*

$$\inf_{s \in \mathcal{F}_D} \|s\|^* = \sup\{-c^T x \mid x \in \mathcal{A},\, \operatorname{dist}(x, \mathcal{K}) \leq 1\}.$$

Proof: Construct the conic convex program

$$\text{CP}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ c \end{bmatrix}, \{0\} \times \mathcal{A}, (\{0\} \times \mathcal{K}) \oplus \mathcal{K}_{\text{norm}}\right), \tag{35}$$

which can be written as

$$\inf\{c^{\mathrm{T}}x \mid x \in \mathcal{A},\ x - u \in \mathcal{K},\ \|u\| \le x_0 = 1\} = \inf\{c^{\mathrm{T}}x \mid x \in \mathcal{A},\ \mathrm{dist}(x, \mathcal{K}) \le 1\}.$$

Using Lemma 1, it follows that the dual of the conic convex program (35) is

$$\inf\{s_0 \mid s \in (c + \mathcal{A}^{\perp}) \cap \mathcal{K}^*,\ \|s\|^* \le s_0\} = \inf\{\|s\|^* \mid s \in \mathcal{F}_D\}.$$

Notice now that (35) is primal strongly feasible, because it has a trivial interior solution $\begin{bmatrix} 1 & 0 \end{bmatrix}^{\mathrm{T}} \in \mathrm{int}\,\mathcal{K}_{\mathrm{norm}}$. Theorem 7 is therefore applicable, and it yields

$$\inf_{s \in \mathcal{F}_D} \|s\|^* = -\inf\{c^{\mathrm{T}}x \mid x \in \mathcal{A},\ \mathrm{dist}(x, \mathcal{K}) \le 1\}.$$

□

Remark 2 *For the case of l_p norms and $\mathcal{K} = \Re^n_+$, the statement of Theorem 10 is known from Todd and Ye [42].*

Remark 3 *Suppose that we have an approximate solution $\hat{x}$, with $\mathrm{dist}(\hat{x}, \mathcal{K}) < \epsilon_1$, $\mathrm{dist}(\hat{x}, \mathcal{A}) < \epsilon_2$, $c^T\hat{x} < -1$. Let Δx be such that $\hat{x} + \Delta x \in \mathcal{A}$ and $\|\Delta x\| < \epsilon_2$. Then, we can invoke Theorem 10 with $x := (\hat{x} + \Delta x)/(\epsilon_1 + \epsilon_2)$ to conclude that*

$$\inf_{s \in \mathcal{F}_D} \|s\|^* \ge -c^T x \ge \frac{-c^T\hat{x}}{\epsilon_1 + \epsilon_2} - \frac{\epsilon_2}{\epsilon_1 + \epsilon_2}\|c\|^*,$$

We will now show that based on an approximate primal solution, viz. some $x \in b + \mathcal{A}$ such that x is 'almost' in $\mathcal{K}$, we obtain a lower bound on the objective value of any 'reasonably' sized dual feasible solution. To the best of our knowledge, this result (Theorem 11) is new.

Theorem 11 *Consider a conic convex program CP$(b, c, \mathcal{A}, \mathcal{K})$. For all $\gamma \in \Re$, there holds*

$$\begin{aligned} &\inf\{\|s\|^* \mid s \in \mathcal{F}_D,\ b^T s \le \gamma\} \\ &\quad = \sup\{-(c^T x + \gamma x_{n+1}) \mid x \in x_{n+1}b + \mathcal{A},\ \mathrm{dist}(x, \mathcal{K}) \le 1,\ x_{n+1} \ge 0\}. \end{aligned}$$

Proof: Recall from (14) and (17) that

$$M_b := \begin{bmatrix} I & -b \\ 0 & 1 \end{bmatrix}, \quad M_b^{-1} = \begin{bmatrix} I & b \\ 0 & 1 \end{bmatrix}. \tag{36}$$

Now, we use a similar argumentation as in the proof of Theorem 10. First, construct the conic convex program

$$\mathrm{CP}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ c \\ \gamma \end{bmatrix}, \{0\} \times M_b^{-1}(\mathcal{A} \times \Re), ((\{0\} \times \mathcal{K}) \oplus \mathcal{K}_{\mathrm{norm}}) \times \Re_+ \right), \tag{37}$$

which can be written as

$$\begin{aligned}&\inf\{c^{\mathrm{T}}x+\gamma x_{n+1}\mid x-x_{n+1}b\in\mathcal{A},\ x-u\in\mathcal{K},\ \|u\|\le x_0=1,\ x_{n+1}\ge 0\}\\ =\ &\inf\{c^{\mathrm{T}}x+\gamma x_{n+1}\mid x\in x_{n+1}b+\mathcal{A},\ \mathrm{dist}(x,\mathcal{K})\le 1,\ x_{n+1}\ge 0\}.\end{aligned}$$

Using Lemma 1, it follows that the dual of the conic convex program (35) is

$$\inf\{s_0\mid s\in(c+\mathcal{A}^{\perp})\cap\mathcal{K}^*,\, s_{n+1}=\gamma-b^{\mathrm{T}}s\ge 0,\, s_0\ge\|s\|^*\}=\inf\{\|s\|^*\mid s\in\mathcal{F}_D,\ b^{\mathrm{T}}s\le\gamma\}.$$

If $b=0$, then (37) has the trivial interior solution $\begin{bmatrix}1 & 0 & 1\end{bmatrix}^{\mathrm{T}}\in(\mathrm{int}\,\mathcal{K}_{\mathrm{norm}})\times\Re_{++}$. And if $b\ne 0$, then $\begin{bmatrix}1 & b^{\mathrm{T}}/\|b\| & 1/\|b\|\end{bmatrix}^{\mathrm{T}}\in(\mathrm{int}\,\mathcal{K}_{\mathrm{norm}})\times\Re_{++}$ is a primal interior feasible solution. Hence, (37) is primal strongly feasible. Theorem 7 is therefore applicable, and it yields

$$\inf\{\|s\|^*\mid s\in\mathcal{F}_D,\ b^{\mathrm{T}}s\le\gamma\}=-\inf\{c^{\mathrm{T}}x+\gamma x_{n+1}\mid x\in x_{n+1}b+\mathcal{A},\ \mathrm{dist}(x,\mathcal{K})\le 1,\ x_{n+1}\ge 0\}.$$

□

Suppose that $x\in b+\mathcal{A}$. If $x\in\mathcal{K}$, then $d^*\ge -c^{\mathrm{T}}x$, as we already knew from (19). If $\mathrm{dist}(x,\mathcal{K})>0$, then x is an approximate solution, and we obtain from Theorem 11 that

$$\inf\{\|s\|^*\mid s\in\mathcal{F}_D,\ b^{\mathrm{T}}s\le\gamma\}\ge\frac{-c^{\mathrm{T}}x-\gamma}{\mathrm{dist}(x,\mathcal{K})}.$$

Remark that Theorem 10 follows from Theorem 11 by letting $\gamma\to\infty$. Theorem 6 can also be seen as an application of Theorem 11 (the converse is true as well, as has just been demonstrated).

10 Conclusion

We have treated conic convex programming duality in a unified fashion. Special attention has been given to conic convex programs that do not satisfy constraint qualifications. It has also been shown how recent duality approaches of [9, 10, 46, 34, 37] fit into the framework. Elaborating on the results of [42], we have also discussed the value of approximate dual solutions.

We believe that duality results under no constraint qualifications have not received enough attention in the past. It is our hope that this paper will help popularize these results in future. In [29], we show that this type of duality relation can be used fruitfully in the design of algorithms whose convergence is guaranteed even in the absence of constraint qualifications.

Our survey is restricted to conic convex programming in finite dimensional real linear spaces. As such, it includes conic convex programming with complex numbers, if a real inner product is used. For instance, we can treat $\mathcal{C}^n$ (the space of complex n-tuples) as a $2n$-dimensional *real* linear space by using the real valued inner product Re $s^{\mathrm{H}}x$. However, due to the lack of ordering of complex numbers, there is no obvious way to generalize duality results to *complex* linear spaces (ordering is crucial in the definition of convex cones, among others). Duality results for convex programming in infinite dimensional real linear spaces have not been discussed in this paper. The strong duality result of Theorem 6 can be generalized to semi–infinite linear and convex programming, see the collective work of [7, 8, 15, 23, 24, 25, 46]. Results for conic convex programming with infinitely many variables and a bounded feasible set are given in [13]; see also the books [3, 19].

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References

[1] F. Alizadeh. Interior point methods in semidefinite programming with applications to combinatorial optimization problems. *SIAM Journal on Optimization*, 5:13–51, 1995.

[2] F. Alizadeh, J.A. Heaberly, and M. Overton. Primal–dual interior–point methods for semidefinite programming: convergence rates, stability and numerical results. Technical Report 721, Computer Science Department, New York University, New York, 1996.

[3] E. Anderson and P. Nash. *Linear programming in infinite dimensional spaces.* John Wiley and Sons, 1987.

[4] A. Ben-Israel. Linear equations and inequalities on finite dimensional, real or complex, vector spaces: a unified theory. *Journal of Mathematical Analysis and Applications*, 27:367–389, 1969.

[5] A. Berman. *Cones, matrices and mathematical programming*, volume 73 of *Lecture Notes in Economics and Mathematical Systems.* Springer-Verlag, Berlin, 1979.

[6] A. Berman and A. Ben-Israel. Linear inequalities, mathematical programming and matrix theory. *Mathematical Programming*, 1:291–300, 1971.

[7] J.M. Borwein. A note on perfect duality and limiting Lagrangians. *Mathematical Programming*, 18:330–337, 1980.

[8] J.M. Borwein. The limiting Lagrangian as a consequence of Helly's theorem. *Journal of Optimization Theory and Applications*, 33:497–513, 1981.

[9] J.M. Borwein and H. Wolkowicz. Characterizations of optimality for the abstract convex program with finite dimensional range. *Journal of the Australian Mathematical Society*, 30:390–411, 1981.

[10] J.M. Borwein and H. Wolkowicz. Regularizing the abstract convex program. *Journal of Mathematical Analysis and Applications*, 83:495–530, 1981.

[11] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear matrix inequalities in system and control theory*, volume 15 of *Studies in Applied Mathematics.* SIAM, Philadelphia, PA, 1994.

[12] W.B. Carver. Systems of linear inequalities. *Annals of Mathematics*, 23:212–220, 1921.

[13] B.D. Craven. Modified Kuhn-Tucker conditions when a minimum is not attained. *Operations Research Letters*, 3:47–52, 1984.

[14] B.D. Craven and J.J. Koliha. Generalizations of Farkas' theorem. *SIAM Journal on Mathematical Analysis*, 8:983–997, 1977.

[15] R.J. Duffin. Infinite programs. In H.W. Kuhn and A.W. Tucker, editors, *Linear Inequalities and Related Systems*, pages 157–170. Princeton University Press, Princeton, NJ, 1956.

[16] M. Epelman and R.M. Freund. Condition number complexity of an elementary algorithm for resolving a conic linear system. Technical report, Massachusetts Institute of Technology, Operations Research Center, Cambridge, Massachusetts, USA, 1997.

[17] Ky Fan. On systems of linear inequalities. In H.W. Kuhn and A.W. Tucker, editors, *Linear Inequalities and Related Systems*, pages 99–156. Princeton University Press, Princeton, NJ, 1956.

[18] J. Farkas. Über die Theorie des einfachen Ungleichungen. *Journal für die reine und angewandte Mathematik*, 124:1–24, 1902.

[19] K. Glashoff and S. Gustafson. *Linear optimization and approximation*, volume 45 of *Applied Mathematical Sciences*. Springer-Verlag, Basel, 1978.

[20] A.J. Goldman and A.W. Tucker. Theory of linear programming. In H.W. Kuhn and A.W. Tucker, editors, *Linear Inequalities and Related Systems*, pages 53–98. Princeton University Press, Princeton, NJ, 1956.

[21] P. Gordan. Über die Auflösungen linearer Gleichungen mit reelen Coefficienten. *Mathematische Annalen*, 6:23–28, 1873.

[22] J.E. Ingersoll. *Theory of financial decision making*. Rowman & Littlefield, Totowa, NJ, 1987.

[23] R.G. Jeroslow. A limiting Lagrangian for infinitely convex optimization in $\Re^n$. *Journal of Optimization Theory and Applications*, 33:479–495, 1981.

[24] D.F Karney. Duality gaps in semi–infinite linear porgramming — an approximation problem. *Mathematical Programming*, 20:129–143, 1981.

[25] D.F. Karney and T.D. Morley. Limiting Lagrangians: a primal approach. *Journal of Optimization Theory and Applications*, 48:163–174, 1986.

[26] E. de Klerk, C. Roos, and T. Terlaky. Infeasible start semidefinite programming algorithms via self–dual embeddings. Technical Report 97–10, Delft University of Technology, Faculty of Technical Mathematics and Informatics, Delft, The Netherlands, 1997.

[27] M. Kojima, S. Shindoh, and S. Hara. Interior–point methods for the monotone semidefinite linear complementarity problem in symmetric matrices. *SIAM Journal on Optimization*, 7(1):86–125, 1997.

[28] C.J. Lin and R. Saigal. An infeasible start predictor corrector method for semi–definite linear programming. Technical report, Department of Industrial and Operations Engineering, The University of Michigan, Ann Arbor, USA, 1995.

[29] Z.-Q. Luo, J.F. Sturm, and S. Zhang. Duality and self-duality for conic convex programming. Technical Report 9620/A, Econometric Institute, Erasmus University Rotterdam, Rotterdam, The Netherlands, 1996.

[30] R.D.C. Monteiro. Primal–dual path following algorithms for semidefinite programming. Technical report, School of Industrial and Systems Engineering, Georgia Tech, Atlanta, Georgia, U.S.A., 1995. To appear in *SIAM Journal on Optimization.*

[31] Y. Nesterov and A. Nemirovsky. *Interior point polynomial methods in convex programming*, volume 13 of *Studies in Applied Mathematics.* SIAM, Philadelphia, 1994.

[32] Y. Nesterov and M.J. Todd. Primal–dual interior–point methods for self–scaled cones. Technical Report 1125, School of Operations Research and Industrial Engineering, Cornell University, Ithaca, New York, 1995.

[33] Yu. Nesterov, M.J. Todd, and Y. Ye. Infeasible–start primal–dual methods and infeasibility detectors for nonlinear programming problems. Technical Report 1156, School of Operations Research and Industrial Engineering, Cornell University, Ithaca, New York, 1996.

[34] M.V. Ramana. An exact duality theory for semidefinite programming and its complexity implications. *Mathematical Programming*, 77(2):129–162, 1997.

[35] M.V. Ramana and R. Freund. One the ELSD duality theory for SDP. Technical report, Center of Applied Optimization, University of Florida, Gainesville, Florida, USA, 1996.

[36] M.V. Ramana and P.M. Pardalos. Semidefinite programming. In T. Terlaky, editor, *Interior point methods of mathematical programming*, volume 5 of *Applied Optimization*, chapter 9, pages 369–398. Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996. Series editors P.M. Pardalos and D. Hearn.

[37] M.V. Ramana, L. Tunçel, and H. Wolkowicz. Strong duality for semidefinite programming. Technical Report 85-12, University of Waterloo, Waterloo, Canada, 1995.

[38] R.T. Rockafellar. *Convex Analysis.* Princeton University Press, Princeton, NJ, 1970.

[39] E. Stiemke. Über positiven Lösungen homogener linearer Gleichungen. *Mathematische Annalen*, 76:340–342, 1915.

[40] J.F. Sturm and S. Zhang. Symmetric primal–dual path following algorithms for semidefinite programming. Technical Report 9554/A, Econometric Institute, Erasmus University Rotterdam, Rotterdam, The Netherlands, 1995.

[41] M.J. Todd, K.C. Toh, and R.H. Tütüncü. On the Nesterov–Todd direction in semidefinite programming. Technical Report TR 1154, School of Operations Research and Industrial Engineering, Cornell University, Ithaca, NY, USA, 1996.

[42] M.J. Todd and Y. Ye. Approximate Farkas lemmas and stopping rules for iterative infeasible–point algorithms for linear programming. Technical report, School of Operations Research and Industrial Engineering, Cornell University, Ithaca, NY, USA, 1994.

[43] A.W. Tucker. Dual systems of homogeneous linear relations. In H.W. Kuhn and A.W. Tucker, editors, *Linear Inequalities and Related Systems*, pages 3–18. Princeton University Press, Princeton, NJ, 1956.

[44] M. Ujvàri. Simple proofs of duality theorems in semidefinite and cone linear programming. Technical report, Department of Operations Research, Eötvös Lorànd University, Budapest, Hungary, 1996.

[45] L. Vandenberghe and S. Boyd. Semidefinite programming. *SIAM Review*, 38:49–95, 1996.

[46] H. Wolkowicz. Some applications of optimization in matrix theory. *Linear Algebra and its Applications*, 40:101–118, 1981.

[47] Q. Zhao, S.E. Karisch, F. Rendl, and H. Wolkowicz. Semidefinite programming relaxations for the quadratic assignment problem. Technical Report 95-27, University of Waterloo, Waterloo, Canada, 1996.